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Psychology of Mathematics Education

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Preface

It has been our great pleasure and honor to organize the Thirty-Third Annual Conference of the Psychology of Mathematics Education North American Chapter in Reno, Nevada. We have a great selection of speakers and papers included in these proceedings that include researchers from Canada, the United States, and Mexico. In addition, several researchers from countries in other parts of the world, such as Turkey, England, and Australia, will present their work at the conference.

This year's conference theme is "Transformative Mathematics Teaching and Learning." The plenary speakers were chosen to address this theme. Research should not only address issues in mathematics education; it should also help us transform mathematics teaching and learning. Plenary speakers include Megan Franke from the University of California-Los Angeles with Discussant James Middleton from Arizona State University, David Reid from Acadia University in Nova Scotia with Discussant Carolyn Maher from Rutgers University, and Marty Simon from New York University with Discussant Douglas Clements from the University of Buffalo.

We would like to thank all authors for contributing papers, as well as the reviewers for taking the time to carefully read the proposals and provide constructive feedback and professional opinions. We would also the University of Nevada, Reno faculty and graduate students, as well as Extended Studies at the University of Nevada, Reno (Shera Alberti-Annuzio, Sheehan Niethold, and Martin Morales), who assisted in conference preparations. Finally, we would like to thank the University of Nevada, Reno for hosting this year's conference.

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TEACHER FOLLOW-UP: COMMUNICATING HIGH EXPECTATIONS TO WRESTLE WITH THE MATHEMATICS

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In-depth analyses of six elementary mathematics classrooms (with varying student achievement) captured student and teacher interactions and documented the critical role of teacher follow-up in shaping opportunities for low-income students of color to engage in algebraic reasoning. This paper details the efforts of teachers as they, contrary to existing school practices, found ways to support student engagement around explanations. Examining the variation in teacher follow-up that existed across classrooms illuminated how one high-achieving classroom teacher's use of follow-up communicated a set of high expectations around mathematical work for students and supported them to wrestle productively with the mathematics.

Across a number of studies, Cognitively Guided Instruction (CGI) research documents the relationship between teachers' knowledge of student thinking and student mathematical achievement. We have found that teachers draw upon their knowledge of students' mathematical thinking as they pose problems, encourage the use of a range of strategies, and support students to share their ideas. We also have evidence of variability in the ways teachers engage in these practices, and we are interested in how that variability relates to student outcomes. We believe that knowing more about the details of teachers' classroom practice within the context of student outcomes can help both to elaborate Hiebert and Grouws' (2007) argument for allowing students to "wrestle" with the mathematics and to address our concern of the kinds of opportunities provided for marginalized students.

Efforts to detail the aspects of teachers' mathematical classroom practice that support the development of mathematical proficiency are not new (see Franke, Kazemi & Battey, 2007; Lampert, 2001; Stein, Engle, Smith & Hughes, 2008; Wood, 1998). For example, researchers have studied how posing rich mathematical tasks, allowing students to solve problems in different ways, and encouraging students to share their strategies may foster learning (Franke, Kazemi & Battey, 2007; Hiebert & Grouws, 2007) or how teachers might orchestrate sharing of student explanations and use students' explanations to highlight a mathematical goal (Lampert, 2001; Forman & Ansell, 2002; Forman, Larreamendy-Joerns, Stein & Brown, 1998). And while evidence shows these practices are not likely in many classrooms, they are even less likely in classrooms of low-income students of color (Anyon, 1981, Ladson-Billings, 1997; Lubienski, 2002; Means & Knapp, 1991). Our project builds on these studies and focuses particularly on tying the classroom practices of teachers working to make use of students' mathematical thinking to student outcomes.

Teacher Practice and Student Learning

This study was a follow-up to a larger experimental study that found teachers who had participated in one year of CGI algebraic thinking professional development knew more about students' algebraic thinking and the students in their classroom performed better on algebraic reasoning assessment items than a comparison group (Jacobs, et al., 2007). Yet, even within the CGI professional development group, variability existed; thus we wanted to understand what teachers were doing in classrooms with higher student achievement to support student learning that may not have been occurring in the classrooms of teachers in the same professional development but with lower achievement outcomes. The teachers of this project were second or third grade teachers with student achievement on our algebraic thinking measure that was consistently high, low, or on average fell to the middle range. We then observed in these classrooms while the teachers were engaged in working on relational thinking, doing so in a way that captured the teacher's interactions with students in whole class settings, as well as all of the talk of twelve randomly selected target students (see Webb et al., 2008, 2009 for details).

The schools participating in this study served predominately low-income students of color and are considered some of the lowest performing in the state (99% Latino and African-American students combined in each school, over 98% students receiving free or reduced lunch, 49-64% English Language Learners, and scored 1-3 out of 10 on California's Academic Performance Index). The district was expected to: use curricula that focused on developing skills over conceptual understanding, "drill" rather than engage in mathematical conversation, and target the students "on the bubble" of proficiency to raise test scores. However, this characterization of the schools only reflects one aspect of the district. At the time of this work, the local district leadership was working with the schools to meet a new vision for developing mathematical understanding, showing what students were capable of, and creating school learning communities. The schools involved in the project had been working hard to meet the needs of students in a variety of ways. The elementary school principals all wanted to participate in our work, seeing algebraic thinking as critical to their students' success and believing that it was appropriate content to take on.

We engaged schools in site-based professional development focused on relational thinking[1], providing professional development first for one group of schools and the following year for a comparison group. Our experimental study showed that the students in both the algebraic thinking group and the comparison group thought relationally and were able to solve a range of algebraic thinking problems. While not all students in the schools understood mathematics in ways we would hope, this study showed that students across the district were capable of engaging in algebraic thinking and the schools were willing to support students in doing so.

Understanding that working on algebraic thinking in the ways we suggested through professional development was contrary to many existing practices, we conducted this follow-up study of six classrooms to understand in detail exactly how teachers were finding space to create opportunities for students to engage in algebraic reasoning.

Classroom Practice

Our observations of these six classrooms found that on the surface they looked much like what we would expect in a CGI classroom. The teachers posed problems that often encouraged more than one strategy, asked students (98% of the time) how they solved the problem, and asked for elaboration of student thinking (76% of the time). These practices are not what we would expect more typically in elementary classrooms, and we would argue that these practices are important for creating opportunities for students to engage in wrestling with the mathematics. These practices alone, however, did not lead to productive student outcomes for all students in these classrooms (See Table 1), suggesting that closer examination of the ways in which teachers engaged students and content was needed to understand opportunities for student learning.

Table 1. Mean student achievement[a] across classrooms

	Group 3 (low-achieving classrooms)		Group 2 (medium-achieving classroom)	Group 1 (high-achieving classrooms)		
	Guo	Collins	Gomez	Lane	Lee	Theissen
Mean	18	27	50	60	75	88
(SD)	(30)	(35)	(46)	(46)	(35)	(25)

[a]Percent of algebraic reasoning posttest problems correct.
Note: Differences across classrooms: $F(5,68) = 5.91$, $p<.001$.

Problems Posed

Looking more closely at the problems teachers posed – taking into account that they were all working on common CGI algebraic reasoning content of relational thinking – we did find that there were differences in the problems posed. If we focus on one teacher from each of the achievement groups—Ms. Guo, Ms. Gomez, and Ms. Lee—we can examine closely how the problems posed were the same and different. Ms. Guo posed the most mathematically powerful sequence of problems that built toward a relational understanding of the equal sign. The majority of Ms. Lee's problems were drawn from ones the students had written. She chose problems that provided opportunities to use relational thinking, but without a particular sequence except when she added a problem to build from student responses. Ms. Gomez's problems often did not lend themselves to using relational thinking, as they focused mainly on multi-digit computation. An examination of the problems teachers posed across classroom observations is one approach to understanding opportunities for student mathematical learning. Yet, the teacher with the "best" problem sequence had the lowest achieving students, suggesting that while the tasks differed in the mathematical opportunities provided for students, they alone did not determine student outcomes.

Student explanations. As we looked further at what was occurring for students in these classrooms, we focused on the explanations students provided to the problems posed. We tracked student answers and explanations, given both to other students and to the teacher, within the whole-class setting and in small-group or pair-share settings. Students were participating in giving explanations in each classroom and for a high percentage of problems. (see Table 2).

Table 2. Overall level of student participation

	Group 3		Group 2	Group 1		
	Collins	Guo	Gomez	Lee	Lane	Theissen
Percent of target students who ever gave an explanation or answer	85	100	100	100	87	83
Percent of problems in which students gave explanations	64	80	88	100	86	70

It was not just the giving of explanations that differentiated classrooms, but rather the kinds of explanations that were given. The degree to which the explanations were complete and accurate varied across classrooms (see Table 3).

Table 3. Work contributed by students during the whole class: Percent of classwork problems with student contributions of each type

	Group 3		Group 2	Group 1		
	Collins	Guo	Gomez	Lee	Lane	Theissen
Correct/complete explanations	55	40	88	100	86	70
Incorrect/incomplete explanations only	9	40	0	0	0	0
Correct answer but no explanation	36	20	13	0	14	30
Incorrect answer but no correct answer or explanation	0	0	0	0	0	0

Note: Numbers in a column do not always sum to 100, due to rounding.

We found that students who gave complete, correct explanations scored better on the student achievement measures designed to measure algebraic thinking (see Table 4). We have examined this finding in a number of ways and each time we find the same pattern: it is not the mere act of explaining that is related to positive student achievement, but rather the giving of correct and complete explanations.

Table 4. Correlations between Student Participation and Achievement Scores

Highest level of student participation on a problem[a]	Achievement Score[b]
Gives explanation	.52***
Correct and complete	.59***
Ambiguous, incomplete, or incorrect	-.01
Gives no explanation	-.41***

[a]Percent of problems in which a student displayed this behavior. Problems discussed during pair-share and whole-class interaction are included. [b]Percent of problems correct. (n=74)

There are two critical points to be made here. First, "correct and complete" is not simply about making an explanation that gets to an end. Correct and complete explanations are about students articulating the steps in their solution in a way that details the mathematical relationships. In explaining a solution to 24 + 19 = 25 + ___, rather than saying the answer has to be 18 "because of the 24 and 25", the student would say (or point to) "24 is one less than 25 so whatever you add to 25 has to be one less than 19". The latter articulation makes explicit the particular relationship between the 24 and 25 and thus the 19 and the unknown. In complete explanations students are articulating in a way that as one listens and watches they can tell exactly what the student did to solve the problem. Second, as this is correlational data one might argue that students in Ms. Lee's class simply knew the math better and that is why their explanations are complete and correct. While we have no data that supports that these students came in knowing more, it is a possibility. What makes it less likely as an argument is that as we examined what teachers did to support students to make correct and complete explanations we found clear differences within and across classrooms. As the data we share will show, when teachers followed up in particular ways, it was more likely that students would produce correct and complete explanations.

Engagement around Explanations

Teacher Follow-Up

As we continued our examination of classroom practice across classrooms with a range of student achievement, we looked closely at the interactions among students and teachers as students gave explanations. We wanted to know how teachers were supporting students to give complete and correct explanations. We focused specifically on teacher follow-up that resulted in the articulation of more student thinking, whether their initial explanations were complete, incomplete, or ambiguous. The teachers with the highest achieving students followed up in ways that led to more explanation being given by the student (see Table 5).

Table 5. Experiences of target students when engaging with the teacher around their contribution during the whole class: Percent of target students who experienced each type of student participation

	Group 3		Group 2	Group 1		
	Collins	Guo	Gomez	Min	Lane	Theissen
More student thinking was provided beyond the initial explanation[a], engagement ended with a correct/complete explanation	8	18	17	92	60	83
More student thinking was provided beyond the initial explanation[a], but the engagement never led to a correct/complete explanation	0	45	17	0	0	0
No more student thinking was provided beyond initially correct/complete explanation	31	55	25	8	33	8
No more student thinking was provided beyond initially incorrect/incomplete/ ambiguous explanation	15	64	0	8	0	0
No more student thinking was provided beyond the student's correct answer	69	27	9	23	60	17
No more student thinking was provided beyond the student's incorrect answer	46	36	0	8	7	0

[a]In most cases, the target students themselves provided the additional thinking; in a few cases, another student in the class provided the additional thinking.

Looking closely at the follow-up, we expected to see that the teachers supporting more complete and correct explanations would be using particular approaches to do so – more probing questions, revoicing, providing tools, and so on. We did not find this to be true. All of the teachers drew upon a repertoire of follow-up moves. It was how they used these moves in relation to the students and the mathematics that seems to support students to provide more explanation, take an incorrect explanation to correct, or an explanation from incomplete to complete.

Returning to our one focus teacher per group, Ms. Guo, Ms. Gomez and Ms. Lee, we found that not only did they draw upon a repertoire of follow-up moves, but that the substance of their follow up was quite different. We examined these interactions in detail to gain insight into the different experiences students had as they solved problems and gave explanations (Franke et al, 2009).

Ms. Guo, the teacher with the lowest achieving students, posed productive problems, asked students how they solved the problems, and while she followed up on student thinking less often than the other teachers, she did follow up. For 45 percent of her students, however, her follow up did not lead to more explanation. Often this was due to her not pressing the student idea and posing a new problem or to confusion about what she was asking.

Problem: a= a; True or False?

Marcus: It's true, true true.
Lindsay: It's a letter. It's an alphabet letter. It's true because A is the same as A.
Marcus: It's true for everything.
Ms. Guo: You guys think that this one is true?
Marcus: Yes.
Lindsay: Yes. Because A and A is the same letters.
Ms. Guo: So they are the same letters, so that makes it true?
Lindsay and Marcus: Yeah.
Ms. Guo: We'll check it out.
Marcus: Everything is true, huh?

Here Ms. Guo followed up on Marcus and Lindsay's thinking and even asked a question that was specific to what students had said. However, she did not press either of the students to articulate what they mean by "A is the same letter as A" and then what that means for the problem they were solving. We do not learn more about either Lindsay or Marcus' thinking as they engaged with Ms. Guo.

In addition, Ms. Guo often followed up but rather than press the student, she revoiced or posed a new problem for the students to consider. In the next example Ms. Guo posed 100 + ☐ = 100 + 50 as a follow-up to a previous unresolved discussion. The class had begun to refer to this idea as "partners", that is, 50 is "partnered" with the other 50. Here Fernando used his notion of needing the same numbers on both sides to argue that 50 goes in the box.

Problem: 100+ ___ = 100 + 50

Ms. Guo:	Ok. One hundred plus box over here is the same as…one hundred plus fifty. What would go inside that box now and why? Fernando?
Fernando:	One hundred…no, the fifty will go right there because it has to be the same number.
Ms. Guo:	What has to be the same number?
Fernando:	One hundred fifty and the other side has to be one hundred fifty too *(unclear whether Fernando is saying 150 or 100 and 50)*.
Ms. Guo:	Ok, so you are adding these together. You said this side has to be *(bell rings)* one hundred fifty?
Fernando:	No.
Ms. Guo:	Oh, what were you saying?
Fernando:	That they have to be those because… 'cause it has to have the same numbers.
Ms. Guo:	It has to have the same numbers. Ok.

Fernando: Fifty, fifty, and one hundred.
Ms. Guo: Ok, this side has a fifty and this side has a fifty. Ok. I see that relation going across. But what if I did this though? What if I did this? If I called…I did one hundred and I called box, B for box. And if I did one hundred plus (*writes: 100 + B = 100 + 75*)…what would B have to be?

It was never clear if Fernando was considering numbers on each side as equivalent quantities across the equal sign or matching to find the same number in the number sentence. Ms. Guo followed up by posing another problem, but not one that would necessarily clarify the mathematical issue of "partners". This issue came up a number of times as the lesson progressed, causing difficulty as students began to match numbers regardless of the position in the number sentence (such as a + a = b + b being always true because "the a's have partners"). The issue is never resolved.

Ms. Gomez, whose problems were often focused on a computational solution and not related to the algebraic thinking, followed up about half of the time and half of that follow-up led to more explanation. Ms. Gomez asked follow-up questions directed at the mathematics, but she did so to help the student through the solution that they could not complete. So the follow-up, while framed as a question, was structured to help the student see the next step rather than to elicit more student thinking. Here Ms. Gomez stopped in while Miguel and Maricela were working on the problem.

Problem: 14 ÷ 2 = 3 x __ + 1

Miguel: Three times…
Ms. Gomez: You are working on number one right?
Miguel: Three times one…
Ms. Gomez: Ok, but you are working on number one… So remember, we always ask ourselves, which of the two sides is complete. The left side or the right side?
Miguel: Left side. *(pointing on paper with pencil)*
Ms. Gomez: Ok, so can you solve the one that's complete? *(Miguel nods)* So, do you think that's a good idea if you solve that first? *(Miguel nods)* Ok. *(Miguel looks up at teacher)* So what does that tell you?
Miguel: That fourteen divided by two equals, is the same as seven?
Ms. Gomez: Ok, very well read. Now so, if this side equals seven, what does this side… Oh my, you've already solved it.

Ms. Gomez wanted to help Miguel with an approach that starts with solving one side of the equation and then using that to help you figure out the other side. This is an example of a problem Ms. Gomez posed that does not readily lend itself to relational thinking, so her approach of solving one side may be best. She did not, however, ask Miguel how he was thinking about the problem nor did she ask him to think about the whole problem, what it is asking and what he knows in relation to what it is asking. If Ms. Gomez's approach made sense to you as a student, it could be very helpful; if not you could be lost. Ms. Gomez did not leave incorrect strategies unaddressed, yet she often did not ask any additional questions when a correct and complete explanation was given.

Ms. Lee, who chose productive relational thinking problems from those her students had written, regularly followed up on students' explanations and regularly supported students to provide additional explanation. She followed up on the mathematical ideas embedded in the student explanations and could always find something productive in what the student had offered, using it to ask a follow up question. Her questions took into account both the mathematics and the student's thinking. Ms. Lee had posed the problem, 11 + 2 = 10 + __. During the whole-class share she followed up on Andrew's explanation by asking him a clarification question and then specific to his explanation as well as the mathematics.

Problem: 11 + 2 = 10 + __

Andrew: I put eleven here, put a two right here, then I plussed it, and it was thirteen. I put take away… take away (*holds fingers up*). I wrote thirteen right here. I put right here a caret and put thirteen. I put here a thirteen. Three, thirteen take away, take away ten and then I minus, I minus (*counting on fingers*)… ten. And then *(inaudible).*

Ms. Lee: Wow. Okay. This is really interesting. Okay, let's look at this. Does everybody understand how you got eleven plus two equals thirteen?

Students: Yeah.

Ms. Lee: Okay. Why did you minus ten? And where did you get that ten from?

Andrew: Cause on the three I added first… I went three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen (*inaudible*).

Ms. Lee: Okay, is there anywhere else in the problem that has a ten?

Andrew: Yes here (*points to the 10 to the right of the equal sign*).

Ms. Lee: There. Okay, does anyone see that connection?

As she followed up with Andrew, she asked a question particular to his explanation and to the mathematics within the problem that has the potential of encouraging him to think relationally. She also worked to connect their interaction to the whole class.

Ms. Lee also, more than any other teacher, asked extension questions about the mathematical properties. In sharing his strategy D'ante stated that he knew when he took zero from 30 it would still be 30. Ms. Lee followed up on this particular idea.

D'ante: Because thirty take away zero is still thirty.

Ms. Lee: Okay, is that true? Is that true that thirty take away zero is still thirty?

Students: Yes. *(a small number of students)*

Ms. Lee: How do you know that's true?

Student: When you have thirty and you don't take away nothing, so it's still thirty.

Ms. Lee: Can you say it louder so everyone can hear you?

Student: Because you have thirty fingers and you don't take away nothing so it's still thirty.

Jennifer: *(loudly, to herself)* 'Cause you don't take away nothing. You have nothing to take away.

Ms. Lee: Is that always true?

This conversation continued for over five minutes and then Ms. Lee took it back to the problem at hand, connecting to why it would help you to use this idea. This type of follow-up was typical in her classroom. Following up on the students' ideas, while following up on a relevant math idea, and pushing beyond the solution given occurred in the highest achieving classes.

Another way to consider the follow up that occurred in Ms. Lee's class is to think about the way that her follow-up sets up expectations for students and the effect this has on opportunities for mathematics learning. In their studies of teachers of African American students, Clark and his colleagues (2009) found that teacher expectations stood out as a major factor in shaping the mathematics learning of students. Battey and Stark (2009) showed how deficit beliefs created impoverished mathematical practices for Latino and African American students. Ms. Lee's actions reflect her belief that students are capable of high-level mathematical work and thus expected to participate as such. Ms. Lee demonstrated follow-up (in ways similar to what Kazemi and Stipek (2001) term "press") to each students' ideas in a way that communicated these high expectations. Ms. Lee did not shy from addressing the mathematical properties or working with variables with her second and third graders. She had the students write the problems they solve together. She asked follow-up questions to everyone (each student shared an explanation at some point across two class sessions), questions that pushed them to articulate their mathematical ideas but in a way that was connected to their own ideas. Ms. Lee found something positive –mathematically—in what each student shared and used it to move the group's mathematical work forward. A growing literature suggests that this focus on what students can do, positioning students competently, matters and matters often for the lowest achieving students (Battey (under review); Boaler & Greeno, 2000; Boaler, 2003; Empson, 2003). In addition, she expected and supported students to do this with each other. This is not only about holding high expectations; it is a set of actions that communicate those expectations.

Student Follow-Up

We are finding that students do the same types of follow-up that their teachers do when they are in small groups or pairs together. In Ms. Guo's class the students in pair share frequently discussed ideas for solving the problem, but they did not complete the solutions nor did they work to make sense of each other's ideas. The goal seemed to be to get ideas out on the table, much like Ms. Guo did, with little to no follow-up to further detail these ideas or move them toward completion. Ms. Gomez's follow-up in both pair-share and whole-class discussions communicated an expectation of completing the problem, often with a particular procedure that the class had been working on. Rarely were there interchanges among students that moved beyond working through a procedure to involve probing of explanations. Students in Ms. Lee's class exhibited further probing of each others' ideas as they shared explanations and worked toward finding more than one way to approach the problem. The ways in which students engaged with each other in pair share reflected the kinds of opportunities created as Ms. Guo and Ms. Gomez engaged students.

As we further examined the interactions in Ms. Lee's class, however, we recognized that there were other potentially productive outcomes occurring through these interactions beyond the sharing of explanations. Students were learning a way of participating in mathematics that included: sharing their ideas to a greater degree of detail, listening and engaging with someone else's ideas, asking mathematical questions based on another's strategy, understanding someone else's ideas in relation to their own, and so on. The following interchange demonstrates both how students follow up like their teacher (Ms. Lee) and how they are learning a range of things beyond how to correctly solve a mathematics problem.

In this example Ana and Michael were working during pair share on a problem posed to their class: 8 + 8 = 15 + ☐. Michael thought the answer to the problem was 15 (likely due to a similar "partner" misunderstanding as in Ms. Guo's class). Ana, who knew the answer was one, worked to get Michael to see that he was not thinking about it correctly.

Problem: 8 + 8 = 15 + ☐

Ana:	It's not... it's not... it's not fifteen, because eight plus... I'll explain it back to you so you can understand that it's one. Because eight plus eight sixteen…
Michael:	I already tried two times.
Ana:	Huh?
Michael:	I already tried two times.
Ana:	It's not... it's not fifteen, because eight plus eight is sixteen. So this one has to... has to equal sixteen—"the same".
Michael:	*(pause)* The equal sign means "the same" and there's eight (*trails off*)...
Ana:	But it, the equal sign means that "the same answer"… It has to be the same answer. Like if... if I put... (*pause)* I could switch it: fifteen plus one is the same as eight plus eight.
Michael:	OH! I know what you're talking about.
Ana:	It's just gonna be sixteen.
Michael:	So it's worth sixteen right here (*gestures to one side of paper)* and sixteen right here *(gestures to the other side).* Ohhh!

Ana's first move to challenge Michael's incorrect idea was to explain that the expressions on both sides had to equal sixteen, but that explanation did not make sense to him. She emphasized the language "the same." Michael agreed with the idea of "the same", but this did not resolve his incorrect answer. Ana persisted. She added that it was not only "the same" but it was "the same answer." As that explanation did not convince Michael, she made yet another attempt to get her point across: she flipped the number sentence around to say "fifteen plus one is the same as eight plus eight." It was this specific move that helped Michael understand the quantity sixteen on both sides of the equal sign, moving him away from his incorrect answer of fifteen.

Ana helps Michael with a way to think about the solution that made sense to him. You can see him gesture in a way that shows you can move the 15 + 1 together and the 8 + 8 together and they are the same because they are both just 16. This interchange exemplifies the impressive caliber of mathematical discourse that occurred in Ms. Lee's classroom. Ana explains; Michael listens and works to make sense. Ana listens to Michael and tries a different approach she thinks might be more successful. Ana finally makes a mathematically powerful move that helps Michael make a connection about the quantities on both sides of the equal sign. There is a sense in watching this interchange that Ana is confident that she can communicate with Michael, and she has the skills and tools to do so. The types of interactions in this exchange occurred in other pairshare episodes in Ms. Lee's class. It is though an interchange that contrasts significantly those in Ms. Gomez's and Ms. Guo's classrooms.

Summary

Freund (2011) found in examining the aspects of mathematical proficiency developed within theses classrooms that developing the types of classroom norms seen in the higher achieving classrooms was not a matter of simply telling kids to work with each other or explain their thinking (all of the teachers did this). It was a combination of norms statements as well as communication of expectations through interaction that determined what constituted engagement around a mathematical explanation (Cobb et al, 2001). We can see through the work shared here and Freund's study that teachers and students together are developing a set of norms that shape participation in ways that allow students to make complete and correct explanations and create a culture of what it means to do mathematics together.

Ms. Guo, Ms. Gomez, and Ms. Lee serve as exemplars of what occurred across the six classrooms with differing student outcomes. Each of these teachers engaged their students in mathematics that countered the prevailing district culture. Each of these teachers was working to reach each of their students and engage them in developing an understanding of algebraic ideas often not addressed in second and third grade and certainly not addressed in the lowest performing schools at these grade levels. Each of these teachers carried out practices that research would contend are productive for developing mathematical understanding. What these three cases do is help us understand the nuance of Ms. Lee's interaction with her students and how that is related to student outcomes. Ms. Lee through her follow-up interactions connected with students' mathematical ideas and pressed students to detail their ideas for themselves and the class. She expected her students to participate in high-level mathematics and worked with them so they would know how.

Conclusions

In understanding the classroom practices that support student achievement in mathematics the data from this project suggest that posing productive mathematics problems while necessary, is not sufficient. Asking students the initial "how did you solve that" is necessary, but not sufficient. Following up on the students' explanation is necessary, but not sufficient. The way one follows up matters. It matters that the follow up presses students to provide more explanation and the explanation gets to something that is complete and correct. It matters that the follow-up is connected to what the student has articulated or shown as well as to the mathematical ideas being addressed in the lesson. It matters that, as the teachers are following up, they are supporting students to learn a way of engaging with the mathematics and creating a culture of doing mathematics that signals that wrestling with the mathematics means asking each other questions, considering others' ideas, and connecting mathematical ideas. It matters that teacher and student interactions create a set of expectations and a way of positioning students that make explicit that each child brings mathematical knowledge to the interactions and can engage in challenging mathematical work. It matters that the interactions highlight persistence, detailing thinking, listening, questioning, as well as the process of getting stuck and unstuck. We would argue that this type of work constitutes creating opportunities to "work and wrestle" with the mathematics in ways that set expectations through action for each of her students.

Endnote

1. Relational Thinking involves children's use of fundamental properties of operations and equality to analyze a problem in the context of a goal structure and then to simplify progress towards this goal. (Carpenter, Franke, & Levi, 2003; Empson, Levi, & Carpenter, 2011)

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UNDERSTANDING PROOF AND TRANSFORMING TEACHING

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This article reviews some key ideas about the nature of proof, kinds of proofs, and the way of reasoning called proving. This provides background for a discussion of the teaching of proof, in which some important aspects of teaching that support students' proving are described, and the ideas of a tool-box of accepted premises and local organization of knowledge are explored as ways to structure teaching. Finally, lessons learned from approaches to teaching proof and problem solving in the past are connected with the earlier discussions of understanding and teaching proof to suggest a transformation of proof teaching into proof based teaching, in which proof is a process of coming to understanding rather than a topic to be taught and learned.

The Reasoning and Proof Standard (National Council of Teachers of Mathematics [NCTM], 2000) advocates that all students should recognize proof as fundamental to mathematics, should read and write mathematical proofs, and should reason in various ways, including the way called *proving*. In this article I will consider what these goals might mean, and how they might be achieved. I will begin by considering the nature of proof and four ways in which it is fundamental to mathematics: as a means to verify, explain, discover and belong. I will give some examples of proofs and compare their formats and functions. And I will examine the types of reasoning involved in conjecturing and proving. This will provide the background for a consideration of the teaching of proof. I will describe aspects of teaching that seem to create a good context for proving in school, and two ideas, a tool-box of accepted premises and local organization of knowledge, that have potential as ways to structure teaching. Finally, I will bring together my discussions of the nature of proof, proofs and proving and the teaching of proof to suggest a possible role for proof in mathematics teaching, as a way to teach rather than a topic to be taught.

Proof

What does it mean to recognize proof as a fundamental aspect of mathematics? How is proof fundamental? It is clear that proof is essential to the work of professional mathematicians. Papers without proofs do not get published. But why are proofs needed in mathematics?

Verification

Fischbein and Kedem (1982) say that proofs are needed because "a formal proof of a mathematical statement confers on it the attribute of a priori universal validity" (p. 128). In other words, proof tells us what is true and what is not. This idea, that mathematical proofs verify mathematical statements, has been recognized and admired for a long time. For example, Descartes commented:

> Of all those who have already searched for truth in the sciences, only the mathematicians were able to find demonstrations, that is, *certain* and evident reasons. (Descartes, 1637/1993, p. 11, italics added)

Mathematical proofs provide certainty. That is one answer why they are needed and why proof is a fundamental aspect of mathematics.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

While many people believe that the essential role of proof in mathematics is verification of theorems, there are objections to this position. Some are fairly abstract, like the observation that Gödel's Incompleteness Theorem tells us that there are true statements in mathematics that have no proofs. Others are completely pragmatic, like Davis's (1972) comment that "most proofs in research papers are unchecked other than by the author ... They are loaded with errors." (p. 259). Obviously proofs with mistakes in them cannot tell us what is true and what is not in any absolute way. Davis concludes that "A derivation of a theorem or a verification of a proof has only probabilistic validity." And there are objections based on looking at how mathematicians actually behave, as opposed to what they say they do. Gila Hanna (1983, p. 70), for example, lists five factors other than a mathematical proof that contribute to the acceptance of theorems, including the reputation of the author and the plausibility of the result.

Explanation

There are other roles proof plays that make proof fundamental to mathematics. "Proof, in its best instances, increases understanding by revealing the heart of the matter." (Davis & Hersh, 1981, p. 151). Proof helps us understand and explain mathematics. Especially in the context of education, this explanatory role has increasingly come to be seen as vital. Gila Hanna was perhaps the first to point out to mathematics educators, at PME 13 in Paris (1989, p. 2-45), that proofs that explain, that show not only that the statement is true but also why it is true, should be favored in mathematics education.

To understand this distinction, consider these two proofs, both of which show that the sum of the first n integers $S(n)$ is $n(n+1)\div 2$:

Initial step
For n = 1 it is true since $1 = 1(1 + 1) \div 2$.

Induction step
Assume it is true for some arbitrary k, that is,
$S(k) = k(k + 1) \div 2$.

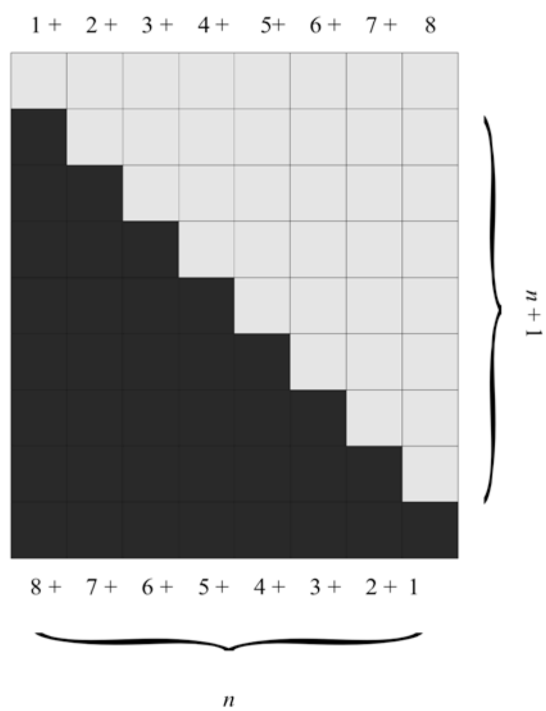
Then consider

$$\begin{aligned} S(k + 1) &= S(k) + (k + 1) \\ &= k(k + 1) \div 2 + (k + 1) \\ &= (k + 1)(k + 2) \div 2 \end{aligned}$$

Therefore, if the statement is true for k it is true for $k + 1$.

Hence, by mathematical induction, the statement is true for all n.
(adapted from Reid & Knipping, 2010, p. 99)

Proof 1



Proof 2

Both proofs require some unpacking, but of a very different kind. To understand Proof 1 the reader must be familiar with mathematical induction as a proof technique, and even then if the technique has been taught formally, without first establishing a link to informal ways of reasoning recursively, the whole process can seem entirely arbitrary. In that case a proof like Proof 1 can be not only non-explanatory, but also unconvincing. Proof 2 offers an example. It does not show that $S(n)$ is $n(n+1)\div 2$. All it shows is that $S(8)$ is $8(8+1)\div 2$. The reader must see that this specific example is in fact a generic example; that the same process would work for any n. But once this is grasped, the proof is explanatory. It gives answers to questions like "Why do you divide by 2?" and "Why do you multiply by $n+1$?"

Discovery

Proof is also fundamental to mathematics as a way of discovering new knowledge. In mathematics education de Villiers seems to have been the first to point this out. In 1990 he noted:

> Proof can frequently lead to new results. To the working mathematician proof is therefore not merely a means of a posteriori verification, but often also a means of exploration, analysis, discovery and invention. (p. 21)

The role of proof in the discovery of non-Euclidean geometries is an historically significant example of proof as a means of discovery. Here is a more simple example:

There are many ways to prove that the sum of any two consecutive odd numbers is even. The simplest is to note that the sum of any two odd numbers (consecutive or not) is even. This verifies the statement and explains it. It is also possible to verify the statement with a proof using mathematical induction, if you believe in it, producing a proof that verifies but does not explain. You could use a generic example: 7+9 = 7+7+2, which must be even because 7+7 must be even and 2 is even. This proof both verifies and explains. Or you could do some algebra: $2n - 1 + 2n + 1 = 2(2n)$ which is even. The final proof lets you discover something more that you were trying to prove. $2(2n)$ is $4n$, so the sum of any two consecutive odd numbers is not only even, it is a multiple of four.

Being a Mathematician

Papers without proofs do not get published. That signals another role of proof in mathematics. Publishing proofs is part of being a mathematician. As Thurston (1995) points out:

> We are driven by considerations of economics and status. Mathematicians, like other academics, do a lot of judging and being judged. Starting with grades, and continuing through letters of recommendation, hiring decisions, promotion decisions, referees reports, invitations to speak, prizes...we are involved in a fiercely competitive system. (p. 34, ellipses in original) In our credit driven system, [mathematicians] also want and need *theorem credits* (p. 36, emphasis in original).

By "theorem credits" Thurston means the social acknowledgement that comes from publishing theorems, and in mathematics results must be published with proofs to count.

This social role of proof is usually implicit. As long as everyone conforms to the norm, it does not become evident. However, when a mathematician publishes without proving theorems, the reaction can be very strong. For example, when Benoit Mandelbrot named and began publishing images of fractals Steven Krantz (1989) published a critique in the *Mathematical Intelligencer* that focused on the lack of definitions, proofs and theorems in Mandelbrot's work. Without these, could fractal geometry be part of mathematics? The exclusion of non-Europeans from the history of mathematics, on the same basis that they did not prove their work (Joseph, 1991), provides another example.

Proofs

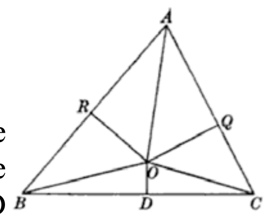
Let us turn now to proofs, as opposed to the concept of proof. What is a proof? The NCTM has a simple definition:

> By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments. (2000, p. 56)

This definition is not far from the first ones published in the North American mathematics textbooks:

> Every statement in a proof must be based upon a postulate, an axiom, a definition, or some proposition previously considered of which the student is prepared to give the proof again when he refers to it. ... No statement is true simply because it appears to be true from a figure. ... [In a proof] are set forth, in concise steps, the statements to prove the conclusion ... asserted. (Beman and Smith, 1899, pp. 19–20, cited in Herbst, 2002)

A proof is something like this:



Given an arbitrary triangle ABC we will show that AB = AC.

Construction

Let O be the intersection of the angle bisector of ∠BAC and the perpendicular bisector of segment BC. (In the case where these lines are parallel it is easy to show AB = AC. This is left as an exercise.) Draw OD perpendicular to BC, OR perpendicular to AB, and OQ perpendicular to AC. Draw OB and OC.

Proof

1	OD = OD	reflexive property of equality
2	BD = CD	OD bisects BC, definition of perpendicular bisector
3	∠ODB ≅ ∠ODC	both are right angles, definition of perpendicular bisector
4	△ODB ≅ △ODC	Side Angle Side congruence property
5	OB = OC	Corresponding Parts of Congruent Triangles are Congruent
6	AO = AO	reflexive property of equality
7	∠OAQ ≅ ∠OAR	AO bisects ∠A, definition of angle bisector
8	∠ARO ≅ ∠AQO	both are right angles, definition of perpendicular bisector
9	△RAO ≅ △QAO	Angle Angle Side congruence property
10	AR = AQ & OR = OQ	Corresponding Parts of Congruent Triangles are Congruent
11	∠ORB ≅ ∠OQC	both are right angles, definition of perpendicular bisector
12	OB = OC	from 5
13	OR = OQ	from 10
14	△ROB ≅ △QOC	Hypotenuse Leg congruence property
15	RB = QC	Corresponding Parts of Congruent Triangles are Congruent
16	AB = AR + RB	segment addition postulate
17	AR + RB = AQ + QC	from 10 and 15, addition property of equality
18	AQ + QC = AC	segment addition postulate
19	AB = AC [Q.E.D.]	transitivity of equality

Proof 3 (adapted from Maxwell, 1959, Chapter II, § 1)

Every statement in the proof is based upon a postulate, an axiom, a definition, or some proposition previously proven. The conclusions follow from the hypotheses by logically rigorous deductions. To make it clear that this is happening, the proof is arranged in two columns, one giving the concise steps of the proof, and the other providing the supporting postulates and so on. This is the famous two column proof format that was invented by two American textbook authors, Arthur Schultze and Frank Sevenoak in the early years of the twentieth century (Herbst, 2002, p. 297). According to Herbst, the two column proof format was an innovation that allowed teachers to better handle some of the challenges they faced. At that time a greater proportion of the population was attending secondary school and taking high school geometry, and at the same time the focus of high school geometry courses had shifted from the content of geometry to the process of proving. Explicit descriptions of what a proof should be (like that of Beman and Smith quoted above) and the general use of the two column format to present all proofs seen in schools, meant that the object of learning, proofs, was explicit. This meant that the teacher and students were clear about the object of the class.

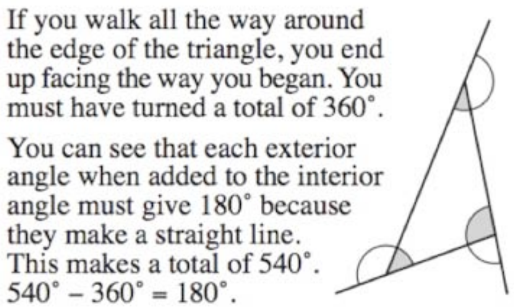
However, writing school mathematics proofs in the two column format means that they no longer resemble mathematicians' proofs. In mathematicians' proofs any step that the reader can be expected to supply is omitted, and explicit reference is made to supporting postulates, definitions and theorems only when they are unusual. If our goal is that "High school students should be able to present mathematical arguments in written forms that would be acceptable to professional mathematicians" (NCTM, 2000, p. 58) then we will have to start teaching written forms that are more like those of professional mathematicians.

Another disadvantage of the two column format is that it shifts the focus away from the statement being proven and towards the form and details of the proof. As a result, students are likely to believe that a proof like Proof 3 would be accepted, and even praised, by their teachers, in spite of the fact that its conclusion is false.

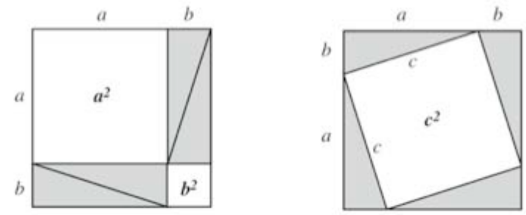
As well as illustrating the two column format, Proof 3 also allows us to revisit the question of the role of proof with a concrete example. Assuming you know that some triangles are not isosceles, you are unlikely to have been convinced by Proof 3. It does not verify anything. Nor is it particularly explanatory or useful for discovering new knowledge (unless you do find it convincing, and then it would lead you to the new knowledge that not only are all triangles isosceles, they are all equilateral, as the same reasoning can be applied to any two sides of the triangle.) What it might be useful for, from a student's point of view, is satisfying school norms for what a proof should look like. A student might expect, in the context of a tradition in which the two column format is highly valued, to be rewarded for producing such a proof. Proof 3 also illustrates a pitfall in Beman and Smith's description of a proof. They require that "No statement is true simply because it appears to be true from a figure." This reflects a general distrust of reasoning using pictures that was dominant in the twentieth century. But pictures can be extremely useful. The simplest way to see what is wrong with Proof 3 is not to go through it line by line looking for a missing or unjustified step. The simplest way to is draw a picture, as accurately as you can, following the steps of the initial construction.

As we have seen above, visual proofs can be both convincing and explanatory, especially in school contexts. Proof 4 (from Hoyles, 1997, p. 12) is another example, and one that works even better as an action proof if you trace a triangle on the floor and walk around it, paying attention to the way that your body turns.

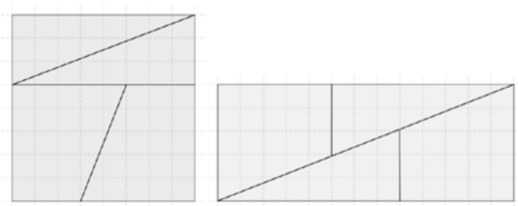
Another well known visual proof is Proof 5, of the Pythagorean Theorem. This kind of proof reflects a very old Chinese tradition of proving using dissections of shapes. Occasionally, such proofs are rejected because of the existence of false proofs like Proof 6, which shows that 64=65. But this makes no more sense than rejecting all two column proofs because Proof 3 exists.



Proof 4



Proof 5



Proof 6

Proving

I have discussed the concept of proof, the texts that are called proofs, and now I will turn to the activity of proving. As the NCTM notes "a mathematical proof is a formal way of expressing particular kinds of reasoning" (2000, p. 56). But what kinds of reasoning should be called *proving*?

We have seen that in the twentieth century there was an emphasis in schools on a form of proofs that made their deductive structure very clear. In addition to the ways in which two column proofs helped teachers, their emphasis on deduction may also have derived from the focus on foundational questions in the first part of the century, which provided axiomatic structures for all branches of mathematics. The work of Polya and Lakatos later in the twentieth century critiqued this focus on axiomatics and deductive logic, pointing out the historical importance of analogies and empirical approaches in mathematical discovery and conjecturing. Their critiques, along with the growing awareness that teaching proving in schools was not very successful, led some jurisdictions, notably England, to put a greater emphasis on pattern noticing and conjecturing. The NCTM's 1989 *Curriculum and Evaluation Standards* also placed conjecturing as an equal partner to verification by deductive reasoning. Hoyles (1997) points out that the emphasis on discovery and conjecturing in England meant that the majority of students saw nothing else. "The majority of students will engage in data generation, pattern recognition, and inductive methods while only a minority, at levels 7 or 8, are expected to prove their conjectures in any formal sense" (p. 9). Duval (1991) goes further, claiming that because arguments leading to conjectures and deductive proving "use very similar linguistic forms and propositional connectives," an emphasis on discovery and conjecturing "is one of the main reasons why most of the students do not understand the requirements of mathematical proofs." (p. 233). However, according to a group of Italian researchers, the relationship between conjecturing and proving is one of "cognitive unity" and their research has shown cases in which students built on what they had learned in making a conjecture to develop a deductive proof. Clearly, the relationship between conjecturing and proving deductively is unclear. However, it is clear that there is a useful distinction to be made between these two kinds of reasoning. The particular kind of reasoning that is called proving is *deductive*. So it is especially important to clarify what deductive reasoning looks like.

When reasoning about a familiar context, even quite young students can produce deductive arguments. For example, consider this argument in favor of the existence of the tooth fairy made by five year olds:

> Kim: My sister says there isn't even a tooth fairy. She says it's our mother.
>
> Teacher: What do you think?
>
> Kim: I think the fairy came in after my sister was asleep. Because my sister said I would get one dime and I got two dimes.
>
> Kenny: I got a dollar. My mom can't spend a dollar because we are saving money for a car. So it has to be the tooth fairy.
>
> (Paley, 1981, p. 40)

Kenny's deductions start with some prior knowledge: He knows he got a dollar for his lost tooth, and he knows that his mother is not spending any money because they are saving for a car. From this knowledge he deduces that his mother did not put the dollar under his pillow and so he rejects Kim's sister's conjecture in favor of Kim's, that the tooth fairy exists. Kenny does not include every step; he never says, "It can't be my mother". He also makes use of a hidden assumption that the only two possible sources of the tooth money are his mother and the tooth fairy. This is typical of children's deductions. Steps are skipped and hidden assumptions are made. But recall that mathematicians also skip steps and use shared assumptions without explicitly mentioning them. Kenny is far from producing a two column proof but he might not be so far from producing a proof acceptable to mathematicians.

Older children can link together simple deductions into chains in which conclusions of one step are used to justify subsequent steps. For example, consider Maya, age 11, who is explaining why the number of squares in a 10 by 10 grid is 10×10 + 9×9 + 8×8 + ... + 1 (Long dashes — represent pauses. Ellipses ... represent omissions).

> Maya: Can everyone see? So you count 1, 2, 3, 4, 5, 6, 7, 8, 9 right? … Since a square—this—any square—the square is 10 by 10 no matter how you turn it, it's always going to be…the same. So you don't have to measure it again. You can go 9 times 9. Do you understand why? Yeah? OK, So you go 9 times 9 like Gino said, 81. Then you can do 3 by 3, 1, 2, 3, 4, 5, 6, 7, 8—and then again you don't have to measure again you know. It's going to be the same. So 8 times 8—64. And you can keep on going.… You can do the 7 times 7—49. And 6 times 6—36. 5 times 5—25. 4 times 4—16. 3 times 3—9.
>
> (Reid & Knipping, 2010, p. 88)

First Maya counts how many 2 by 2 squares will fit along the top edge. Then she deduces that the number of 2 by 2 squares down either side will be the same, because the square is the same length on all sides. So the number of 2 by 2 squares is 9 times 9. For the 3 by 3 squares there are 8 along the top and the same reasoning gives 8 times 8. "And you can keep on going." The same reasoning gives the number of all the remaining sizes of squares.

Deductive reasoning is not something children need to be taught, although clearly contexts in which deductive reasoning is useful will encourage them to develop their reasoning further. However an important step toward proving is the formulation of that reasoning. It is necessary for students to become aware of their own reasoning and to explore ways to express that reasoning so that it can be understood by others. Maya is taking steps in that direction as she explains her reasoning to her classmates. In the next section I will discuss ways to support he development of students' proving in classrooms.

I have discussed the role of proof, some kinds of proofs, and the nature of proving. I would like now to turn to the theme of the conference, transformative mathematics teaching and learning, and consider the teaching of proof. How might the goals of the NCTM's Reasoning and Proof Standard be realized? In what contexts do students learn to prove? How might teaching be transformed by a renewed attention to proof?

Teaching proof

Efforts to teach proof in the twentieth century occurred either in the context of a high school geometry course, or, in the New Math era, in a unit on logic. The NCTM is clear about one thing we learned from this experience:

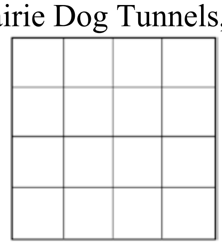
> Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in geometry. (2000, p. 56)

But how *should* proof be taught? In this section I will consider first five aspects of teaching that Vicki Zack and I identified as contributing to students' proving in her classroom and in others'. I will then turn to two ideas that I believe help to clarify what it means to teach proof.

In Reid and Zack (2009) we describe how five key aspects of Vicki's teaching contributed to her grade 5 students' proving. They are: a focus on problem solving, allowing sufficient time, encouraging conjecturing, expectations concerning the nature of students' communications, and the teacher's expertise in listening closely and seizing opportunities to provoke discussion.

Problem Solving

Problem solving was at the core of the mathematics curriculum in Vicki's classroom. Non-routine problems were integrated into the everyday mathematics lessons, and a problem solving approach was also taken to other areas of the curriculum. The Problem of the Week tasks were especially rich as contexts for proving. These differ from the other non-routine problem-solving tasks used in the class in that the children were asked to write detailed explanations at two or more points in the problem solving process. For the most part Vicki used the same problems year after year. This allowed her to make small adjustments in phrasing or in follow up tasks that further supported the development of the children's reasoning (Brown, Reid & Zack, 1998). Four of the Problem of the Week tasks have received the most attention in our research. They are Prairie Dog Tunnels, Handshakes, Decagon Diagonals and Count the Squares (see Figure 1).



Find all the squares in the figure on the left. Can you prove that you have found them all?
What if ... ? [A five by five grid is given as a figure]. Can you prove you have them all?
Extension: What if it were a 10 by 10 square? What if it were a 60 by 60 square?

Figure 1: The Count the Squares problem

The prompts for Prairie Dog Tunnels, Handshakes, and Decagon Diagonals are:

- Nine prairie dogs need to connect all their burrows to one another in order to be sure they can evade their enemy, the ferret. How many tunnels do they need to build?
- If everyone in your class shakes hands with everyone else, how many handshakes would there be?
- How many diagonal lines can be drawn inside a figure with 10 sides? [Figures were provided of a triangle, square, pentagon and hexagon, labeled: 3 sides, 0 diagonals; 4 sides, 2 diagonals; 5 sides, 5 diagonals; 6 sides, 9 diagonals]
  How many diagonal lines would there be in a 25-sided polygon?
  How many diagonal lines ... in a 52 sided polygon?

In these tasks the students used common techniques such as making organized tables, using diagrams, and searching for patterns in sequences and using differences. The first three problems involve finding the sum for the whole numbers from 1 to n. In solving them the equivalence of $(1+2+3+4+\ldots+n-1)$ and $n(n-1)/2$ was discovered through a combination of empirical testing and deduction. Empirical strategies such as drawing all the possible diagonals were used to establish correct answers when n was small. These gave way (as n increased) to reasoning deductively either that each person shakes one less hand than the previous person (the first shakes $n-1$ hands, the next $n-2$, and so on) or that each of the n people shakes $n-1$ hands but that counts each handshake twice so the correct number is $n(n-1)/2$. The fact that these two methods of reasoning both correctly solve the problem established their equivalence.

Generally, the students did not spontaneously use symbols. Generalizations were most often expressed as procedures tied to the structure of the problem, for example, "Multiply the number of people by the number of handshakes each person does, which is one less, and divide by two because you counted all the handshakes twice." Vicki then nudged the students to go further, asking them if they could express this idea by using a letter to represent any number of people shaking hands. In some years expressions like *n*(*n*–1)¸2 were written using variables. For example, Micky recorded this proof of a formula for finding the number of diagonals in a polygon:

> If you can find all the diagonals possible from one [vertex] you can figure out the whole amount of inside diagonal lines.... I know that a [vertex] connects with all of the other [vertices] except for 3, itself the [vertex] to the left and right. You subtract 3 from the amount of total sides, ... here's the rule: (Z = no. of sides)...Z – 3 ´ Z ¸ 2 = no. of diagonal lines in figure.
>
> (from Micky's Math Log, May 16, 1994, described in Zack, 1995)

Micky's proof has several steps. He is explicit about one: that the number of diagonals from each vertex is Z – 3 (because each vertex is connected by diagonals to all the others, except itself and its neighbors). However, he leaves the justification for another step implicit: Every diagonal is counted twice, so the formula ends with dividing by two.

Time

In Vicki's class the children were given a great deal of time to experiment with, think through, discuss and refine their understandings. Each Problem of the Week was assigned on a Monday. The students worked independently on the problem during class time (an extended 90 minute class period) and wrote a detailed description in their Math Logs. On Tuesday, she allowed time for the students to review their Logs if they chose to do so and at times prompted some students to reflect upon their explanation (for example, to clarify their thinking, elaborate on a diagram or add a diagram). Wednesday was the day the discussions took place, again in a 90-minute session. They worked first in pairs (or in a group of three), and then came together in a group of four or five. In these small groups they compared solutions and discussed further, and then reported to the half-class, with more discussion following. On occasion, if the discussions in the small group or the class warranted, class time on Thursday and Friday was also used to allow the discussions to come to a fruitful conclusion. This means that the exploration of the problems occurred over a significant period of time, almost four hours per task. This allowed for conjectures to be made and explanations sought without being artificially cut short by time constraints.

Conjecturing

The problems used allowed the students to make hypotheses and discover solutions, which they then proved in order to verify and explain. The processes of conjecturing and proving were intertwined in two ways. Proving made use of insights gained through the explorations that led to conjectures (For example, cut-out squares of different sizes were used to support the counting of squares, but they were also used in proving. See Maya's proving above and Zack, 2002). Conjectures were also used as the basis for proving (For example, one child conjectured that the number of squares would always be a multiple of 5, unless the size of the grid was a multiple of 3. He later rejected answers conjectured by his peers if they contradicted his general rule; see Reid, 2002).

Expectations

The groundwork for proving laid during the year included an expectation that the children would be looking for patterns, and that they could be nudged to think about the mathematical structure underlying the pattern (Zack, 1997). In addition, there were expectations that everyone's answers should be considered and that answers should not be changed without discussing how they arose and what might be the source of an error.

This valuing of clarity, explanation, and attention to others supported the expression of the students' thinking, whether it involved proving or not, but also made that thinking available to others to question. At the close of each session, Vicki distributed a sheet entitled "Helpful explanations/Helpful ideas" and asked the children to note any ideas or explanations they found helpful, to tell why, and to credit the peer(s) who helped. She found that the children became increasingly aware of the contribution others had made to their understanding, and at times could indicate how they have reshaped others' ideas to make them their own (See Zack & Graves, 2001 for further elaboration).

Expertise

In terms of her mathematical background, Vicki could be considered a typical elementary school teacher in that she describes her background in formal mathematics as weak. She is not an expert on the nature and role of mathematical proof, and has not formally proven anything herself since high school. However, her research interest in how meaning is constructed in dialogue led to a close look at the children's ways of expressing their ideas, and then in turn at issues of convincing and proving.

Working with the children's ways of making meaning is central to Vicki's teaching. Her expertise lay in listening closely, recognizing potentially fruitful avenues and seizing opportunities to provoke discussion. There was a constant expectation for explanation (e.g., asking "Explain how it works and why it works."), and for generalization (Will it always work? Can you construct a general rule?). Vicki discovered the richness of the mathematical tasks in large measure due to the children's ways of solving the problem. The tasks lent themselves to algebraic thinking and generalizing and proving. The proving arose from the tasks, even though there had been no 'a priori' objective to teach proof and proving.

The five aspects of teaching described above, problem solving, sufficient time, conjecturing, expectations for communication, and teacher expertise, seem to be important in creating a context for proving, even where no explicit intent to teach proof existed. However, in higher grades where the teaching of proof might become more structured, more attention must be paid to two key ideas: the tool-box, and local organization.

The Tool-box

Mathematicians' and children's proofs make reference to, or assume without stating, other theorems or assumptions. Things are taken for granted. Even in the work of professional mathematicians there are theorems that are used without their proofs being read, and even without any source being known.

> Within any field, there are certain theorems and certain techniques that are generally known and generally accepted. When you write a paper, you refer to these without proof. You look at other papers in the field, and you see what facts they quote without proof, and what they cite in their bibliography. You learn from other people some idea of the proofs. Then you're free to quote the same theorem and cite the same citations. You don't necessarily have to read the full papers or books that are in your bibliography. Many of the things that are generally known are things for which there may be no known written source. As long as people in the field are comfortable that the idea works, it doesn't need to have a formal written source. (Thurston, 1995, p. 33)

Netz (1999) discusses the omission of references to theorems and assumptions in classical Greek proofs. He calls the set of theorems and assumptions that can be used without comment the "tool-box". For example, Archimedes can assert that two segments are of the same length because they are radii of the same circle. He does not have to make any reference to this justification. He simply states that they are the same length and leaves it to the reader to figure out why (Netz, 1999, p. 172). As his readers were all members of a cultural community for whom the same tool-box was taken for granted, this was acceptable.

Similarly, Proofs 1, 2, 4 and 5, above, make use of tools from my tool-box which includes rules governing algebraic symbolization and manipulations, the idea of mathematical induction, and even implicit rules about using generic examples that have probably never been stated anywhere, but which are followed by those who consider such proofs acceptable.

Netz argues that the results proven in Euclid's Elements constitute most of the classical Greek tool-box. One might imagine that the study of the Elements was undertaken as the beginning of a scholar's mathematical education and so thereafter mathematicians could assume that anything in Euclid could be used without comment or reference. However, Netz suggests that the contents of the tool-box could become known in another way.

> The very fact that an argument was made, without any intuitive or diagrammatic support for that argument, must have signalled for the audience that the argument was sanctioned by the Elements. Once this is the expectation, the need to refer explicitly to the Elements declines, which would in turn support the same tendency: the regular circle in which local conventions are struck without explicit codification. (p. 232)

In other words, the process might resemble that described by Thurston, above; the reader comes to know, by observing what things are taken for granted in proofs, what is in the tool-box, without even being told explicitly.

In a mathematics classroom, determining what is in the class tool-box can be done in various ways. One approach sometimes taken is to inform the students that they must forget everything they already know and start from the axioms and definitions given in the textbook. But experience tells us that the students do not forget everything they know, and the axioms and definitions given in the textbook are never complete themselves. What results is the class pretending to base its arguments on the given axioms and definitions, while being guided by their prior knowledge. The tool-box is supposed to be limited to what the textbook allows, but is actually larger.

Another approach is for a teacher to start presenting proofs without establishing what is in the tool-box, so that, as Netz says happened for the Greeks, the contents of the tool-box become known to the students through the making of arguments without stating their justifications. If a proof is based on an assumption that the measures of angles can be added, but this is never stated or justified, then it must be part of the tool-box.

The idea of the tool-box can be usefully connected with another idea, that of local organization.

Local Organization

Freudenthal (1971) discusses the introduction of proving and claims that proving must begin with what he calls “local organization” as opposed to the “global organization” of an axiomatic system. In a globally organized system the definition of parallelogram would be part of the tool-box and would either be explicitly taught or would become known through its use in proofs. Freudenthal describes another approach: A discussion, for example, of the properties of parallelograms can begin by simply listing all those that are apparent to the students. Similar lists might be made for rectangles and rhombuses. In examining such lists, Freudenthal claims, “There are a host of visual properties which ask for organization. Here starts deductivity; rather than being imposed it unfolds from local germs. The properties of the parallelogramme become deductively interrelated” (p. 424). Finally, one property emerges as a definition from which the others can be deduced. This is local organization. It can be extended as the properties of parallelograms are related to the properties of rectangles, rhombuses and squares.

So rather than a definition being given at the outset, the students’ proving determines which property is a definition and which ones are consequences of it. There is a local organization of mathematical knowledge, but no global system into which that knowledge fits. This is necessary, Freudenthal states, because, “a student who never exercized organizing a subject matter on local levels will not succeed on the global one” (p. 426). It is at the local level that proving and defining are learned, before being used (perhaps much later) to define and prove in an axiomatic system.

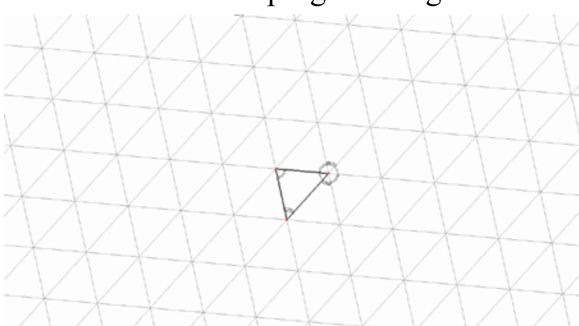
Freudenthal’s implications for teaching are clear:

> In general, what we do if we create and if we apply mathematics, is an activity of local organization. Beginners in mathematics cannot do even more than that. Every teacher knows that most students can produce and understand only short deduction chains. They cannot grasp long proofs as a whole, and still less can they view substantial part of mathematics as a deductive system. (p. 431)

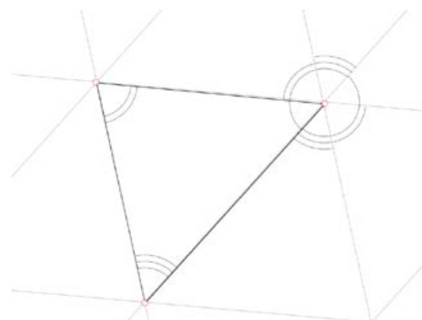
Local organization and use of an implicit tool-box raise the question for teaching of what must be made explicit, what requires proof, and what can be left implicit. Freudenthal’s examples (in *Mathematics as an Educational Task*, 1973) offer a resolution to this problem. The decision as to what to leave implicit, what to make explicit and what to prove must be made in order to develop students' understanding of mathematical concepts and ability to apply them. In cases where the proof brings a new understanding of the concepts involved, the proof is useful. Assumptions that have unexpected implications will need to be made explicit in exploring those implications. Assumptions that would seem obvious and trivial to the students if made explicit can safely be left implicit.

Proof 7 is another visual proof that the angle sum of a triangle is 180 degrees. It depends on an assumption that any triangle can be used to tile the plane. It is clear that the six angles around a vertex total 360 degrees (although this depends an another hidden assumption, that angle measures can be added, and a hidden definition, of degrees). The congruence of the marked angles depends on the fact that all the triangles are congruent. All of the hidden assumptions here are likely to be accepted without question by students, and making them explicit would be pointing out the obvious. In contrast, the typical two column proof depends on properties of transversals that are not so obvious as assumptions, and if proven depend on the even less obvious parallel postulate.

Proof 7 makes use of a minimal tool-box, and puts together its elements in a way that proves an important mathematical property of plane triangles, a property which in turn contributes to many other proofs of significant results. As it is based on assumptions that are more obvious than the conclusion it verifies and explains the result, in a way that the traditional two column proof does not. A similar comment could be made about Proof 4, although unless it is actually enacted the fact that one turns 360 degrees in tracing around a triangle may not be obvious to those who have not done LOGO programming.



Proof 7



Proof 7 detail

Teaching Proof versus Proof Based Teaching

The NCTM opened the twenty-first century with a call for increased attention on mathematical proof and reasoning, for all students. Reform movements at the beginning of the twentieth century also called for mathematics teaching, and especially high school geometry teaching, to focus much more on proof, at a time when parallel reforms were allowing many more students to go to high school. In North America this resulted in the invention of a new proof format, the two column proof, along with other pedagogical innovations designed to help teachers teach proof.

> To make student proving possible, a system of resources had to be developed and coordinated with a norm for accomplished proofs. The integration of all those elements produced a stable geometry course oriented toward students' learning the art of proving embodied in the two-column format. However, that stability came with a price – that of

> dissociating the doing of proofs from the construction of knowledge. (Herbst, 2002, p. 307)

Instead of school proofs being like mathematicians proofs in that they gave insight into mathematics, school proofs became another topic to be covered, disconnected from all others. Proofs became an object, rather than a process of learning.

To avoid following the same path now, we must tread carefully. We must ensure that we see proof as fundamental to mathematics as a way to develop understanding of mathematical concepts, and as a way to discover new and significant mathematical knowledge. Proof cannot be limited to the format of proofs, and to the role of verification of knowledge (for which there is probably good empirical or other evidence already).

We have recent experience with a comparable reform effort, the increased focus on mathematical problem solving after the 1970s. At first, problem solving was an object of teaching. Texts added new chapters on problem solving, including Polya's four stages, sets of heuristics and even larger sets of strategies to help teachers and students learn to be better problem solvers. Problem solving was a huge area for research in mathematics education. Unfortunately, a lot of the research indicated that making problem solving an object of mathematics teaching wasn't making anyone better problem solvers (Schoenfeld, 1987, p. 30). Perhaps the people with the best success in developing problem solving in students, the mathematics faculty who coached students for the Putnam exam or for various Olympiads, had it right. "Students don't learn to solve problems by reading Polya's books, they said. In their experience, students learned to solve problems by ... solving lots of problems." (p. 30). Some decades on now, problem solving is taking on a different role in mathematics teaching. Textbooks for new teachers and some curricula and school mathematics textbooks are advocating that problems not be an object of mathematics teaching, but instead be the means by which mathematics is taught.

I would like to conclude by suggesting that today proof teaching and research is in a similar state to problem solving in the 1980s. Much research is being done, but proof is still seen as a topic to be taught. In the case of problem solving we have moved towards seeing it as a way of teaching, and there are early hints that something like proof based teaching, akin to problem based teaching, might emerge in the next decades. It will be interesting to see.

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STUDYING MATHEMATICS CONCEPTUAL LEARNING: STUDENT LEARNING THROUGH THEIR MATHEMATICAL ACTIVITY

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This paper describes a program of research aimed at better understanding mathematics conceptual learning. Rather than studying the products of learning (conceptual steps), it focuses on the process (mechanism) of students' learning – learning that occurs through their mathematical activity in the context of a sequence of mathematical tasks. Intensive design research investigates subtle shifts in thinking that lead to the development of new abstractions.

Introduction

In Simon (1995), I introduced the idea of a "hypothetical learning trajectory (HLT)" to characterize important aspects of pedagogical thinking. The HLT consists of (1) the educator's goal for the students' learning, (2) the mathematical tasks that will be used to promote the students' learning, and (3) hypotheses about the process by which the students will learn. The paper emphasized the interdependency of the learning tasks and the learning process. Much research is still needed to support all of the three components. However, my colleagues[1] and I have embarked on a research program to investigate particular aspects of the conceptual learning process, which also implies careful attention to tasks. Whereas considerable important work has been going on investigating social interactive aspects of learning that take place during class discussions, our research focuses on cognitive learning that can take place when students are engaged in a series of mathematical tasks in small groups and individually. In this paper, I describe this emerging research program.

What Do We Mean by "Studying Learning?"

> In spite of the animated theoretical debate that has taken place on the nature of abstraction, little experimental research is available. … We surmise that the lack of experimental evidence is due to the difficulty of observing the processes of abstraction (as opposed to the products, for which there is more evidence). (Hershkowitz et al., 2001, p. 197)

Because our research focuses on learning through interaction with a task, we use primarily cognitive (constructivist) constructs as the basis for this work.[2] The focus of this work is on the process by which abstractions are developed.[3] Much of the important research in mathematics education over the last 30 years has been referred to as "research on mathematics learning." Given that this language is already widespread, it is difficult to denote the differences between our research program and the research that has been labelled "research on mathematics learning." I will try to articulate two key distinctions. First, most of the work subsumed under this heading has characterized static mathematical student understandings (e.g., Steffe, 2003) or a hierarchy of student understandings (schemes, classroom practices) (e.g., Cobb, McClain, & Gravemeijer, 2003). Whereas this work has been foundational to our work, it is not what we mean by "studying learning." Our focus is in understanding the subtle shifts in thinking that account for the transition between two consecutive schemes (or understandings or classroom practices). Our goal is to explain how the scheme or understanding has come about as students interact with a sequence of mathematical tasks. Second, when learning is explained, it is often explained using broad concepts such as reflective abstraction, generalizing assimilation or negotiation of meaning. For us, these are not sufficiently nuanced to be useful for instructional design. Siegler (1996) argued,

> The standard labels for hypothesized transition processes: assimilation, accommodation, and equilibration; change in M-space; conceptual restructuring; differentiation and hierarchic integration; are more promissory notes, telling us that we really should work on this some time, than serious mechanistic accounts. (p. 223)

And diSessa & Cobb (2004, p. 81) pointed out, "Piaget's theory is powerful and continues to be an important source of insight. However, it was not developed with the intention of informing design and is inadequate, by itself, to do so deeply and effectively." Our research program is aimed at moving from Piagetian constructs as a source of insight and broad hypothesis toward an explication of students' mathematics learning through engagement with tasks that can support instructional design.

What is the Nature of a Mathematical Concept?

Understanding the nature of mathematical concepts and understanding how they are learned are necessarily intertwined. Each has implications for the other. Thus, investigating how mathematical concepts are learned requires a useful characterization of the nature of concepts, but as will become clear, the nature of a concept involves considering aspects of how it is learned. To make some key distinctions about mathematical concepts, I use the following example presented in Simon (2006, pp. 4-5):

> In a fourth-grade class [students 11-12 yrs old], I asked the students to use a blue rubber band on their geoboards to make a square of a designated size, and then to put a red rubber band around one half of the square. Most of the students divided the square into two congruent rectangles. However, Mary cut the square on the diagonal, making two congruent right triangles. The students were unanimous in asserting that both fit with my request that they show half of the square. Further, they were able to justify that assertion by explaining that each of the parts was 1 of 2 equal parts and that the two parts made up the whole. I then asked, "Is Joe's (rectangular) half larger; is Mary's half larger, or are they the same size?" Approximately a third of the class chose each option.[4]

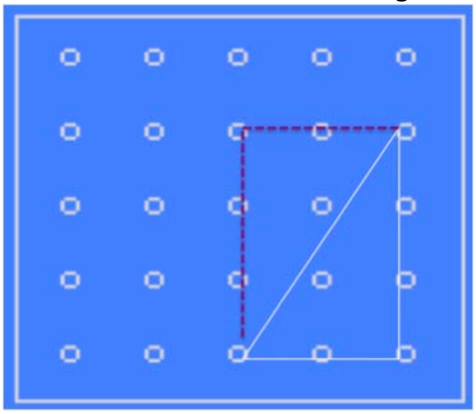
When I share this scenario and ask current and prospective mathematics educators about potential instructional interventions for the two thirds of the class who did not recognize the equality of the area of the two parts, the overwhelming majority suggest having students cut up the triangular half and superimpose it on the rectangular half (and additional activities of this type). Let us examine this response.

No matter how many such experiences are created for the student, these experiences will never teach the students the critical concept involved. Readers of this paper likely know that the areas of the two different shaped halves are equal. They know this, not because they have cut up and superimposed one half on the other half; they know that the two halves *must* be equal. They know the logical necessity of their equality. Having students cut and superimpose the halves may convince the students *that* the halves have equal area, but the activity does nothing to help them learn that the halves *must* have the same area.

The cutting up and superimposing activity could be done with different fractions and different shaped parts. Students could come to believe that the same fraction with a different shape has the same area. This is what I have called an "empirical learning process" (Simon, 2006). In an *empirical learning process*, the student introduces a number of inputs and sees a pattern in the outputs. However, an empirical learning process *never* produces a mathematical concept. A mathematical concept involves knowledge of the logical necessity of an idea. I have used Piaget's (2001) term *reflective abstraction* to identify learning processes that lead to mathematical concepts. The choice of this term will become clearer in the subsequent discussion.

Let's consider a second example (Heinz et al, 2000). Ivy is a sixth-grade teacher (students 11-12 yrs old) who is committed to her students learning mathematics with understanding. In this episode, she was beginning the teaching of the area of triangles. Following is an outline of her lesson as it unfolded:

1. Students worked in groups to find the area of a right triangle (legs 2 and 3 units) on a geoboard.
2. The class discussed their strategies. Completing the rectangle was a popular strategy.



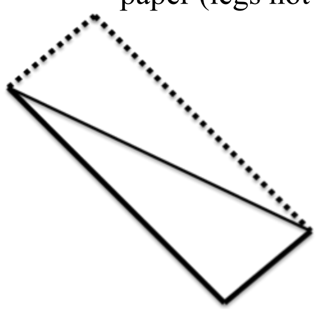
3. Students worked in groups to find areas of other right triangles on their geoboards. Following Ivy's instructions, they recorded measures of the base, height, and area for each triangle in a chart.
4. Ivy convened the class and recorded in a large chart the base, height, and area of the different triangles contributed by the students.
5. Ivy then directed,

> Look at how these numbers are in this chart with our areas . . . and see if you can figure out a pattern that you can use every time using the numbers [measures of base and height] to come up with the area. . . . There is something that you can do to these [measures of] the bases and the heights to get the area. (Heinz et al., 2000, p. 94)

Ivy's encouragement of students to find a pattern in the "numbers" is a promotion of an empirical learning process. In her lesson, she treated the geoboard work as if it was a black box that turned inputs (measures of the legs of the triangles) into outputs (measures of the areas) for the purpose of finding a pattern in the numbers. Finding a pattern in this way does not result in knowing the logical necessity of the relationships expressed in the formula for the area of a triangle.

At this point, I anticipate the following objection, "Denoting patterns is a key aspect of doing mathematics." Mathematics has been called "the science of patterns" (Steen, 1988; Devlin, 1996). We do not want to eliminate attention to patterns in mathematics classrooms. However, the discussion here is about explaining the process of learning mathematical concepts, not of doing mathematics more broadly. Empirical identification of a pattern never, in itself, results in conceptual understanding. It may be the trigger for some other activity that results in such learning. I argue that identifying patterns empirically is neither necessary nor sufficient for conceptual learning. Further, in some situations it might not even be appropriate. In our example, looking for a pattern among the legs of a triangle and the area is neither optimal for learning nor does it provide a useful model of mathematics.

Consider now this modification of Ivy's lesson. I begin the lesson as Ivy did in order to highlight particular contrasts in students' opportunities to learn, even though more effective lessons might be designed for this subject matter. The sequence follows:

1. Students work in groups to find the area of a right triangle (legs 2 and 3 units) on a geoboard.
2. The class discusses their strategies. Completing the rectangle is a popular strategy.
3. Students work in groups to find areas of other right triangles on their geoboards. *No recording chart*.
4. Students are given a ruler and asked to find the area of right triangles drawn on plain paper (legs not parallel to sides of paper).
5. 
6. Students are given the measurement of the legs of right triangles involving larger numbers for the dimensions. They are asked to find the area of each triangle (without drawing) by mentally running the process they did on paper.
7. Students are asked to write a generalization (algorithm) for how to calculate the area of a right triangle given the measures of the sides.

What do we see in this example? If we accept that the lesson sequence could promote reinvention of an algorithm for the area of a right triangle, we can see that it does so *without* employing an empirically generated pattern. The sequence is designed so that students can explain the appropriateness of each step in the algorithm based on the relationship between a right triangle and a particular related rectangle. I will engage in further analysis of this lesson in the next section.

Evolving Constructs for Studying Learning during Engagement with Mathematical Tasks

If we are convinced that empirical learning processes alone do not result in conceptual learning, then the important question is, *how might we think about learning that develops conceptual understanding?* The modified lesson discussed above is not an exemplary lesson and the intended learning is not particularly impressive. However, the simplicity of the lesson, allows it to be used for exemplification of particular ideas.

Students are able, without difficulty, to find the area of right triangles on the geoboard by completing the rectangle and taking half of it. For the purpose of this paper, we are not concerned with where the idea came from to complete the rectangle, only that carrying out the strategy with understanding is not beyond their current conceptual knowledge. The geoboard work developed an activity from which the learning could take place. It became a *process* in the APOS sense (Dubinsky & McDonald, 2001). That is, students reached a point where they were able to anticipate the whole process, not needing to stop and determine each next step. [5]

Note that for an elementary student, adding a rubber band to complete the rectangle, determining the number of little boxes contained in the rectangle (perhaps by counting) and taking half of that number is still pretty far from knowing how to compute the area of a right triangle given the measures of its legs. In the modified lesson, the diagrams of right triangles are given next. This changes the context, putting additional demands on the student. First, the student needs to draw the rectangle that produces two congruent triangles. When the geoboard was used, the choice of rectangle probably occurred without discrimination, because the geoboard is arranged as an orthogonal array and the triangles were arranged with the legs parallel to the sides of the geoboard. When working with the diagram, the students are able to produce appropriate rectangles based on their anticipation of a rectangle made up of two congruent triangles. Once again their strategy becomes a process as they reach a point where they do not need to think about how to draw the additional lines to complete the rectangle. The second difference in the paper context is the unit squares do not appear on the paper. Therefore, the students need to use their prior knowledge of measurement and computation of the area of a rectangle.

The mental run problems offer a third context, again changing the demands of the task. These problems are aimed at fostering abstraction of the relationships involved. On the geoboard or on paper, the student could determine the length of the rectangular sides using visual clues after having completed the rectangle. For the mental run problems, the students have no rectangle to visually examine. They need to anticipate the dimensions of the rectangle based on the dimensions of the triangle. They are able to do this based on an anticipation of how they constructed the rectangle (on paper) from the triangle. At this point they should have the abstractions that can be represented by $A = \frac{1}{2} l_1 l_2$, where l_1 and l_2 are the legs of the right triangle.

What can we take from this example?

1. This was not a problem solving activity. That is, at no point were the students facing a real problem. They moved smoothly through the tasks. Problem solving is an important part of mathematics. However, there are ways of teaching concepts that do not depend on students spontaneously solving the “next” problem.
2. Related to #1, this lesson sequence was not constructed to foster cognitive conflict. Rather, it was anticipated that the students would have the knowledge to operate at every point in the lesson when they reached those points.
3. The lesson sequence was designed to foster conceptual learning without hints or leading questions on the part of the teacher.
4. The learning was not the result of an empirical learning process. Students were not involved in finding a pattern in the numbers. Students came to understand the logical necessity of the relationships represented by the formula above.

If the learning is not the result of empirical learning, problem solving, or cognitive conflict, and if the teacher did not lead the student to the idea, how can we understand the learning process that was hypothesized in this lesson? (See Simon, et al, 2010 for the analysis of a teaching experiment involving division of fractions.) I will provide a brief, rudimentary explanation. We use three constructs adapted from Piaget's (2001) description of reflective abstraction: *goal-directed activity*, which can be mental and/or physical, *reflection*, which we understand as humans' innate ability (and tendency) to recognize (not necessarily consciously) commonality in their experience (von Glasersfeld's, 1995), and *abstraction*, which involves a learned anticipation. Using these constructs we view the students' hypothesized learning as the development of abstractions through reflection on their goal-directed activity (consistent with Piaget's reflective abstraction). If you return to the description above, at each point the students came to anticipate a process that coordinated particular aspects of their extant knowledge. In particular, the students came to anticipate:

1. the right triangle as half a rectangle divided diagonally,
2. the requisite rectangle formed by creating sides at right angles to the legs of the triangle,
3. the sides of the resultant rectangle having the dimensions of the legs of the right triangle,
4. the area of the triangle determined using a known formula for the area of a rectangle and halving the result.

But what does it mean that they "came to anticipate?" The student, faced with a task, engages in activity to meet their goal. Initially, the student engages step by step, determining the next action as they complete the preceding one. As they encounter similar situations, they at some point begin to notice a commonality (reflection) in their *activity* used to meet their goal. This results in anticipation of the needed actions. The sequence of actions no longer need to be assembled, the whole activity sequence is anticipated in response to recognizing the situation. Finally, the anticipation of the process reaches a point at which not all of the activity needs to be carried out in order to arrive at the intended result. The student no longer needs to create or imagine the rectangle to find the area of the right triangle.

Vision of a More Scientific use of Tasks for Engendering Conceptual Learning

I offered the example above to provide a concrete basis for discussing the particular vision for mathematics education that underlies our research program. Overall, my colleagues and I support the shared goals of the mathematics education community: equal opportunity, deep understanding, strong problem solving abilities, and competence in communicating mathematical ideas. However, our research program is focused on a specific goal that can advance the larger goal of deep understanding of mathematics. This particular goal is an improved ability to engineer task sequences that can foster particular understandings for a diverse set of students. We use the term "foster" to emphasize that teaching cannot *cause* learning. However, mathematics teaching has the potential to make conceptual learning likely.

To consider the vision, I invite engagement in the following thought experiment. A highly competent teacher gives her mathematics class a problem whose solution requires the mathematics to be learned. The students are asked to work in small groups, provided with a rich set of representations to work with, and asked after considerable work time to discuss their work with the class as a whole. A couple groups generate solutions to the problem, but the majority are not able to generate solutions (although there is considerable variability in their efforts). During the class discussion, different students talk about their work and one of the students who solved the problem presents her solution. After some questions and paraphrasing by the teacher, many of the students who did not solve the problem, seem to understand the solution presented.

I have probably described many classes with which you are familiar. Now I ask you to make a prediction. Who will more likely retain the new mathematics, apply it appropriately, and build on it – a student who independently generated the solution or a student who came to understand it in the context of the class discussion?[6] I think most mathematics educators would predict that the student who generated the solution to be more successful with the concept in the future. There is a difference in the cognitive demands of generating a solution versus understanding a solution (analogous to active versus passive knowledge of a language). This difference is not generally discussed.[7] Further, one can see this distinction as an equity issue. Who are the students who usually solve the mathematical tasks during small group time? It is the students who are more advanced in their mathematical knowledge. In the common scenario I described, the more-advanced students continue to work novel problems, while the less-advanced students struggle to follow the explanations of solutions given by the more-advanced students. This phenomenon can work against narrowing the gap between less-advanced and more-advanced students and even contribute to increasing the gap.

The hypothetical class described is not an unusual one. Particularly when a difficult new idea is the focus, generally few students generate the mathematics on their own. Sometimes no one in the class is able to generate a solution prior to the class discussion. This is where our research program comes in. What if we had the research-based knowledge so that typically 80% or more of the students could generate the new mathematical ideas through their engagement with the mathematical tasks? Would this improve the quality of student learning? Would this improve the quality and impact of the class discussion? Would this increase access to high-quality learning for more students? – an issue of equity.

The vision behind our research program is just that: understand how students learn through their mathematical activity, so that pedagogical design principles can be created that guide task sequence design. As a result, task sequences would be designed to engage students in the particular activity that allows them to abstract the intended new ideas. Currently, mathematical tasks often discriminate in favor of those who have the appropriate prior experience and against those who do not have it. The vision is to provide instructional task sequences that are designed to build up the requisite experience.

One might ask, “Don’t we do this already?” Let’s consider a couple of frequently used approaches. The use of manipulative materials and other physical or iconic representations is often thought to create the experiential base for conceptual learning. These tools are useful and have potential beyond their current uses. However, there is insufficient theory guiding the use of physical and iconic representations. Teachers often have students initially solve problems using these representations. The students produce solutions that are isomorphic, *in the mind of the knowledgeable adult*, to formal solutions. However, these “concrete solutions” in themselves do not produce conceptual understanding and do not embody the vision articulated in the last paragraph.

A second approach is the pedagogical use of a *series* of problems. This is an attempt to create a series that allows students to generate ideas, but where the steps from one problem to the next are within the students’ grasp, that is novel problems that are only a small change from the prior one. Although there is not much theory to guide the development of such series, these series at their best might approximate what is described by the vision presented. However, it seems that in many cases, key problems in the sequence are only solved by a minority of the students.

In contrast, in our work, the intention is to identify (goal-directed) activities in which students are already capable of engaging, from which they predictably can abstract the new ideas. There is no point at which we *hope* that they can do or see something that is key to the new mathematics. There is no point at which their learning is based on solving a novel problem that they might not be able to solve. The vision involves a shift from provoking learning to engineering learning (albeit not in a deterministic sense).

One of the basic tenets of this vision is that children are born with an ability to learn concepts and they do this through their activity. Let us take two well-known examples. First, most people have probably observed that children, after learning the rules and basic moves of a game (e.g., chess, video games), often develop sophisticated concepts for playing the game without instruction from anyone. They are certainly more sophisticated than learning simply what move has a positive effect and what move has a negative one. Do we understand how these concepts are developed? Could this ability be harnessed?

This second example involves mathematics learning. One of the greatest mathematical learning achievements in the life of any individual is the development of a concept of number. This is generally accomplished without professional teachers although parents figure prominently. There are many types of activities in which parents can and do engage their young children. However, there is one that is ingenious – counting. Counting activities, in which children engage in coordinating the touching of an object with the reciting of the next number name in the learned number name sequence, result in major conceptual gains. The child is not trying to solve a problem or even trying to learn something mathematical. They are most likely trying to imitate the adult, enjoy the play, and gain the adult's approval. Counting is a specific activity that leads to important abstractions of number concepts. Our vision is to understand the process underlying learning to count and an array of other mathematical concepts and to develop pedagogical design principles based on this understanding.

How Does This Work Potentially Contribute to and Fit into Mathematics Instruction?

In classrooms in which students initially work in small groups (and/or individually) and then convene in whole-class discussions, we envision our work informing the design of sequences of tasks for students' small group and individual work. Envisioned task sequences would provide the experience (requisite activity and opportunities for abstraction) that would allow students to have significant insight into the mathematics at hand. The purpose of the classroom discussion would not be to *bring* students to the intended ideas, but rather for students to share their insights developed through their work with the tasks, make new ideas explicit, provide justifications, and develop shared language and symbols for the ideas (as in *situations of institutionalization* (Brousseau, 1997). Teachers, in addition to leading class discussions, can interact with small groups of students who are in the process of abstracting a new idea.

It may not be necessary to employ this type and level of didactical engineering to all conceptual learning in the classroom. However, there are particular mathematical ideas taught in elementary and secondary school that many students never learn or never learn well. It is towards these pedagogical challenges that this approach is particularly directed.

I have described this approach to fostering learning as not involving problem solving. Certainly, the mathematics classroom must promote and engage students regularly in problem solving. However, having an approach to foster new concepts that does not rest on each student's ability to solve novel problems would be a useful pedagogical tool.

Current Project

In our last project, we developed a methodology for studying students learning. The methodology is an adaptation of teaching experiments with individual students (Simon, et al., 2010). We are now embedding that methodology in a 5-year project, Measurement Approach to Rational Number (MARN).[8] The project combines two main goals, continuing our inquiry into students' learning through activity and developing approaches to teach some of the most intractable concepts in the domain of fractions and proportional reasoning. Therefore we are studying learning in the context of fractions and decimals, and we are using our emerging understanding of how students learn through their activity to design tasks for promoting fraction and ratio understanding. The design of instruction builds on a foundation of measurement concepts modifying and extending the approach of Davydov and his colleagues (c.f., Davydov, & Svetkovich, 1991). Our hypothesis is that measurement-based activities can provide the specific basis for students' abstractions in these key mathematical areas.

MARN Methodology

The MARN design research will occur in two phases, individual teaching experiments and whole-class teaching experiments. The purpose of the first phase is to develop knowledge of promoting intended abstractions through task sequences. The resulting task sequences and hypothesized learning processes will constitute a significant part of the hypothetical learning trajectories for the whole-class teaching experiment.

As mentioned, the methodology we developed for studying students' learning through their mathematical activity involves individual teaching experiments. The design of the project is also predicated on the potential value of the individual teaching experiments to inform the whole class work on learning fractions and ratio. Our expectation of this potential is not shared by many in the mathematics education research community.[9] The MARN Project gives us a chance to examine this methodological issue empirically.

Rationale for Individual Teaching Experiments

> How do changes in children's thinking occur? Focusing on change . . . will require reformulation of our basic assumptions about children's thinking, the kinds of questions we ask about it, our methods for studying it, the mechanisms we propose to explain it, and the basic metaphors that underlie our thinking about it. (Siegler, 1996, p. 218)

Our goal for studying learning (understanding the mechanism by which conceptual understanding develops) is seen by many as impossible. Indeed it is both difficult and the success of any study is uncertain. The methodology for data collection is critical to producing a set of data from which warranted inferences about the learning process can be made.

We have found that the most important feature of the data collection is that it produces reasonably continuous evidence of the student's activity (mental and physical). Whereas observing physical activity is relatively straight forward, inferring mental activity requires attention to physical activity in conjunction with encouraged regular verbalization on the part of the student. Inference of students' mental activity has been part of qualitative research for several decades. What is different in this research is trying to infer the ongoing evolution of the thinking as opposed to key snapshots.

The main obstacle to continuous evidence of students' activity is instances when the students are listening to someone else who affects their subsequent activity. For this reason, we found that we could not use even 2 students together in a teaching experiment. When one person watched and listened to the other pursuing an idea, it often changed the flow of the observing student; however the researchers had no evidence of what went on in the mental activity of that student. For similar reasons the researcher avoids hints, suggestions, and leading questions. This data collection approach also allows us to focus on what aspect of the abstraction process can take place without extensive interaction.

Conclusions

The research program I have described is based on a simple idea: the more we understand about the learning of mathematical concepts, the better we will be able to promote conceptual learning. Bransford, Brown, and Cocking (2000, p. 221) asserted, "A scientific understanding of learning. . . provides the fundamental knowledge base for understanding and implementing changes in education. Our research program focuses on students' learning through their mathematical activity in the context of series of mathematical tasks. It rests on two important claims. First, learning can be studied directly and to productive ends Second, learning can be engineered in ways that do not depend on novel problem solving on the part of students. Third, engineering learning opportunities that allow students to build abstractions upon their mathematical activity (engendering the requisite activity) would increase equitable access to high-quality conceptual learning. Because this research program pushes the boundaries of what is generally believed to be possible, this program of research, meets with skepticism. In particular, the following questions are raised about the research.

1. Can data be produced to make useful inferences about the learning process?
2. Can empirically based constructs be developed which are useful for a variety of mathematical concepts for a variety of students in a variety of contexts?
3. Can design principles for task and task sequence creation be developed across concepts?
4. Can analyses of individual concept learning usefully inform design for classroom lessons?

Our research program represents a conjecture that the answers to these questions are, "Yes." As mentioned this is an extremely difficult and uncertain undertaking. Let me speculate (optimistically) about the potential payoff of this endeavor.

Contribution to Research

Design research is a productive and increasingly used methodology in mathematics education. It figures prominently in the generation of learning progressions, an exciting and growing area of current research. The problem with design research is that researchers' ability to study the learning of particular concepts is limited by their ability to promote the learning they wish to study. A greater understanding of students' learning through their mathematical activity (developing abstractions) and concomitant design principles for generative task sequences could provide a basis for more consistently productive design research.

Contribution to Curriculum Development

Whereas recent curricula tend to increase students' opportunities to learn, they tend to be uneven (within each curriculum) in quality. Part of the unevenness may be the variation among the authors, but much of the unevenness seems to be due to lack of task design principles grounded in an understanding of student learning processes. Strong lessons reflect an author's insight into a particular area. However, there is insufficient theory to guide the majority of lessons.

Contribution to Teaching and Teacher Education

In the context of ongoing mathematics education reform efforts, teachers seem to have derived both constraints on and recommendations for their teaching. As a result they try to refrain from telling students what they want them to learn and giving answers. They tend to use a variety of strategies: classroom discussions, small groups, non-routine problems, multiple representations, manipulatives and computer tools. However, none of this gives them a clear understanding of how to help students learn a new concept. As a result, teachers often fall back on strategies that do not differ *fundamentally* from traditional teaching, such as asking leading questions (trying to get students to say what the teacher would say if they were not avoiding telling) or having the student who already has the concept tell the others. Classroom discussion is a powerful tool. However teachers are too often working from a vague plan of conducting a classroom discussion so the majority of the students, who do not understand the new concept, understand it. Teacher education aims at strategies, skills, and dispositions, but neglects, for the most part, to help prospective and practicing teachers understand fundamentally new ways to promote the learning of particular concepts.

The research program described is aimed at providing a partial basis for filling the voids in each of these areas. However, while remaining optimistic, some tempering is in order. It is reasonable to expect that what is learned through basic research should be useable by researchers (e.g., in design research). However, the effect on teaching and teacher education will require additional research and development. What a researcher learns after 10+ years of intensive investigation is not necessarily accessible to practitioners.[10] We must ask of every product of basic research, "What part or version of this can practicing teachers understand with a realistic level of support?" And another set of questions not often asked about adult learners (e.g., teachers, doctoral students) is, "How can we understand the learning trajectory with respect to these new ideas (support experiences and learning process)?"

Finally, there is good reason why few research groups endeavor to study learning directly. It is difficult and success in any particular study is uncertain (Simon et al, 2010). The ambitious program of research that I have outlined will require the efforts of many research groups who take part in shaping the program and contributing results. Further, as a community, we will need to continue to make a strong case for support of basic research in mathematics education in an era dominated by a demand for "results now."

Endnotes

1. Ron Tzur and I collaborated on the initial work of this research program. We built upon his dissertation work and my prior research. Other collaborators have included Luis Saldanha, Evan McClintock, Tad Wattanabe, Gulseren Karagoz Akar, Ismail Zembat, Karen Heinz, and Margaret

Kinzel. Our current project includes Barbara Dougherty, Zaur Berkaliev, Arnon Avitzur, Nicora Placa, and Jessica Tybursky.

2. We use a social lens to examine interactive aspects of the classroom and the constructs of social and sociomathematical norms to examine opportunity for learning

3. Focus on *abstraction* is often seen as the province of constructivist theoretical approaches. However, Russian activity theory, derived from the theory of Vygotsky, uses abstraction as a key construct (c.f., Davydov, 1990).

4. In case the reader is sceptical about whether there was a shared meaning for "bigger," the same problem has been done in interviews in which the squares represented cookies that the interviewee liked and the question was "Which piece would you rather eat?" and the reasons for the answer were probed. The same phenomenon was observed.

5. This process could be investigated and explicated further, but it is not within the aims of this paper.

6. Answering this thought question empirically would present significant methodological challenges.

7. There is widespread conviction that engaging students in developing new mathematical ideas is more effective than giving them the ideas in a lecture. However, given classrooms in which students are engaged in generating ideas, there is little or no attention to the differential contributions of students and the effect on the quality of their learning.

8. The Measurement Approach to Rational Number (MARN) is supported by the National Science Foundation under grant no. DRL-1020154. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

9. Case in point was some protracted arguments on this point during our first project advisory board meeting.

10. Often other researchers struggle to deeply understand the results of lengthy, focused investigations.

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EXPLAINING STUDENT PERFORMANCE THROUGH INSTRUCTION

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Lack of satisfaction with the quality of the mathematical knowledge of non-mathematics majors has led to the design of new curricular materials. In this study, a mixed methods explanatory design was used to compare the performances of two groups of engineering majors enrolled in two courses on differential equations and investigate their written work in light of differing curricular and instructional approaches.

Introduction

Calls for curriculum and instructional reform have been expanded to include collegiate mathematics education (American Mathematical Society, 2011). Lack of satisfaction with the quality of mathematical knowledge of students completing service courses in the mathematics department has raised considerable concerns regarding how these courses are taught. Among many such courses, Differential Equations (DEs) entertains a particularly prominent role since it caters to a variety of clients from engineering areas. Skills fostered in a traditional theory-driven DE course have been claimed to be of little value to these degree programs. For instance, in a survey, Varsavsky (1995) found that engineering faculty value skills like modeling over technical competencies like differentiation and integration. Indeed, engineering and physical science faculty recommend that service courses in mathematics be made more relevant to their students and suggest incorporating an engineering viewpoint (Czocher, 2010; Pennell, Avitable, & White, 2009; Varsavsky, 1995). These perspectives have motivated the design of new curricular materials that aim to frame student learning in meaningful contexts (eg, Rasmussen and King, 2000). Despite such curriculum development efforts, little is known about the actual impact of such curricula on student learning. The primary motive of the present study was to address this gap. To this end, I compared the performances of two groups of students enrolled in two different sections of an introductory course on DEs to see what differences, if any, existed in their work on various types of tasks due to exposure to differing curricular and instructional approaches. One section followed a standard commercial textbook while the other used a reform-based curriculum with a conceptual orientation that was built around contexts central to engineering fields.

Context and Background: The Case of Differential Equations

The study of DEs is a unique point in the trajectories of engineering and physical science majors. In some instances, it serves as a capstone to the calculus sequence. On the other hand, an introductory course on DEs might be the first time that these students are exposed to material that is specialized for their disciplines. Many introductory undergraduate courses strip the equations of their natural contexts in order to treat the equations abstractly and then treat DEs deductively. That is, a general equation is presented, its solution is derived, and applications from the physical sciences related to the target equation can then be handled by manipulating the generic solution.

Researchers have argued that curriculum and instruction must align in order to support students' transitions to studying advanced mathematical topics. The study of DEs poses special challenges to building and implementing supportive curricula since the "switch from conceptualizing solutions as numbers to conceptualizing solutions as functions is nontrivial for students" (Rasmussen, 2001, p. 67), and students' documented difficulties with function extend to the form of the initial conditions (ICs) (Raychaudhuri, 2008). Others have shown a related aversion to the use of boundary conditions (Black and Wittmann, n.d.).

"Unifying" views of introductory DEs range from the use of linear operators to physical approaches that emphasize formulation (see West, 1994) to models and derivations from first principles (Myers, Trubatch, and Winkel, 2008). Scholars have studied student performance on procedural and conceptual components of DE knowledge (Arslan, 2010) and on contextual and de-contextualized problems (Upton, 2004), whose analyses graded student responses as either "right" or "wrong;" a method too coarse to account for different heuristics or representational schemes potentially nurtured in instruction. Donovan (2004), in a pair of case studies, demonstrated the variety in students' conceptualizations of linear DEs and their representations, but did not address the bases for student reasoning. In contrast, Black and Wittmann (n.d.) studied students' reasoning strategies and identified important connections between the physical and mathematical interpretations of the DE model that influence student performance. Bingolbali, Monaghan, & Roper (2007) found that the teacher's proclivity for a particular interpretation of derivative was adopted by students. Indeed, it is likely that the student is heavily influenced by the values placed on types of knowledge (Hedegaard, 1998).

These considerations guided the development of the curricular materials whose impact on learning was examined in this work. The reformed curricula, Baker's (n.d.) text, supports a unifying solution strategy with physical reasoning by showing how a guess-and-check heuristic for formulating the solution mimics the physical system's response to the forcing term in the DE. Additional considerations, regarding the contextual domains provided for students' investigations, motivated the development of examples and contexts used in the text, as well as how content was sequenced. The curricula are described more fully in a later section of this article.

Methods

Participants

The participants were 51 undergraduate students enrolled in an introductory DE course and the two postdoctoral lecturers who taught them. The course is intended for non-mathematics majors and all 51 students were engineering majors. Of the 51 volunteers, 30 were in Lecture 1 (L1) which used a custom edition of *Elementary Differential Equations and Boundary Value Problems* by Boyce and DiPrima and 21 were in Lecture 2 (L2) which used "An Introduction to Differential Equations for Scientists and Engineers," a set of course notes written by a member of the math department faculty. There were 25 males and 5 females in L1 and 15 males and 6 females in L2. Typically, students were in the end of their freshman or sophomore years, depending on their level of high school mathematics preparation. However, many of the students had earned enough college credit to have junior standing. The students carried, on average, between 10 and 11 credit hours in addition to their course on DEs, the equivalent of two or three additional classes. The mean grade point average (GPA) for the participants was 3.29 (SD = 0.44) and their mean mathematics GPA was 3.14 (SD = 0.61). Their collegiate mathematical preparation, in terms of coursework, was uniform. All had completed single- and multivariable calculus, but not linear algebra. In addition, all had completed at least the first two quarters of their engineering and physics sequences. The two groups had similar backgrounds relative to the number of incoming credit hours, the number of quarters enrolled, the number of credit hours earned, the number of credit hours carried, overall GPA, math GPA, and the number of math, science, and engineering credit hours taken.

Both lecturers held postdoctoral positions and both were familiar with their respective curricula. L2 had some mathematics education experience and coursework during his graduate studies. L1 was a nonnative speaker of English.

Data Collection Instrument

Three tasks were created toward the end of the observation period from material common to both lectures and embedded in the groups' respective final exams. To establish content validity, the items were drafted and revised five times with input from both lecturers, and finally from the course coordinator. This type of negotiation when designing data collection instruments that measure student learning has precedence in mathematics education (Boaler, 2008). The values of parameters were different for the two classes since final exams were administered on different days. The problems were written with the conventions used in each class, eg, derivatives were denoted by y' in L1 and by *dy/dt* in L2. Values for parameters were selected in order to simplify calculations while maintaining structural similarity between the matched problems. Students in both classes were allowed graphing calculators. A brief description of each problem is offered below.

Problem 1 (P1): A first-order linear, constant coefficient, nonhomogeneous mixing problem.

Part (a) asked the students to find the amount of contaminant in a tank for any time t, supposing that initially the tank was full of pure water. In part (b), the flow of the contaminant is turned off at time $t = \tau$. The students were asked to find the amount of contaminant for times $t > \tau$.

L1 wanted his students to derive the differential equation, while L2 did not want to test that knowledge. Both lecturers decided to abbreviate P1, cutting out a third part that asked the students to interpret their solutions in terms of the physical situation. Thus, this problem is contextually situated, but is not an application of DEs to an engineering context. Since these tasks were not identical for the two groups, I do not compare the students' scores on this item beyond including it in the total score for the tasks. However, some information, such as students' handling of the ICs, is intact and relevant to the analyses presented here.

Problem 2 (P2): A second-order linear, constant coefficient, nonhomogeneous DE:

$4u''+u = 4e^{-t/2}$. Given the DE, part (a) asked students to find the general solution. In part (b), students were asked to suppose that $u(0) = u_0$ and that $u'(0) = 1$ and then to find a value of u_0 so that the amplitude of the steady state solution was 5.

The model in P2 represents a system that oscillates with no dampening, subject to forcing that is like an exponentially decreasing pulse. The focus is on the connection between ICs and long-term behavior of the system.

Problem 3 (P3): Separation of variables. Students were asked to use the separation of variables method to replace $m(t)u_t - n(x)u_{xx} = 0$ with a pair of ordinary DEs.

In both lectures, PDEs were most commonly treated with constant coefficients. P3 was designed to indicate whether students were able to use the method of separation of variables in a novel setting or if their knowledge was limited to a sequence of steps specific to the case $m(t) = n(x) = 1$.

Data Collection and Analysis

Since this study followed a mixed methods explanatory design (Creswell & Plano Clark, 2011), quantitative analysis of the exam tasks was followed with qualitative analysis of the students' responses. Quantitative data were generated through the grading of exams for accuracy. Tasks were graded independently from the lecturers so that data transformation decisions would not affect students' grades. The quantitative design was a prospective causal-comparative study and an ANCOVA was selected to analyze the total scores on the constructed tasks while controlling for the students' prior mathematics achievement, as measured by their math GPAs. Other numerical transformations involve frequencies and percentages. Qualitative data were generated through the observation of 24 (out of 29) sessions of each lecture and the detailed inspection of the students' responses to the tasks.

Student responses were examined first to establish what kinds of solutions they attempted. For each problem, a rubric was created to account for the level of difficulty of each stage of the solution process, and each stage was subdivided into steps. For example, one step of solving the DE by the method of integrating factor would require evaluating an integral. This step would not be broken down to diagnose the student's performance on integration by parts in order to avoid assessing prerequisite knowledge.

Each step of a student's response was assigned a value of "correct" (1) or "incorrect" (0), but responses were graded so that step $n+1$ was graded for consistency with step n. This decision prevented minor mistakes from compounding and so errors in student thinking could be diagnosed instead of tabulating wrong answers. There were a total of 14 steps in P1, 13 steps in P2, and 7 steps in P3. The value of each task, and so its parts, were scaled to 14 points to assign equal weight to each in the total score.

24 class sessions of each section was observed over one academic term. Lectures were 48-minutes long and met three days a week. Two 48-minute recitation sections were held on the other two days. Field notes included the instructors' and the students' comments, the instructors' board work, and my overall impressions of how lessons progressed. In analyzing observational records, I focused on lesson content, context, and pedagogy (Saroyan & Snell, 1997), noting lesson structure, the number and quality of examples used in each session, the number of connections made among topics made within and outside of mathematics, interactivity among the instructor and the students, and the mathematical behaviors that were modeled by the instructor.

Curriculum and Instruction

Both instructors used a traditional lecture format and both classes treated the same set of topics. L1 used Boyce and DiPrima's text (T1), which intended to provide exposure to the theory of differential equations with "considerable material on methods of solution, analysis, and approximation" (Boyce & DiPrima, 2009, p. vii). T1 follows an exposition-example-exercise format, where new topics are introduced formally through definitions and formulae, organized around analytic techniques with topics grouped into modules to allow for flexibility in usage. Theorems are stated precisely in symbols and are sometimes proved rigorously and sometimes through example. T1 does not assume familiarity with linear algebra, but many theorems rely on linear operators. Exercises tend to drill for procedural fluency and theoretically oriented problems have step-by-step directions or ask only for verification. In either case, the solution path is evident from the problem statement. Separate sections are devoted to applications, such as mechanical vibrations, and these are placed after the sections that develop techniques for solving the relevant DEs.

L2 used Baker's (n.d.) text (T2) which uses a modeling approach and the author's goal is to draw on common problems in the practice of science and engineering to motivate the creation and solution of DEs. It follows an example-exposition-exercise format where physical considerations precede mathematical formalism in each section. T2 does not contain "theorems," but instead offers "principles" which are written in English. Some principles are proven rigorously and some are justified through examples. Among the rigorously proved theorems, T2 uses English explanations while T1 relies on symbolic proof. There are few truly concrete examples or exercises, as most have at least one parameter. In comparison to T1, there are fewer exercises and these tend to be less technically difficult and less computationally oriented. Only one solution technique is presented throughout T2: guess-and-substitute (G&S), which is similar to the method of undetermined coefficients.

L1 followed the development of topics from abstract to concrete. The lectures were content-driven and L1 made every effort to convey the material on the syllabus. He drew examples from the text's exposition or from exercises that were similar to the assigned homework problems, with a focus on computation. Between one and seven examples were presented each lecture, with an average of between three and four per lecture. The examples followed presentation of theory. L1 held a goal orientation toward problem solving in that examples were considered complete when an answer was reached. He selected examples to maximize variety in technical complexity and sequenced them logically so that the adjustments in parameters from one example to the next were evident. The steps taken in each example were clearly labeled in order to provide guidelines to help structure student thinking about the problems. Taken together, these pedagogical choices yielded high intra-lesson coherence. Mathematics was communicated through symbolic representations and summary formulas. Students were encouraged to ask questions, and they did so regularly but infrequently.

L2 followed the development of topics through the text, but he did not regularly devote class time to explicit discussions of theory. Lectures were context-driven, in that all abstractions were derived through applications. Class time was spent on the structure of the physical problem, recognizing and articulating assumptions, deriving a model from first principles, and justifying its appropriateness, so the sessions had high inter-lesson continuity. Each example modeled a simplified real-life physical problem and each took between one-half and three sessions. L2 focused on building students' awareness of symbols and formulae as tools to represent quantities and relationships. The pace of the session was driven by L2's questioning, but the students were highly interactive, both posing and answering questions, while pursuing systematic exploration of the relationships between physical properties and their reflections in the model. Examples rarely ended in formulae; instead L2 would ask metacognitive questions. Thus, L2 held a process-orientation toward modeling. He wrote very little on the board, instead communicating mathematics verbally.

Findings

Numerical Results

P1 was missing from one student's exam in L1 and so his data are not included in the numerical analyses. Students in L1 scored a mean of 27.55 (SD=1.76) out of the 42 points available (Range: 3.08, 42). The mean score in L2 was 31.89 (SD = 1.90, Range: 6.38,42). A fixed effects ANCOVA model was selected with Lecture as the independent variable and Math GPA as the covariate. All statistical tests were performed at the $\alpha = .05$ level and the data satisfied all assumptions for the model. A homogeneity of slopes test revealed no significant interaction between Lecture and Math GPA. The ANCOVA summary is shown in Table 1. The main effect for Math GPA is statistically significant $(F_{GPA} = 27.191, df = 1,47, p < .001)$ and the adjusted main effect for Lecture is also significant $(F_{Lec} = 5.972, df = 1,47, p = .018)$ with a moderate effect size and showing only moderate power (partial $\eta^2 = .113$, observed power = .668). Thus, when taking prior mathematics achievement into account, the L2 student performed better on constructed tasks.

Source	Sum of Squares	*df*	Mean Square	*F*	Sig.	Partial η^2	Observed Power
Lecture	324.366	1	324.366	5.972	0.018	0.113	0.668
Math GPA	1476.928	1	1476.928	27.191	0.000	0.367	0.999
Error	2552.881	47	54.317				
Corrected Total	4259.339	49					

Table 1. ANCOVA summary table

Solution Strategies

Three techniques can be used to solve the DE in P1: the methods of integrating factor, separation of variables, and G&S. The G&S method is the only method appropriate to solving the DE in P2. In L1, the method of undetermined coefficients was presented in the context of second-order linear DEs of the form**Error! Not a valid embedded object.**. After two lessons, *f(t)* was allowed to be nonzero and L1 explained the method symbolically and procedurally. The students, in the text and in class, were given the form of the particular solution as $t^s e^{\omega t}\left[\sum A_k t^k \cos\beta t + \sum B^k t_k \sin\beta t\right]$, in which they had to set the parameters *s*, ω, and β to match the form of each of the summands of *f,* and then apply the DE to determine the weights A_k and B_k. In L2, the method was explained physically as follows: when written in standard form, the right hand side of the DE represents the external force applied to the physical system (the "forcing term") and that the system responds to the forcing (the "response") in kind. Thus, the form of the response (ie, the solution) must match the form of the forcing. Table 2 illustrates the frequencies of solution strategies used by the students, from left to right, they are: method of integrating factor, separation of variables, guess-and-substitute, and did not attempt (DNA).

	Int. Factor		Separation		G&S		Other		DNA	
	L1	L2	L1	L2	L1	L2	L1	L2	L1	L2
App (a)	26	–	2	1	–	20	1	–	–	–
App (b)	4	–	16	–	–	18	1	–	6	3
Concept (a)	2	–	–	–	28	18	–	–	–	3

Table 2. Chosen solution strategies

Almost all of the L2 students used the G&S strategy. This is not surprising since it was the only strategy they were shown, but they did use it more successfully than L1 students used the other methods. Again, this is not surprising since the L2 students had more practice with the method in various settings. What is surprising is that the L2 students were able to successfully adapt the G&S strategy to the procedural problem, more often than did the L1 students. Many L1 students either set $m(t) = n(x) = 1$ or else incorrectly applied the equation to the guess $u(x,t) = X(x)T(t)$ indicating that the "substitute" part of the strategy was not as prominent for the L1 students. The G&S method highlights the relationship between the solution and the DE by framing the DE as a condition on the solution in much the same way as an algebraic equation is a condition upon its solution. This perspective may be lost with techniques that reduce the relationship between the DE and the solution to a sequence of steps.

Initial Conditions

In P1 part (b), the IC does not correspond to the level of contaminant at time zero, but rather to when the flow of contaminant is halted. Thus, the solution function is piecewise differentiable. Nine students (31%) in L1 handled the ICs correctly in this case as compared with 13 (62%) of the students in L2. The most common mistake committed was to neglect the constant of integration and was made solely by L1 students. The second most common mistake was not ensuring that the pieces of the solution were matched at τ, caused by using $t = 0$. In P2, the ICs determine specific properties of the long-term behavior of the solution so that amplitude is formulated in terms of the unknown ICs. Sixteen students (53%) of the students in L1 handled the ICs correctly as opposed to 19 students (90%) in L2. Many students who arrived at one correct IC did not find the other since they did not extract both roots of the condition squared. However, the most common mistake, among both groups, was to apply the ICs only to the homogeneous solution. The second most common mistake was to apply the ICs to the steady state solution. This points to an interesting conception, since it amounts to setting $t = 0$ after letting $t \to \infty$, but it is not pursued further here.

Both lectures introduced ICs as “another condition on the solution to the DE.” ICs are an important part of the DE model, since they contain parameters that determine how the transient solution adjusts to the steady state solution. One key difference between how the two lectures treated ICs was that in L2, ICs were driven by context and each time were derived from the physical situation during the derivation or the discussion of the model. In contrast, the ICs in L1 were either stated outright at the beginning of the initial value problem or were selected by considering the neatness of the solution.

Discussion and Conclusions

The students in L2 were more successful with the G&S strategy, which can be attributed to a number of reasons. First, there were fewer strategies from which they could choose when solving problems. In L2, guessing was an allowed heuristic. Lastly, the G&S was developed on physical grounds, which allowed for alternative ways of understanding the solution strategy (Harel & Koichu, 2010). One could also argue that since G&S does not require memorization then the strategy is easier to access and implement. In a test-taking situation, the take-the-best heuristic (see Gigerenzer, 2008) might be the only accessible approach to the test taker. In this light, L1 students needed a greater amount of time to search through all possible integration techniques in order to recognize a discriminating cue. The fast-and-frugal nature of the G&S heuristic favors the L2 students whose allowable solution-space includes the G&S method, which mimics the familiar educated-guess-and-check strategies.

In P1 and P2, students needed to attend to the ICs in novel ways. In P1, the ICs join solutions from distinct, non-overlapping time periods. The difficulty exhibited here by L1 students may be less an indicator of correctly applied ICs and may instead be symptomatic of their discomfort with piecewise functions. In comparison, L2 students' success in handling ICs suggests that contextual treatment of ICs both places them appropriately within the solution procedure and strengthens conceptual connections among the components of the mathematical model in a sensible way. Alternatively, L2 would close each example, not with a formula, but by modeling how to justify the solution process by matching properties of the solution with assumptions made at the outset of the example. The looking-back heuristic thus reinforces ICs as relevant to the model. The application orientation of L2 and its text provides a foundation for relational understanding (Skemp, 1987) and legitimizes physical intuition as a way of understanding (Harel & Koichu, 2010) when working with ICs.

Overall, the students in L2 performed better on the constructed tasks than did the students in L1. While there may be a number of extraneous factors contributing to this finding, evidence from a closer analysis of the students' responses reveals two main reasons for the difference in performance. The students in L2 were more adept at treating ICs and they were more successful in selecting and applying solution techniques. While results merit further investigation, such as a larger sample size and more varied instrumentation, they do point to the potential of conceptually- and contextually-oriented curriculum and instruction for enhancing student learning of DEs.

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HOW ARE GRAPHS OF TWO VARIABLE FUNCTIONS TAUGHT?

This is a study about how graphs of functions of two-variables are taught. We are interested in particular in the techniques introduced to draw and analyze these graphs. This continues previous work dedicated to students' understanding of topics of two-variable functions in multivariable calculus courses. The model of the "moments of study" from the Anthropological Theory of the Didactic (ATD) is used to analyze the didactical organization of the topic of interest in a popular calculus textbook, and in a typical classroom presentation. In so doing we obtain information about the institutional dependence of findings in previous studies.

Antecedents

Despite its importance, there are not many published articles in the mathematics education research literature that deal with the particularities of functions of two variables. The first published article we found that explicitly treats functions of two variables is by Yerushalmy (1997). In it he insisted on the importance of the interplay between different representations to generalize key aspects of these functions and to identify changes in what seemed to be fixed properties of each type of function or representation. Kabael (2009) studied the effect that using the "function machine" might have on student understanding of functions of two variables, and concluded that it had a positive impact in their learning. In other work, Montiel, Wilhelmi, Vidakovic, & Elstak (2009) considered student understanding of the relationship between rectangular, cylindrical, and spherical coordinates in a multivariable calculus course. They found that the focus on conversion among representation registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge. Martínez-Planell and Trigueros (2009) investigated formal aspects of students' understanding of functions of two variables and identified many specific difficulties students have in the transition from one variable to two variable functions. Using APOS theory, they related these difficulties to specific coordinations that students need to construct among the set, one variable function, and R^3 schemata. Finally, in a study about geometric aspects of two variable functions, Trigueros and Martinez-Planell (2010) concluded that students' understanding can be related to the structure of their schema for R^3 and to their flexibility in the use of different representations. They gave evidence that the understanding of graphs of functions of two variables is not easy for students, that it can be related to the structure of students' schema for R^3, and in particular, that intersecting surfaces with planes, and predicting the result of this intersection, plays a fundamental role in understanding graphs of two variable functions and was particularly difficult for students.

The way students are taught, and the way mathematical topics are introduced in the textbooks used by students plays an important role on what they learn. In this study we analyze the way graphs of functions of two variables are presented in a widely used textbook, and in standard university classrooms. Our research questions for the part of the study we present here are:

- How is the topic "graphs of two-variable functions" introduced in a widely used textbook?
- How is this topic taught in a university class?
- Are conversions among representations favored?
- What relationships can be found between the above mentioned students' difficulties, the presentation used in the textbook, and the selected classrooms?

Theoretical Framework

In this article we incorporate Anthropological Theory of the Didactic (ATD) as a tool for the epistemological analysis of the textbook and classrooms. In this theory the mathematical activity and the activity of studying mathematics are considered parts of human activity in social institutions (Chevallard, 1997; Bosch and Chevallard, 1999). The theory considers that any human activity can be explained in terms of a system of *praxeologies*, or sets of practices which in the case of mathematical activity constitute the structure of what is called *mathematical organizations* (MO). Mathematical organizations always arise as response to a question or a set of questions. In a specific institution, one or several techniques are introduced to solve a task or a set of tasks. Tasks and the associated techniques, together form what is called the *practical block* of a praxeology. The existence of a technique inside an institution is justified by a technology, where the term "technology" is used in the sense of a discourse or explanation (*logos*) of a technique (*technè*). The technology is justified by a theory. A theory can also be a source of production of new tasks and techniques. Technology and theory constitute the *technological-theoretical block* of a praxeaology. Thus a praxeology is a four-tuple ($T/\tau/\theta/\Theta$) (tasks, techniques, technologies, theories), consisting of a practical block, (T/τ), the tasks and techniques, and a theoretical block, (θ/Θ), made up of the technological and theoretical discourse that explains and justifies the techniques used for the proposed tasks. Typically, a praxeology gives raise to new praxeologies as new problems are explored, techniques are generalized and different ones are introduced, the range of application of technologies expand, and the theoretical basis grows to encompass more general phenomena. This gives raise to mathematical organizations consisting of interrelated sets of praxeologies.

Within an educational institution a mathematical praxeology is constructed by a didactic process or a process of study of a MO. This process is described or organized by a model of six moments of study (Chevallard, 2002) which are: *first encounter* with the praxeology, *exploratory moment* to work with tasks so that techniques suitable for the tasks can emerge and be elaborated, *the technical work moment* to use and improve techniques, the *technological-*

theoretical moment where the technological and theoretical discourse takes place, the *institutionalization moment* where the key elements of a praxeology are identified, leaving behind those that only serve a pedagogical purpose, and *evaluation moment* where student learning is assessed and the value of the praxeology is examined. It is important to clarify that the order of the moments is not fixed. It depends on the didactical organization in a given institution, but independently of the order it can be expected that there will be instances where the class will be involved in activities proper to each of the "moments".

This didactic model is used in this study to describe the mathematical organization related to functions of two variables presented both in the textbook used by students and in the work done in class. This description will be helpful when looking for relations between results of the analysis and students' difficulties and constructions found in the literature. The use of results of this endeavor can be helpful in the design and analysis of activities of a didactic sequence in terms of their institutional suitability with the purpose to help students understanding the concept of two variable functions.

Methodology

This study is related to a project being conducted by the authors in two universities in different countries. A textbook was selected by the researchers to be analyzed considering that it is used in both universities involved in the study, and widely used in other universities. Two researchers independently reviewed the text in terms of the theoretical framework and negotiated their findings until agreement was reached. One of the researcher observed several classes where the topic was introduced by different teachers, took notes about the way functions of two variables was taught by different teachers and interviewed some of them. After transcription of observations and interviews, data was analyzed again by two researchers and results negotiated between them. Results obtained were compared with those found in previous studies and with the genetic decomposition suggested in Trigueros and Martinez-Planell (2010) in terms of the constructions this model proposes for the learning of the topic in order to look for possible relations of constructions found to be made by students and the way functions of two variable are introduced in the text and in classrooms.

In this study we concentrate on results related to graphs of two variable functions. For this purpose, we take into account the analysis of the selected textbook and the classroom observations and interview from one of the teachers who represent the way most of the observed teachers taught this topic to their students.

Analysis of a Textbook Based on the ATD Moments of Study

Graphs of functions of two variables are introduced in courses of multivariable calculus to help students construct a richer mental model to reason about them, and to illustrate the important concepts of multivariable differential and integral calculus.

In the selected text, Calculus: early transcendentals by James Stewart, 6th edition (2006), multivariable functions are introduced in the text in chapter 14 devoted to partial derivatives. However, fundamental planes (that is, planes parallel to the coordinate planes), considered as a prerequisite to understand these functions, are met before, in Section 12.1, where the three-dimensional coordinate system is introduced. This is done through the introduction of the tasks of graphing the planes $z=3$ and $y=5$. This can be considered a *moment of the first encounter* with graphs of two variable functions. These tasks remain isolated since they are not afforded any special role in relation to their use in understanding other subsets of R^3 (by forming intersections, for example), and are not met again until the exercises at the end of the section. Its only connection with a mathematical organization is their appearance in a section devoted to three-dimensional space. There is another *moment of the first encounter* for the topic of graphing functions of two variables in Section 12.6 where the graphs of quadric surfaces are shown for the first time. Even though quadric surfaces are not always graphs of functions of two variables, the task of graphing an ellipsoid with equation $x^2 + y^2/9 + z^2/4 = 1$ is introduced, and the technique to do the task explained: *"By substituting $z = 0$, we find that the trace in the xy-plane is $x^2 + y^2/9 = 1$, which we recognize as an equation of an ellipse"*, and immediately generalized to a families of traces: "*In general, the horizontal trace in the plane $z = k$ is $x^2 + y^2/9 = 1 - k^2/4$* ... " . We found another example of a *moment of first encounter* in Section 14.1 where after defining functions of two variables, the text presents the first examples of graphs of functions. A linear function and the top half of a sphere are graphed by recognizing their types of equations (no use of traces), then a graph is generated by a computer without using or mentioning traces, and finally an elliptic paraboloid is presented, one that had been presented before, by making reference to its prior appearance.

We can observe an absence of questions generating the need for graphing functions, nor the importance of understanding how to graph them. These facts, together with the isolation of the tasks presented, are not conductive to a mathematical activity where techniques arise in a productive way in terms of students' learning.

In the *exploratory moment* in Section 12.6, the tasks of graphing five quadric surfaces are introduced, perhaps not as an end in and of itself, but as a means to establish the technique of traces that expectedly is to be further developed with the continued exploration of other tasks. Although at first sight this number of might seem quantitatively adequate, the examples and their accompanying explanation require students from the outset to recognize and place in space a family of curves, a task that has not been introduced before. The tasks in this exploratory moment are not adequate to prepare students to use traces as a technique to draw the graphs of two variable functions. The exploration continues in Section 14.1, where the text mentions the use of traces in computer generated graphs and shows four such graphs with hardly any comment. Here, the technique of using traces to draw graphs of function is related to tasks done with technology, but the way it is presented makes it difficult for students to interpret how the technique works in this mathematical organization.

As we can see the textbook's moment of exploration does not present students with opportunities to encounter relevant tasks which can help them make sense of what the traces shown are about, they also are not given real opportunities to explore the tasks in order to find regularities or properties which can help them make sense of the technique that is being introduced.

To examine the *moment of practice of the technique* we found that in Section 12.1 there are some exercises that make direct or indirect use of fundamental planes to describe regions in three-dimensional space. Only a few of them are assigned exercises in the syllabus of both courses. Further and more importantly, these exercises do not provide the opportunity of exploring the result of intersecting fundamental planes with subsets of R^3 and to relate these intersections with the technique of traces introduced before. Then, in Section 12.6, relatively few exercises at the end of the section require using sections to produce the graph of a surface, and only six of them are assigned in the courses' syllabus. In Section 14.1 we found some exercises that require using cross-sections to draw the graph of a two variable function, most of them are presented in terms of matching problems. This kind of problems would be useful to exercise the technique of drawing graphs of surfaces if they required students to justify their selections using cross-sections. As they are presented, students attempt using other strategies, frequently without success, and hence these tasks do not really give students the opportunity to practice the technique introduced. Exploratory moments are not integrated and systematic; the text has hardly any task where interpretation or justification of the technique is needed. It introduces other techniques, i.e. recognition of the algebraic form of quadric surfaces, but unrelated with the use of traces.

We can say that the text does not provide enough opportunities for the students to work on the practical block of the praxeology and no ground is set to develop consistently the theoretical part of the praxeology related to graphing functions and converting flexibly among different representation registers.

As discussed in Chevallard (2007), the *moment of development of technology and theory* is closely interrelated with each of the other moments of study. This is clearly seen to be the case in this topic. The technology of using traces or cross-sections to draw the graph of a function of two variables is introduced in the moment of first encounter and developed with scant opportunities to do task explorations, as discussed above. In the book, there is not an explicit discussion of the fact that substituting a number for a variable in an equation with three variables corresponds to intersecting a fundamental plane with the graph of the equation. Hence, there is hardly an explanation about cross-sections, projections, and contours. The examples discussed in Section 12.6 assume that students can readily recognize families of curves and place them in space; the reader is left to make sense by him or herself, or resort to a memorized table of surfaces and formulas to answer questions about graphs of two variable functions.

From the point of view of ATD, the lack of a technology to make sense of the technique introduced leaves the mathematical organization ungrounded. Students may not understand why the graph of a function of two variables is important, why it is a surface and how to make sense of even computer generated graphs. The mathematical organization constructed consists of isolated ideas. It does not have coherence.

Consider now the *moment of institutionalization.* From the point of view of the two educational institutions, as documented in the course syllabus, this is a topic to be mastered. Differential and integral calculus of functions of two variables is to a large extent based on reducing problems of functions of two variables into problems of functions of one variable by holding a variable fixed and analyzing the resulting one-variable function. The textbook can be considered as the reference for the institutionalization of those elements of the mathematical organization that are considered as mathematical objects. However, as discussed before, the way graphs of functions are presented does not provide any ground for students to be aware of the importance of this idea. Moreover, quadric surfaces are not used again in any substantial way after their introduction in Section 12.6, their importance and their role is never clarified. They remain isolated from the rest of the mathematical organization being presented in multivariable calculus. Even if it appears that the intention was to use them, as a means to introduce sections in graphing surfaces, this goal is not achieved as we have seen. They are not reinforced in Chapter 14 where functions of two and three variables are introduced. The *moment of institutionalization* of this topic is not clear.

The *moment of evaluation* of what is learned by students is done by the instructors teaching the course. Although the textbook includes some review questions and a test, for the most part they are not related with graphing functions of two variables, and they are not used by the teachers. If we review the courses' syllabus we find that students are evaluated using three or four partial examinations and a final examination. In one of the universities the topic of graphing surfaces comes late in the semester of Calculus II, and is not included in a partial examination, so it accounts for no more than 3% of the final grade but in Calculus III the topic is evaluated more thoroughly. At the other university the topic is covered at the beginning of Calculus III and is evaluated in a partial examination. Despite the fact that the analysis of graphs of two variable functions by considering its restriction to fundamental planes is pervasive throughout the study of calculus of two variables, its introduction through the textbook and in evaluations, does not stress that importance. Students get the impression that this is not an important topic and may not pay attention to it. There is not a clear presence of the *moment of evaluation* of the technique itself in the textbook.

Analysis of the Teaching of the Graphs of Two Variable Functions

As can be expected, professors in their class complement the information given in textbooks. We now use the moments of study to analyze the data we obtained from observing the classes of a particular professor during the two weeks he devoted to the teaching of functions of two variables, as an illustration of how the teachers work in class, since we found out many similarities in the work of all the teachers observed, even though they teach in different countries. Again, we focus here on their graphs.

After a brief discussion of one variable functions where the teacher emphasizes graphs of these functions, in particular linear and quadratic functions, he introduces functions of two variables as an extension of functions of one variable and asks students how they would graph this type of function and what kind of geometrical object the graph would be. After discussion with the whole group the teacher leads the students to the fact that the graph should be a surface. He introduces the equation of a plane in R^3 and its relationship to the graph of the plane, using fundamental planes in the analysis. Students are given a few tasks where they have to draw planes parallel to the coordinate planes and to find their equation. The *moment of the first encounter* with two variable functions is presented in terms of the relationship between functions of one and two variables and the question of how the graph would look like. Fundamental planes are introduced but there is no justification of why they are needed.

Later on, the class as a whole discusses quadric surfaces; the tasks for discussion consist in finding the conic curves that combined give raise to different surfaces. Then tasks are given to the students where they have to find intersections of a surface with fundamental planes in order to graph a given function and find its domain and range.

Thus the *moment of task exploration* consists of a series of tasks designed for the students to be aware of the difficulties involved in graphing functions of two variables and how they can use some strategies, such as using fundamental planes, including intersections with the coordinate planes to help them doing so. Graphs of functions are also linked with other properties of functions of two variables. An example of a task is: draw the graph of the function $z=2-y^2$ by using different planes of the form $x=k$. It is interesting to note that when students use the planes, the equation remains the same, so they have to discuss the difference between the graph of the function and the intersection curves. The teacher then gives homework where students can use computer generated graphs to relate the equation of quadric surfaces with their graphs. In the moment of task exploration students work with graphing functions of two variables using fundamental planes. The technique is introduced as a means to facilitate graphing the functions and it is also linked to the domain and range of the given functions. We consider that up to this point, the teacher is constructing the practical part of the praxeology. We noticed, however that this moment is not expanded to the homework; it goes back to graphs and equations of quadric surfaces without connection to the introduced tasks.

The *moment of practice of the technique* is quite restricted. After introducing the technique the teacher goes back to the definition of function of two variables and gives students some tasks where they have to find domain, range and graph of several functions. Students can either use what they have reviewed about quadric surfaces and their graphs, or fundamental planes. Most of the tasks are related to quadric surfaces.

Students do not have enough opportunities to work with the technique, although it seems clear to them that the use of fundamental planes is related to finding these graphs.

While working with the technique there is some explicit discussion of the fact that substituting a number for a variable in an equation with three variables corresponds to intersecting a fundamental plane with the graph of the equation, and some explanation about contours and projections. However, since the number of tasks worked by the students is reduced, in our opinion, the *moment of the development of the technology* is not well developed, this may leave students with a superficial idea of the importance of the technique, although they may be able to use it with simple functions. The technological-theoretical block of the praxeology is not well developed in this class.

The professor works with other examples when finding graphs of functions, and each time he makes clear that fundamental planes, contours and projections are useful to understand how the function “behaves”. This repeated emphasis can be considered as the *moment of institutionalization* of the topic we are concerned with.

In the first partial exam the teacher asks students to draw the graph of a simple function using fundamental planes. He also asks a question where students have to find the intersection of a fundamental plane with a given surface and they have to graph the resulting curve in the plane. This can be considered *the evaluation moment*.

Discussion and Conclusions

Results of the previous analysis show that even though curriculum and teachers underline the importance of the graphs of functions of two variables in relation to their study, the components of the praxeology are presented in an isolated and incomplete form in the textbook used and in the analyzed classrooms: its *practical block* is presented in a very superficial way, there are very few opportunities for students to do *tasks* where they can establish relationships among different representations, and the *technological-theoretical block* is absent. This presentation is bound to be ineffective, in terms of learning. Its importance is also not communicated to the reader. We can also see that although professors, as demonstrated by the one described here, present this praxeology with more elements, including more work on *tasks* relating representations, in terms of the *moments of study,* this presentation is also incomplete. It consists of quite isolated elements, since the *technique* introduced is not practiced enough and is not clearly related to a *technological* discourse.

In previous research Trigueros and Martínez-Planell (2009, 2010) investigated the relationship between students’ notion of subsets of Cartesian three-dimensional space, and their understanding of graphs of two-variable functions with students who had taken courses as the one described here and used the textbook as a reference, using APOS theory and Duval’s theory of representations as a theoretical framework. Results from these studies showed that understanding of two variable functions is not easy for students and can be related to the structure of their schema for R^3 and to their flexibility in the use of different representations. In particular, it was shown that students who had completed a multivariable calculus course present many difficulties with constructions involving fundamental planes and surfaces in Cartesian space. Most of the interviewed students had difficulty relating information about two variable functions in different representation registers, did not use fundamental planes to draw graphs, showed confusion between surfaces, curves and solids in space and only one of them was found to be able to intersect surfaces with planes and predict the result of this action. Results also demonstrated that most students have many difficulties understanding functions of two variables, their domain and range, and that the generalization of understanding of one variable functions to two-variable functions, in particular in the case of graphical representation, is not direct. The study showed, for example, that many students do not readily convert the action "substituting $z = 0$" , which is a *task,* into that of intersecting a fundamental plane with the surface, so they are not ready to consider families of traces. They need more opportunities to work on tasks that help them interiorize actions into processes, and thus build the necessary *techniques*.

Results of the present study, together with those of the previous ones, show a more complete and detailed picture of observed phenomena. It is not surprising that students do not learn how to use traces to draw graphs of two variable functions. It is also comprehensible why they have difficulties to distinguish different subsets of Cartesian space and do not clearly understand the importance of being able to analyze graphs of these functions. Results obtained from the analysis of the *moments of study* of the mathematical activity associated with this topic demonstrate that not enough opportunities are given to students to master the *techniques* and *technologies* needed to analyze and use graphs of two variable functions. So, an effort needs to be done to balance activity in class so than all the moments of study are present in the study of these functions and help students deepen their understanding. It is true that students can use technology to see the graphs of functions, but without the necessary tools to make sense of the information contained in them, they may not "see" what the teacher intends to show.

Taken together, results of both studies give information about what needs to be done. We have started some work in this direction. We have designed sets of activities aimed at helping students to construct a better understanding of functions of two variables. Activity sets are available on the internet at http://math.uprm.edu/~rafael/. The design of these activities follows the constructions modeled in the refined genetic decomposition described in Trigueros and Martinez-Planell (2010), and also takes into account the characteristics of the two universities where the studies took place.

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INVESTIGATING RELATIONS BETWEEN ABILITY, PREFERENCE, AND CALCULUS PERFORMANCE

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The goal of the present study was to report an instrument designed to determine students' mathematical performances and preference for visual or analytic thinking for the calculus derivative and antiderivative tasks as well as examine the relationships among students' cognitive style, cognitive ability, and mathematical performance in calculus. Data were collected from 150 Advanced Placement calculus students. The results suggest that the instrument is measuring an important component of cognition and has the potential to be a measure of performance and preference for visual thinking in calculus.

Introduction

Researchers have been interested in identifying the preference and ability components of cognitive style for several decades (e.g., Clements, 1979; Bishop, 1980, 1989; Hegarty & Waller, 2005). The visualizer-verbalizer distinction in particular has been an area of interest for researchers in various disciplines (e.g., Hadamard, 1945, Kozhevnikov, Hegarty, & Mayer, 2002; Paivio, 1971; Richardson, 1969, 1977). The goal of the present study was to report an instrument designed to determine students' mathematical performances and their preferred mode of thinking for the calculus derivative and antiderivative tasks as well as examine the relationships among students' cognitive style, cognitive ability, and mathematical performance in calculus.

Background

Krutetskii (1976) identified types of mathematical giftedness based on students' preferences for two cognitive processes: verbal-logical or visual-pictorial. Following the work of Krutetskii (1976), Moses (1977), Lean and Clements (1981), Suwarsono (1982), and Presmeg (1985) have recognized that individuals could be placed on a continuum (i.e., degree of visuality) according to their preference for visual processing. In designing instruments —Problem Solving Inventory (PSI) and Mathematical Processing Instrument (MPI)—consisting of algebra word problems to determine students' preferences, and in their work, Moses and Suwarsono defined visuality as the extent to which a learner prefers to use visual processes to solve mathematics problems. That is, visualizers are considered as learners who prefer to think with images and visual strategies, and analyzers (or verbalizers) as learners who prefer not to think with images and visual strategies when there is a choice on a specific task. In this study, we used the visualizer-verbalizer distinction as a lens to determine students' preferred mode of thinking. However, in the remainder of this paper, we have used the terms "analytic" and "analyzer" interchangeably to describe verbal-logical processing or verbalizers.

There have been studies of cognitive abilities and styles in mathematical performance in different content areas. Battista (1990) with high school students found that spatial visualization and logical reasoning were significant factors of geometry achievement and geometric problem solving, and that spatial visualization was related to the use of visual and analytic problem solving strategies (i.e., analytic, visual without drawing, and visual with drawing). A similar finding was reported by Ferrini-Mundy (1987), who found a correlation between spatial ability and certain aspects of calculus. However, other research has shown divergent perspectives. Studies including MPI, developed by Suwarsono (1982) and later modified by Presmeg (1985) as a measure of visualizer-analyzer cognitive style have found either a weak relationship or no relationship between either mathematical performance or cognitive abilities and mathematical visuality. For instance, Galindo-Morales (1994) compared mathematical visuality indicated by MPI and performance of students enrolled in three calculus courses using different instructional approaches (i.e., graphing calculator, Mathematica, and no technology) and concluded that there was no significant relationship between the degree of visuality and calculus performance in any of the three groups. Hegarty and Kozhenikov (1999) administered the MPI to measure sixth grade students' problem solving performance and preference for visual thinking. Their results revealed that mathematical visuality did not correlate with problem solving performance and was negatively associated with the cognitive abilities—verbal ability, nonverbal reasoning, and spatial ability.

Our contention is that calculus requires visual thinking and adequate understanding of visual representations, and that aspects of visuality (or visual imagery) that play an important role in calculus performance may not be measured accurately by existing questionnaires consisting of tasks that do not involve calculus. Moreover, research has shown that the nature, complexity, or novelty of a task influences the degree of visuality (or visual imagery) a student uses when solving the task (e.g., Dean & Morris, 2003; Lowrie & Kay, 2001; Massa & Mayer, 2006; Paivio, 1971; Richardson, 1977). Although various tasks and questionnaires have been designed to measure cognitive styles and learning preferences related to the verbalizer-visualizer distinction (e.g., Mayer & Massa, 2003; McAvinue & Robertson, 2006-2007; Riding, 2001), no adequate instrument for Calculus is available. Thus, there is a need for a calculus instrument designed to determine students' mathematical visuality. We believe this demands research examining the role of cognitive abilities and styles in calculus performance. The present study extends existing research on cognitive styles by examining the relationships among calculus students' preferred modes of thinking, cognitive abilities, and mathematical performances and provides measures of visualizer-analyzer style dimension and mathematical performance in calculus.

Method

Participants

The participants were 169 high school students who were enrolled in Advanced Placement (AP) calculus courses at four high schools in two school districts in Central Florida in the United States at the time of the study. All 169 students agreed to participate in the study. Nineteen students who failed to take all tests were excluded from the data. Of the 150 students, 55 percent of the students were males, and 45 per cent were females.

Materials

The six tests, measuring spatial orientation (Cube Comparisons (CC) and Card Rotations (CR)), spatial visualization (Form Board (FB) and Paper Folding (PF)), and logical reasoning (Nonsense Syllogisms (NS) and Diagramming Relationships (DR)) abilities, are part of the KIT of Reference Tests for Cognitive Factors (Ekstrom, French, & Harman, 1976). Cognitive style tests consisted of a revision of Mathematical Processing Instrument for Calculus ([MPIC], Haciomeroglu et al, 2009) and a modified version of Mathematical Processing Instrument

([MPI], Suwarsono, 1982). The students' scores on the Advanced Placement (AP) Calculus Exam were collected from teachers at the end of the study.

Spatial Ability Measures

The Cube Comparisons Test consists of 21 items and requires the participant to view two drawings of a cube and determine whether or not the two drawings can be of the same cube. The Card Rotations Test consists of 10 items, each of which presents a two-dimensional figure and eight other drawings of the same card. The participant indicates whether each of the eight cards, without reflecting, is the same or different from the original figure. The Form Board Test consists of 24 items. Each item presents five shaded drawings of pieces and requires the participant to decide which of the shaded figures, from two to five, can be used to make the given geometric figure. The Paper Folding Test consists of 10 items each of which illustrate folds made in a square sheet of paper and a hole punched in it. The participant selects one of the five drawings that shows the position of the holes when the paper is completely unfolded.

Logical Reasoning Ability Measures

The Nonsense Syllogisms Test consists of 15 items. Each item is a formal syllogism, in which statements are nonsense and cannot be solved by reference to past learning. The participant determines whether conclusions drawn from the statements show good reasoning. The Diagramming Relationships Test consists of 15 items. In each item, the participant selects one of five diagrams, which illustrates the interrelationships among sets of three objects.

Cognitive Style Measures

Two cognitive style tests, a revision of Mathematical Processing Instrument for Calculus ([MPIC], Haciomeroglu et al, 2009) and a modified version of Mathematical Processing Instrument ([MPI], Suwarsono, 1982), were used to determine the degree to which students preferred visual or analytic thinking.

The MPIC and the MPI consist of two parts. The first part of each instrument is a test consisting of mathematical tasks: there are 10 derivative and 10 antiderivative tasks (i.e., 7 graphic and 3 algebraic tasks in each test) on the MPIC and 8 algebra word problems on the MPI. The second part is a visualizer-analyzer questionnaire consisting of a visual and an analytic solution for each task on the MPIC and at least 3 or more visual or analytic solutions for each task on the MPI. Upon completion of each test, the students were given the visualizer-analyzer questionnaire and were asked to choose for each task a method of solution that most closely describes how they solved the tasks.

In this study, the MPIC was used to measure the students' preference for visual thinking and mathematical performance for derivative and antiderivative tasks presented graphically or algebraically. Thus, it yielded two performance and two visuality scores for each student: performance (P-G) and visuality (V-G) scores from 14 graphic derivative and antiderivative tasks, and performance (P-A) and visuality (V-A) scores from 6 algebraic derivative and antiderivative tasks. The MPI was used to measure mathematical visuality, but not performance because it consisted of algebra word problems and may not reflect the differences in their mathematical performance. The internal reliability of the MPI visualizer-analyzer questionnaire was 0.225. The internal reliability of the V-G (14 graphic tasks) and V-A (6 algebraic tasks) visualizer-analyzer questionnaires were 0.918 and 0.71 respectively.

Calculus Performance Measures

Three calculus performance scores were included in the analyses. The students' scores on the AP Calculus Exam were collected from teachers at the end of study. The AP Calculus Exam is an important standardized test. High school students who perform well can earn college credit and advanced placement. It covers differential and integral calculus topics, and scores are reported on a 5-point scale (5 is the highest and 1 is the lowest). The students' calculus performance was also assessed by the MPIC Derivative and Antiderivative tests presented graphically and algebraically. The internal reliability of graphic (P-G) and algebraic (P-A) tests were 0.801 and 0.36 respectively.

Procedure

All students received standardized instructions and were tested in groups of 12 to 30 in their classrooms. All participating students gave their informed consent and were debriefed at the end of the study. Four school visits were made during semester, and the tasks were administered in the following order: At the first visit, Form Board, Card Rotations, and Diagramming Relationships were administered. At the second visit, Paper Folding, Cube Comparisons, and Nonsense Syllogisms were administered. The students had completed MPI test prior to the third visit, and they were first given MPI questionnaire and then MPIC Derivative test and questionnaire at the third visit. At the fourth visit, MPIC Antiderivative test and questionnaire were administered. The students were willing to participate in the study and enjoyed most of the tests under classroom conditions. We were unable to administer fewer tests per day due to the time restrictions. Results might have been higher under research conditions. The students were given 8 minutes for FB, 4 minutes for ND and DR, and 3 minutes for CC, CR, and PF. Completion of MPIC and MPI was not timed. The total scores for CC, CR, FB, and NS tests were determined by subtracting the number of incorrect answers from the number of correct answers. Since there were 5 response options for each item on PF and DR, the total scores were determined by subtracting one-fourth the number of incorrect answers from the number of correct answers.

Scoring of MPIC and MPI

In determining preference for visual or analytic thinking, the primary goal was to identify the students' methods as visual or analytic; whether their answers were correct or incorrect mattered less than their method(s) in measuring mathematical visuality. On the MPIC visualizer-analyzer questionnaire, to determine the students' visual preference scores for the derivative and antiderivative tasks, they were given a score of 0 for each analytic solution and 2 points for each visual solution. If a solution does not give any indication of method or both methods were used, a score of 1 was given. On the MPI visualizer-analyzer questionnaire, to determine the students' visual preference scores for the algebra word problems, they were given a score of 0 for each analytic solution and 1 point for each visual solution. Thus, for the derivative, the antiderivative, and the MPI questionnaires, the total possible scores were 20, 20, and 8 points respectively.

In assessing students' performance on the MPIC Derivative and Antiderivative tests, the students were given a score of 0 for each incorrect answer and 1 point for each correct answer. Thus, for each of the two tests, the total possible score was 10 points. To illustrate the use of the MPIC, we give an example of one of the derivative tasks (see Figure 1) and the corresponding item in the questionnaire. We consider thinking as visual when individuals prefer to use visual methods and thinking as analytic when individuals prefer not to use visual methods when there is a choice on a specific task. Analytic solutions are generally equations-based. An analytic solution to a task presented graphically typically may involve translation to an equation, computing the integral of the equation, and then using this new equation to draw the antiderivative graph.

Graph is shown. Please sketch the graph of the derivative.

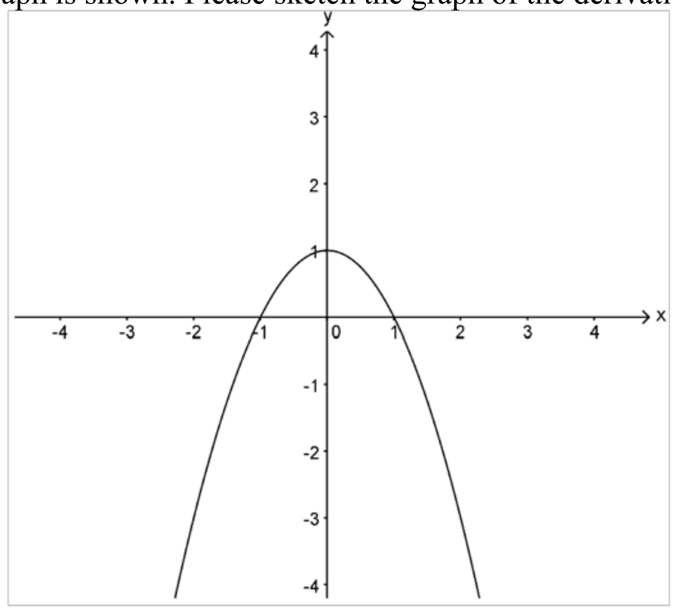

Figure 1. Derivative task.

We observed that instead of estimating equations precisely, analytic students referred to basic groups of functions such as linear, quadratic, or cubic functions and their derivative graphs associated with odd or even powers of x respectively. The following is the analytic solution given on the questionnaire for the derivative task in Figure 1: *I estimated the equation of the graph (or recognized the equation of the graph). For example: This could be the graph of* $f(x) = -x^2$ *so I computed the derivative as* $f'(x) = -2x$ *and drew the derivative graph using this equation.*

Visual solutions are image-based. They are able to visualize the changing slopes of tangent lines to the function and accordingly are able to construct an entire derivative graph with no need to consider individual parts of equations at critical points or intervals. These individuals are able to determine the shape of derivative graphs based on their visual estimates of slopes. The following is the visual solution given on the questionnaire for the derivative task in Figure 1: *From the graph I estimated the slopes (or the slopes of tangent lines) at various points on the graph of the function and used this to draw the graph of the derivative. For example: The slopes of tangent lines are positive and decreasing as x approaches 0 from the left. The slope is zero at x = 0 because the graph of the function has a horizontal tangent line at (0, 1). The slopes of tangent lines are negative and decreasing as x approaches positive infinity.*

For the tasks presented algebraically, we consider thinking as visual when students prefer to draw the graph of the given function on paper (or in mind) and estimate the slopes of tangent lines at various points on this graph to draw a possible graph of the derivative or antiderivative. On the other hand, we consider thinking as analytic when students prefer to calculate the derivative or integral, and used this equation to draw a possible graph of the derivative or antiderivative. For instance, one of the algebraic tasks requires sketching a possible graph of the antiderivative, given $f'(x) = 3x^2 + 1$. An analytic solution involves computing the integral as $f(x) = x^3 + x^2 + c$ and drawing the graph of f(x) using this equation, whereas a visual solution involves drawing the graph of $f'(x) = 3x^2 + 1$ on paper (or in mind) and using the y values to estimate the slopes to draw the graph of the antiderivative.

Results

Means and standard deviations for each of the ten measures appear in Table 1. In order to determine between cognitive styles as assessed by the MPIC and the MPI visualizer-analyzer questionnaires and the other variables, Pearson product-moment correlations were computed. The correlations between cognitive styles, abilities, and mathematical performances are presented in Table 2.

Correlational Analysis

There was a significant correlation between the three measures of calculus performance. The correlations between MPI and the other two measures of cognitive style V-G and V-A were non-significant and negative. Of the three measures of cognitive style, V-A significantly correlated with AP and P-G. There was a significant but small correlation between V-A and P-G. The MPI had non-significant negative correlations with the three performance measures. The correlations between the three measures of cognitive style and the measures of spatial and logical reasoning abilities were either negative or non-significantly low.

Among the spatial ability measures, only FB had a significant correlation with P-A. CC and CR had the lowest correlations with the performance measures. FB, PF, NS, and DR significantly correlated with AP and P-G. The correlation between CC and P-G was significant, but CR was correlated neither with AP nor with P-G. It can be seen from the correlations of cognitive ability tests, except CR and FB, the four measures of spatial ability significantly correlated with each other. The two measures of logical reasoning ability significantly correlated with each other. DR correlated three of the four measures of spatial ability, CC, FB, and PF, whereas NS only correlated with FB, suggesting that FB was the only spatial ability measure correlating with both measures of logical reasoning ability.

Table 1
Means and Standard Deviations of Measures

Measure	Label	M	SD	n
1. AP Calculus Exam	AP	2.69	1.55	150
2. MPIC Derivative & Antiderivative Tests - Graphic	P-G	0.46	0.26	150
3. MPIC Derivative & Antiderivative Tests - Algebraic	P-A	0.24	0.18	150
4. Cube Comparisons Test	CC	9.99	4.70	150
5. Card Rotations Test	CR	59.01	15.35	150
6. Form Board Test	FB	7.74	5.68	150
7. Paper Folding Test	PF	6.52	2.23	150
8. Nonsense Syllogisms Test	NS	2.55	4.50	150
9. Diagramming Relationships Test	DR	8.69	3.74	150
10. MPIC Visualizer-Analyzer Questionnaire - Graphic	V-G	1.09	0.67	150
11. MPIC Visualizer-Analyzer Questionnaire - Algebraic	V-A	0.60	0.56	150
12. MPI Visualizer-Analyzer Questionnaire	MPI	0.62	0.18	150

Table 2
Correlation Matrix for Twelve Measures

Measure	1	2	3	4	5	6	7	8	9	10	11	12
1. AP	—											
2. P-G	.62*	—										
3. P-A	.42*	.54*	—									
4. CC	.23	.28*	.20	—								
5. CR	.16	.24	.04	.50*	—							
6. FB	.38*	.40*	.28*	.45*	.23	—						
7. PF	.33*	.33*	.15	.36*	.35*	.47*	—					
8. NS	.30*	.40*	.17	.20	.14	.27	.10	—				
9. DR	.36*	.40*	.23	.34*	.18	.37*	.30*	.41*	—			
10. V-G	.31*	.51*	.18	.09	.08	.19	.17	.18	.16	—		
11. V-A	.11	.28*	.11	.02	.01	.15	.08	.12	.11	.40*	—	
12. MPI	−.08	−.08	−.05	.07	.04	.12	.09	−.08	.11	−.09	.06	—

*$p < .05$.

Conclusions

This study contributes to the existing research by examining the relationships between cognitive styles, cognitive abilities, and mathematical performances in calculus. The correlational matrix revealed that spatial orientation ability, measured by Card Rotation and Cube Comparisons tests, did not correlate with calculus performance. Unlike spatial visualization and logical reasoning ability, spatial orientation seems to be unrelated to calculus performance although visualizing mathematical objects from different perspectives is crucial to understanding calculus. The significant correlation between spatial visualization and calculus performance could be partially attributed to the tasks that require sketching derivative or antiderivative graphs; however, this trend is also evident in consideration of correlations with AP test scores. The MPIC and MPI visualizer-analyzer questionnaires did not correlate with spatial ability and logical reasoning ability measures, suggesting that cognitive abilities do not influence students' preference for visual or analytic thinking, and vice versa. This is consistent with previous research (Hegarty & Kozhevnikov, 1999; Lean & Clements, 1981; Moses, 1977; Suwarsono, 1982). Krutetskii (1976) also observed that gifted students do not possess strong spatial abilities and might prefer not to use visual methods.

A factor analysis on the twelve variables in Tables 1 and 2 provides interesting results. Using the varimax rotation, eleven of these variables load onto four easily interpretable factors: a calculus performance factor with AP (0.582), P-G (0.721), P-A (0.612); spatial ability factor with CC (.717), CR (.639), PF (.523), FB (.495); logical reasoning factor with DR (.76), NS (.416), and cognitive style factor with V-G (.729), V-A (.522), P-G (.426). A modified version of Mathematical Processing Instrument ([MPI], Suwarsono, 1982) was used to measure the calculus students' visual preference. The MPI did not did not load on any of the four factors and did not correlate significantly with any measure. On the other hand, the MPIC test and questionnaire regarding derivative and antiderivative tasks presented graphically loaded substantially on the cognitive style factor and correlated significantly with calculus performance measures, suggesting that the MPIC is measuring an important component of cognition. Our results are consistent with those of Galindo-Morales (1994), who reported that visuality as assessed by the MPI was not related to calculus performance. However, when calculus derivative and antiderivative tasks were used to measure mathematical performance and visuality, significant correlations can be found. Moreover, most calculus students have acquired a deep conceptual understanding of mathematics and might have characteristics that distinguish them from others (Ferrini-Mundy, 1987).

Our work with AP calculus students has generated new information about ability, style and mathematical performance in calculus. We believe that analyses of data obtained with the MPIC have produced results worthy of continued study, and that the MPIC has the potential to be a measure of performance and preference for visual thinking in calculus.

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HOW DO STUDENTS CREATE OPPORTUNITIES TO LEARN MATHEMATICS?: REPRESENTING STUDENTS IN RESEARCH ON CURRICULUM USE

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The purpose of this paper is to argue for increased attention to students' experiences in research on mathematics curriculum. I call for examining students' roles in the process of curriculum use to complement studies of student data to assess the outcomes of curriculum use. I present a framework that illuminates students' roles in the process of curriculum use.

Opportunities to learn are considered to be the strongest influence on what students learn (Hiebert & Grouws, 2007). One way to think about creating opportunities to learn is through a process of curriculum construction. When students are represented in studies of mathematics curriculum, student performance (and, more rarely, attitudes or dispositions) is assessed as an outcome indicator. However, students are less frequently represented in research studies as explicitly part of the process of constructing curriculum. The purpose of this paper is to argue for an increase in research on how students play roles in creating opportunities to learn mathematics. In this paper, I present a framework for the ways in which researchers have represented or might represent students as part of the *process* of curriculum construction, beyond representing students as outcomes of teachers' curriculum use.

The need for more complex representations of students in research on mathematics curriculum has been requested previously. Smith and Star (2007) argued for broader conceptualizations of impact of mathematics curriculum. They recommended that researchers look beyond achievement outcomes to include students' disposition measures as outcomes and to examine interactions between indicators of impact when studying the effects of curriculum. The Research Advisory Committee [RAC] of the National Council of Teachers of Mathematics [NCTM] (Confrey et al., 2008) called for examining subgroups of students to assess *for whom* various curriculum materials and approaches are more and less effective (and why). The RAC suggested that a more complex representation of students would involve accounting for race, socio-economic class, and gender when studying the effects of curriculum materials. Erickson and Shultz (1992) noted that when researchers have investigated students' experiences in classrooms, they rarely accounted for subject matter. In this paper, I extend these calls for more complex representations of students in research on mathematics curriculum by emphasizing a focus on students' roles in the *process* of creating opportunities to learn mathematics.

Why is it valuable to consider how students are represented in research on mathematics curriculum? In the context of post-No Child Left Behind, so much emphasis is on "what works." Students are represented in terms of achievement scores, as outcomes in the process of implementing instructional innovations (including particular curriculum materials or teaching practices). Yet we realize that students are more than their test scores. Classrooms are social contexts, and all of the participants – including students – affect the unfolding of social events. As the social event of a lesson unfolds, it usually does not do so exactly as a teacher planned; changes are often made to the lesson in the moment, as the participants respond to one another. In this way, among others, students can play a role in constructing what can be learned during a lesson.

In this paper, I present a framework for characterizing how students influence their opportunities to learn mathematics. I take up the following questions in this paper: How have students been represented in research on mathematics curriculum? How *might* students be represented in research on mathematics curriculum? What are the affordances and constraints of these different constructions? My analysis of current research on mathematics curriculum suggests the need for further research on the role of students in the *process* of curriculum use.

Theoretical Framework

To describe my general perspective on research on mathematics curriculum, first I define what I mean by "curriculum." Then, I share some of the purposes for studying mathematics curriculum use in past research, and I discuss how students have been represented in this area of research. The Mathematical Tasks Framework (Henningsen & Stein, 1997) is presented for the insights it provides about how students are constructed and might be constructed in research on curriculum.

Curriculum

Following Stein, Remillard, and Smith (2007), I view curriculum broadly as "the substance and content of teaching and learning – the 'what' of teaching and learning (as distinguished from the 'how' of teaching)" (p. 321). I view curriculum construction as a dynamic & flexible process. Opportunities to learn mathematics in a classroom involve teachers and students participating with written curriculum materials and each other (Remillard, 2005). These learning opportunities are shaped by the tasks in the written materials, how teachers implement the tasks, and how students engage with the tasks.

Purposes of Research on Mathematics Curriculum

Mathematics educators whose research addresses mathematics curriculum have examined (1) comparative effects of new curriculum materials (evaluation studies) and (2) how features of curriculum and pedagogy shape these effects and influence students' learning (Star & Smith, 2007). Across studies that evaluate the effects of curriculum materials, the focus has been either upon demonstrating that newer materials ("Standards-based" materials designed to help teachers implement lessons aligned with the NCTM Standards) do no harm or can more effectively promote students' learning of content aligned with NCTM Standards, compared to other curriculum materials (e.g., Huntley et al., 2000).

Research on mathematics curriculum has evolved toward representing teachers' roles with greater complexity. Researchers from the QUASAR project illustrated that teachers' implementation of mathematics tasks could transform tasks that were written to be challenging to become less demanding or teacher could maintain the level of challenge through task implementation (Henningsen & Stein, 1997). Tarr et al. (2008) studied the effects of the use of middle grades curriculum materials along with instructional practices (Standards-Based Learning Environment) on student achievement; they found that curriculum materials converged with teachers' instructional practices to foster students' achievement. Across studies, results suggest the importance of teachers' roles in constructing students' opportunities to learn from curriculum materials.

There is a need to represent students in research on curriculum with greater complexity as well, including how students help create opportunities to learn from particular curriculum materials. In a recent review of the research literature on how curriculum influences student learning (Stein, Remillard, & Smith, 2007), students' roles are discussed for less than a page (p.

355), which suggests a lack of research on their role in the process of curriculum use. If students are represented typically at the "end" of the curriculum use process, specifically in terms of what they have learned, students may be represented as having a relatively passive role in curriculum use – as receiving the curriculum. Following Goodlad (1979), it is possible that the teachers' instructional approaches and written curriculum materials (or, in Goodlad's terms, the intended, perceived, and operational curriculum) may not be aligned with what students experience (the received curriculum). Thus, capturing students' perspectives in relation to curriculum use would provide insight on constructing opportunities to learn mathematics. Not only might students receive a curriculum that may not be aligned with what the teacher intends or implements, students' engagement with a mathematical task can influence their learning.

Stages of Curriculum Use and Students' Roles

Stages in the Mathematical Tasks Framework (Figure 1) provide insights about how students have been represented in research on curriculum use. The first stage is the mathematical task as represented in curricular/instructional materials, or the written curriculum. Teachers then plan to use written tasks in the form of lessons. Teachers implement these lessons, which would be the second stage – mathematical task as set up by the teacher in the classroom. The third stage represents students' interactions with mathematical tasks – mathematical task as implemented by students in the classroom. The fourth stage represents students as *outcomes* of curriculum use.

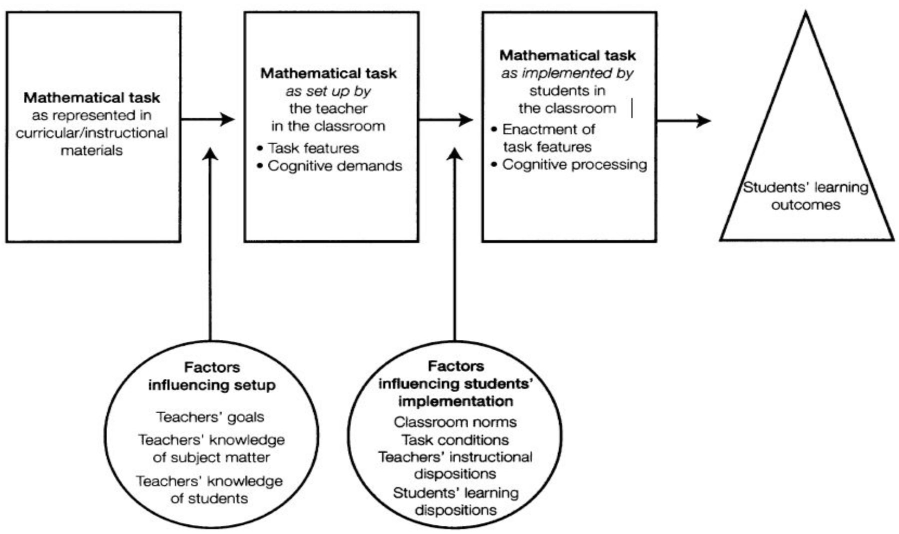

Figure 1. Mathematical Tasks Framework (Henningsen & Stein, 1997, p. 528)

Note that students also appear as factors influencing how teachers set up tasks (teachers' knowledge of students) and how students implement tasks (students' learning dispositions). The Mathematical Tasks Framework suggests that students' roles in the process of curriculum use include influencing teachers' planning and implementation of mathematical tasks, students' engagement with tasks, and, potentially (if there was an arrow creating a cycle from student learning outcomes to teachers' implementation of the task again in the future) students could influence how teachers revised their lessons based upon these mathematical tasks.

The idea that students play a role in influencing opportunities to learn is not new. Borasi (1990) wrote about students as the "invisible hand operating in mathematics instruction" (p. 174), and she illustrated that students' conceptions and expectations can influence their learning. Erickson and Shultz (1992) argued for the need to capture variation in students' subjective experiences of curriculum. Although some mathematics education researchers have examined how student engagement shapes their opportunities to learn (e.g., Nasir & Hand, 2008), these studies are either set outside of school or not in relation to tasks in specific curriculum materials. To support mathematics educators in moving toward more complex representations of students in research on mathematics curriculum, there is a need for a framework to describe students' roles in the process of curriculum use.

Methods

To develop this framework, I purposefully selected research literature to examine for this paper. I used the following criteria to select articles: To discuss how students have been represented to date, I drew upon peer reviewed publications. To consider how students *might* be represented, I drew upon a wider range of texts and looked beyond peer reviewed publications to include conference papers and book chapters. The Mathematical Tasks Framework provided a structure for locating research on students' experiences with curriculum use. I examined research with representations of students in the second stage (mathematical task as set up by the teacher), the third stage (mathematical tasks as implemented by students), and, if there could be a feedback loop between student learning outcomes and selection of mathematical tasks, I examined research that represented students in the process of teachers revising opportunities to learn as well.

Results

In this section of the paper, I explore how students have been represented in the process of curriculum use within a framework (Table 1), and components of this framework serve as inspiration for how students might be represented in future research on the process of curriculum use in mathematics education.

Processes	How *are* students represented in curriculum research?	How *might* students be represented?
When *teachers plan* lessons based upon curriculum materials	• Students' mathematical thinking as resources to draw upon when designing opportunities to learn. • Students' funds of knowledge as resources to draw upon when designing opportunities to learn.	• Students' voices as a factor that influences selection of curriculum materials or other aspects of curriculum design.
When *teachers implement* lessons based upon curriculum materials	• Students as texts to be "read" by teachers during instruction (to inform in-the-moment revisions) • Students' engagement as a factor that influences cognitive demand of mathematical tasks (maintenance or decline).	• Students' engagement as a factor that influences *increase* in cognitive demand.
When *students implement* mathematical tasks	• Variations in how students engage with the same mathematical tasks.	• In terms of strengths in students' engagement in balance with opportunities to improve their engagement.
When *teachers revise* lessons based upon curriculum materials	• Students' mathematical thinking as a factor that influences revisions of opportunities to learn.	• Students' voices (attitudes or other perspectives toward curriculum) can be a factor that influences revisions of opportunities to learn.

Table 1. Students' Roles in Constructing Opportunities to Learn

Students as Resources when Planning Lessons

Students play a role in the process of constructing opportunities to learn when they are draws upon as resources for teachers when selecting tasks or designing and planning to implement mathematical tasks in their classroom. In *Child and the Curriculum*, Dewey (1902) argues for "psychologizing" the curriculum. This process involves constructing educative experiences for learners that originate from the child's present experiences and move into the logic of academic disciplines. Research on curriculum design (designing, selecting, and planning curriculum) is conducted in the spirit of psychologizing the curriculum when it represents students as resources to inform the process.

Some ways that students have been represented as resources in the process of curriculum design or planning to implement tasks has been in the form of local instruction theories (Gravemeijer, 2004) or students' "funds of knowledge" (Bartell et al., 2010). Local instruction theories (Gravemeijer, 2004) describe a reflexive relationship between hypotheses about how students learn, think, and reason about a given topic and the means of instructional support to help students understand that topic. Students' thinking is a resource for task design with the goal of moving students' thinking toward a particular mathematical learning goal. Some researchers promote a view of curriculum design that draws upon home and community-based funds of knowledge (knowledge, skills, and experience found in students' homes and communities). Bartell and colleagues (2010) describe how pre-service elementary teachers designed lessons that built on their own students' funds of knowledge after learning more about the students' lives (e.g., participating in a tour of the students' communities led by community members). In this research, students play a role in curriculum construction because the opportunities to learn in these pre-service teachers' classrooms would differ or be adjusted depending on what they learned about their students' lives outside of school.

Additionally, students could be represented in research on the process of curriculum design in settings where curriculum materials (or tasks or content goals) were chosen (at least in part) based upon students' input. How frequently are students consulted as stakeholders in the process of selecting and designing opportunities to learn? After all, education is *for* students (Levin, 2000), so their voices about the materials and tasks could be represented by research and considered when making instructional decisions. For instance, researchers could investigate the criteria students use to select mathematics curriculum materials or tasks.

Students as Resources when Implementing Lessons

Teachers' interactions with students could influence how teachers implement tasks in the classroom. For instance, teachers "read" their students when implementing mathematical tasks, which can lead to improvised curriculum construction, such as changing tasks or ending a task early (Remillard, 1999). Additionally, mathematical tasks have been observed to either maintain their level of challenge and difficulty (maintain cognitive demand) or become less challenging (cognitive demand decreases) as teachers and students interact (Henningsen and Stein, 1997). Students may pressure teachers to reduce ambiguity in order to reduce their anxiety about being successful on a challenging task. Students shape their learning opportunities as they influence teachers' implementation of tasks.

Future research could examine whether and how students can play a role in raising the level of cognitive demand of a task. It is possible that students could push the level of cognitive demand higher if teachers are open to students' mathematical wonderments. As students have opportunities to pose their own mathematical problems (Brown & Walter, 2004), they may start asking increasingly challenging questions. Students may make connections that the teacher had not intended, perhaps even during a lesson that was not headed toward making connections, taking a lesson that was initially focused on procedural fluency toward a lesson that examines meanings behind and connections between procedures.

Students' Implementation of Mathematical Tasks during Lessons

Prior research has examined how students engage differently with tasks from the same curriculum materials. Two studies of seventh grade students in classrooms that used NCTM Standards-based curriculum materials (Lubienski, 2000; Jansen, 2008) illustrate variations between how different students take up the same mathematical tasks. SES differences appeared to moderate how students engaged with open, contextualized mathematical tasks as set up by the researcher-teacher (Lubienski, 2000). Higher SES students engaged with more confidence and with an eye toward the intended mathematical ideas when working on open, contextual tasks. Lower SES students preferred more external direction and approached problems in a way that led to missing some of the intended mathematical points. Jansen (2008) examined how students' beliefs moderated their participation in classrooms with Standards-based curriculum materials. Students who believed whole-class discussions were threatening avoided talking about mathematics conceptually, yet these students participated by talking about mathematics procedurally. Additionally, students' beliefs about appropriate behavior during mathematics class appeared to constrain whether they critiqued their classmates' solutions. Waddell (2010) investigated African-American elementary school students' engagement in classrooms that implemented Standards-based materials over three years and identified ways in which the student's engagement converged with and diverged from the patterns of engagement promoted by their teachers' instructional practices. These studies illustrate that students' engagement and participation, as moderated by SES, cultures, or their beliefs, influences their opportunities to learn from mathematical tasks.

It is helpful to examine a range of patterns in student engagement with mathematical tasks, because teachers would benefit from increased awareness of variations in students' engagement with tasks. Teachers can more effectively support students' engagement if they have knowledge of how students might engage. When researchers investigate student engagement, one dilemma involves how to discuss ways to improve students' engagement without placing blame upon students for not participating productively. (Henningsen and Stein (1997) appear to navigate this dilemma by focusing upon how teachers can influence students' engagement.) Future research should continue to uncover different ways in which students engage with the same mathematical tasks, especially highlighting positive qualities of students' engagement that lead to productive learning opportunities. Results of such studies could support more teachers as they read their students while implementing tasks. Knowledge about how students engage can help teachers as they intervene to support students to engage productively (or encourage students when they do engage productively).

Students as Resources when Revising Lessons

Teachers can use their knowledge of their students to revise their lessons based upon their curriculum materials. Referring back to Gravemeijer's (2004) local instruction theories, these theories are considered to be revisable. Teachers can develop and take these theories as conjectures about how and why curriculum materials or mathematical tasks (and instructional moves to use when implementing the tasks) are effective, then test and modify them in their own classrooms. When teachers learn or develop new understandings about students' mathematical thinking, evidence suggests that they may change their instruction (Fennema et al., 1996). Thus, students are represented in the process of research on mathematics curriculum revisions by descriptions of their mathematical thinking and their influence on curricular (and instructional) decisions.

It is also possible to imagine research in which opportunities for students' learning are revised based on students' feedback. Some researchers have attempted to solicit students' perspectives about their experiences with particular curriculum materials. In a report written by high school students and their teacher (Holt et al., 2001), students shared their perspectives on their experiences in classrooms with Standards-based curriculum materials. These students reported an appreciation of the participatory aspects of these classrooms and reports that the teachers' supportive stance toward the materials improved their experience. Bay, Beem, Reyes, Papick, and Barnes (1999) assessed over 1,000 middle school students' reactions to Standards-based curriculum materials after a year of use by asking students to write letters about their experiences with the materials. Students were generally positive about the materials and appreciated the hands-on activities, real-life applications, and collaborative work. Students' perspectives on curriculum materials have been investigated, but researchers could follow up on this line of work by studying the effects of implementing students' recommendations or the selection of materials chosen with students' input.

Discussion

Just as complex representations of how teachers use curriculum materials provide insight on creating opportunities for students to learn, additional representations of students are needed in research on mathematics curriculum. Currently, students are represented primarily as outcomes of curriculum use. Researchers could continue to investigate the role of students as a factor that influences teachers' planning and implementation of curriculum, as a factor that influences revisions of lessons, and as a diverse group that engages differently with similar tasks. Advances in research on mathematics curriculum could include studies of students as active participants as decision-makers about curriculum and representations of students' strengths when engaging in mathematical tasks (including ways in which students play a role in raising the cognitive demands of tasks).

Explicitly investigating and highlighting students' roles in the process of curriculum construction can provide insight into social processes of learning to consider not only whether but *how* the use of curriculum materials shapes learning. Although there are political pressures to use curriculum materials that are proven to be "effective," it is important to understand the conditions that lead to productive uses of these materials. Students' roles in the process of constructing opportunities to learn are conditions that mathematics educators need to understand more fully to gain a broader perspective on how curriculum materials support students' learning. For these reasons, I hope for more complex representations of students to play an increasingly central role in future research on mathematics curriculum materials.

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MIDDLE SCHOOL ALGEBRA FROM A FUNCTIONAL PERSPECTIVE: A CONCEPTUAL ANALYSIS OF QUADRATIC FUNCTIONS

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This study presents a model of three conceptual advances in understanding quadratic functions based on a teaching experiment with 6 8th-grade students. Using a covariation approach, students investigated quadratic growth as a coordinated change of x- and y-values. Qualitative analysis yielded three major shifts in students' understanding: a) developing an understanding of the first and second differences for y as rates of growth with Δx implicit; b) explicitly coordinating Δx with the constant second differences; and c) coordinating covariation and correspondence views through meanings for the parameter "a" in $y = ax^2$.

Objectives

Calls for "algebra for all" have grown in frequency in recent years, influencing district policies nationwide. For instance, the National Mathematics Advisory Panel (2008) recommends that districts prepare to enroll increasing numbers of students in algebra by Grade 8. Similarly, the NCTM (2000) Principles and Standards calls for including algebraic ideas throughout the K-12 curriculum and recommends that middle school students in particular focus on learning concepts in algebra. Successful implementation of "algebra for all", however, depends on finding ways to help students understand fundamental algebraic concepts such as equality, the use of variables, and functions. Research suggests that traditional courses focused on strategies for manipulating symbols, simplifying expressions, and solving equations yield poor results in overcoming students' well-documented difficulties in understanding algebraic relations (e.g., Knuth et al., 2006). These limitations have led to efforts to expand notions of what constitutes school algebra, and one major set of recommendations emphasizes a functional perspective as a central concept for organizing algebra instruction (Schliemann, Carraher, & Brizuela, 2007), with the early introduction of functional relationships in the elementary and middle grades. Placing functions at the center of algebraic reasoning can support students' abilities to make sense of quantitative situations relationally and provide an important foundation for future success in mathematics.

Given the importance of functional understanding for developing algebraic reasoning, one of the critical challenges remains better understanding how students' early function conceptions develop. As Asquith et al. (2007) noted, the challenges in learning more about students' reasoning "are particularly relevant at the middle school level, at which time the transition from arithmetic to algebraic thinking is arguably most salient" (p. 250). This paper presents an investigation of middle school students' emerging understanding of quadratic growth, presenting a model of three conceptual advances students experienced when studying quadratic function. It closes with a discussion of the implications for algebra understanding.

Students' Understanding of Quadratic Functions

Quadratic functions represent the basis of the more advanced mathematics to come at the secondary level and as such can act as a transitional topic for supporting students' developing algebraic reasoning. However, attempts to effectively introduce quadratic relationships have proved difficult. Students struggle to understand the role that the parameters "a", "b", and "c" play in $y = ax^2 + bx + c$, and have difficulty describing the effects that changing the parameters can have on a function's graph (Zazkis, Liljedahl, & Gadowsky, 2003). Studies have also documented students' tendencies to inappropriately generalize from linearity in order to make sense of quadratic relationships (Chazan, 2006), including using linear interpolation and attempts to find a slope and a point (Ellis & Grinstead, 2008; Eraslan, 2007). The phenomenon of generalizing from linearity suggests a need to consider introducing non-linear functions earlier in students' algebraic reasoning. One aim of this study was to identify the concepts that middle school students developed when investigating quadratic relationships from the perspective of reasoning about quantities.

Conceptual Analysis and the Development of Models of Students' Thinking

Conceptual analysis as a tool in mathematics education can be employed to satisfy a number of different goals. One can develop a conceptual analysis in order to specify the mental operations required to obtain a particular set of concepts or to analyze ways of understanding a body of ideas based on describing the coherence between their meanings (Glasersfeld, 1995; Moore, 2010). Thompson (2008) identified four uses of conceptual analysis: a) to build models of what students actually know at a specific time and in specific situations, b) to describe propitious ways of knowing for students' mathematical learning, c) to describe ways of knowing that might be problematic to students' understanding of important ideas, and d) to analyze the coherence of various ways of understanding a body of ideas. The purpose of this study is compatible with (a); its aim is to build a model of what students understand about a particular type of quadratic growth. In so doing, the model introduces advances in understanding quadratic growth that may be favorable for fostering a deeper understanding of functional relationships.

Following Glasersfeld's theory of radical constructivism (1995), the analysis presented here is based on the understanding that a student's knowledge is fundamentally unknowable, and thus any conceptual model is simply a researcher's tool for making sense of the student's mathematics. From this perspective it becomes important to refine tentative models over time. The use of the teaching-experiment methodology (Steffe & Thompson, 2000) supported the creation, testing, and revision of models of students' mathematics over multiple iterations. One aim of the teaching experiment was to investigate the viability of an introduction to quadratic function that emphasized the covariation approach (Confrey & Smith, 1995) within a quantitatively-rich setting. Within this approach, students examine quadratic growth as a coordinated change of x- and y-values. An open question was whether students could make quantitative sense of the phenomenon of constant second increases for y coordinated with uniform increases for x, and whether this understanding could support a more robust view of quadratic function that could be meaningfully connected to the correspondence rule $y = \mathrm{a}x^2$.

Methods

Participants and the Teaching Experiment

The study occurred at a public middle school with 6 8th-grade students, whose teachers identified them as high performers (2 students), medium performers (2 students), or low performers (2 students). Students across a range of performance were included in order to create a heterogeneous group in terms of mathematical backgrounds, knowledge, and skills. The students participated in a 15-day teaching experiment, which met for 1 hour a day. The students worked with a computer simulation of growing rectangles in Geometer's Sketchpad, in which they could manipulate the size of the rectangle. By adjusting the rectangle's height, the length would adjust automatically, preserving the height/length ratio (Figure 1):

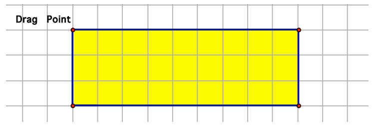

Figure 1: Image Capture of a GSP growing rectangle script

The students explored relationships between the heights, lengths, and areas of various rectangles by using scripts in Geometer's Sketchpad and by creating their own drawings, tables, graphs, and equations. The author taught the sessions, which were observed by two project team members who offered reactions and commentary after each session.

Data Sources and Analysis

All sessions were videotaped and transcribed, and additional data sources included students' written work and the author's written reflections on each session. Data were analyzed via the constant comparison method with an open and axial coding technique. Retrospective analysis (Steffe & Thompson, 2000) of the videotapes supported the creation of an initial model of each student's evolving understanding of quadratic growth. The development of the initial models of conceptual change led to the identification of 4 major categories of interest across all students: (a) students' meaning for the first differences; (b) students' meaning for the constant second differences; (c) the coordination of the increases in height with the increases in area; and (d) the relationships between the constant second differences, the rectangle's dimensions, and the parameter "a" in $y = ax^2$. Identifying the students' operations for each category led to the development of the model presented below.

Results: A Three-Stage Model of Students' Conceptual Development

Three major conceptual advances occurred during the teaching experiment that shifted the students' evolving understanding of quadratic growth. Figure 2 provides an overview of each of the conceptual advances and the sub-categories of shifts that occurred within each stage.

Stage 1: Understanding differences as rates of growth with Δx implicit	*1a*: 1st Differences as the rate of growth of the area
	1b: 2nd Differences as the rate of rate of growth of the area
Stage 2: Understanding differences as rates of growth with Δx explicit	*2a*: There is a relationship between Δx and the 2nd differences
	2b: Connecting the 2nd differences to the rectangle
	2c: Identifying how Δx is related to the 2nd differences
Stage 3: Coordinating the covariation and correspondence views	*3a*: Viewing "a" as change in length per 1-unit change in height
	3b: Viewing "a" as the ratio of the change in length to the change in height
	3c: Understanding "a" as ½ the 2nd differences (Δx implicit)
	3d: Understanding "a" as ½ the 2nd differences (Δx explicit)

Figure 2: Three stages and sub-categories of conceptual advances

The following table is one student's record of a 2 cm by 3 cm rectangle:

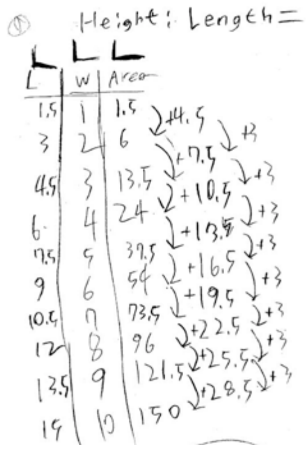

Figure 3: Table of length/width/area values of a growing rectangle

All of the students' discussions about quadratic growth relied on the growing rectangles context. The students created tables of data to record the height, length, and area values they observed in Geometer's Sketchpad such as the one seen in Figure 3. The students' propensity to represent data in well-ordered tables in which the height or the length values increased by uniform 1-cm increments encouraged a focus on what is often referred to in textbooks as the "first differences" and the "second differences" for *y*. The first transition the students experienced was one in which they shifted from understanding these values as differences to understanding them as rates of growth.

Stage 1: Understanding Differences as Rates of Growth With Δx Implicit

1st differences. The students initially focused only on the differences between successive area values without coordinating with the way the height or the length grew. For instance, when describing the first differences for height/area table of a square that grew by 1-cm increments, Jim explained, "it goes 1 and then 3 and then 5 and then…and then you go to 3, 5, 7,…just keeps going." Attention to coordinating the growth in area with how the height grew was absent from Jim's description. The teacher-researcher asked the students to draw diagrams of growing rectangles that depicted the increases in area. Figure 4 shows Ally's 2 x 3 rectangle that grew to become a 3 x 4.5 rectangle and then a 4 x 6 rectangle:

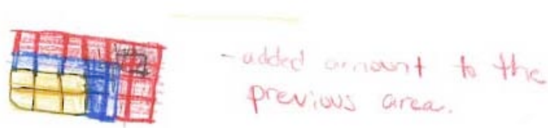

Figure 4: Ally's depiction of the additional area produced as the rectangle grows

Ally explained, "So we added 7.5 to this part [pointing to the 2 x 3 rectangle], we added 10.5 to this part [the 3 x 4.5 rectangle], and then, because it's the difference, it's [the second differences] how many more squares you had to add to this one [the 3 x 4.5 rectangle], instead of compared to this one [the 2 x 3 rectangle]." In this depiction Ally began to coordinate the number of squares making up the additional area to each time the rectangle "grew". The increments were not explicit for Ally, but she and the other students began to attend to the fact that the first differences represented the growth of the area for each increase in the rectangle's size. The next day, Jim more generally stated that the first differences represented "How many new squares it's gaining every time it grows." Jim's use of the term "every time" suggests that he was coordinating the growth in area with some growth in height and length, but did not explicitly attend to the unit of growth.

2nd differences. The students initially described the second differences as "the second level sort of thing", or "the new outside area." Ally's picture in Figure 4 helped her explain that "every time it grows it adds 3", but it was unclear whether she saw this as the rate at which the rate of growth of the area grew.

A continued emphasis on creating drawings helped the students clarify the meaning of the second differences. For instance, Bianca explained that they represented "the area of the next shape minus the area of the previous shape", or the "amount added to the amount added to the area." Jim noted that the students should name this value, stating, "I think we should give it names, like, the amount added to the amount added is so confusing!" Ally suggested "Difference in the Rate of Growth" and the students settled on this term, eventually shortening it to the "DiRoG". Jim characterized the DiRoG as, "So the rate of rate of growth is how many square units it's gaining from the rate of growth."

When the students created a table for the height, length, and area of a 1 x 2 growing rectangle, Jim found the DiRoG to be 4 square units, and then exclaimed, "it's going up by the rate of the rate…the rate that the rate of growth is growing!" Elaborating, he said, "When I add that new shelf thing [referring to increasing the size of the rectangle], there's 4 left over instead of 3." At this point Jim began to connect the DiRoG to an increase in the rectangle's size, but the unit of increase remained implicit. The connection to the manner in which the rectangle grew was limited to conceiving of how the area grew "each time", rather than for a specific value for which the height or length increased.

Stage 2: Understanding Differences as Rates of Growth With Δx Explicit

Δx and the 2nd differences. If the students were not coordinating the DiRoG with the value of Δx, they would likely not anticipate that the DiRoG would change for a table in which Δx was something other than 1. In order to test this prediction, the teacher-researcher asked the students to create tables for a 2 cm x 5 cm growing rectangle, anticipating that some students would increase the height value increased by 1 cm, and others would increase the height value by 2 cm. This did occur, for instance, Daeshim created a table in which Δx was 2 cm, and Jim created a table with Δx as 1 cm.

The students argued about whether the DiRoG should be 5 cm^2 or 20 cm^2 until Jim realized that it depended on Δx: "He's [Daeshim] going by 2's, but I'm going by 1's." After a class discussion in which the students agreed that the DiRoG could legitimately be 5 or 20, Jim asked, "So your rate of growth can change no matter what?" At this stage the students understood that the DiRoG depended on how the rectangle grew, but had not yet determined how the DiRoG was dependent on Δx.

Connecting the 2nd differences to the rectangle. Because the students struggled to determine how Δx would predict the DiRoG, they decided to consider the DiRoG in relationship to the rectangle's dimensions. Daeshim drew a picture to show that the DiRoG would be twice the area of the original rectangle for a 4 cm by 14 cm rectangle (Figure 5):

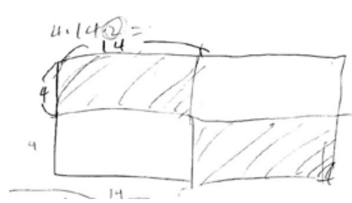

Figure 5: Daeshim's drawing of a growing 4 cm x 14 cm rectangle

Daeshim iterated the height so that the next rectangle became an 8 cm by 28 cm rectangle. He explained that this effectively created an additional area equivalent to 3 of the original rectangles, and repeating this process again would produce an additional area equivalent to 5 of the original rectangles, so the DiRoG, representing the difference in the rate of growth of the area, would be equivalent to the area of two of the original rectangles. Daeshim produced a corresponding table in which the Δx value was 4, and the DiRoG was 112 cm^2.

In contrast, Jim decided that the DiRoG would be equivalent to twice the length of the rectangle when the height was 1 cm: in this case, that would be 7 cm^2. He explained, "You reduce the height until 1, and then you can just multiply the area, or length, by 2. Because at 1, length and area are the same." He was aware that this depended on the rectangle growing in 1 cm increments for the height. Bianca agreed with Jim's method, but characterized it differently, stating "It's the length when the height is 2." Jim and Bianca imagined a rectangle that was a 1 cm x L cm rectangle. In this case reducing a general H x L rectangle to 1 cm for the height would produce a length of L/H. The area of that rectangle would be L/H, and thus the DiRoG would be $2 \times L/H$, which is equivalent to twice the length (or area) of a rectangle when its height is 1 cm.

Δx and the 2nd differences. The different table configurations for the 4 cm x 14 cm rectangle led to the two DiRoG values, 7 cm^2 and 112 cm^2. Tai explained that both values were correct, "Because one the height is growing by 4, and for the other one, height's growing by 1." Daeshim then explained that when the DiRoG, 7, is multiplied by the square of the Δx value in the other table, 4, the result is 112: $7 \times 4^2 = 112$. In general terms, if the Δx value increases by h units instead of 1 unit, the DiRoG must be multiplied by h^2. After working with many different table configurations, the students decided that this relationship was true, but were not able to explain why. It was at this point that shifting to a correspondence view became necessary to move the students' thinking forward.

Stage 3: Coordinating the covariation and correspondence views

Viewing "a" as the change in length per 1-unit change in height. The students examined a table of height and area values and had to predict the area when $h = 82$ (Figure 6).

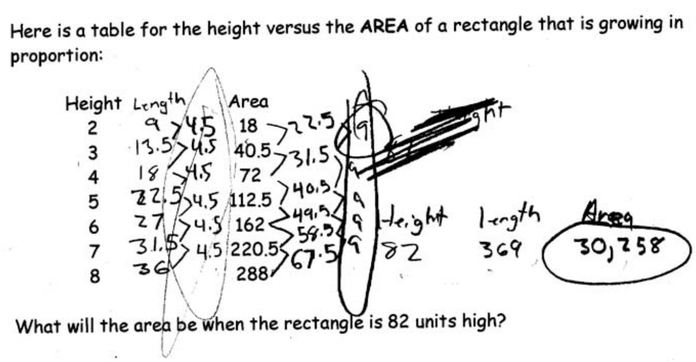

Figure 6: Student's work on a far prediction problem

The students introduced a third length column and found length values by dividing the area by the height. Tai explained, "The area divided by the height is the length. And if you can find out the length, then for this then you can find out the area." The constant increase of 4.5 cm in the length helped the students create a general strategy. Jim explained, "It would go over 4.5 for every time you go up the height 1." Jim determined that he could find the area by multiplying the length by the height, i.e., "$n \times 4.5 \times n$", which the students then shortened to $4.5n^2$. Jim and the other students' ongoing attention to coordinated changes even as they identified a general correspondence rule marked the beginning of their coordination of the covariation view and the correspondence view.

Viewing "a" as the ratio of the change in length to the change in height. Once the students created equations in the form $y = ax^2$, they began to examine connections between "a" and the quantities height and length. The parameter "a" can be thought of as the ratio of the rectangle's length to its height. However, this view did not gain purchase with the students, who were entrenched in a dynamic view, preferring to think about how the heights, lengths, and areas changed as the rectangle grew. Tai explained, "The number in the front is always the difference in the length divided by the difference in the height." Daeshim formalized this as "dL/dH", where "dL" and "dH" referred to the constant differences in successive length and height values. The students' ability to relate "a" to a coordinated change in height and length values served to further connect the covariation and correspondence views.

Understanding "a" as ½ the 2^nd differences (Δx implicit). The students noticed that the "a" in $y = ax^2$ was half the DiRoG. This is true only for tables in which Δx is 1, but the students did not initially attend to this limitation. After creating the correspondence rule $A = 0.75h^2$, Jim explained, "The difference of the rate of growth, half of that is here [points to .75]." After working with multiple tables, the students agreed with Jim's conclusion and Bianca formalized it as "DiRoG/2 = a". At this point, the students understood the parameter "a" in two ways: as half the DiRoG (although this depends on the height growing in 1-cm increments), and as the ratio of the change in length to the change in height. This led Bianca to realize that they could determine the DiRoG by finding $\Delta L/\Delta H$ and multiplying it by 2. However, the students could not easily explain why their generalizations were true, and they were not aware of the fact that their generalizations were limited to the case in which height increased by 1 cm.

Understanding "a" as ½ the 2^nd differences (Δx explicit). In order to help the students re-focus their attention on the change in height values, the teacher-researcher introduced tables with varying Δx values. The students examined a new table with a Δx value of 4 in which the correspondence rule was $y = 4.5x^2$. The students were instructed to consider, "What does the 4.5 have to do with the DiRoG?"

It was through a re-consideration of the quantities height, length, and area that the students eventually began to connect the DiRoG, the "a" value, and the Δx value. The students initially predicted that "a" would be equal to DiRoG/2. They quickly determined that this was incorrect and at that point recalled Daeshim's conjecture that if the Δx value increases by h units instead of 1 unit, the DiRoG must be multiplied by h^2. Bianca realized, "Length over height! It's length over height, times DiH (Δx) squared, times 2 equals the DiRoG of the area." Bianca realized that "a" was the ratio of the length to the height of the rectangle. She knew that typically, twice this value is the DiRoG, but she also realized she needed to compensate for the Δx value being 4 instead of 1. Multiplying by Δx^2, which she called "DiH", provided the correct compensation. Eventually, the students formalized this connection as "a = (DiRoG/2) × DiH^2".

Discussion

The students experienced little difficulty transitioning to tables that did not increase by uniform height values. The images of the changing height and length values in the growing rectangle supported their ability to make sense of these tables and create correspondence rules. Moreover, the students were able to make meaningful connections between correspondence rules and graphical representations by relying on images of coordinated changes between height, length, and area. For instance, all of the students correctly predicted that the parabola for $y = 5x^2$ would be narrower than the parabola for $y = 0.5x^2$, because the former represented a larger rectangle at each specific height value that would add more area with each increase in height.

The students' conceptions about quadratic growth were not without limitations. It was challenging for them to coordinate the growth in area with corresponding growth in height and length, and the students did not appear to reach a point at which they viewed the quantities as varying continuously rather than in repeated discrete increments. Nevertheless, the students developed a number of important threshold concepts (Meyer & Land, 2003) during the course of their investigations. They conceived of the first and second differences as rates of growth, and coordinated the growth of area with growth in height and length values. Their coordination of the covariation view with the correspondence view also fundamentally changed their understanding of the role of the parameter "a", and aided their abilities to create meaningful correspondence rules to represent data.

Lobato et al. (2009) remarked that a common view of conceptual analysis is that it should be based on analyzing the understanding of mathematically sophisticated adults, if participants are included at all. When student thinking is mentioned, its value is relegated to providing a window into the psychology of mathematics rather than as a source for articulating the substance of mathematics. But analyzing student reasoning, particularly at the middle-school level when students confront complex functional relationships for the first time, can inform the construction and refinement of a set of conceptual learning goals. The results of this study suggest that a covariational approach to quadratic relationships can provide a foundation for understanding the nature of quadratic growth and can support a meaningful transition to correspondence relationships.

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AN ANALYTIC FRAMEWORK FOR REPRESENTATIONAL FLUENCY: ALGEBRA STUDENTS' CONNECTIONS BETWEEN MULTIPLE REPRESENTATIONS USING CAS

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To better support students' conceptual understanding of algebra in the information age, we need an improved understanding of how students interpret connections between multiple representations using CAS as a representational toolkit. This research centers on the development of an analytic framework for categorizing students' connections between multiple representations, an indicator of students' representational fluency. Videotaped task-based interviews with high school algebra students solving equations using CAS were analyzed to generate the proposed analytic framework, which has a hierarchical structure based on the direction and purpose of students' translations between representations.

Purpose of the Study

Each mathematical representation is a glimpse into a version or phase of a particular mathematical object, and when taken together, multiple representations offer complementary perspectives of a mathematical object, which can help to reveal its structure. Flexibility in multiple representational approaches is an indicator of more sophisticated mathematical competencies (Brenner et al., 1999). Despite an emphasis on multiple representations in standards and curricula, school algebra students' difficulties with translating between multiple representations have been well documented (e.g., Dreyfus & Eisenburg, 1996). In technology-intensive approaches, some researchers have found that school algebra students can use multiple representations in solving tasks and are successful in translating between multiple representations (Ruthven, 1990) while others attest to the persistent difficulties students face in using multi-representational approaches with flexibility (Huntley & Davis, 2008).

CAS-intensive trends in school mathematics, and algebra in particular, that were pioneered in the late 1980s and 1990s have finally infiltrated contemporary curricula and classrooms (e.g., Davis & Fonger, 2010). Congruous with the perspective that the coordination of multiple representations is an indicator of conceptual understanding, issues of linking or connecting mathematical representations are significant concerns. Indeed, although CAS environments can act as representational toolkits, Heid and Blume (2008) report that, "students do not necessarily connect representations when operating in a multiple representation environment" (p. 98). Indeed, Heid and Blume (2008) articulate a "need to better understand how students move between, connect, and reason from multiple representations" (p. 98).

With the integration of CAS in school algebra, researchers and practitioners alike need to know how high school students link multiple representations while using CAS technology (Arbaugh, et al., 2010). In addressing this gap, the purpose of this study is to gain insight into the ways in which students connect multiple representations during task-based partner interviews using TI-Nspire CAS. Specifically, this study seeks to answer the following research question: When solving equations using CAS as a representational toolkit, how can students' connections between multiple representations be characterized?

Theoretical Framework

Janvier (1987) defines the psychological conversion made from one type of representation to another as a translation process. In other words, the meaning of a source representation is interpreted in reference to a target representation perspective. Adapted from Huntley, Marcus, Kahan, and Miller (2007), a useful framework for investigating multiple representations is the "Rule of Four" model, which involves symbolic, verbal, graphic, and numeric representation systems, with arrows indicating translations between them. Morgan, Mariotti, and Maffei (2009) posit, "converting between different systems of representation is a critical cognitive activity for developing understanding of a mathematical object" (p. 247).

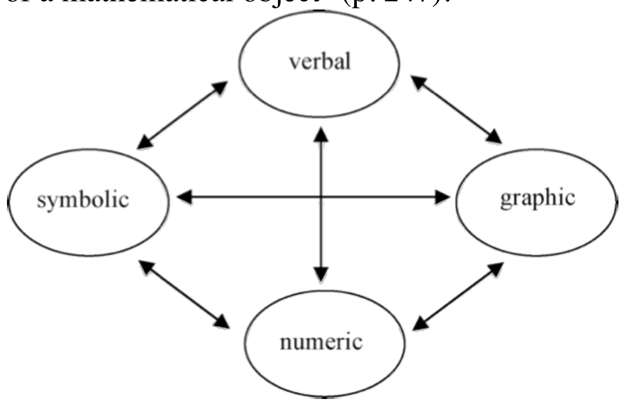


Figure 1: Rule of Four illustrating translations between representations.

The design of the TI-Nspire CAS Touchpad (OS v2.0) houses representations on separate types of "Pages" with a pre-determined structure and main representation for each (including Calculator, Notes, Graphs, Table). For clarity in determining students' use of representations, the Page type and corresponding prominent representation are integrated: symbolic-calculator (S), verbal-notes (V), graphical-graphs (G), and numeric-table (N).

In this context, the construct of representational fluency serves as a tool to characterize students' multi-representational activity. Inspired by Sandoval and colleagues (2000), representational fluency (RF) is the ability to construct, interpret, translate between, and link multiple representations. It is implicit in this definition that both the construction of representations on CAS (inscriptions) and discourse about these representations (e.g., interpretations) are of importance for accessing students' RF. Specifically, I focus on the connections (links) students make between representations to be an indicator of students' RF.

For purposes of this study, a student(s) is said to *make a connection between multiple representations using CAS as a representational toolkit* if they give a correct interpretation of multiple, mathematically equivalent, representations that are evident in their CAS activity or reflection on CAS activity or inscription(s). In other words, for a student to make a connection they must verbalize that they are coordinating information (i.e., invariant features of the object in question as evident in mathematically equivalent representations) in their interpretation of one representation in terms of another, or a pair of representations.

Methodology

Participants and Context

The study was conducted at a Midwest high school that drew accelerated students from several area high schools for mathematics and science coursework only. The school and classroom context was chosen based on the requirement that students have access to TI-Nspire CAS technology, and use this technology regularly in their second year algebra classroom. Of the 25 students in the targeted class, four volunteered to participate in the study and were interviewed. These ninth grade students had taken algebra in middle school and would take geometry after passing their current algebra class. By the beginning of the data collection for this study, students had been using CAS as a regular part of their mathematics instruction for seven weeks.

On three separate occasions prior to interviews with students, the researcher observed the enacted curriculum of students' classroom and took field notes on students' and teachers' use of CAS with regards to multiple representations and use of the adopted textbook, the third edition of the University of Chicago School Mathematics Project *Advanced Algebra* (Flanders et al., 2010). The main purpose of these site visits was to inform the design of instrumentation for the CAS-based task structured interviews; it was necessary to understand the enacted and written curriculum in this classroom to design tasks that involved accessible yet non-routine mathematics and familiar CAS functionality. The classroom observations, reviews of the written curriculum, and conversations with the teacher verified that all student participants had opportunities to learn mathematics through a variety of representations. Moreover, through the use of hypothetical or imperative language, the teacher and textbook often recommended or expected that students use CAS to create representations.

Data Sources and Instrumentation

The main data sources were digital video recordings of task-based interviews conducted with two pairs of students. The teacher determined pairs of students based on students' ability to work well together while engaged in mathematics tasks. Coincidentally, same sex pairs were formed (one male, one female), and the teacher reported that despite the high-ability of all students at this school, there was some variability in these four students' mathematical abilities. On two separate occasions during Fall 2010, each pair was interviewed for 50-minutes during class time. Partner interviews were conducted based on the rationale that richer data would be generated than what might transpire in interviews with individuals. In partner situations, students' interactions with each other and their CAS were perceived to be more authentic to their classroom experiences in which tablemates were observed to regularly communicate about mathematics and CAS technology.

During the interviews, students were prompted to solve two equations, Task R and L (see Figure 2). Both tasks were presented in an initial verbal representation, while Task L also included a graphic representation. These tasks were designed so that various constraints and affordances of the initial representation(s), the context, the mathematics, and/or the technology, might prompt students to construct and/or translate between representations.

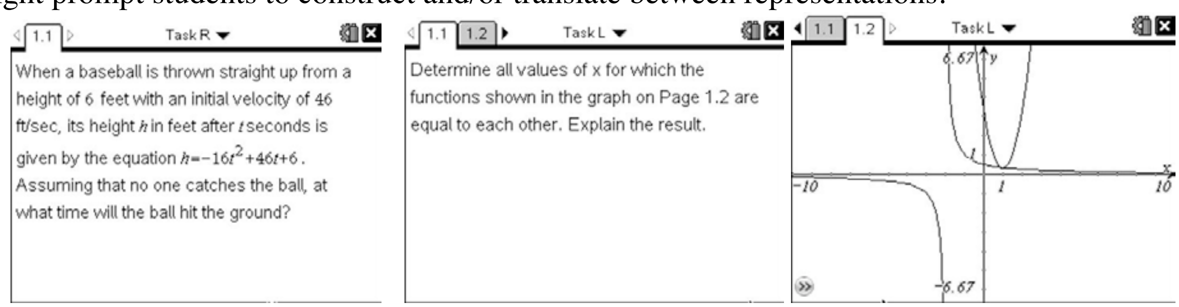


Figure 2. Tasks presented to pairs of students on CAS and in paper form.

Each task was stored and saved as a TI-Nspire document that each student accessed electronically on a handheld CAS. Participants were encouraged to use CAS for the entire interview, but they were also provided pencils and paper copies of screen shots of the tasks (as shown in Figure 2). Data from saved electronic documents of completed tasks and paper and pencil work (if any) were collected at the close of each interview.

During the task-based interviews, participants were encouraged to follow a "think aloud" protocol, openly conversing with one another throughout the interview, providing verbal explanations to accompany their CAS activity as they completed each task. To guard against the researcher changing the cognitive demand of the tasks, a collection of interview prompts was prepared to provide parameters for the interactions between the researcher and participants. The overall intent of these prompts was to elicit further explanations from the students regarding the meaning of their solution approaches with respect to multiple representations.

For instance, "Linking Probes" were given after students had considered the solution and/or solution process from multiple representations (e.g., "How is what you see/did here the same or different from what you see/did here?"). In some cases, students considered multiple representations on their own; in other cases, the researcher encouraged the use of multiple representations through probes (e.g., "Could you solve this in another way?"). In other words, if the students did not self-prompt the use of multiple representations or got stuck in an approach for several minutes, the researcher suggested students consider an alternative approach. Although it is possible that these prompts encouraged students to elicit connections between multiple representations that would not have otherwise been verbalized, this was deemed appropriate because the focus of the study was on the connections students were able to make, rather than on those they chose to verbalize.

Data Analysis

Two video files per interview—one per student in a pair that captured each student's CAS screen—were synchronized into one timeline for video analysis using Studiocode (SportsTec, 1997-2010). The merging of video files allowed for a data analysis method that accounted for both individual and taken-as-shared understandings (cf. Cobb & Yackel, 1996) of the pair of students while they solved equations using CAS. Seed ideas for an analytic framework for linking multiple representations were developed a priori to data analysis, yet the cyclical process of coding, developing, and refining an analytic framework occurred in several stages.

Initial rounds of analysis involved coding and memoing using a grounded-theory inspired approach in which the code categories and descriptions were developed in response to student data. Instead of inventing new terminology in all cases, a conflation of existing terminology was determined to be more beneficial. Specifically, the analytic framework was shaped by the analysis of data and was also purposefully crafted as an amalgamation of existing categorizations and descriptions of students' connections from the literature. A mathematics educator who was familiar with the study and related literature critiqued a refined version of the analytic framework. The entire data set was then reexamined and all instances of connections were coded, memoed, and transcribed, allowing segments of data to be revisited as the analytic framework was being developed. With all connections instances coded, emerging categories, sub-categories, and definitions for the framework were further refined into a hierarchical structure.

Results

The proposed framework for representational fluency is hierarchical in terms of the direction and number of students' translations between representations and the nature of their connections with respect to the perceived goals of students' problem solving activity. Specifically, connections are categorized to be *uni-directional*, *bi-directional*, *multi-directional*, or *abstract*. Both bi-directional and multi-directional connections necessarily involve uni-directional connections. The difference between bi-directional and multi-representational connections is that the former involve a pair of representations (to and from two distinct representations), and the latter involve more than two different representations (and may include a bi-directional pair). Abstract connections go beyond specific reference to types of representation to generalizing the underlying mathematical objects/principles (e.g., equations/equality). Table 1 outlines the framework levels with brief descriptions; connections can be categorized as *one* of: I, IA, IB, II, IIA, IIB, IIC, III, or IV. Transcript excerpts are discussed next to exemplify select levels and sub-levels including: *uni-directional justification*, *bi-directional reconciling*, *multi-directional connection*, and *abstract connection*. The examples were chosen to distinguish between justification and reconciling codes, and also to give more detail on each of the four directional categories, highlighting the hierarchical nature of the framework.

Level of Connection	Brief Description
I. Uni-directional Connection	Translation; interprets meaning of a given source representation in reference to a target representation (Janvier, 1987).
IA. Representational Resourcefulness	Uses a representation to overcome a barrier (Jon Davis, personal communication, 11/18/2010).
IB. Uni-directional Justification	"Use representations as justifications for other claims" (Sandoval et al., 2000, p. 6).
II. Bi-directional Connection	Translation and complementary translation (Janvier, 1987).
IIA. Bi-directional Justification	Pair of representations are used to (dis)confirm an approach (Sandoval et al., 2000).
IIB. Bi-directional Reconciling	Coordinated activity; checking the solution between two representations (Kieran & Saldanha, 2008).
IIC. Reflection on Reconciled Objects	Reflection on the compatibility of a result between a pair of representations (Kieran & Saldanha, 2008).
III. Multi-directional Connection	More than two representations are related by translation processes.
IV. Abstract Connection	A generalization is made across different representations.

Table 1. An analytic framework for representational fluency based on sophistication of connections between multiple representations.

In a *uni-directional justification* connection type, students "use representations as justifications for other claims" (Sandoval et al., 2000, p. 6). In other words, a representation is used to confirm or disconfirm a conjecture or solution approach in another representation. This code is not the same as checking the end result or product of a solution approach against another (i.e., reconciling), instead, the emphasis is that some information from a source representation is used to inform the solution approach in a target representation before the solution is obtained. For example, students from the first interview, attempted to solve Task R using the graph and relate the points where the graph crosses the x-axis to the verbal problem situation in which it only makes sense to have a positive value for time.

Researcher: So how can you use this graph to solve?

Student A: …There are there's two points where it crosses the x-axis [taps finger on desk

twice].
Student B: Right.
Student A: So it'd have to be the positive number and not the negative number because you can't have a negative time.
Student B: True.

In the exchange above, a contextually-based verbal representation is used to justify what is reasonable for a solution on the graph. This is not considered representational resourcefulness because the students were not stuck in using the graph, thus the reference to the verbal representation was to inform their graphic solution.

At the second level of the framework, instances of *bi-directional reconciling* are specific events of coordinated activity in which the student(s) is moving back and forth between two representations. For example, the results of equation solving processes are reconciled between a pair of representations in the second interview while students completed Task L.

(1) Student C: Oh, wow, there's more to the equation. [re-executes solve command on calculator page: solve(1/(x+2)=3*(x-1)^2+0.3,x), ENTER, yields x=-1.96245 or x=0.873499 or x=1.08895]
(2) Student C: … [traces x=1.09 on graph] Yeah look at that, it works now. I got, I got them to equal at 1.09 just like it does in the equation [looks at calculator page, mistakenly points to x=-1.96245].
(3) Student D: At x=1.09? [traces on graph]
(4) Student C: Oh wait. It's very close to 1.09, it just rounds up [comparing approximated values on calculator screen, x=1.08895 and rounded values on graph; continues to compare other values]
(5) Student D: [traces near x=1.11…x=1.08 on graph] So they're basically the same.

In the above excerpt, both students reconciled the results between the symbolic and graphic representations. Student C reconciled the solution from symbolic to graphic (lines 1-2), back to symbolic (line 4), a complementary translation. The utterance from Student D in line 5 is interpreted to mean that the solution of 1.09 was reconciled to be the same in both the graphic and the symbolic representations, taken as evidence of a bi-directional connection in which the solution is checked between two representations.

A student(s) is said to make a *multi-directional connection* when more than two different representations are related by translation processes. The example below from the first interview is a taken-as-shared multi-directional connection between graphic, verbal, and symbolic representations.

(1) Student B: It's [the graph] actually telling you it would hit the ground at 3 seconds, where as with the calculator you can make so many mistakes when figuring out the problem …
(2) Student A: I'm going back and I'm putting three in for the answer and I'm seeing if it comes out zero [typing 0=-16(3)^2+46(3)+6 then ENTER in the calculator page yields “true”]. Which it says it's true, so—
(3) Student B: [types h(3), ENTER, yields 0] Yep.
(4) Researcher: So what does that help you to understand about the problem?
(5) Student A: That the point that we got is the time it took for it to hit the ground.
(6) Student A: And then when we plugged it back into the calculator it told us that the equation is equal to zero and that's what we were looking for.
(7) Student B: Yup, and it basically reassured the fact that the graph, that that point was right.

Consistent with the definition, lower levels in the framework are evident within the dialogue of the above example. Student A interprets the graphical solution in terms of the verbal representation, a translation (line 1). Both students reconcile the solution obtained from the graphic representation with a symbolic representation (lines 2-3, 7). Third, the students reflect on the results that had been reconciled between the graphs and calculator pages and relate this to the verbal representation (lines 4-6). In sum, the students have related more than one pair of representations by a translation process. The fact that uni-directional and bi-directional connections are identified within the multi-directional connection is also evidence of the hierarchical nature of the framework, yet instances were not double-coded.

A student(s) is said to make an *abstract connection* when they make a generalization across different representations. In particular, a student demonstrates flexibility in solving equations from multiple representations and is able to generalize the process of solving equations from a functions-based perspective in which an equation is viewed as two expressions, interpreted as functions, which can be viewed from symbolic, graphical, and/or numeric representations. At this level of a connection, the notion of equality is understood from multiple representations. For example, the students in the first interview were asked to reflect on the fact that they had obtained a solution using the graphs page, but hadn't obtained a solution using the calculator page for Task R. This led Student B to generalize that "Because the graph is just a symbol of the equation [switches from Graphs page to Calculator page] or like the diagram of the equation so you should have been able to get the same thing." Moreover, when solving a given equation, Student B articulated that, "If you did it correctly you should have gotten the same answer." So at this point in the interview, even though the students had only successfully solved the task using the graphic representation, Student B was able to articulate that the same solution should be obtainable using a symbolic representation. This is evidence of a generalization across different representations.

Discussion

Students' difficulties in connecting representations of algebraic objects have been well documented. The definition of a connection and the analytic framework for representational fluency proposed here are the building blocks for future research aimed at understanding students' strengths in connecting representations and for instruction designed to foster richer connections among multiple representations using CAS. Using connections between multiple representations as an indicator, students' representational fluency can be categorized in a hierarchical manner per the direction and purpose of students' translations between representations. The results and examples discussed above can be illustrated using the Rule of Four framework, showcasing the four distinct levels in the proposed analytic framework (see Table 2). By teaching topics in algebra using pairs and sets of representations, and emphasizing bi-directional and multi-directional connections through the use and reflection on CAS inscriptions, students may develop representational fluency and in turn come to a more robust conceptual understanding of the mathematical object(s) in question.

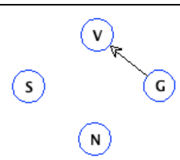
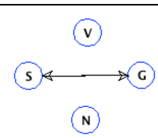
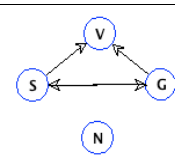
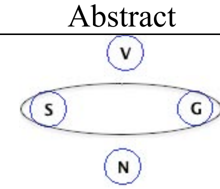
Uni-directional	Bi-directional	Multi-directional	Abstract
			

Table 2. Illustrations depicting levels of connections using Rule of Four framework.

The proposed framework and schematic representations mask some of the intricacies of the nature of students' connections as others have studied them. It would serve well to use the framework for analyzing data from a larger sample size. Specifically, the multi-directional connection category might be expanded to account for students' problem solving goals, analogous to the subcategories for the bi-directional level. Additionally, future research might employ the use of the analytic framework to elucidate types of opportunities afforded by written curricula to make connections between multiple representations with CAS. This framework would also be useful in the design of instructional intervention or tasks aimed to foster sophisticated connections between representations through the use of directionally linked dynamic technology environments.

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CRITICAL MOMENTS IN GENERALIZATION TASKS. BUILDING ALGEBRAIC RULES IN A DIGITAL SIGN SYSTEM

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We report findings of research that deals with secondary students' interaction with a digital microworld, eXpresser. eXpresser is designed to support students in constructing general figural patterns of square tiles and expressing the algebraic rules that underlie them. Our focus is on the sense production that emerges as students use eXpresser. In order to analyze student productions, we adopt a semiotic perspective based on a broad notion of mathematical sign systems. We identify in the analysis, critical moments that illustrate the emergence of algebraic expression in eXpresser's alternative sign system. These moments are constituent elements of the process of deriving a general rule and support the development of the notions of dependent and independent variable.

Introduction

The findings of various studies that deal with pattern recognition and expression have led to conceiving of mathematical generalization as a gateway to learning algebra for pre-algebraic students. This approach seeks not only for students to become competent in recognizing regularities in numerical or figurative sequences, but that they also can imbue the corresponding algebraic expressions with meaning and sense (Noss & Hoyles, 1996, Mason, 2008; Bednarz, 1996; Kieran, 2007; Lee, 1996). Many research experiments have used this generalization approach to algebra while working with digital learning environments, and these have been shown to help students along that path, such as studies undertaken using Logo (Hoyles & Sutherland, 1989) and using spreadsheets (Sutherland & Rojano, 1993). Yet one of the greatest obstacles reported concerns the move from recognizing and analyzing a pattern to expressing it symbolically. The MiGen project (1) has developed a microworld, *eXpresser*, designed specifically to support students in generalizing rules based on structure, in this case of the structure of figural patterns of square tiles (Noss et al., 2009). The *eXpresser* microworld includes construction objects and a set of actions that can be carried out on them so that students can build models made up of these patterns and test their generality through animation.

In this paper we focus on the potential of students building a rule that represents the structural regularities of a pattern, while also being able to exploit a symbolism for exploring the performance of the pattern in general. This symbolism is closely tied to the nature of the sign systems made available by *eXpresser* for rule-building. The affordances of *eXpresser* features were directly used in the study reported here in order to design generalization and symbolization activities, which were investigated with groups of secondary school students, aged 11-13, in England and Mexico (2). The study aims to analyze the sense production processes that were triggered when students were on the path toward building a rule, which represented the structure

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of the patterns that together made their model. Episodes of in-class experimental sessions in which the students were interacting with *eXpresser* were analyzed from the basis of a semiotic perspective in which the notion of Mathematical Sign System (MSS) plays a central role (Puig, 2008).

Theoretical Perspective: Mathematical Sign Systems and Sense Production

Since mathematical texts are produced with a heterogeneous set of signs, the notion of Mathematical Sign System (MSS) dealt with in Filloy et al. (2010) and used in our analysis entails not only the signs considered as specifically mathematical, but also those pertaining to natural language, figures, diagrams, as well as the signs of digital learning environments. Hence that notion of MSS does not refer to a set of mathematical signs organized in a system, but rather to a system of signs –specific to mathematics or not- that is of a mathematical nature (i.e. the system is conceived as of a mathematical nature rather than the individual signs). It is crucial to take the system as a whole because the use (or uses) of the system is the responsible for meaning and sense making. Moreover the notion is not limited to socially established sign systems, but includes signs, sign systems or strata of sign systems that students produce in order to make sense of what is presented to them in a teaching model, in an interview situation or in a mathematics task.

Thus, sense production in mathematics activity comes about by way of chains of reading and transformations of actions that are undertaken on texts expressed in an MSS (Puig, 2003). Thus in the particular case of work with *eXpresser*, our interest lies in analyzing the sense production of students related to the expression of rules that correspond to figurative models built in an environment in which they interact with its MSSs through reading and transformation actions. In *eXpresser,* students have access to increasingly abstract set of MSSs, as evident below.

***eXpresser*: A Digital Microworld to Express Generality**

In *eXpresser*, students are asked to construct a general model and to develop a rule that determines the total number of tiles in the model. The students construct the model by visualizing its structure and determining appropriate patterns of repeated building blocks that together make up the model. The students have to make explicit rules to calculate the number of tiles in each pattern, which are colored if and only if the rule is correct and then combine the rules for the different patterns to obtain the total number of tiles in the model. Of course there is more than one way to construct a model, each of which leads to a "different" rule for the same model.

Figure 1 shows a snapshot of the *eXpresser* with what is known as the Train-Track model already constructed (shown top left of the screen labelers My World). A C-shaped building block (A) has been created (by dragging and dropping separate tiles and grouping them) and is repeated by placing each repetition two squares across and zero places down (When making a pattern, the translations across and down for each repetition of a building block, as well as an initial number of repetitions has to be specified). When the C-shaped building block (A) is made, its properties are shown in an expression (B). The building block is repeated as many times as the value of a variable called ‘Model Number’ (D), in this case 4.

Patterns are colored by calculating and then allocating the *exact* number of colored tiles to its construction. In the case of the pattern made of C-shapes, using the expression for construction (B) and the number of times the building block is repeated, the rule for the total number of tiles in this pattern is ‘7×Model Number’, in Figure 1 (E). As students build their constructions in ‘My World’, a second window is seen alongside (the ‘General World’). This mirrors exactly My World until the student unlocks a number in their model, the metaphor for making a variable, at which point the General World chooses randomly a 'general' value for the variable. In Figure 1, the value of the Model Number in the General World is randomly set to 9, resulting in a different instance of the model (F). The General World is only colored when students express *correct* general rules in the 'Model Rule' area of the screen (G). Students cannot interact directly with the General World. They are, however, encouraged to click the play button (H) to animate their general model to test its generality.

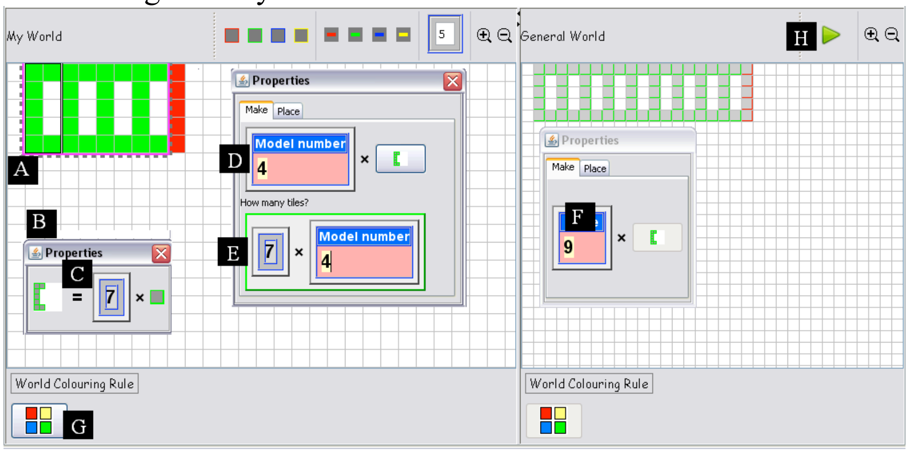


Figure 1. Constructing a model and expressing its rules in *eXpresser*. (A) Building block. (B) Expression of construction of the building block. (C) The number of tiles of the building block. (D) The number of repetitions of the building block. (E) Number of tiles required for the pattern, with general rule. (F) Any variable used in 'My World' takes a random value in the 'General World'. (G) A general rule that expresses the total number of tiles in the whole model. (H) Play button for animating patterns.

The construction objects in *eXpresser* are structurally related to the elements of the symbolic rule as follows: a) the number of building blocks is associated with the variables and constants; b) the number of tiles in each block is associated with the coefficients of the variables in the expression; c) the patterns constitute the terms of the expression; d) and the model is associated with the complete expression. In particular terms in the task referred to in this article, the students were asked to build a model associated with a symbolic expression that had linear and constant terms. In other words, the model had to be made up of several patterns that together would serve to represent a Train-Track model, which, if properly built, could be "extended" as student wanted. The students were asked to use different colors for each of the patterns that made up the model, in order to highlight the way they perceived the structure. The students were also required to answer questions that are aimed at provoking reflection on the model-formula structural relations that they had constructed.

Experimental Work with eXpresser

The data presented in this paper refer to the work of 11 to 13 year old students (14 Mexican and 22 English students) who began to study algebra in secondary school. Although the students worked on several activities throughout the study, this report only includes an analysis of the Train-Track activity. Data was compiled from videotapes of classroom sessions, written reports of the students on worksheets and computer files produced by the students registering their interactions with the software. Our attention was focused on analyzing the role of the different sign systems proposed in *eXpresser* in the sense production processes. In England, the task was presented fully computerized and students answered some reflective questions presented through digital worksheets. In Mexico, the students worked with *eXpresser* and with paper and pencil worksheets.

Results – MSSs in eXpresser

In building models, the students moved from working with MSSs associated with visual perception to more abstract MSSs, such as numerical and in some cases algebraic MSSs. During the process it was possible to see how the students made sense of the elements of a symbolic expression by associating them with the specific elements that made up their figurative model. Thus the generalization processes consisted of searching for a structure that could be transformed from the figurative model to a symbolic expression by means of the structure of the task-pattern, and then a rule could be derived based on the perceived structure. Data analysis made it possible to identify critical moments of generalization activity, which characterized aspects of work in *eXpresser* related to the nature of the different MSSs available in the setting. These aspects include, *inter alia*: a) constructing building blocks from unit tiles; b) unlocking numbers in order to vary them; c) numerical articulations among the different patterns that make up the model; d) building model rules using the *eXpresser* language; e) building different patterns based on the perceived structure to produce different models; f) analyzing the relations between the particular construction and the general models that it is possible to see in *eXpresser*. In this paper we focus only in the affordances of *eXpresser* that were designed to aid the transition from pattern construction to arithmetic expression and its transformation to algebraic expression (aspects b) and d), for which we have used analyses of student working on the Train-Track activity.

Building Blocks, Patterns and Models

A building block is an object that belongs to a MSS that is more abstract than that of individual tiles, albeit both are of a figurative nature. First, the students must become aware of the need to determine a building block in order to make a pattern. Analysis of the videotapes and of the files saved automatically by the software when the user interacts with it enabled us to follow up on student actions in their attempts to build their patterns and models. Although the students had no trouble defining the building blocks, records of their work in both countries showed that some had a tendency to simply produce the pattern tile by tile. In this case, they were helped by redirect their attention to the building block they had been in fact implicitly repeating while dragging tiles, thus helping the students identify their own building blocks (see Noss et. al., 2009). Ten different forms of constructing the Train-Track model were identified, see Table 1.

Locked numbers
Unlocked numbers

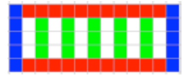
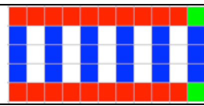
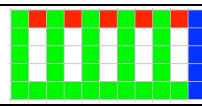
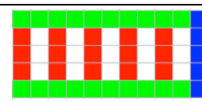
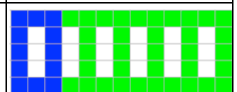
A	B	C	D	E
1Mexican & 2 English students	4 Mexican & 3 English students	3Mexican students	1Mexican student	1Mexican & 9 English students
				
Two patterns: $5n + 2(n - 1)$	Three patterns: $3n + (2n - 1) + (2n - 1)$	Three patterns: $5n + (n - 1) + (n - 1)$	Five patterns: $3n + (2n+1) + (2n+1) + 5 + 5$	Two patterns: $7n + 5$
F	G	H	I	J
1 English student	1 English student	1 English student	4 English students	1 English student
				
Three patterns: $5n + 2n + 5$	Three patterns: $4n + 3(n+1) + 2$	Three patterns: $6n + n + 5$	Three patterns: $4n + 3n + 5$	Two patterns: $7n + 12$

Table 1. Different forms of constructing the Train-Track by students participating in the study. To better communicate each model's structure, in the bottom line the corresponding algebraic rules are shown (*n* is the number of one of the building blocks in the model). It should be noted that this not mean that the participant students could produce such an expression in the algebra MSS.

From Arithmetic Expressions to Algebraic Rules

One important step in the generalization process toward the use of an abstract MSS is that of deriving a symbolic rule (arithmetic or algebraic) that makes it possible to obtain the overall number of tiles in a pattern (dependent variable) in terms of the number of building block repetitions (independent variable). The microworld provides immediate visual feedback when, for a given pattern, the production rule is correct: that is by coloring all of the tiles used in the particular construction (see Figure 2).

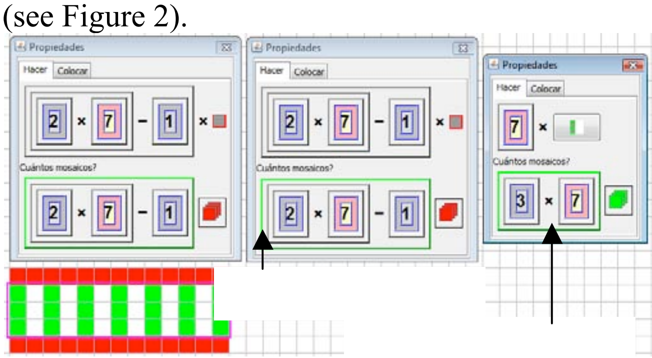


Figure 2. Arithmetic rules for the three patterns that constitute the model (top (red) line), bottom (red) line and vertical (green) line)

Different types of numerical expressions (texts) can be seen in the image on the left of Figure 2, including: locked numbers (constants) and unlocked numbers (generalized numbers) in which the values can change. Identification of the role played by these types of numbers in the figurative model is essential to the construction of algebraic rules. Although indicating the overall number of tiles that constitutes a pattern is sufficient for it to be colored, a rule must be provided in order for the pattern to remain colored for *any* number of repetitions of the building block (so that the Train-Track can, in turn, be extended or reduced). Most students in both Mexico and England used initially a specific number to indicate the number of tiles in each pattern. Despite introductory tasks in *eXpresser*, the persistence of this student strategy shows (unsurprisingly given previous research) a common reluctance to articulate a relationship and to express it in a general way. To overcome, the teacher (3) had to intervene and encourage students to use a building block. The intervention relies on repeatedly changing the variable in the number of repetitions of the building block and therefore 'messing-up' the coloring of the pattern– drawing on the vocabulary used by students in a dynamic geometry setting. The visual feedback challenged students to find a general rule that according to Gregg (one of the students in the English trials) "tells it [the *eXpresser*] how many tiles it needs for any number in the pink [unlocked] box". However, some students (9 in Mexico) when answering the worksheets sometimes but not always made a numerical formula for some of the patterns in the model. For example, Figure 3 shows the work of Sofia indicating two formulas: one with an operation (7×5) and other with a constant number (6) in the case of model C in Table 1. Sofia also forgot to indicate the arithmetic expression for one of the patterns, since the last one was not built with the pattern construction tool, she used the tool copy-paste to include it in the model.

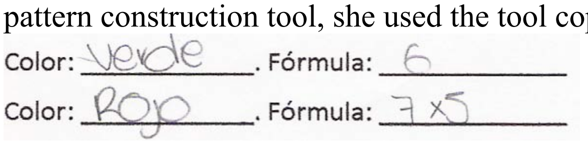


Figure 3. Construction of one numerical formula and one constant to indicate the number of tiles in two patterns (Verde – Green; Rojo – Red)

Eventually, most of the students managed to make symbolical formulae for their patterns, though many were isolated and not interrelated. For example, in the worksheet in Mexico, Diego constructed a formula with two variables (a×5 + 2b), for the model (C) in Table 1 he did not seem to see the relationship between a and b ($b = a - 1$) and it was only after teacher intervention that he (and another two students) were able to construct a formula with one independent variable (see Figure 4).

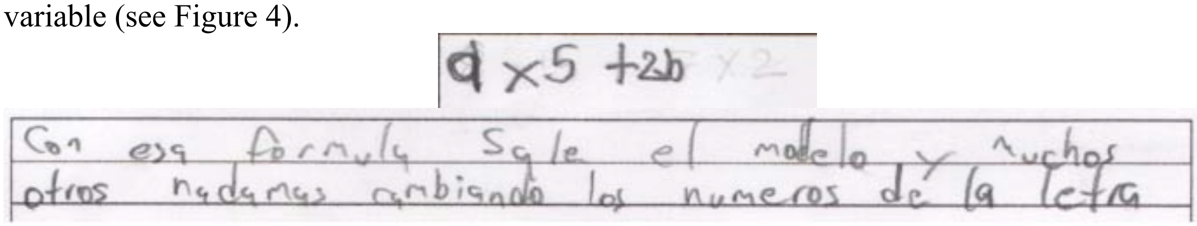


Figure 4. A student's work on expressing the formula to calculate the number of tiles in the Train-Track model. Student writing: "With this formula I get the model and many more when I change the numbers of the letter"

An effective intervention capitalizes on drawing students' attention to the General World. Models that use patterns with more than one independent variable, are 'messed-up' in the General World since, for each variable, the microworld chooses random values likely to be different so the models do not maintain their structural coherence. Figure 5 shows Connor's work, an English student; he has unlocked both the 5 and the 3 as the number of repetitions in his two patterns, but has not specified any explicit relationship between them. The teacher asked him many times to choose several different random values for the number of green building blocks and find the number of red building blocks. Connor painstakingly repaired the model each time when it was no longer colored. He had implicitly identified a relationship but could not express it in general terms. After several repetitions of the same prompt, when the value of repetitions of the green block was 8 and that of the red had to become 15 (2×8 – 1), Connor verbalized his rule: "You times the 8, you times the number of green tiles by 2 […] and then take away 1", and used *eXpresser* to represent explicitly the relationship between the two patterns in his model.

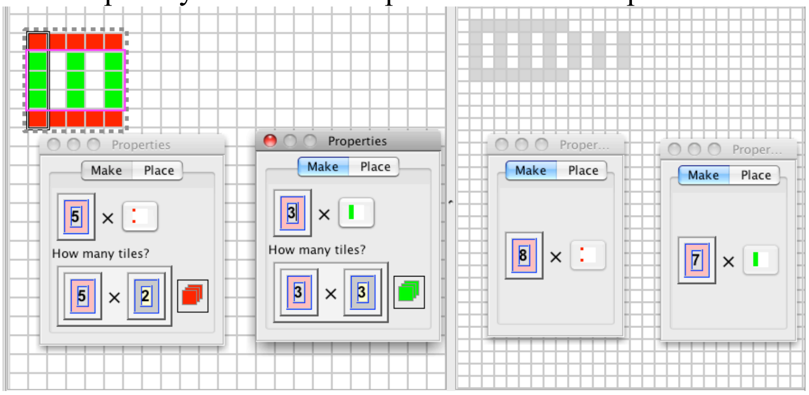


Figure 5. Connor's model is messed-up in the General World

In *eXpresser*, numbers are perceived as objects to which a value can be assigned, that can be locked or unlocked and given a name. Hence, in their capacity of being concrete objects, numbers are associated with one single screen position. If that number is required in order to construct a rule (simply being represented by the same variable is sufficient in the MSS) then it must be a part of two different relations, thus forcing the student to make a copy of it and not simply defining another number (or variable) that has the same value. As such, in Figure 2, the object-number shown with a numerical value of 7 was copied five times in order to construct and relate the rules for the three patterns that make up the model. Determining this fact is one of the most important characteristics of building a formula in the MSS of algebra because the subsequent symbolization only consists of assigning a name to that number.

Differentiating specific numbers from general numbers and number-objects means that *eXpresser* substantially changes the activity of mathematical expression of patterns and thus makes the passage from generalized numbers to use of letters more natural within the digital environment. As such, students are able to reflect upon establishing relations among objects that, despite belonging to different strata of MSSs, all have a figurative referent that enables the students to imbue those relations with sense. We assert that this characteristic made it possible for the majority of students participating in the study to express a rule corresponding to the Train-Track model in an algebra like MSS. Figure 4 shows part of the worksheet of Diego on which he has expressed the formula for calculating the overall number of tiles in the Train-Track model, as well as his interpretation of that formula (expressed in words).

Students in England were asked to reflect on their derived rules in *eXpresser* by writing some arguments to support their rules' correctness or not. A 12 year old student, Nancy, whose model/rule is shown in Figure 6, wrote: "My rule is correct because each "block" has 7 squares. So however many blocks there are, there are 7 squares for each one so you multiply the number of blocks by 7. But, at the end there is another block to finish the pattern off. In this block there are 5 squares so you add the number of squares (the blocks multiplied by 7) to the final block (the 5 squares). This rule should apply to this pattern each time". Nancy showed a good understanding of each term in her derived rule and seemed to understand the generality of her rule. She was able to identify how the coefficients in her rule were derived and expressed clearly what her variable was, i.e. the number of blocks. In her own words, the term ‘any’ was referred to as ‘however many’.

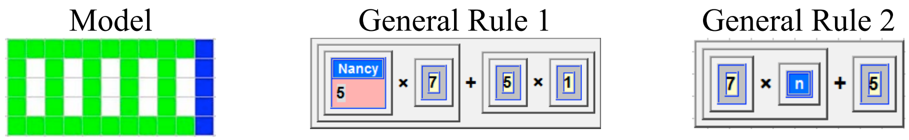


Figure 6. Nancy’s model and rules in eXpresser

Another 12 year old student, Connor (model (B) in Table 1), wrote: “My rule is correct because it worked without fault and the color stayed on the General World. (6×3) + [(6×2)–1] ×2 this is my rule. The 6×3 is the green tiles and how many of the green blocks we need times the how many tiles are in each block. The 6×2–1 is the red tiles and the -1 because otherwise the red tiles would go over how many you need and ×2 at the end is because I’ve got 2 lots of the red”. Even though Connor might think generally, he still used the specific value 6 for his variable when he wrote down his general rule- a generic example maybe. It is evident that he also had a clear understanding of each term in his rule supporting his rule’s correctness by reference to the feedback from the *eXpresser*, i.e. his model stayed colored for any model number.

Discussion

The availability in the environment of an increasingly abstract set of MSS strata to work with specific numbers, general numbers and number-objects made it possible for the students participating in the study to formulate rules that generated different patterns and to inter-relate them by way of one single independent variable. This was we argue a critical moment in a generalization task and helped students to develop and distinguish the notions of independent and dependent variable. We assert that the relationship among these generative rules is the result of the acts of reading and transforming that the students undertake on their own productions in *eXpresser* when trying to formulate one single rule to calculate the total number of tiles in the figurative model that was made up of different patterns. Production of sense arises in those acts with respect to expression of the rule in an algebra-like MSS. Nonetheless some of the students were able express generality in their own and *eXpresser*’s language, but still found it difficult to express it algebraically, a move supported by later collaborative tasks.

Endnotes

1. The MiGen project is funded by the ESRC/EPSRC Technology Enhanced Learning programme; Award no: RES-139-25-0381). For more details about the project see http://www.migen.org.
2. The experimental work in Mexico is funded by CONACYT (Grant No. 80359).
3. In England, the students were supported by either the teacher or the intelligent support component of the MiGen system, which provides prompts based on students’ interactions and helps them reflect and decide about their next step in solving the task

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FROM SPECIFIC VALUE TO VARIABLE: DEVELOPING STUDENTS' ABILITIES TO REPRESENT UNKNOWNS

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Third- through fifth-grade students participating in a classroom teaching experiment investigating the impact of an Early Algebra Learning Progression completed pre- and post-assessments documenting their abilities to represent or describe unknown quantities. We found that after a sustained early algebra intervention, students grew in their abilities to represent related unknown quantities using letters as variables.

Algebra has historically served as a gateway to higher mathematics that—due to high failure rates—has been closed for many students. These failures have been due in large part to a narrow treatment of algebra as an exercise in symbol manipulation without much regard for the symbols' underlying meanings (Kaput, 1998). Mathematics education researchers (e.g., Davis, 1985; Kaput, 1998; Olive, Izsak, & Blanton, 2002) have argued that addressing this issue requires us to view algebra not as an isolated eighth- or ninth-grade course, but as a continuous strand spanning the entire K-12 curriculum. This is not to be interpreted as a call to shift traditional instruction in symbol manipulation to earlier grades, but rather as one to consider broadening our notions of what it means to think algebraically and introducing elementary school students to important ideas of algebra in the context of their study of arithmetic.

In response to this call, we are presently drawing from research findings and curricular resources in the area of early algebra to develop an Early Algebra Learning Progression (EALP) organized around five "big ideas": 1) Generalized Arithmetic, 2) Equations, Expressions, Equality, and Inequality, 3) Functional Thinking, 4) Proportional Reasoning, and 5) Variable.

We are conducting a one-year classroom-based study in grades 3-5 to gather initial efficacy data regarding the impact of EALP-based classroom experiences on elementary students' developing understandings of these big ideas. The focus of this paper will be our initial findings regarding students' developing understandings of particular aspects of variable. We will share pretest and mid-year assessment data documenting students' performance using variables to represent unknown quantities. We will additionally have end-of-the-year posttest data ready to share at the conference.

Theoretical Framework

While traditional treatments of algebra often present the subject as one about "manipulating symbols that do not stand for anything" (Kaput, 1999, p. 134), we must acknowledge that much of algebra's power comes from the ability to represent unknown and varying quantities succinctly and manipulate these expressions without constant reference to their underlying meaning. The concept of variable must thus play a critical role in early algebra (Schoenfeld & Arcavi, 1988). Schoenfeld and Arcavi argue that rather than asking students to practice symbol manipulation and solving for unknowns, teachers should encourage students to view variables as shorthand tools for expressing already-understood ideas about varying quantities.

Despite its importance, variable is a concept with which many students struggle. Documented difficulties include believing that variables stand for names, labels, or attributes (e.g., *h* stands for height, *w* stands for weight) (Booth, 1988; Clement, 1982; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; MacGregor & Stacey, 1997; Weinberg et al., 2004) and being unable to operate with or even consider unknown quantities rather than specific values (Blanton, 2008; Carraher, Schliemann, & Schwartz, 2008).

While variables can take on several meanings in different mathematical contexts (Kieran, 1991; Küchemann, 1978; Usiskin, 1988), in this paper we focus on students' abilities to use variables to represent unknown quantities. We draw from Carraher et al.'s (2008) work, in which third-grade students were asked to represent the number of candies in two sealed boxes—one of which had three additional candies resting on top. The majority of the students initially assigned particular values to the amounts in each box (e.g., 3 and 6), suggesting they were unable to work with indeterminate amounts. Given a similar context, Blanton (2008) found that the presence of unknown quantities led students to consider such problems unsolvable. By drawing attention to multiple possible solutions to the *Candy Box* problem and encouraging students who refrained from assigning particular values to contribute to the discussion, Carraher et al. found that students' representations—which included drawings, tables, and verbal comments—were enriched and that eventually variable representations (e.g., N and $N + 3$) were accepted.

Kinzel (1999) asserts that students' difficulties with algebraic notation originate from narrow conceptions of variable (e.g., viewing the symbol as a label) and that if algebra is to be viewed as a meaningful representational tool, further attention should be given to symbolizing and interpreting variables in classrooms. Carraher et al.'s (2008) work illustrates that young elementary students are capable of representing unknown quantities in sophisticated ways when these representations build on their informal representations and understandings.

Method

Participants

Participants include 301 students in grades 3-5 from two elementary schools in southeastern Massachusetts. The school district in which these schools reside is largely white (91%) and middle class, with 17% of students qualifying for free or reduced lunch. Six classrooms (two from each of grades 3-5 and all from one school) are serving as experimental classrooms and 10 classrooms (four grade 3, four grade 4, and two grade 5, all from both schools) are serving as control classrooms.

Classroom Intervention

Students in the experimental condition are participating in an EALP-based classroom teaching experiment for approximately one hour each week for the majority of one school year. A member of our research team—a former third-grade teacher—is serving as the teacher during these interventions. Additionally, each lesson is observed by a member of the research team as a way to identify characteristics of students' thinking and issues of instructional design. The project team also meets twice weekly, once on-site to discuss initial observations about the teaching experiment, and once off-site to connect initial findings to the proposed EALP and revise the EALP. A typical one-hour lesson consists of a "jumpstart" at the beginning of class to review previously-discussed concepts, followed by group work centered on research-based tasks aligned with our EALP (see Figure 1 for an example). These tasks are designed to encourage students to reason algebraically in a variety of ways and justify their thinking to themselves and their classmates. All classroom tasks and assessment items presented in this paper are adapted from Carraher et al. (2008).

> Jack and Ava both have a box of candy. Each box contains the same number of pieces of candy, but we don't know how many in each. Ava is given four more pieces of candy.
>
> a) How could you represent the number of pieces of candy Jack has?
> b) How could you represent the number of pieces of candy Ava has?

Figure 1. The Candy Boxes task

Students in the control condition are participating in their usual classroom activity with their regular classroom teachers. District-wide, all classroom teachers are using "Growing with Mathematics" (Iron, 2003) curriculum materials. This curriculum does not include a particular focus on early algebra or tasks similar to the ones included in our intervention.

Data Collection

A pretest and (identical) posttest along with a shorter "mid-year" review were designed to measure students' understandings of algebraic topics identified across the five "big ideas" in the EALP. The constructs measured on our assessments are closely linked to the EALP. Item construction was research-based where multiple internal reviews of all assessment items were conducted by the authors in order to ensure consistency, coherence, and fidelity to that of the constructs mentioned in the EALP.

From these assessments, we will focus in this paper on two tasks—the *Piggy Bank* task from the pretest/posttest (see Figure 2) and the *Silly Bands* task from the mid-year review (see Figure 3)—that investigated students' abilities to represent unknown quantities in relation to other unknown quantities.

> Tim and Angela each have a piggy bank. They know that their piggy banks each contain the same number of pennies, but they don't know how many. Angela also has 8 pennies in her hand.
>
> a) How would you describe the number of pennies Tim has?
> b) How would you describe the total number of pennies Angela has?

Figure 2. The Piggy Bank task

> Carter and Jackson each have the same number of silly bands, but we don't know how many they have. Carter earns 3 more silly bands for cleaning her room.
>
> a) How would you represent the number of silly bands Jackson has?
> b) How would you represent the total number of silly bands Carter has?

Figure 3. The Silly Bands task

The pretest was administered to all participants prior to the start of the teaching intervention in early September 2010. The posttest will be administered at the intervention's conclusion, in May 2011. In this paper, we will be able to share results of the pretest and the mid-year review administered in late November 2010, after eight intervention lessons. Because the purpose of this mid-year review was to offer the research team formative feedback on the impact of the intervention, it was only administered in the experimental classrooms. As such, our focus in this paper will be exclusively on the experimental students' growth representing unknown quantities as evidenced in their performance on the aforementioned assessment items. A more complete experimental-control comparison will be included in the presentation as we will have posttest results to share by that time.

Data Analysis

Each item on the pre-test and mid-year review was first scored dichotomously (i.e., correct or incorrect). A response was scored as correct if a variable expression was written that correctly conveyed the given relationships (e.g. Jackson has J silly bands; Carter has $J + 3$ silly bands), if a correct statement was made about the relationship between the items in question (e.g., "Carter has 3 more silly bands than Jackson" or even "Carter has more silly bands than Jackson"), or if a student stated a need to use a variable but did not specify a particular one (e.g, "I would represent Tim's number of pennies with a letter"). In addition, there were a few instances in which students wrote equations rather than just expressions that were scored as correct (e.g., "Jackson has $a + 3 = b$"). Equations such as $a + 3 = a$ were scored as incorrect because the same variable was used to represent two different quantities.

Next, each item on the pre-test and mid-year review was given an appropriate strategy code. For part a on both assessments, responses were assigned the code *Letter* if a student used a variable to represent the unknown quantity (e.g., "Tim has x candies"), *Value* if a student assigned a specific value to the unknown quantity (e.g., "4 candies"), and *Picture* if a student drew a picture to represent the unknown quantity. Student responses in which the quantities in the task were compared using words and specific quantities (e.g. "Tim has 8 fewer pennies than Angela") were coded as *Comparison with Quantity.* Responses in which comparisons were made using words but no specific quantities (e.g., "Tim has fewer pennies than Angela") received a *Comparison without Quantity* code.

Responses to part b were coded in the same way as part a, with additional attention paid to whether the responses were related to part a or independent of part a. For example, if a student responded that "Tim has x pennies" in part a and "Angela has $x + 8$ pennies" in part b, this second response would receive the code *Related Letter*. If, instead, a student responded that "Angela has y pennies," this second response would receive the code *New Letter*. "I don't know" or blank responses received a *No Response* code, and all other responses were coded as *Other*. A chi-square analysis was conducted to check for a significant association between the correctness of responses on the pre-test as compared to the related mid-year review item.

To assess reliability of the coding procedure, a second member of the research team coded a randomly selected 20% sample of the pre-test data. The mid-year review data was fully scored by two coders. Pre-test scoring agreement between coders was 92% for both correctness and strategy on part a and 90% for correctness and 92% for strategy on part b. Mid-year scoring agreement between coders was 87% for correctness and 82% for strategy on part a and 81% for correctness and 84% for strategy on part b. All differences in scoring were discussed by the coders and resolved.

Results and Discussion

In this section, we report experimental student results from the *Piggy Bank* task (see Figure 3) and the *Silly Bands* task (see Figure 4). There was a significant association between the correctness of responses for all items on the pre-test as compared to the mid-year review. For part a, $\chi^2(1) = 85.85$, $p < .001$ where it was 15.08 times more likely for a participant to get *Silly Bands* part a correct on the mid-year review than to get *Piggy Bank* part a correct on the pre-test. For part b, $\chi^2(1) = 52.33$, $p < .001$ where it was 7.46 times more likely for a participant to get *Silly Bands* part b correct on the mid-year review than to get *Piggy Bank* part b correct on the pre-test. Both results were statistically significant.

Percent of students who gave correct response and percent of students who used a variable correctly on part a on Piggy Bank and Silly Bands tasks

	Grade 3	Grade 4	Grade 5
Piggy Bank (Pre-test)	(n = 39)	(n = 42)	(n = 42)
Gave correct response	5.1%	23.8%	33.3%
Used variable correctly	0%	0%	0%
Silly Bands (Mid-year)	(n = 39)	(n = 41)	(n = 41)
Gave correct response	79.5%	80.5%	80.5%
Used variable correctly	61.5%	63.4%	68.3%

Table 1. Part a results

Percent of students who gave correct response and percent of students who used a variable correctly on part b on Piggy Bank and Silly Bands tasks

	Grade 3	Grade 4	Grade 5
Piggy Bank (Pre-test)	(n = 39)	(n = 42)	(n = 42)
Gave correct response	5.1%	23.8%	35.7%
Used variable correctly	0%	0%	0%
Silly Bands (Mid-year)	(n = 39)	(n = 41)	(n = 41)
Gave correct response	66.6%	70.7%	68.3%
Used variable correctly	51.3%	48.9%	58.5%

Table 2. Part b results

From Tables 1 and 2, one can see that students initially struggled to produce correct representations or descriptions of unknown quantities, even with very liberal criteria. None of those students who did produce a correct representation or description used variables to represent the unknown. However, a majority of participants at all grade levels were able to use a variable correctly on part a and close to over half were able to use a variable correctly for part b by the time of the mid-year review. Furthermore, participants tended to use the same variable in part b of the *Silly Bands* task that they used in part a, showing an ability to consider the relationship between two unknown quantities. Tables 3 and 4 show the percent of students using the previously-discussed strategies in response to the pre-test and mid-year review tasks.

Percent of students using each strategy in response to part a on Piggy Bank and Silly Bands tasks

	Grade 3		Grade 4		Grade 5	
Strategy	*Piggy Bank* (Pre-test)	*Silly Bands* (Mid-year)	*Piggy Bank* (Pre-test)	*Silly Bands* (Mid-year)	*Piggy Bank* (Pre-test)	*Silly Bands* (Mid-year)
Letter (as variable)	2.6%	89.7%	0%	75.6%	0%	85.4%
Value	33.3%	5.1%	45.2%	12.2%	42.9%	4.9%
Picture	0%	0%	0%	2.4%	0%	4.9%
Comparison with Quantity	5.1%	5.1%	11.9%	9.8%	23.8%	4.9%
Comparison without Quantity	0%	0%	11.9%	0%	7.1%	0%
Other	5.1%	0%	14.3%	0%	16.7%	0%
No Response	53.9%	0%	16.7%	0%	9.5%	0%

Table 3. Part a strategies

First, in response to part a, note that participants demonstrated no prior knowledge of representing an unknown quantity using variables, preferring instead to assign specific values. However, by the time of the mid-year assessment, at least 75% of students at all grade levels attempted to represent the unknown quantities using a variable, with the majority of them doing so correctly. We had predicted that students would use a pictorial representation (e.g., a piggy bank) or an empty box to represent an unknown amount, especially prior to instruction. However, only a small number of these pictorial and box representations surfaced. The decline in *No Response* codes was dramatic, especially in grade 3.

Percent of students using each strategy in response to part b on Piggy Bank and Silly Bands tasks

	Grade 3		Grade 4		Grade 5	
Strategy	*Piggy Bank* (Pre-test)	*Silly Bands* (Mid-year)	*Piggy Bank* (Pre-test)	*Silly Bands* (Mid-year)	*Piggy Bank* (Pre-test)	*Silly Bands* (Mid-year)
Related Letter	2.6%	59.0%	0%	53.7%	0%	61.0%
New Letter	0%	28.2%	0%	22.0%	0%	26.8%
Related Value	25.6%	2.6%	35.7%	9.8%	35.7%	4.9%
New Value	18.0%	5.1%	11.9%	4.9%	2.4%	0%
Related Picture	0%	0%	0%	2.4%	0%	2.4%
New Picture	0%	0%	0%	0%	0%	0%
Comparison with Quantity	5.1%	0%	16.7%	7.3%	21.4%	4.9%
Comparison without Quantity	0%	5.1%	7.1%	0%	9.5%	0%
Other	7.7%	0%	14.3%	0%	16.7%	0%
No Response	41.0%	0%	14.3%	0%	14.3%	0%

Table 4. Part b strategies

In response to part b, we noted a shift in responses from assigning a specific value on the pre-test (related or non-related to the response in part a) to using a variable on the mid-year review. Well over half of participants across the grades wrote an expression using the same variable used in part a, with the majority doing so correctly. No students left this task blank or responded with "I don't know" on the mid-year review.

One particular category of response caught our attention. Some participants used variables but represented the unknown quantities with equations rather than expressions. For example, in response to the *Piggy Bank* task, one student represented the number of Angela's pennies as $n + 8 = n$. Such a response recalls Booth's (1988) assertion and Carraher et al.'s (2006) similar finding that students may struggle with "lack of closure" and feel the need to "simplify" or come to a "single term" answer. Such students appear to have difficulty holding operational and structural views of expressions simultaneously. As Linchevski and Herscovics (1996) suggest, expressions such as $n + 8$ are difficult for students because the addition operation cannot be performed. The expression must be viewed as the result of the operation and not merely a process. This tendency among some students to write incorrect equations was present even after several months of our instructional intervention, with 10.3%, 4.9%, and 2.4%, in grades 3, 4, and 5, respectively, answering in this manner on the mid-year review.

Conclusion

As Blanton (2008) asserts, "It is as children work with symbols that they acquire meaning for them. They will experience a natural progression in their thinking that begins with a limited understanding of symbols and symbolizing" (p. 66). While students initially exhibited great difficulty representing unknown quantities in a general way (i.e., without assigning a specific value), over half of our participants at each grade level exhibited the ability to use a variable to represent an unknown quantity by the time of the mid-year review. While we have indications that several students had difficulty viewing variable expressions as objects not in need of further simplification, overall the majority of students seem to be learning to use variables in meaningful ways that were unknown to them at the start of the school year.

Endnotes

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AMBIGUITY OF THE NEGATIVE SIGN

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The goal of this study was to understand errors that student make when simplifying exponential expressions. College students enrolled in four college mathematics courses were asked to simplify and compare such expressions. Quantitative analysis identified three persistent errors: interpreting negative bases, negative exponents, and parentheses. Qualitative methods were used to examine why they made these errors. Analysis indicated that students frequently misinterpret the negative sign when attached to the base of an exponential expression. We theorize that students' concept image of the negative sign must move beyond rule models to correctly interpret numbers in exponential expressions.

Students entering post-secondary schools often need frequent reminders on how to use algebraic properties while solving problems in pre-calculus and calculus classes. Algebra provides the foundation to advanced mathematical thinking; and proficiency in algebraic manipulations is essential for students if they want to enter STEM careers (Liston & O'Donoghue, 2010). The terminology and rules of algebra contain little meaning to many students and appear arbitrary (Demby, 1997; Kieran, 2007). Rules are memorized and are considered to be similar to the rules of a game. Many students have difficulty keeping track of memorized rules that appear to be contradictory at times.

Interpreting the equal sign is one example that has been studied extensively. For example, Knuth et al. (2006) found that an operational conception of the equal sign, the idea that the equal sign was a signal to do something, was more commonly held among middle school students than a relational conception of the equal sign. Students with a relational conception understood the equal sign as representing congruence between the expressions on either side of the equal sign. They found middle school students with the relational understanding were much more successful at solving algebraic equations. Thus, the misconception of the equal sign as a prompt to do something interfered with students' mathematical development.

Carraher and Schliemann (2007) described the difficulties students have bridging arithmetic to algebra and in particular interpreting mathematical symbols. Kieran (2007) expanded those notions by further discussing the development of algebraic thinking in middle and high school. She noted that considerable research exists that describes the ways in which students use variables, expressions, and equations. However, there is relatively little research on students' interpretation of the negative sign which is interpreted as either an operation (subtract) or a negative number. This research paper is part of an on-going study in which the authors are examining pedagogical approaches and interventions that increase precalculus students' achievement. Here we investigate how college students interpret a negative sign that is embedded in exponential expressions.

Previous Research

Students must overcome numerous obstacles to become fluent in algebra, including the interpretation of operations implied by the positioning of symbols next to each other (Lee & Messner, 2000). Research (Chalouh & Herscovics, 1983) on concatenations indicated that many students had difficulty interpreting both mixed numbers where addition is implied or algebraic expressions of the form *3a.* After many years of high school experiences learning that multiplication was implied in concatenations (*3a*), some college students applied multiplication with mixed numbers. They simplified $3\ \frac{2}{3}$ to be $3 \times \frac{2}{3} = 2$. Clearly, these students over generalized one set of meaning that is appropriate for expressions like *3a.* Matz (1980) noted that many students did not know how to interpret a negative sign and exponents. For example, students frequently confuse whether -3^2 is 9 or -9.

Vlassis (2004) examined how middle school students interpreted negativity. The eighth-grade students conceptualized negativity as a process, linked to the binary operation of subtraction. Negative nine made sense to these students, for example, in an expression such as $n - 9$ but -9 was more difficult to explain. She concluded that the different uses of the negative sign are counterintuitive and an obstacle for students to overcome.

Teachers often use a textbook to guide their instruction; Lee and Messner (2000) examined opportunities for students to work with the negative sign in sixth- through ninth-grade mathematics curricula. Lee and Messner found that textbooks typically provide multiple opportunities to work with positive and negative integers but seldom offer experiences with negative fractions, exponents, decimals, and mixed numbers. To support students' understanding of $-n$, some textbooks use colored chips in which red is negative and black is positive (wikiversity, 2010). For example, to combine -9 and $+4$, students take nine red chips and four black chips. The four black chips annihilate four of the red chips with the result of five red chips, negative five. Here, students develop the notion that the negative sign indicates an entity that neutralizes a positive chip with the net result of zero.

We theorize that students encounter the notion of additive inverse but it is not always clearly articulated or connected to the idea of an identity. Without discussion on the different interpretations of the negative sign, many students have difficulty interpreting the negative sign in concatenations. Kieran (2007) suggests that additional research is needed to understand the reasons for this difficulty.

Theoretical Framework

A mathematical concept can be thought of as a complex web of ideas constructed by each individual from mathematical definitions and mental constructs (Tall & Vinner, 1981; Vinner, 1992). Tall and Vinner described these two components as concept definition and concept image. A concept definition is a set of words that is used to specify the concept. The definition may be phrased in language accepted by the mathematical community or in everyday language taught by teachers who create short mnemonics to help students remember procedures.

For example, teachers and textbooks often use the phrase *opposite of* rather than the mathematical term *additive inverse* to describe the relationship between a number, its negative and the additive identity, zero. A concept image includes "all of the mental pictures and associated properties and processes." (Tall & Vinner, 1981, p. 152). For additive inverse, the image may include $4 + (-4) = 0$ as (a) four jumps to the right and then 4 jumps to the left on a number line or (b) two numbers positioned equidistant but on the opposite side of zero on a number line. Thus, the concept of additive inverse is a synthesis of the concept definition and images associated with it. Vinner (1992) suggested that individuals create idiosyncratic images that may interfere with the development of new concepts.

Concepts develop over time and are important for advanced mathematical thinking (Tall, 1991). They allow for generalization, inference, different levels of abstraction, cognitive economy, and communication (Palmeri & Noelle, 2003). Palmeri and Noelle theorized that a concept image becomes more sophisticated through three stages: rule model, prototype model, and exemplar model. During the stage of the *rule model*, conceptual rules are carefully worded definitions that are internalized by students and used to carry out procedures or to determine whether an object is a member of a set. The *prototype model* indicates the stage in which an individual recognizes nuances that distinguish individuality among the members of a set. For example, if we consider the set of all triangles, members may look quite different with different side lengths and angles; however, all have exactly three sides. These images become prototypes that allow the individual to check whether an object is a member of the set. Through the prototype model, the concept image becomes more sophisticated and includes both a concept definition and a variety of concept images that illustrate diversity. The *exemplar model* enables individuals to synthesize information about both the commonality and variability of category exemplars and indicates a structural understanding of the concept. Through the exemplar model, the concept image becomes yet more complex and the concept definitions and images are intertwined. For example, students in this stage might recognize how changing the coefficients of a function changes features of its graph. The individual compares the symbolic representation with prototype images and manipulates the images in such a way to describe the effect of the new coefficients. The individual is aware of the nuances between different members and can modify them at will through an understanding of the underlying mathematical structure.

Methods

For this study, we assessed 904 students enrolled in college algebra, pre-calculus, and first and second semester calculus for their ability to simplify and compare exponential expressions. The assessment contained 19 problems involving exponential expressions. Quantitative methods were used to identify the problems that a large number of students missed across the four class levels. There were four problems in which more than 40% of students at all levels solved incorrectly. We labeled these problems as indicators of persistent errors.

To understand why students consistently made these errors, we sought to interview 50 students. However, only 19 students agreed to the interview with a $10 stipend paid for the half-hour interview. These 19 students were enrolled in college algebra, pre-calculus or calculus 2. During the interview, students were asked to read and resolve each of the four problems and explain their reasoning aloud as they solved them. Questions were asked to help identify the concept definitions and images that students used while solving the problems. Conceptual cross-case matrix analysis (Miles & Huberman, 1994) was used to characterize students' explanations to identify commonalities among student explanations.

Findings

Through quantitative and qualitative analysis, three persistent errors emerged around the following concepts: interpretation of a negative sign preceding a base; the role of parentheses in an exponential expression; interpretation of a negative exponent. This paper is based on our qualitative analysis of the first two errors which are related. Students' difficulty distinguishing between a negative number raised to a power and the additive inverse of a number raised to a power became evident by examining their explanations of three problems. Two of the problems required students to simplify exponential expressions and the third problem involved comparing two exponential expressions.

Interpretation of a Negative Sign Preceding a Base

Two problems on the assessment required students to simplify an expression where either a negative sign preceded the base or was part of the base. One asked students to simplify $(-8)^{2/3}$ and the second to simplify $-9^{3/2}$. All of the students who incorrectly simplified the first problem read, "Negative eight raised to the $\frac{2}{3}$ power." Most of the students who incorrectly simplified the second problem read, “Negative nine raised to the $\frac{3}{2}$ power.” Other variations of the second problem were, “I’m not sure how to read that [pointing to $-9^{3/2}$]. It’s different from that [pointing to $(-8)^{2/3}$] because it has parenthesis.” and “I never know whether the negative sign is included.” The reading of these two problems with a negative sign was similar. Students read both problems as if the negative sign was part of the base. They did not differentiate when the negative sign indicated the additive inverse or a negative number. Only one student who misread the expression proceeded to solve the problem correctly. She wrote on her paper, “ $-9^{3/2} = -\sqrt{9^3} = -\sqrt{9 \times 9 \times 9} = -\sqrt{81 \times 9} = -\sqrt{729}$.” She did not finish simplifying the expression because she “always used a calculator.” It is interesting to note that she did not find the square root of nine before cubing it. We interpreted her solution strategy as a reliance on rules, first raise the base to the indicated exponent then find the root of the number.

All of the other students who misread $-9^{3/2}$ and attempted to solve it made the same error. They interpreted the negative sign as attached to the nine before any power was taken. They wrote “$\sqrt{-9^3}$” or stated that the answer was an imaginary number and could not be solved. Four students further simplified $\sqrt{-9^3}$ and their answers resulted in: $3i$, $i\sqrt{729}$, or $(3i)^2 = -81i$. Clearly, they made additional errors in simplifying the expression; however, the error we were most interested in was the initial one of interpreting $-9^{3/2}$ as $(-9)^{3/2}$ rather than the additive inverse of $9^{3/2}$.

Two students correctly interpreted the expression, $-9^{3/2}$. One pre-calculus student correctly stated the steps to follow to simplify it, “Take nine then cube it, then square root it, then make it negative.” She read the problem by translating it into the procedure to simplify the problem rather than the more precise reading that a calculus student made, “Take the negative of nine to the $\frac{3}{2}$ power.” Both of these students recorded the correct answer.

Only one student we interviewed recognized that taking the square root first and then cubing the result would make the computation easier. It appears most students learned an order of operation that begins with applying the power indicated by the numerator in the exponent followed by computing the root indicated by the denominator in the exponent regardless of the efficiency of this order.

Role of Parentheses in Exponential Expressions

The assessment required students to compare two expressions, $(-17)^8$ and -17^8. Five of the 19 students interviewed had a correct answer and gave a similar explanation, “Negative 17 to the eighth [pointing at $(-17)^8$] is larger than negative 17 raised to the eighth power [pointing at -17^8] because the [even] power of a negative number is positive. Negative 17 to the eighth [pointing at $(-17)^8$] is a positive number and larger than negative 17 to the eighth [pointing at -17^8].”

In contrast, the students who were incorrect read both expressions as “negative 17 to the eighth” claimed that these two numbers were equal. Six students stated, “Parentheses don’t matter.” When probed, they explained that parentheses matter only when there are more terms. With additional terms, the parentheses indicate what operations should be completed first. One student reasoned, “Parentheses tell you the order of operation, what needs to be done first. You only need them for something like $4(17-9)$. They tell you that you have to subtract before multiplying.” The other students who gave incorrect answers explained that both expressions were raised to an even exponent and thus both were positive and that –17 and 17 were the same distance from zero, thus the same. Clearly, the concept image of $(-17)^8$ and -17^8 were identical because –17 and (–17) had identical meaning.

The probing questions that we asked about whether the two numbers were the same prompted one calculus student to retrieve a kernel of forgotten mathematics. Reflecting, he explained that -17^8 could be thought of as -1×17^8. Thus it was a negative number and smaller than the other expression. Immediately after completing this problem, he said, “I made a mistake. I don’t know why I did that. I saw the negative sign in that problem [pointing to $-9^{3/2}$] and the square root and automatically wrote it off.” At this point, he went back and correctly simplified the previous expression, $-9^{3/2}$. We suspect that the reason for his mistake was not an oversight, rather it indicated an important misconception that was linked to his concept image of negative numbers. The negative sign was irrevocably linked to a number and was interpreted as an indicator of a negative number unless it followed a number or variable.

Discussion

Consistent with findings of previous researchers (Chalouh & Herscovics, 1983; Lee & Messner, 2000; Matz 1980), college students in this study had difficulty interpreting the negative sign in concatenations. Many students did not distinguish a negative number raised to a power from the additive inverse of a number raised to a power. Their concept image was limited to a rule model in which they carried out a procedure to interpret any number with a negative sign, regardless of other mathematical symbols that were next to it. The concept image of $-n$ was limited to a negative number and its meaning did not change when it was raised to an exponential power.

Parenthetically, none of the students that we interviewed used the language of additive inverse when discussing problems on the assessment. Conceptualizing -9 as an additive inverse of 9 suggests a relational interpretation; however, students frequently failed to extend this relational understanding to more complex expressions. In particular, they failed to recognize the symbol $-9^{3/2}$ as the additive inverse of $9^{3/2}$. However, in an operational context such as $1 - 9^{3/2}$, students recognized that $9^{3/2}$ was to be subtracted from the number one. In this mathematical context, students first simplified the exponential expression then found the difference.

When asked if there is an inconsistency between these two interpretations, these college students explained that “…the minus sign means to minus when a number precedes it.” Here it is clear that many students may not think of the negative sign as indicating an additive inverse that can be applied to more complex expressions. Without recognizing the mathematical meaning of an additive inverse that is linked to the concept of an additive identity, these students did not develop a more sophisticated concept image of the negative sign. In addition, this may explain why students do not recognize n^{-1} as the multiplicative inverse of n.

Removed from a procedural context, students tend to misinterpret the symbol $-9^{3/2}$. Negative numbers become mathematical objects that are located on the number line. The associated concept image of a negative number is a dot on the number line to the left of zero, equidistant from zero as its positive counterpart. Students fail to separate the negative sign and the base, it becomes an inseparable object. This was made explicit by several of the students as they explained the meaning of -17^8 and $(-17)^8$. Students interpreted both expressions as a negative number that is raised to the power 8. To them, the parentheses were unnecessary because the negative sign and 17 are combined to form an inseparable object, -17. They correctly recognized -17 and (-17) as the same mathematical object but erroneously extended this to -17^8 and $(-17)^8$. The rule for interpreting a negative sign was resistant to recognizing that its meaning can change when other mathematical symbols are added.

The concept image that these college students developed was different from the eighth-grade students in Vlassis's (2004) study. Vlassis found that the eighth-grade students interpreted negativity as the operation of subtraction. They saw the negative sign as an indication to conduct a procedure. In contrast, the college students in our study interpreted negativity as a mathematical object or entity with a distinct location on a number line. These students associated the negative sign as an operation when it was preceded by a number. In this case they interpreted it differently. Thus, in one sense, these college students were moving toward a concept image with two definitions. We theorize that simultaneous recognition of these two concept rules would lead to a prototype model in which students recognize similarities and differences in the concept images. Thus, they could select the appropriate concept image for a given context. Integration of these images would lead to an exemplar model and students could flexible apply these images in different concatenations. Students using an exemplar model would understand the function of the negative sign to indicate the multiplicative or additive inverse of a number and extent this meaning to more complex situations.

We theorize that students' concept image of negative numbers is not sufficiently well developed to enable them to recognize the negative sign as a relational concept (additive inverse) when the number in question involves an exponent. However, if the number is embedded in an operational context, they generally interpret it correctly.

From our experience, this problem continues to show up in advanced mathematics classes. Students interpret $-x$ as a negative number regardless of the value of x. For example, if students are told that (a, b) is a point in Quadrant II then asked, "In which quadrant does the point $(-a, b)$ lie?" they fail to interpret $-a$ as a positive number. We theorize that it is critical for students to develop a more sophisticated concept image of $-x$ that includes the concept definitions of a negative number and the additive inverse to correctly interpret the negative sign in concatenations.

Implications

The goal of this research project was to understand the fundamental reasons why students have difficulty simplifying exponential expressions. We found that a limited concept image of a negative number was a confounding factor. This inflexible image led students to misinterpret the negative sign in concatenations. We suggest that teachers need to introduce the language of "additive inverse" and "multiplicative inverse" into the class room to help student develop a more sophisticated understanding of the negative sign. In addition, students need more time investigating the interpretation of the negative sign in different concatenations.

We found it interesting that eighth-grade students in Vlassis's (2004) study created a concept image of negativity as an operation. As students mature this image seemed to be replaced by negative numbers as entities unless it was preceded by another mathematical expression . Additional research is needed to understand how students' concept of negativity develops over time. The goal is to help students create more sophisticated concept images that are based on exemplar models rather than rule-based models. We agree that colored chips can be helpful to students as they investigate the meaning of the negative sign but over reliance on one interpretation of the negative sign can hinder students' development of a more sophisticated concept image. They need to spend more time on different interpretations of the negative sign in concatenations. This can be accomplished by creating mathematical contexts that produce cognitive conflict forcing students to reexamine their prototypes for consistency. Continued research is needed to better understand the reasons behind persistent errors that students make when interpreting the negative sign in this and other contexts.

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A STUDENT TEACHER'S SUPPORT FOR COLLECTIVE ARGUMENTATION

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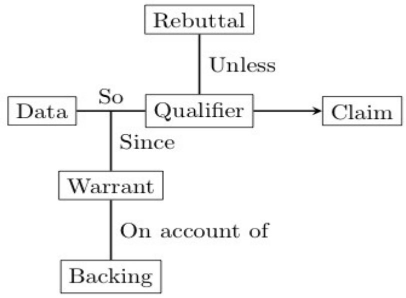
We observed a student teacher's practice as she taught one unit of instruction. Using a modification of Toulmin's (1958/2003) diagrams, we analyzed her support for collective argumentation by examining questions she asked, parts of arguments she contributed, and the kinds of warrants her students contributed. Our findings suggest reasons she was identified as an exemplary student teacher and support moves that may be applicable to other teachers' practice.

As part of a larger study, we observed several student teachers as they each taught one unit of instruction. Our research team identified one of the student teachers, Bridgett (a pseudonym), as exhibiting practices more commonly seen in experienced teachers. Bridgett's facility for engaging in frequent, prolonged, and mathematically significant discourse with her students presented an opportunity for investigation into how collective argumentation might be supported by teachers. In this paper, we characterize her teaching moves in support of collective argumentation to answer the following question: In what ways did an exemplary student teacher support collective argumentation as she taught a geometry unit?

Background

Recent work in mathematics education has highlighted argumentation as an important part of classroom discourse. Building on Toulmin's (1958/2003) model of argumentation, mathematics educators, following Krummheuer (1995), examined collective argumentation in classroom settings. Collective argumentation involves people arriving at a conclusion, often by consensus. This avenue of research is an extension of Toulmin's work, as he examined argumentation in the traditional sense of one person convincing an audience of the validity of a claim. Some work in mathematics education examines individual construction of arguments (e.g., Hollebrands, Conner, & Smith, 2010; Inglis, Mejia-Ramos, & Simpson, 2007), but we build on studies addressing collective argumentation. Current work in collective argumentation involves examining student learning through this lens (Krummheuer, 2007) as well as examining how ideas become "taken as shared" (Rasmussen & Stephan, 2008).

An argument, as described by Toulmin (1958/2003) and currently used in the field, involves some combination of claims (statements whose validity is being established), data (support provided for the claims), warrants (statements that connect data with claims), rebuttals (statements describing circumstances under which the warrants would not be valid), qualifiers (statements describing the certainty with which a claim is made), and backings (usually unstated, dealing with the field in which the argument occurs). Toulmin conceptualized an argument as occurring with a specific structure (see Figure 1) in which these parts of arguments relate to one another in specific ways. In practice, arguments are often more complicated in that, for example, statements offered as data may also need support, thus functioning as both data in one argument



and a claim in a sub-argument[1].

Figure 1: Diagram of a Generic Argument (adapted from Toulmin, 1958/2003)

Our research answers Yackel's (2002) call to examine the teacher's role in collective argumentation by focusing on both the parts of arguments he or she provides and the teaching moves that prompt or respond to parts of arguments provided by students. Previous research has shown that facilitating mathematical discussions, of which collective argumentation is a part, is difficult for teachers, particularly when students have been engaged in tasks that may have multiple solution paths (Hufferd-Ackles, Fuson, & Sherin, 2004; Stein, Engle, Smith, & Hughes, 2008). Hufferd-Ackles et al. suggested creating a "math-talk learning community" (p. 81) in a classroom as one way to ameliorate the difficulties of facilitating productive discourse, while Stein et al. described five practices that are useful when facilitating discourse particularly around cognitively demanding tasks. Our research examines the role of one student teacher as she and her students engage in collective argumentation.

Methodology

We observed and videotaped one exemplary student teacher, Bridgett, for seven consecutive class periods, covering the bulk of a geometry unit within an integrated and accelerated freshman mathematics course. Members of the research team transcribed each of the video recordings. Field notes, class video recordings, and relevant written class materials supplemented the transcripts. We identified and diagrammed episodes of argumentation within the transcripts, purposefully ignoring non-mathematical classroom talk and arguments that were largely definitional (such as deciding how many sides a nonagon has) or pedagogical in nature. Individual or paired members of the research team created diagrams, and the entire team vetted each proposed diagram after intense discussion and frequent

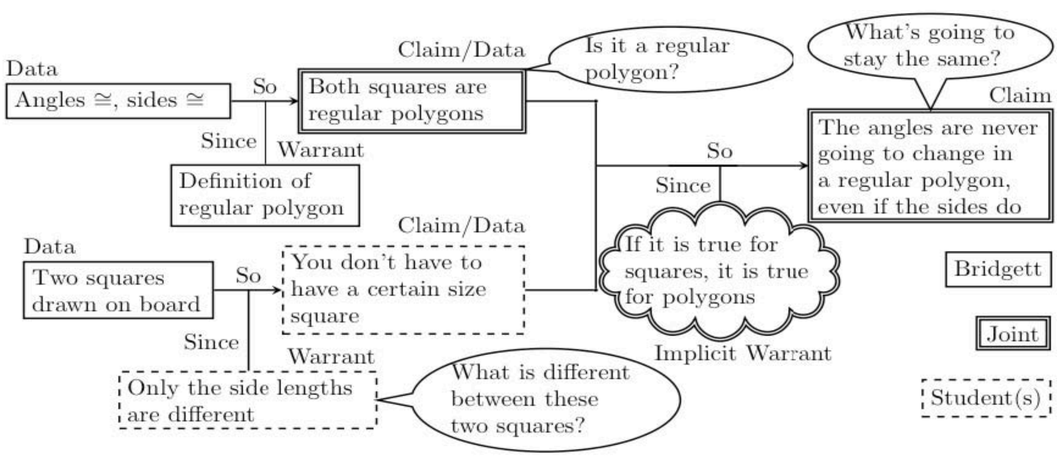


Figure 2: Diagram of Argument about Regular Polygons

We used a modification of Toulmin's (1958/2003) diagrams to document the collective argumentation. We color coded[2] the diagrams to record whether Bridgett, the students, or both Bridgett and students together contributed a given component (as in Conner, 2008) and recorded Bridgett's actions in support of argument components. These practices allowed for a finer nuance in identifying who contributed a specific part of an argument and how that contribution came about. See Figure 2 for an example of Bridgett's support for argumentation. In addition, when a part of an argument was implied but not directly stated by the class, we included it in the diagram, labeling it as an implicit part. Most of these parts were warrants (see the "cloud" in Figure 2 for an example of an implicit warrant.

We categorized Bridgett's questions, other support and direct contributions across all the diagrams. We analyzed Bridgett's supportive acts in light of the structure of the arguments and their mathematical context. Our identified patterns of support constitute the core of the present results.

Results

Bridgett engaged in many different activities to support the collective argumentation in her class. She asked questions that prompted students to provide various parts of arguments, she responded to students' contributions in various ways (sometimes verbally acknowledging the contribution, sometimes recording the contribution on the board for future use), and she directly contributed parts of arguments. On a more global scale, she negotiated norms in her class that applied to all members that involved supporting claims with data and warrants. Our analysis of Bridgett's support for argumentation is ongoing; this report focuses on the parts of arguments prompted by Bridgett's questions, the particular kinds of warrants that were contributed by students, and the parts of arguments that Bridgett herself contributed. Through the analysis of these aspects of Bridgett's support for argumentation, we aim to provide a description of what is possible, even in a student teacher's classroom, and to describe the moves that Bridgett used that might be productive in other teachers' classes.

The episode of argumentation chosen to illustrate our points in two of the following sections (Figure 2) occurred during a whole class discussion about polygonal angle formulas. The class was engaged in an attempt to derive the formula for the measure of the interior angles of (regular) polygons. The class had already discussed various ways to partition particular polygons into triangles, in order to sum all the interior angles, and the observation was made that each angle in the polygon needed to be congruent if a general formula for the interior angle measures was to be found. Bridgett recognized that many of the students had only an intuitive understanding of regular polygons, and that the definitional properties of regularity were not explicit in some of the students' minds. Therefore, she explored with the class what should be true of "regular" polygons.

Questions as Support for Collective Argumentation

As a method to analyze Bridgett's support, our research team identified the various questions Bridgett posed that supported collective argumentation within the classroom. Examining her questions across the collection of arguments, we recorded 290 questions from Bridgett that directly supported a part of an argument. We then analyzed who contributed the various parts of the arguments and found that her questions prompted a total of 156 student contributions, 119 contributions from both the students and Bridgett, and 15 contributions from Bridgett. This first level of analysis reflects two important norms within Bridgett's support for collective argumentation. First, Bridgett's support facilitated a significant amount of student contribution. Second, due to the significant amount of student contribution, Bridgett rarely needed to respond to her own questions; rather, the support she provided by asking appropriate questions elicited meaningful student participation. These findings seem to substantiate our assumption that classified Bridgett as an exemplary student teacher.

To further our analysis, we differentiated among the various parts of arguments prompted by Bridgett's questions. For example, her questions prompted more contributions of data than any other part of the argument. She asked 82 questions that prompted students to provide data, 67 that prompted data to which both Bridgett and her students contributed, and 6 that resulted in Bridgett herself providing data. Included in these numbers are parts that were coded as data alone and parts that were coded as both data and claim, as illustrated in Figure 2. The statement, "Both squares are regular polygons" was coded as both data and claim because it first was a claim, made by Bridgett and her students together, that the squares on the board were regular polygons. It was then used, together with "You don't have to have a certain size square" as data for the main claim in the argument. Although this large proportion of data supported by Bridgett's questions corresponds to the proportion of data provided within the arguments, this finding demonstrates how Bridgett's questions required students to provide the facts needed to support their development of mathematical claims. Bridgett's questions prompted many of the parts of arguments that were contributed, but not all contributions were prompted by questions. In the next section, we explore similarities and differences in student contributions of warrants prompted and unprompted by Bridgett.

Student Contributions of Warrants

Many of the questions posed by Bridgett supported students in making their reasoning explicit. Bridgett also supported students' reasoning in other ways, such as marking a diagram or encouraging the students to continue their thought processes. Of the 57 instances in which Bridgett provided any support for a warrant, 48 of the warrants were explicit and only 2 could be attributed solely to Bridgett. In contrast, of the 175 warrants that were unsupported, 63 warrants were explicit and Bridgett provided 15 of the explicit warrants. Thus, when Bridgett provided support for a warrant, students were more likely to make their reasoning explicit, either in collaboration with the teacher or on their own. However, when Bridgett did not provide support, the students were less likely to make their reasoning explicit, and Bridgett may have contributed the appropriate reasoning. These conclusions align with Toulmin's (1958/2003) suggestion that warrants are typically left implicit unless specifically requested, or, in our case, supported by the teacher. In addition, these conclusions speak to the importance of the teacher's role in collective argumentation; specifically, assisting students in making their reasoning available to the class.

Although it was typical for students to leave their reasoning implicit in the absence of support, there were 25 instances in which the students made their reasoning explicit without specific support from Bridgett. For this reason, we focused our analysis on the types of warrants provided by students with and without Bridgett's support. Our analysis yielded three categories of warrants: mathematical concepts (i.e., definitions, theorems, and properties), procedures, and observations. Of the 24 explicit warrants provided exclusively by students in conjunction with support from Bridgett, 8 were categorized as concepts, 12 were categorized as procedures, and 4 were categorized as observations. One example of an argumentation episode in which Bridgett provided support for a students' warrant was made while the class was reviewing homework. Bridgett posed the question to the class, "Is it possible to have a triangle with side lengths 3, 4, and 7?" The class responded, "No." Bridgett then asked, "Okay, why not?" One student reasoned, "[Be]cause the two smallest sides have to be as long as the longest side. [Be]cause 3 plus 4 equals 7 and 7's not larger than 7." This argument is illustrated in Figure 3a. The student's warrant was categorized as concepts because the student's reasoning made use of the triangle inequality theorem although it was not precisely stated. Bridgett supported the warrant by asking the question, "Why not?"

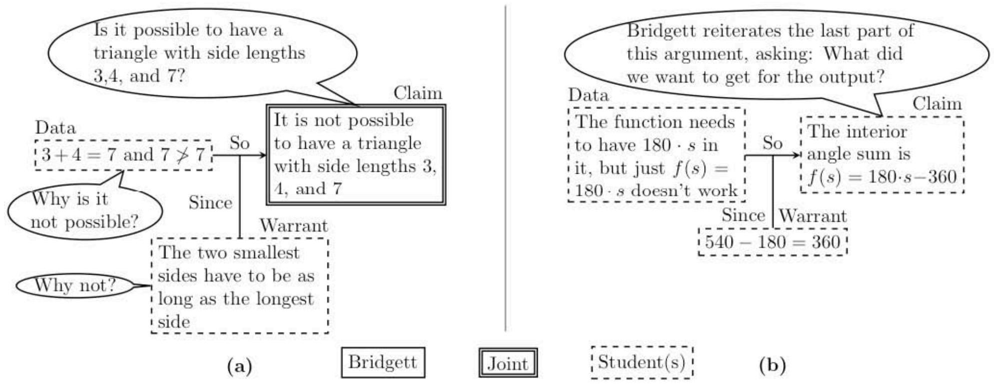


Figure 3: Arguments about (a) Triangle Side Lengths and (b) Polygon Angle Sum

In comparison, when students provide explicit warrants in the absence of support from Bridgett, 8 were categorized as concepts, 12 were categorized as procedures, and 5 were categorized as observations. For example, one episode (see Figure 3b) in which Bridgett did not provide support for a student's explicit warrant was made while the class determined the function for the sum of the measures of the interior angles of a polygon given the number of sides. A student was at the board explaining how he developed his formula. Claims that had entered into the collective up to this point include the linearity of the function, the slope of the function is 180, and 180(3) = 540 but the sum of the interior angles of a triangle is 180 degrees. The student claimed the function needed to have 180 as the slope, but 180 "doesn't work out right." He then stated, "But then I just subtracted 180 from 540 and it equals 360. Yeah. So, subtract 360." While making this statement, he wrote on the board $f(s) = 180s - 360$. The student's warrant was categorized as a procedure. Although Bridgett provided support for the claim, she did not provide support for his warrant.

Students provided the same types of warrants with the same approximate frequency, regardless of the presence of support. We draw two conclusions from this comparison. First, it appears that Bridgett negotiated classroom norms regarding the appropriate types of justification and provided support for warrants involving each of the types of reasoning. Second, the minimal amount of variation in the number of warrants in each category suggests Bridgett was not selective in the types of warrants she supported. Her choice of when to provide support may be due to many factors, including the complexity of the argument, the perceived mathematical abilities of the student(s) providing the claim, or the context of the argument. Further analysis is warranted and ongoing. Although we can only speculate as to why Bridgett may have chosen to provide support in particular circumstances, we can analyze the contributions she makes to an argument.

Bridgett's direct contributions

Bridgett directly contributed only 100 parts of arguments (out of more than 800 parts), slightly less than one per episode of argumentation, during the seven days included in this analysis. In addition, we attributed an additional 15 parts of arguments to her (coded as implicit contributions). Of her 100 explicit contributions, 59 were coded as data, with only seven of these being coded as both data and claim. When a part of an argument serves as both data and claim, it is either presented as an intermediate claim that serves as data for a later claim, or it serves as a claim's support (data) that is later challenged. It is significant that most of Bridgett's data contributions were coded exclusively as data because it is indicative of the purposes they served in the arguments. In many cases, when Bridgett introduced a problem to the class, her contribution was coded as data because it included information necessary to solve the problem (and/or specified what the problem was). For instance, in the argument diagrammed in Figure 2, Bridgett started the episode of argumentation by asking, "What is different between these two squares?" after she had drawn two squares on the board. The two squares drawn on the board were coded as data because they were used in this way in the argument; they were attributed to Bridgett because she drew them. In fact, without these two squares, the argument would not have occurred. It is different in contribution than the student's contribution that is coded as claim/data. The student's claim/data used Bridgett's data to make a claim that was then used as data for the final claim. Thus, while Bridgett's drawings of squares initiated the argument, the crux of the argument depended on claims and data contributed by students.

The other data attributed to Bridgett in this argument was more aligned with the data provided by the students and was characteristic of the other kind of data provided by Bridgett. Bridgett pointed out that their definition required that all angles were congruent and all sides were congruent. This was not challenged by anyone. Perhaps in this simple case there was no possibility of a challenge, but in general, when Bridgett contributed a piece of data, there was no challenge by students that would provoke a sub-argument. The one major exception to this rule was an argument initiated by Bridgett after a student disagreed with her claim that a circle was not a polygon. In this one episode, in which almost all of Bridgett's data/claim contributions occurred, Bridgett engaged the class in a proof by contradiction, a proof method with which the students were not particularly familiar, and she was challenged at various points in the argument by her students. This episode, although different in several respects from typical classroom discourse in Bridgett's class, showed us several important things. First, students were not afraid to challenge Bridgett's statements, even though they chose not to do so under normal circumstances. Second, when the class was doing something that was clearly unfamiliar, Bridgett contributed more parts of the argument. Finally, norms in this class applied to both the teacher and the students. The students held Bridgett to the same standards of argumentation, in particular, justifying claims. This unusual episode shows the consistency of the kinds of data contributed by Bridgett. Bridgett's data remained unchallenged except when students were very unfamiliar with the method of argument and strongly disbelieved her claim.

Discussion and Conclusions

The expansion of Toulmin's (1958/2003) model provides researchers an effective tool for the examination of collective argumentation (Conner, 2008). Through the systematic dissection and classification of what often feels like "messy" discourse, we are able to more clearly discern patterns and recognize the contributions of the various actors in collective argumentation. Important considerations, such as by whom claims are made, who provides justification for those claims, and the extent of student and teacher contributions, are quantified, which allows for a more focused qualitative analysis. It is through this analysis method that we are able to pinpoint characteristics of Bridgett's teaching that suggest that it is exemplary.

Though defining "good teaching" is beyond the scope of this paper, Bridgett's student-centered classroom aligns with the view of effective mathematics teaching espoused by many teacher preparation programs and policy documents (Wilson, Cooney, & Stinson, 2005). She is unusual in this regard, as previous studies have found that such practices, and the beliefs that drive those practices, often diminish once prospective teachers enter the classroom (e.g., Eggleton, 1995). Bridgett's emphasis on and willingness to flow with students' contributions greatly enriched the mathematical discourse in her classroom. The students eagerly participated in discussions by making claims and providing data, and they also demonstrated their acceptance of the resulting learning style by holding Bridgett to the same standard of justification by demanding warrants and providing rebuttals to claims they considered dubious. In short, Bridgett's teaching was exemplary from many points of view, especially in light of her status as a student teacher.

An important characteristic of good mathematics teaching is the ability to develop a classroom environment that engages students in doing mathematics (Wilson et al., 2005). Lampert and Cobb (2003) explained the importance of mathematical discourse within such an environment, "classrooms [can] not be silent places where each learner is privately engaged with ideas… [learners] need to talk or write in ways that expose their reasoning to one another and to their teacher" (p. 237). Lampert and Cobb emphasized the important role of the teacher in facilitating and supporting such productive mathematical discourse; however, the authors contend that much less is understood regarding the teacher's role. The case of Bridgett provides a glimpse into some of the important aspects of the teacher's role in supporting mathematical discourse through collective argumentation. Our detailed analysis of the collective argumentation within Bridgett's classroom introduces examples of teacher moves that promote student engagement in doing mathematics. The questions Bridgett posed as well as the additional support she provided encouraged her students to significantly contribute to the development of mathematical arguments. Therefore, Bridgett's teacher moves promoted collective argumentation in the truest sense of the word *collective*, for the entire classroom continually worked together to arrive at mathematical conclusions.

The foregoing analysis has revealed many aspects of Bridgett's mathematical discourse with her students. The types of data she provided and the situations in which she required warrants seem consistent with a high level of student-centered discourse. However, it is worth asking if the relationship is causal. Would mimicking Bridgett's argumentation style lead to similar outcomes for other teachers with other students? A definitive answer to this question is currently unknown, as there are several other characteristics that might reasonably have contributed to Bridgett's teaching style and subsequent classroom interaction. Two possible mechanisms not examined here, but included as part of the larger study whose final results are forthcoming, are Bridgett's belief structure (Cooney, Shealy, & Arvold, 1998) and her beliefs about mathematics, teaching, and proof (Conner, Edenfield, Gleason, & Ersoz, 2011). Nevertheless, Bridgett's propensity to request and expect warrants from her students, as well as her reluctance to directly provide intermediate data and claims in arguments, are practices that could conceivably enhance the mathematical discourse in other classrooms by encouraging fuller student participation, resulting in more authentic argumentation. Further research on the transferability, as well as the causality, of argumentation mechanics is needed in order to confirm or disprove this hypothesis.

Endnotes

1. Toulmin (1958/2003) called these "preliminary arguments" or "lemmas" (p. 90). We use sub-argument to indicate that these may not come before the other argument in terms of time and to avoid lemma, which has a specific mathematical meaning.

2. Color is represented by line style in diagrams in this paper.

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TEACHER PERSPECTIVES ON MATHEMATICS CONTENT AND PEDAGOGY: DESCRIBING AND DOCUMENTING MOVEMENT

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In this paper we report on a study examining how teacher perspectives on mathematics content and pedagogy changed after the teachers' schools participated for one year in the Mathematics Coaching Program. Teacher narrative responses to questions about student work samples provided qualitative data for analysis. Questions asked for teacher input on student thinking and instructional decisions. Responses reveal teacher change in the area of the program goals of movement toward an integrated procedural/conceptual perspective on content and a learner responsive perspective on pedagogy.

Purpose of the Study

We know that teachers bring to their classroom practice many factors that influence their pedagogy. They bring knowledge, skills, and understandings of mathematics that impact student learning (Adler & Davis, 2006; Ball, Lubienski, & Mewborn, 2001; Fennema & Franke, 1992; Hill & Ball, 2004; Hill, Rowan, & D. Ball, 2005). They also bring into their classrooms perspectives on the nature of mathematics and how it should be taught. Perspectives on mathematics include whether it is a procedural or a conceptual activity, whether it is necessary to know mathematics both conceptually and procedurally, and whether there is some combined way to know mathematics (Baroody, Feil, & Johnson, 2007; Star, 2005). Perspectives on mathematics pedagogy include whether it is better taught with a teacher-directed or more learner-responsive approach, or if one can use and apply both approaches. In this paper we report on a study exploring teacher change in terms of perspectives on mathematics and pedagogy with teacher participation in a professional development project. We examine these particular perspectives because the program serving as the context for this work has among its goals to support teacher movement along a continuum from a strict and superficial procedural perspective on content to a more richly connected and integrated procedural/conceptual perspective; and along the pedagogy continuum from a teacher-directed to a learner responsive perspective.

In the following paragraphs, we define the perspectives on mathematics and pedagogy that are central to this work, and review the literature on which those definitions are based. We then describe the methods utilized in our study, including the context of the work, the nature of the data, and data analysis procedures. Finally, we discuss findings and closing thoughts.

Theoretical Framework

Supporting teachers' practice toward students' mathematics learning necessitates consideration of multiple concepts brought to the teaching of mathematics. The area of teacher mathematics content knowledge already has a deep base in the literature. What we bring to the discussion are teachers' perspectives on mathematics content and pedagogy is a distinction between perspectives and what is generally understood in the literature as 'disposition'. Scholars define dispositions as traits that lead a person to follow certain choices or experiences (Damon, 2005) or as tendencies to exhibit frequently a pattern of behavior directed to a broad goal (Katz, 1993). For Gresalfi and Cobb (2006), the word "disposition" encompasses ideas

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about, values of, and ways of participating with a discipline, identifying with it, and how it is realized in the classroom. A dictionary definition of disposition describes it as “a person's inherent qualities of mind and character” (New Oxford American Dictionary, 2005). Similarly, a perspective is defined as “a particular attitude toward or way of regarding something; a point of view; the state of one’s ideas” (Oxford Dictionaries, 2010). Perspective, although sometimes related to attitudes and beliefs in the literature, is usually recognized as being a result of experience, and that it can change, and that it influences practice. (e.g. Ross, 1986; Ross & Smith, 1992; Zeichner & Tabachnick, 1983).

In our work we did not chose to name the mathematics and pedagogy concepts we study dispositions because definitions of dispositions as cited above suggest less the likelihood that they can change than does the meaning of perspective. As educators, we believe that the perspectives teachers bring to bear on mathematics and pedagogy are not “inherent” but have been learned through lived experiences in and out of school. Additionally, our work in professional development relies in part on the belief that professional development can result in teacher change in terms of perspectives on content and pedagogy.

Mathematics Content and Pedagogy Perspectives

Contrary to what many in the general public believe, the content of mathematics is much richer than only arithmetic or computation; and learning mathematics content with understanding is a more complex endeavor than merely knowing the “how to do” of mathematics (Heibert et al., 1997). These and other perspectives are on the opposing “sides” of what Jon Star (2005) calls the “so-called math wars” (p. 404), about which he cites Judith Sowder’s statement that “Whether developing skills with symbols leads to conceptual understanding, or whether the presence of basic understanding should precede symbolic representation and skill practice, is one of the basic disagreements” (1998, as cited in Star, 2005). Star clearly advocates for procedural understanding, but does so from the position that both procedural and conceptual understandings are viewed too simplistically: Conceptual understanding as rich and concrete and procedural as superficial and lacking connections. Star suggests a framework where both knowledges are studied for their rich, connected, and deep relationship and integrated qualities.

Baroody, Feil, and Johnson (2007) suggest a conceptualization that is consistent with Star’s “recommendation to define knowledge type independently of the degree of connectedness” (p. 123). Baroody et al. propose the following definitions of procedural and conceptual knowledge: a) Procedural knowledge refers to “mental ‘actions or manipulations’, including rules, strategies, and algorithms, for completing a task” (de Jong and Ferguson-Hessler, p. 107 as cited in Baroody et al., p. 123); b) Conceptual knowledge is “knowledge about facts, [generalizations], and principles” (de Jong and Ferguson-Hessler, p. 107 as cited in Baroody et al., p. 123).

Baroody et al. (2007) distinguish their conceptualization with degrees of depth/superficiality, connectedness, and mutual dependence/independence and note: “depth of understanding entails both the degree to which procedural and conceptual knowledge are interconnected and the extent to which that knowledge is otherwise complete, well structured, abstract, and accurate” (p. 123). We take our content perspectives from this literature, and assign a range of conceptualizations, from procedural to integrated procedural/conceptual, to form a continuum of perspectives on mathematics content for our study.

The NCTM, particularly by way of its standards publication (2000), puts forth a vision of school mathematics “where all have access to high-quality, engaging mathematics instruction. There are ambitious expectations for all … Knowledgeable teachers have adequate resources …

The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding" (NCTM, 2000, p. 3). This vision, which includes learning mathematics with understanding (Heibert et al., 1997), has been embraced by much of the mathematics education community. It is in direct contrast to what one might describe as traditional teacher-directed mathematics instruction that often includes less visible engagement in the learning process. That is not to say that students are not cognitively engaged; but it does mean that students are not communicating mathematical ideas, not drawing on reflective practices and deep understanding. Instead, the teacher directs students on what to do, on what and how to learn, playing the dominant role; and students respond to the teacher by following instructions (Eccles, Midgley, & Alder, 1984; Gmitrová & Gmitrov, 2003).

Student-centered instruction is that which is "[d]esigned to elicit and build on students' ways of understanding mathematics" (Empson & Junk, 2004, p. 122) and is often problem-based (Ma & Zhou, 2000). Student-centeredness includes the teacher talking less and the learner talking more, with the learner doing the mathematical thinking and having opportunities to self-correct and generate knowledge through rich mathematical practices.

The project context of our work had a goal to move teachers beyond student-centered instruction to learner-responsive pedagogy (LRP). As in student-centered pedagogy, the LRP teacher makes decisions based on the learner's interests and focused on the learner's active engagement in the lesson. But LRP includes two additional distinctive qualities a) a shift, or expansion of each constituents' responsibilities and roles in the learning process from teacher as authority to authority shared by teacher and learner; and b) action: a resulting and deliberate instruction based on teacher knowledge of learner thinking and understanding. On-going analysis of student thinking is a fundamental component of LRP so instructional decisions can be made in direct response to the learners' understanding and action in making pedagogical choices based on the learner's needs is an intentional move. These pedagogy perspectives form a continuum from teacher-directed to learner responsive perspectives in our study.

Methodology

Participants, Sampling, Context, and Data Source

Participants in this study are teachers in schools enrolled in a mathematics coaching program during the 2008-2009 academic year. All are certified or licensed teachers, and are credentialed to teach in any of grades one through eight; some also are credentialed to teach kindergarten. They teach in elementary, intermediate, or middle schools served by the coaching program. Teachers were free to choose whether or not to participate in the research, and are solicited to allow access to their data from the coaching program for research and evaluation purposes.

In the 2008-2009 project year, 143 teachers consented to participate in the research. Of the 143 consenting teachers, 100 responded to student work items in the autumn and again in the spring of the academic year, as pre- and post responses. From the set of 100 participants with both pre and post responses, we created a purposeful, random sample (Patton, 2002) of 20 participants for analysis. We based our sampling strategy on the following principles:

a) Include all responders with narrative responses on all items making the sample purposeful in its inclusion of only full sets of extended responses (Patton, 1990).
b) Randomize within the purposeful sample to assured a representative set.
c) Limit participation to 20 participants: 20 participants, each with 20 response in each of two administrations generated 800 data points for analysis, a significant number of data points

for qualitative analysis, given the practicalities of time constraints, collaborative coding, and inter-rater reliability goals of the work (Patton, 2002).

The context of the study is the Mathematics Coaching Program, a statewide program training mathematics coaches to work in elementary, intermediate, and middle schools. Program goals include teaching movement : a) toward richly connected procedural/conceptual perspective on mathematics and away from a strict (and superficial (Star, 2005)) procedural perspective; and b) toward a learner-responsive and less teacher-directed pedagogy perspective.

The data source used for our analysis is a questionnaire generating teacher analyses of student work. Participants provided narrative responses to two questions for each of ten student work samples. One question asked for teacher interpretation of the student's thinking and the second asked for teacher suggestion of next instructional moves. For example, one item provides the following student work: Bobby was given the problem 17 – 9 = __ and solved it as follows: 17 – 9 = 17 – 10 – 1. Teachers were asked to a) Provide a rationale to describe what Bobby was thinking; and b) provide an explanation of what to say or do to help Bobby further his thinking.

Data Analysis

Participant extended responses were coded through two lenses on the autumn (pre) and spring reviews (post), and also compared for pre and post results. Responses on student thinking were coded for content on a continuum from Procedural to Conceptual to Integrated Procedural/ Conceptual perspectives. Responses about instruction were coded on a continuum from Teacher Directed to Problem/activity-based and Student Centered to Learner Responsive Pedagogy. See the abbreviated codebook of Table 1 for the list of codes. One or more content codes was assigned to each student thinking response; and one or more pedagogy codes to each instruction response. Multiple researcher reviews of sample responses, collaboration on coding assignments, and comparisons to coding by an outside reviewer (Morse, Barrett, Mayan, Olson & Spiers, 2002) helped reach inter-rater reliability goals of 85% in the analysis.

Code	**Mathematics Content Perspective**
IPC	**I**ntegrated **p**rocedural/**c**onceptual perspective
C	**C**onceptual perspective
P	**P**rocedural perspective
IO	**I**ncorrect/**o**ther
Code	**Mathematics Pedagogy Perspective**
LRP	**L**earner **R**esponsive **P**edagogy
PSC	**P**roblem/activity-based and **S**tudent-**C**entered instruction
TD	**T**eacher **D**irected
O	**O**ther does not clearly belong to any of the other categories

Table 1. Abbreviated Codebook

Results: Movement in Content and Pedagogy Perspectives

Table 2. includes results of the coding analysis of participant perspectives on mathematics content, showing movement or lack thereof on a P to C to IPC continuum. Data points are best viewed as clustered around or tending toward a particular position, allowing for some variance in the content and pedagogy perspectives, while still describing a location.

As the data in Table 2 shows, 25% of the participants tended to exhibit positive movement from P to IPC. Consider an example teacher response: Jenny uses the following method to find

28% of 60,000 mentally. 20% is 1/5 and 1/5 of 60 is 12, so 20% of 60,000 is 12,000. One percent of 60,000 is 600, and that times 8 is 4800. So the answer is 12,000 + 4,800, which is 16,800. We coded a teacher's response of "When finding answers to problems mentally it is easier to break it down into easier chunks," as P because the response only commented on what one would do. Later in the year, that same teacher responded "Jenny broke apart the problem into easier chunks. She understands the relationships between percents and fractions and understands that you can divide 60 by 5 to show 1/5" coded IPC because it included both procedural and conceptual component; and that conceptual and procedural components are connected.

Another problem showed a toothpick aligned with a ruler between 8" and 10.5" and a student's response that the toothpick measured 10.5" in length. We coded a response of "I think she doesn't understand that you start at the beginning of the ruler to measure" as a P response, while a second response of "She seems to be able to read the ruler but is struggling to understand how to measure when an object doesn't begin at 0" was coded as an IPC response because of the teacher's analysis that notes the concept of the starting point of a measure.

On a division of fraction problem where the student used a mathematically valid alternative algorithm to find a correct answer, negative movement is exampled by a teacher's analysis of "It seems she made common denominators." This was coded C because it simply mentioned a conceptual component. It did not suggest a procedure, and thus could not be P or IPC. Later, that same teacher responded with "Common denominators are used for adding or subtracting fractions. She doesn't understand that you divide fractions by multiplying the reciprocal of second fraction," coded as P because it focused on the procedure, without explaining meaning.

Positive Movement	Percent of participants	Negative Movement	Percent of participants	No movement	Percent of participants
P to C	5%	IPC to P	15%	Remain P	20%
P to IPC	25%			Remain IPC	10%
				Remain P & IPC	25%

Table 2. Participant Movement on Mathematics Content Continuum

Table 3 includes the proportional results of the coding analysis of the qualitative data on participant perspectives on mathematics pedagogy. These results show percentages of movement on the TD to PSC to LRP mathematics pedagogy continuum. In Table 3 25% of the participants revealed positive movement from TD to PSC, and PSC to LRP. An example of such positive movement from Bobby's 17-9= __ problem cited above starts with "By using his own explanation I could: 1. verify his mistake as I see it and 2. allow him to discover his own error and then he could recognize his error in the future" which is coded PSC because it does focus on helping the student realize his error. Later in the year that same teacher's response becomes "I would have him solve both sides using pictures or models and compare his answers. Using this method Bobby could see that his process is wrong. He can visualize the need to add that 1 back in the equation." This end of the year instructional suggestion is coded LRP because the teachers knows and helps the student discover his errors, by having the student compare and reflect upon his own work. The teacher and the student share the authority in the experience.

An example of the 15% of the participants who showed negative pedagogical movement is as follows: A teacher's first response about Bobby's problem was, "Bobby I like how you rounded the 9 to 10. Now we have to subtract 17-10=7 and add 1 back to get to the 9. Let me show you with our cubes what I would do." We coded this response TD, but with expectation of movement at the post administration because of the potential in the use of manipulatives. However, at the post administration, the same teacher responded, "If Bobby explains his answer to me then I would correct him when he explains -1 and tell him he needs to +1 because the problem was 17-9 and show him with base 10 blocks - 9 cubes is less than 10 cubes or 1 long (10 cubes).” Bobby’s use of manipulatives was clearly still from a teacher directed perspective, and perhaps even more TD in the language of “I would correct him,” “tell him,” and “show him.”

Positive Movement	Percent of participants	Negative Movement	Percent of participants	No movement	Percent of participants
TD to PSC	20%	LRP to PSC	10%	PSC	40%
PSC to LRP	5%	More TD	5%	TD	15%
				LRP	5%

Table 3. Participant Movement on Mathematics Pedagogy Continuum

Analysis by Mathematics Content Strands

An additional review of the data by mathematics content strand revealed interesting results for two different NCTM content standards: number and operations; and data analysis and probability. Two of the items drawn from the number and operations content standard provided student work showing unusual solutions. The first was one in which the student reduced the fractions to common denominators and divided along the numerators, using a mathematically valid approach. Seventeen out of twenty teachers did not accept this as a valid method, insisting that the student should have used the “invert and multiply” strategy. One other responded that the student “got lucky.” The second item is a three-digit subtraction problem for which the student developed his own alternate algorithm. Thirteen of the twenty teachers refused to accept this student’s mathematically valid solution as a legitimate solution. In both of these items, although the teachers who doubted the solutions did not necessarily lack content knowledge, they were unwilling to accept the alternative strategies. This suggests a reluctance to value student thinking, which would hinder the use of a learner-responsive pedagogical approach.

Two items in the sample drawn on the data analysis and probability standard also revealed interesting findings. One item included student interpretation of a graph that had no labels or numbers. The student explanation described a representation of distance against time, but every teacher in the sample of 20 viewed the graph as representing only speed against time. Hence no teacher interpreted the student’s explanation as correct. On a different problem, focusing on probability, 20% of the teachers responded with thorough explanations revealing an understanding of the mathematics; but most offered responses that were clearly incorrect or with what we might call “non-answers” circumventing the topic and suggested little to no knowledge of the relevant content. In both cases, data suggest that even those with overall PSC or LRP pedagogical perspectives did not know this particular mathematics well enough to question students through explorations or help students come to a mathematically valid understanding.

Closing Discussion

As noted earlier, the MCP context for this research study has among its goals to support teacher movement in mathematics content and mathematics pedagogy perspectives. We find that the program impacts teacher perspectives and that our coding helps document and describe that movement. That we can capture even small movement with data on only one year in the MCP suggests a useful methodology in capturing the subtleties of incremental change. As opposed to definitive, consistent, permanent positions, teachers are positioned in-between categories, tend toward a position, or contribute data that shows only slight movement toward a position. In the day-to-day work that the MCP coaches do with teachers, being able to identify subtle changes and the nuances of individual teacher's perspectives is critical to the coaches' work. That a set of teachers may fill in many different positions on the continua does not suggest more codes are needed, but, rather, that the continua represent the realities of teacher growth. They are practical and useful tools for describing fluid movement, being flexible enough to capture the teachers' sometimes daily and often small, incremental changes in perspectives. Many teachers also are likely to be positioned differently for some content than for others, so connecting this work to teacher content knowledge can reveal additional directions for professional development.

Finally, this work has implications for equity pedagogy (Erchick, Dornoo, Joseph, and Brosnan, 2010). A teacher's strict and superficial procedural perspective on mathematics limits students' opportunities for the rich mathematical learning of the integrated procedural/conceptual perspective; and examples of limitations of content knowledge that emerged in this work also hinder students' access to the mathematics. A teacher directed perspective as we define it in this study is akin to Friere's " 'banking' concept of education" (1973; 1989, p. 58), where the teacher transmits, deposits, and "the scope of action allowed to the students extends only as far as receiving, filling, and storing the deposits" (p. 58) and does not allow for the shared authority, and the accompanying learning, that is necessary for Learner Responsive Pedagogy.

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THE INFLUENCE OF A TEACHER'S DECENTERING MOVES ON STUDENTS ENGAGING IN REFLECTIVE THINKING

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In this paper, we discuss how a teacher's decentering moves influence the discourse in her classroom. Research has revealed decentering to be a valuable construct in characterizing the interactions in a professional learning community. Our research explores the role of decentering in a teacher's ability to support student learning in the classroom. Specifically we focus on how a teacher's decentering moves influences her students' propensity to reflect on their thinking and solutions. Our findings illustrate how a teacher's improved decentering ability supports student reflection and achieving a conceptually oriented classroom.

Introduction

Researchers have found the construct of decentering to be a useful construct in analyzing a facilitator's ability to support mathematical discussions in a professional learning community (PLC) of secondary teachers (Carlson, Moore, Bowling, & Ortiz, 2007). The authors observed that as a facilitator increased her or his propensity and ability to discern the thinking of her or his peers, the mathematical discourse in the PLC became more substantive and conceptual. In this paper we adapt decentering to analyze a Precalculus teacher's interactions with her students. We discuss how a teacher's decentering actions relate to her students' opportunities for reflecting on their responses and thinking. We also characterize transitions in students' tendency to engage in reflective thinking in relation to the teacher's decentering actions.

Theoretical Framework

Piaget (1955) described decentering as the act of adopting a perspective that is not one's own. Steffe and Thompson (2000) built on Piaget's work in describing how one individual adjusts his or her behaviors in order to influence another individual, or a group of individuals, in a particular manner. Specifically, Steffe and Thompson used decentering to characterize the interactions betweens teachers and students.

A teacher decentering assumes that the student's behavior has a rationality of its own and tries to discern the mental actions driving the student's behaviors. As the teacher decenters, they build a model of the student's thinking, and then base his or her actions on this model. To contrast, a teacher not decentering interacts with students assuming that her or his thinking and the student's thinking are identical. Over the course of a conversation, a non-decentering teacher may notice that the student's thinking is different, but the teacher does not construct a model of the student's thinking, therefore acting in a non-decentered way.

Researchers (Carlson, et al., 2007) found decentering to be a key tool for facilitators to achieve successful discourse in secondary mathematics and science PLCs. When characterizing the PLC facilitators' decentering actions, the authors determined various Facilitator Decentering Moves (FDMs) in order to classify the PLC interactions. The decentering moves ranged from the facilitator acting entirely in a non-decentered way (FDM1), representing the lowest end of the spectrum, to the facilitator building a model of PLC members' thinking and how the members were interpreting her or his actions while adjusting her or his actions based on these models

(FDM5), representing the highest end of the spectrum.

Teuscher, Moore, and Carlson (2011) made further connections between a teacher's level of decentering and the student discourse in the classroom. In studying one teacher's use of research based curriculum, the authors noted that the teacher's question types and purposes changed as the teacher began to focus on her students' thinking and understanding of the concepts independently of her way of thinking about the tasks (decentering). We believe that student knowledge is unique to each individual and we conjecture that a decentering teacher will support her or his students' learning by enabling the students to build and reflect on their thinking and actions. We predict that classroom discourse will also shift toward more opportunities for student reflection as a teacher is decentering. When teachers promote student reflection and thinking, the students are placed in situations that Hiebert and Grouws (2007) call *opportunities to learn*.

Hiebert and Grouws (2007) suggested that the type of curriculum influences opportunities for students to learn and we conjecture that the curriculum may influence teachers' pedagogical decisions. A curriculum emphasizing a conceptual orientation may influence a teacher's pedagogical goals to promote an environment of conceptual discourse. On the other hand, a curriculum emphasizing a procedural orientation places value in students gaining procedural fluency. In characterizing these two goals of learning, procedural and conceptual, Hiebert and Grouws (2007) note two types of teaching patterns: teaching for skill efficiency and teaching for conceptual understanding. Teaching for skill efficiency means providing students with opportunities to practice and master mathematical procedures quickly and accurately. Teaching for conceptual understanding means providing students with rich mathematical tasks that support them making mental connections between mathematical ideas, procedures, and facts. Teaching orientations impact the nature of classroom discourse, promoting either a calculational or conceptual orientation (Thompson, Philipp, Thompson, & Boyd, 1994).

Stigler and Hiebert (1999) noted that even though an element in the system (in this case the teacher) may change, the system itself resists sudden change and will "rush to repair itself" due to cultural scripts. The system they refer to includes teachers, students, administrators, and parents. Cultural scripts are unwritten "rules" or behaviors that are shared by members of a society; in the context of teaching and learning, it is the expected role of teachers and students. Due to these constraints in achieving classroom shifts, we hypothesize that a teacher's decentering ability and corresponding changes in her or his students' activity will occur in gradual shifts.

For the present study, we investigate the following research questions:

1. How does students' discourse shift toward more opportunities for reflection as a teacher's decentering levels increase?
2. How does the nature of student reflection change when the teacher promotes and the curriculum supports a conceptual orientation in mathematics?

Methods

Claudia is a secondary mathematics teacher at Rover High School. Previous to the study, Claudia had a total of 20 years teaching experience including elementary, junior high and high school. At the time of this study, Claudia was assigned to teach Precalculus for the second time in her career. All mathematics and science teachers at Rover High School participated in a National Science Foundation Math and Science Partnership (MSP) project (No. EHR-0412537) designed to support secondary mathematics and science teachers in improving their instruction and their students' learning. The project leaders designed and implemented school-based interventions including graduate courses, workshops, and leadership for school-based PLCs at

Rover High School to support teachers in using inquiry-based instructional methods and a conceptually oriented curriculum. To support Claudia's efforts, the project leaders gave Claudia the opportunity to use a newly developed research-based conceptually oriented Precalculus curriculum (Carlson & Oehrtman, 2010), which we will refer to as the Pathways curriculum. In this curriculum, covariational reasoning, rate of change, proportionality, and problem solving are emphasized.

Claudia's two Precalculus classes used the Pathways curriculum and were videotaped daily to observe the interactions and discourse between Claudia and her students. Videos were digitized and viewed by the research team, with select videos analyzed to examine shifts in Claudia's interactions with her students. Three exemplary videos were selected for coding: one from the first week of class (August), one from five weeks into the school year (September), and one from five months into the school year (December). The research team coded the teacher-student interactions for students' mathematical reflections and the degree to which Claudia was decentering while interacting with her students.

To code student opportunities for reflection, the last two levels of descriptor C.9 "The teacher encouraged students to reflect on the reasonableness of their responses" from the Middle Tool Observational Tool was used (Reys, 2004; Tarr, et al., 2008). We first coded each teacher-student interaction that provided students' the opportunity to reflect on the reasonableness of their or someone else's responses. The second level of coding was within each interaction, the teacher moves depending on the whether the teacher encouraged or discouraged conceptual understanding (see Table 1) by either promoting a calculational or conceptual orientation (Thompson, et al., 1994).

Student Opportunity	Teacher moves
Student reflection on answers	1. The teacher asked students if they checked whether their answers were reasonable but did not promote discussion that emphasized conceptual understanding, or 2. The teacher encouraged students to reflect on the reasonableness of their answers, and the discussion involved emphasis on conceptual understanding.

Table 1. Student Opportunities (adapted from Reys, 2004)

To characterize Claudia's decentering, we used the construct of the facilitator decentering moves (Carlson, et al., 2007) in the context of interactions between Claudia and her students. Table 2 displays the construct of facilitator decentering moves (FDM); in the context of our study we referred to the teacher-student interactions as teacher decentering moves (TDM).

Decentering Codes	Description
TDM1	The teacher shows no interest in understanding the thinking or perspective of a student with which he/she is interacting.
TDM2	The teacher appears to build a partial model of a student's thinking, but does not use that model in communication with the student. The teacher appears to listen and/or ask questions that suggest interest in the student's thinking; however, the teacher does not use this knowledge in communication.
TDM3	The teacher builds a model of a student's thinking and recognizes that it is different from her/his own. The teacher then acts in ways to move the student to her/his way of thinking.
TDM4	The teacher builds a model of a student's thinking and acts in ways that respect and build on the rationality of this student's thinking for the purpose of advancing the student's thinking and/or understanding.
TDM5	The teacher builds a model of a student's thinking and respects that it has a rationality of its own. Through interaction the teacher also builds a model of how he/she is being interpreted by the student. He/she then adjusts her/his actions (questions, drawings, statements) to take into account both the student's thinking and how the teacher might be interpreted by that student.

Table 2. Characterization of Teacher Decentering Moves (adaption from Carlson, et al., 2007)

Results

In this section we discuss the quantitative and qualitative results from the analysis of Claudia's three classroom videos. Each classroom session lasted 50 – 54 minutes. Table 3 displays the number of minutes of class time that was coded as a teacher-student interaction in which students reflected on their responses. The percentage of student contribution to these reflective interactions is listed after the minutes, measured as the time Claudia was not speaking during the interaction. Each instance of student reflection was coded according to whether or not the discourse promoted conceptual understanding (e.g., making connections between mathematical ideas).

Reflection type	August (minutes)	September (minutes)	December (minutes)
Did not empathize conceptual understanding	4.43 (28.91%)	0.00 (0.00%)	0.00 (0.00%)
Emphasized conceptual understanding	1.87 (28.91%)	5.39 (46.62%)	12.69 (52.10%)

Table 3. Teacher-student interactions categorized by reflection type

The percentage of total class time in which interactions included student opportunities to reflect on answers was 11.50% in August, 9.90% in September, and increased to 25.3% in December. The percent increase of time spent in these interactions, determined as the difference in minutes between August and December relative to the total class time in August, was 11.93%. The increase in student reflection on their mathematical responses is evident, but the table only conveys the percentage of class time for which this occurred. In August, the 11.50% spent in reflection was based on interactions that consisted of Claudia questioning her students during a whole class discussion. When students worked in small groups during the same lesson, the research team did not observe opportunities for student reflection. In both the September and December videos all student reflections occurred during work in small groups as the teacher circulated and visited with groups.

Although the time in which opportunities for student reflection increased from August to December, this data alone does not illustrate the quality or nature of these questions. In an attempt to provide a deeper characterization of Claudia's questioning, we analyzed her classroom videos to determine the role of decentering in her questioning, as well as transitions in the nature of her questioning and student thinking elicited by her questioning. The following three excerpts give examples of student reflection in the context of Claudia's interactions with her students.

In Excerpt 1a (August) Claudia read student responses to a recent quiz on average rate of change and asked students to determine what was missing or incorrect in the responses. In the transcript, '…' is used to indicate one speaker interrupting another speaker.

Excerpt 1a

C: Listen to this one and I'm going to tell (you?) the comment I wrote. Average rate of change is the overall distance, divided by the amount of time that tells you the average amount of miles per minute from p1 to p2. Now what's wrong with this answer? What is it actually telling me?
S1: How to find it.
C: So it's telling me the what? How to find it, but what about how to find it?
S2: The math.
C: The math, the formula, okay? So when it says what's the meaning of the average rate of change? It doesn't mean tell me in words what the formula is, it means: tell me what it means.

In Excerpt 1b (August) Claudia discussed with the class the meaning of the average rate of change of a diver's height from the water with respect to time using a table of values. Claudia and her students changed the time interval to half a second on the calculator and discussed the meaning of a rate of change of zero between two successive values in the table.

Excerpt 1b

C: Okay, so that would mean that what would happen, the person jumps off the board?
S1: Floats there for a second.
C: Okay, so is the diver suspended in mid-air for a half a second?
(Some students in class say no.)
S2: He wasn't suspended but…
S3: …went back in.
(Some students in class are making gestures of a concave down parabola.)
C: Oh, so he went up and back down, okay, but we're not seeing that right here in this table. But the way you could find out is change your interval again and what could we change it to this time?
S4: One-quarter.

13	C: Point two five, or you could do smaller, you could do point two, whatever.

In Excerpts 1a and 1b, we characterize Claudia's actions at the TDM1 level because she was focused on students arriving at a specific answer with no indication that she was interested in the thinking or perspective behind the students' responses (Excerpt 1a: lines 6, 8-10; Excerpt 1b: lines 9-11, 13). Although she initiated the opportunity for student reflection (Excerpt 1a: lines 1-4; Excerpt 1b: lines 1-2, 4), meaningful discussion did not follow (Excerpt 1a: lines 5-7, Excerpt 1b: lines 5-8) and Claudia proceeded without pushing the students for further explanation (Excerpt 1a: lines 8-10, Excerpt 1b: lines 10-11). The opportunity for student reflection in Excerpt 1a did not promote conceptual understanding (lines 6-8). Although Claudia made an attempt to promote conceptual understanding in Excerpt 1b (lines 1-2, 4), she did not try to create a model of student thinking. Claudia instead questioned her students in ways that suggested she was seeking an answer, which did not promote deep or meaningful reflection.

In Excerpt 2, which occurred during a September class, students were in groups of three to four students. Claudia gave her students a task stating: "A local hotel currently rents an average of 28 rooms per night. The hotel management estimates that for every \$5,550 spent on hotel renovations they will be able to rent an additional 6 rooms each night. Sketch a graph that represents the relationship between the number of rooms that can be rented in terms of the amount of money spent on renovations." (Carlson & Oehrtman, 2010, Module 2) This question was part of a unit on linearly and proportionality. Claudia circulated the room monitoring the students' activity and asked questions of students in different groups as they engaged in the task. Claudia's conversation with two students is presented in Excerpt 2.

Excerpt 2

1	C: So if you can pay for one room at a time that would mean, as soon as you pay the
2	next 925 dollars you would get an additional room. Right?
3	S1: Correct.
4	C: Okay, what kind of relationship is that?
5	S1: (*thinking*)
6	S2: Step function.
7	C: (*Laughing*) No.
8	S2: It is not proportional.
9	S1: It is not proportional and it is not linear because…
10	C: Okay, why is it not proportional?
11	S1: Because you start at 28, and you can't spend negative money on the next room.
12	C: Are the changes proportional though?
13	S1: No.
14	C: So for each additional room built are they always going to pay nine hundred?
15	S1: I don't know, we are trying to think about how to describe this.
16	S2: Well you presume, because they're not only going to build half a room.
17	S1: If for every apple you need one orange and you have half an apple then you need
18	half an orange, but this is not the case here because you spend half the money
19	here you are still going to get one, or you can't spend half the money here.
20	C: Okay, are you assuming that you have to spend the 5,550 dollars in your graph, is
21	that what you are assuming?

In Excerpt 2, we characterize Claudia's interactions with her students as TDM3 level because she made moves to guide Student 1 to her own way of thinking (lines 1-2, 4, 10, 12, 14).

Specifically, she was prompting the students to describe that there was a proportional relationship between the two quantities. Although her questioning (lines 1-2, 4, 10, 12), initiated opportunities for her students to reflect on their solution it did not support their reflections. In providing these opportunities, however, the students' utterances indicated that their thinking was different than Claudia (lines 6, 8, 9, 11, 13), and they persisted because they were convinced of the reasonable of their response. As the interaction progressed, Claudia's question in line 14 gave the students another opportunity to more fully verbalize their thinking. They justified why the situation with the hotel renovation was not proportional based on their understanding of proportionality (lines 16-19). By the end of the excerpt, Claudia began to reflect on the rationality of the student's thinking (lines 20-21). Immediately after Excerpt 2, Claudia allowed her students to describe their way of thinking, and this led her to accept their solution as a viable alternative to be respected (TDM4). She then asked Student 1 to present her solution to the class as one of the acceptable approaches to solving the problem.

In Excerpt 3 (December), students were in groups of three to four students discussing a problem about the relationship among the arc length, radius and angle. The context of the problem is a bug traveling on a fan blade with the unit of measure in radians and students are to explore the relationship between arc length, radius, and angle. The concept emphasized in the previous lesson was that one radian of angle measure corresponded to the arc measure of one radius length. Claudia circulated the classroom monitoring students' discussions as they engaged in the task. In Excerpt 3 Claudia was conversing with students in one group regarding the question: How does the angle measure change if the radius of the fan is changed, but the distance the bug travels (0.765 radians) is not changed?

Excerpt 3

C: Okay, in radians is not changed though. You are thinking of the arc length, you are thinking of the distance the arc length.
S2: That's not a distance the bug traveled that's the angle of the distance.
C: But can we take an arc length and change it to a radian measure?
S1: Yeah.
C: Okay, so?
S2: But that isn't the distance that the bug traveled.
C: But if the arc length of an angle is given, can we find the distance the bug traveled?
S2: If it is given in radians.
C: Yes, if you have a radius and an angle you can, so you are just saying that the way the question reads, how does the angle measure change if the radius of the fan is changed, but the distance the bug travels in radians is not changed? Is because of how it is worded, he is thinking the arc length. Okay so if the…
S2: But the bug travels would be, they say the units that they give you to use and the quantity they ask you to measure do not match.
C: True.
S2: That would be like, what is the gas mileage in liters, and it's like (*inaudible*).
C: (*laughs*) Okay yeah, okay so let me look at this for a minute – how does the angle measure change if the radius of the fan is changed, but the distance the bug travels, 0.765 radians, is not changed? Well can you say that a linear measurement is 0.765 radians? Can I say that?
S1: No, but that is what it says here.

24	C:	What is 0.765 radians, what does that represent for any circle?
25	S1:	The openness of the angle.
26	C:	But what does the actual 0.765 radians measure, what does that mean?
27	S1:	How– it's proportional to the radius so it is that much of the radius around the
28		circle.
29	C:	Okay, so we could say that this is 76.5% of one radian in length? Could we say
30		that?
31	S1:	Yeah.
32	C:	S2 what do you think? So if the measurement is 0.765 radians then the length of
33		the arc is 76.5% the length of the radius.

In Excerpt 3, we characterize Claudia at the TDM4 level. She acknowledged understanding how Student 2 interpreted the question and made moves to help both Student 1 and Student 2 reach a common understanding of the question, based on her model of the students' thinking about the question (lines 1-2, 11-14). Claudia's questioning (lines 4, 6, 8-9, 21-22, 24, 26, 29-30, 32-33) initiated opportunities for student reflection (lines 3, 5, 7, 10, 15-16, 18, 23, 25, 27-28, 31). Specifically, she prompted the students to reflect on a misconception that the radius length could be only used to measure the angle subtended by an arc, as revealed through the comments of Student 2 (lines 15-16, 18). Claudia made moves to resolve this misconception by linking mathematics concepts from the prior lesson in regards to the correspondence between radian measure of the circle and its radius. The questions Claudia asked (lines 21-22, 24, 26, 29-30, 32-33) required the students to continue to reflect on reasonableness of their thinking until both students reached a common understanding (which was reached after this excerpt).

Conclusions

Hiebert and Grouws (2007) identified two critical features of classroom teaching that foster conceptual understanding : (1) explicit attention must be made to the connections between facts, procedures, and ideas of mathematics; and (2) students must "struggle" with mathematics, not in the literal sense, but in the sense that they need to put an effort in making sense of the mathematics. Excerpts 2 and 3 display Claudia supporting her students' thinking by questioning them in ways that led to their reflecting on the mathematics (Excerpt 2: lines 6, 8, 9, 11, 16-19, Excerpt 3: lines 3, 5, 7, 10, 15-16, 18, 23, 25, 27-28, 31). In these excerpts Claudia modeled a higher level of decentering (TDM3, TDM4) when compared to Excerpts 1a and 1b (TDM1), in which no decentering was present. Claudia's higher level decentering actions enabled mathematically rich conversations emphasizing conceptual understanding while generating more opportunities for student reflection. Also, classroom discourse shifted from a calculational orientation towards a conceptual orientation as students were asked to explain their thinking. Stigler and Hiebert (1999) had noted a system resists sudden changes due to cultural scripts. We had hypothesized that the changes in classroom norms would be gradual. Just as Claudia's progression in decentering was gradual as the semester progressed, so too was the students' propensity to reflect on their thinking.

Future studies should continue to investigate teachers' decentering abilities and how these abilities support student learning. Such studies should also include a focus on the attributes (e.g., content knowledge, attitudes, and beliefs) of a teacher that support shifts in her or his decentering abilities. Although one of the observational scale descriptors was adapted from the Middle School Observational Tool (Reys, 2004), the other descriptors should offer further insight in the decentering actions of a teacher and the learning and actions of students.

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TEACHER EDUCATORS' DISCOURSE MOVES IN SUPPORTING PRESERVICE ELEMENTARY MATHEMATICS TEACHERS' LEARNING

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Teacher Educator's questioning strategies are essential in maintaining the level of cognitive demand of tasks for preservice teachers' learning of mathematics for teaching. Classroom discourse has been known to maintain the level of cognitive demand of tasks. However, qualitative differences exist in the use of discourse. This study focuses on comparing instructors' questioning strategies with two groups of preservice elementary teachers using the same cognitively challenging task on surface area, where in one group the level of the cognitive demand of the tasks is maintained while in the other it is lowered. We highlight effective questioning strategies during whole group discussion.

Introduction

The purpose of this study is to investigate the role of instructor's questioning strategies in facilitating discourse in mathematics methods classes for preservice elementary teachers. We analyze the role of questions used by teacher educators in setting up a classroom community that emphasizes justification and explanation (evidence and warrants). Our analysis targets specific types of discourse that teachers use and how they affect the flow of the discourse within the whole class discussion. While classroom norms have been found to affect the quality of discourse (Cobb, 1999), the role of the teachers is key to orchestrating the whole class discussion. What and how the teachers ask questions affect the course of the discussion, whether building towards key mathematical understandings as opposed to merely sharing ideas. This paper contributes to the literature in extending our understanding in supporting teacher educators to facilitate whole-class mathematical discussions with preservice elementary teachers. We share an analysis of two instructors who were using the same cognitively demanding task designed to support preservice elementary teachers' understanding of surface area and generalized across prisms and cylinders.

Theoretical Framework

The theoretical framework of this study is derived from a sociocultural perspective, whereby mathematics teaching and learning is inherently social and embedded in active participation in communicative reasoning process (Lerman, 2001). Lerman (2001) deems that learning is a consequence of social interactions, which "along with physical and textual interactions can cause disequilibrium in the individual, leading to conceptual reorganization" (p. 55). He argues that "consciousness is constituted through discourse" (p.88) and associates speaking mathematically with learning mathematics and learning to think mathematically. Communication in mathematics classrooms provides opportunities for students to engage with ideas, refine understandings, and share insights and strategies. Walshaw and Anthony's (2008) review of current literature on discourse indicates that "students' active engagement with mathematical ideas will lead to the development of specific student competencies and identities" (p. 516). Rich mathematics discourse is at the center of constructing and connecting knowledge in mathematics. In classrooms where students explain and defend their ideas, analyze and evaluate the ideas of others, justify solutions, and explore multiple perspectives through conversations with peers, students deepen their own conceptual understandings and further their learning in mathematics. Classroom discourse has the potential to develop and deepen students' conceptual understanding of mathematics. The quality and type of discourse are crucial to helping students think conceptually about mathematics (Kazemi & Stipek, 2001; Lampert & Blunk, 1998; Nathan & Knuth, 2003; van Oers, 2002; Van Zoest & Enyart, 1998).

Teacher discourse, i.e. what teachers say and how they say it, has a significant influence on how and what students learn (Knott et al., 2008). Teacher discourse can influence both the amount and the quality of learning that takes place, and may often inadvertently lower the level of the mathematical task from cognitively demanding to one of rote application of procedures (Stein et al., 2000). Particularly during whole class discussion, in the collective act of abstraction that occurs for students during the course of the lesson, the teacher facilitates and choreographs the mathematical discourse through the use of a wide range of meta-mathematical discourse moves (Knott et al., 2008). They are meta-mathematical in a sense that "they do not directly supply mathematical content but rather they are about mathematics, and employ the language or 'register' of mathematics" (Knott et al., 2008, p. 95). For example, teachers use moves such as steering, probing, redirecting, clarifying, validating, prompting, rephrasing, re-voicing, and generalizing during the interaction with students engaging in the mathematical tasks. Although these discourse moves are not directly mathematical content discourse, they are important in making mathematics learning happen in the classroom, by encouraging students to participate in shaping their own learning.

The use of discourse with preservice teachers poses additional challenges to mathematics educators. Specifically, preservice teachers have experienced mathematics lessons throughout their schooling and often enter their preparation program with an expectation of learning to become better at what they think mathematics teachers do (Nichol, 1999). One recommendation to rectify this problem is to focus on using problems and dilemmas of practice "as springboards for investigation of mathematics teaching and learning" (Nichol, 1999, p. 48). Morrone et al. (2004) found that instructors who consistently pressed for understanding and used scaffolded discourse to facilitate preservice elementary teachers' learning when using a series of challenging tasks were able to generate mathematical knowledge about content and teaching.

In this study we analyze how these discourse types might be associated with the development of preservice elementary teachers' mathematical explanation and justification. We attend to instructors' tactical moves that encourage preservice elementary teachers to attend to the content by tracking teacher educators' questioning strategies and noting the kinds of opportunities to engage with content that are offered to students during whole class discussion (Gresalfi & Williams, 2009).

Methods

Participants and Setting

This study is part of a larger project, the Elementary Preservice Teachers Mathematics Project (EMP), developing a series of cognitively demanding mathematical tasks (Stein et al., 2000) for preservice elementary teachers. The tasks were developed across several mathematical topics (Number Theory, Fractions, Ratios and Proportions, Geometry, and Geometric Measurement) where the questions were appropriately scaffolded and supported by classroom discourse in both small group and whole class discussion. In the pilot study, participants consisted of 32 undergraduate students at a major New England University, majoring in elementary education, special education, or deaf studies. The participants were in two classes taught by two different instructors, who were doctoral candidates in mathematics education in their third year of the program. Both instructors strove to develop their students' understanding but used different methods. Preservice teachers in the two classes first worked on the mathematical tasks in small groups followed by whole class discussions. However, the two sections differed during the whole class discussion. In one section, the instructor only went over answers whereas in the other section, the instructor also included a discussion about the big ideas of the key concepts and procedures.

The study was conducted midway through the second semester of a two-course sequence for elementary education majors. Both sections had previously experienced using discourse and sense-making to discuss mathematical problems in small groups and to work toward justification. The task being used in this study was part of the EMP geometric measurement strand. It focused on surface area and covered two class periods. The preservice teachers examined the lateral and total surface area of prisms on the first day and examined similar ideas around cylinders with an emphasis on drawing connections between prisms and cylinders on the second. Specifically, the participants were introduced to a different way of finding the surface area of prisms and cylinders by considering the lateral surfaces and bases separately. They discovered the perimeter of the base of a prism or cylinder multiplied by the height of the figure is equivalent to the area of the lateral surface area. This study focuses on a question (Figure 1) in which the preservice teachers were asked to determine which of three figures (two prisms and cylinder) had the largest surface area. By this point, they should have noted that the lateral surfaces areas had the same dimensions and so the base areas were the only difference. Prior to the whole class discussion, the preservice teachers had explored and discussed the questions in their small groups both with and without the instructor.

15. The following nets show the lateral surface rectangle but not the bases of a triangular prism, a square prism, and a cylinder, respectively. Which one has the greatest surface area? Explain.

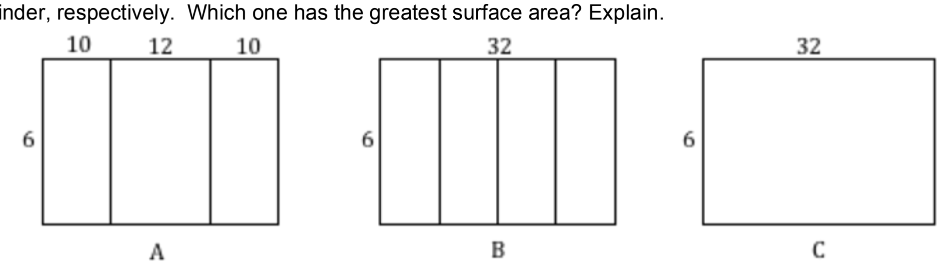

Figure 1. Question 15 from the EMP Task on Surface Area

All lessons were videotaped and transcribed. The tasks were coded using the Instructional Quality Assessment Academic Rigor (IQA-AR) Rubric for Potential of the Task (Boston & Smith, 2009) to assess the levels of cognitive demand of the mathematical tasks as they were intended to identify questions at a high level. The transcripts were coded using the Instructional Quality Assessment Academic Rigor (IQA-AR) Rubric Implementation of the Task (Boston & Smith, 2009) by pairs of coders to ensure inter-rater reliability.

Using the rubrics, we identified questions that have high levels of intended cognitive demand. We then analyzed the transcripts for instances when the level of cognitive demand was maintained or dropped during the implementation. Question 15's level of cognitive demand was high but then was dropped in the first group and maintained in the second group.

In order to analyze and compare instructors' discourse moves, we adapted existing analytical frameworks (Fraivillig, Murphy, & Fuson, 1999; Knott, Sriraman, & Jacobs, 2008) to attend to teachers' discourse moves, especially the use of questioning strategies. We deemed it necessary to adapt these frameworks because there are differences in context between K-12 and preservice education settings. Table 1 shows the analytical framework used in our study.

Table 1. Analysis Framework for Teacher Educator's Discourse Moves

Purpose of Move	Questioning Strategy	Description
Eliciting	Probing	To elicit students' prior knowledge
	Rephrasing/Re-voicing	To validate students' mathematical thinking
	Prompting	To provide background knowledge
Supporting	Steering	To move the discourse in a particular direction based on students' engagement in and demonstrated level of understanding of the task
	Re-directing	To steer the discourse back when the discourse is moving towards irrelevant or incorrect mathematical assumptions
Extending	Challenging	To demand explanation and justification of students' claim
	Generalizing	To push students to move beyond the particular to the general case

Findings

Comparing the transcripts from the two instructors orchestrating the whole class discussions around question 15, we identified differences in their use of tactical moves in framing the whole class discussion, orchestrating discourse, and connecting and extending the preservice teachers' thinking. Table 2 shows the frequencies of each of the discourse strategies used by each instructor during their whole class discussion around question 15. Both instructors used moves that elicit and support preservice teachers' thinking, but the second instructor included moves that extend preservice teachers' understanding (Challenging and Generalizing) to think beyond what was explicitly asked in the question, and pushed them to explain and justify their mathematical thinking.

Table 2. Frequencies of whole class discourse moves for question 15

Discourse Moves	**Instructor 1**	**Instructor 2**
Probing	13	17
Rephrasing/Re-voicing	1	0
Prompting	0	0
Steering	1	3
Re-directing	0	0
Challenging	1	6
Generalizing	0	1

Framing of the Whole Class Discussion

The way an instructor frames a whole class discussion affects how the discourse that follows will unfold. The first instructor (I1) in this study immediately steered the preservice teachers' attention to the shapes of the bases and dismissed the lateral surface area to be unimportant. Instructor 2 (I2) begins the discussion with an open question, "What did you guys discover in question number 15?" setting up a tone of inquiry, followed by a probing question, eliciting preservice teachers' knowledge of the three lateral surface rectangles.

Instructor 1	Instructor 2
On question 15 let's start and just talk about how we would go ahead and find the bases of those 2 shapes. Before we do that, we were asked which has the greater surface area. Was it necessary for us to find the lateral surface area of each of those shapes?	Let's talk about question number 15, where maybe I think the difference in calculating the bases comes out. What did you guys discover in question number 15? We had 3 lateral surface rectangles. What was the first one? What kind of prism was the first one?

Orchestrating the Whole Group Discussion

Both instructors led preservice teachers in a whole class discussion as they validated solutions for finding the area of the bases in question 15. They used discourse to help preservice teachers synthesize the information. Although both instructors were eliciting preservice teachers' solution methods and leading similar discussion, Instructor 2 used tactical moves that established important norms for discussion in the classroom, which maintained the level of cognitive demand of the task and pushed for clarity. Instructor 2 employed the Challenging and Generalizing tactical moves which led preservice teachers to make claims and provide justifications. Instructor 1 relied more on Probing and Steering moves that led to brief preservice teacher responses, often without justification.

In one episode, Instructor 1 asked a closed question that led to an answer, without justification, and steered the discussion back to question 15 and asked, "So which one of our three shapes is going to have the largest surface area?" The preservice teachers responded to his question with a one-word answer, "cylinder," followed by the instructor providing them with an explanation, in which he was relating and connecting the mathematics ideas that emerged from the problem.

Instructor 2 led the preservice teachers to step back and think about the purpose of the task by asking, "What's the big idea here? Why did I have you guys calculate the area of these bases? What do you notice?" The emphasis on the "big idea" and the push for explanations encouraged the preservice teachers to consolidate their ideas.

I2:	So, what's the big idea here? Why did I have you calculate the area of these bases? What do you notice?	Generalizing
S9:	In order to find the area of each of the bases, you have to find out which one has the greatest surface area or the greatest lengths of total surface area.	
I2:	Oh, why is that?	Challenging
S9:	Because the prisms and cylinders that we're making out of paper, they all have the same lateral surface area. And so the only differences are the area of the bases.	
I2:	Can someone just repeat what S9 just said? She just made a really good point.	Re-voicing

S10: You only need to find the area of the bases because all of the prisms have the same lateral surfaces rectangle, so that measurement is going to be the same. So by finding the measure of both of the bases or just one of the bases, you can find out which one has the biggest surface area.

Connecting/Extending Preservice teacher Thinking

We found that the instructors' questioning strategies played an important role in extending preservice teachers' thinking and leading them to make connections. In both of these classroom exchanges, the instructors encouraged preservice teachers to make connections to their past experiences in working with the idea of determining the figure with the greatest area with fixed perimeter. Instructor 1 steered the preservice teachers to make the connection, and when a preservice teacher was unsure of her connection to the geoboard to explain why, with a fixed perimeter, a circle would have the largest area, there was no push for clarity. There was no clear indication that the preservice teachers were able to generalize that the circle would have the greatest area given that the perimeter was fixed. When the preservice teacher (S2) suggested that a square would have the largest area, the instructor steered the conversation to address that the figure should not be limited to rectangles (including squares), but could extend to circles.

I1:	If we have a fixed perimeter, which rectangle has the largest area?	Probing
S2:	The one that's closest to a square.	
I1:	The one that's closest to square if we are using tiles. If we are not using tiles then we would want to find the perfect square. So in a similar case here we are looking for the most ideal shape here, which is a circle. …if I'm taking my piece of paper, pretend this is a 6 by 32 piece of paper for a second [folds paper], we want to find the smallest surface area possible, we could just fold this in half and have the surface area on the top and bottom have zero right here. As we think about it and get further and further away from this line we are going to a circle, we are getting a larger and larger surface area as we move forward from that direction.	Re-directing

However, another preservice teacher's (S11's) choice of using a geoboard for the explanation, presented a challenge to explain why the circle would have the greatest area, but the instructor's choice of discourse move failed to provide support to push for clarity in the preservice teacher's explanation and justification.

S11:	One way I thought about it was like if you were using a geoboard, the closer it got…I don't know it's hard for me to explain, but you could tell with something that has the same perimeter as a circle it's not going to be as large of an area because it won't fill up the space as much. I don't know, maybe that doesn't make any sense at all.	
I1:	So you're saying that a circle wouldn't be as large?	Rephrasing
S11:	No, a triangle wouldn't be as large because it wouldn't fill up the same…I don't know. I mean it makes sense in my head.	

In contrast, the preservice teachers in Instructor 2's class connected their past experiences and suggested several strategies (using strings, physically fitting prisms and cylinder, and using drawings) to explain why the circle would have the greatest area.

S5 I was thinking the day we were doing strings, for all of them we have to form the base, we have the same length of string and we form it in 3 different shapes. So it's cool that even though the perimeter is always going to be the same, the circle is always going to have the greatest area according to like the way we set it up…

I2: Interesting. What do you guys think about what S5 just said? Do you think she's correct? — Challenging

S1: We took the cylinder and put the rectangular prism in it…it should fit inside it and there should be the square with little arcs, pieces missing from it, showing that it would have the same perimeter or circumference, but with a bigger area for the circle because the square can fit inside of it.

The instructor further challenged them to generalize the idea by introducing a hexagonal prism and pressed the preservice teachers to provide justification.

I2: S3 has the circle drawn around the square, and the square is drawn around the triangle. So what does this imply? What if I gave you guys the same lateral surfaces rectangle here, another 6 by 32, and I said make me hexagonal prisms? Where do you think its total surface area would fall amongst these 3? — Generalizing

S3: Between the square and the circle.

I2: Why must it be this way? — Challenging

S4: We discussed that it must go between the square and the circle because the more sides you add to the figure, the closer it gets to a circle. So whereas the square has 4 sides, the hexagon has 6, and it just keeps breaking down the paper into closer of a circle.

S2: So if you would fit the hexagon into the cylinder, like the hexagonal prism into a cylinder, there would be less area left over on the outside than the square.

Finally, the instructor steered back the conversation to highlight the connections that the preservice teachers had made.

I2: Okay … S5 mentioned something that kind of relates back to a task we did in the beginning with area and perimeter, do you remember with string? How does this connect to that, some of the big ideas we did in that task with the string? S2 had mentioned it and I want to make sure everyone kind of makes that connection because making that connection is really important. — Steering

S4: Is the connection just that we're dealing with a fixed length [makes string with hands], like with a fixed length of string, and forming it into different shapes? The same thing with the fixed length of the piece of paper?

I2: S5 what do you think, you brought it up? Is what S4 saying what you were trying to say? — Challenging

S5: Yeah. I mean it's a general idea. They kind of brought up the same idea with fitting the paper. It's interesting that clearly constructing the figure with the exact same shape and as we said the lateral surface area is the same but…and the perimeter is the same but we're using one of the lengths from the lateral surface, but it creates a completely different area for each figure.

These interactions led preservice teachers to make claims and warrants about their solutions. Big ideas and connections were emphasized. The discourse in Instructor 2's classroom maintained the level of cognitive demand of the task.

Conclusions

When working on mathematical tasks, preservice teachers tend to focus too much on getting the "correct answer" and fail to step back and look at the big ideas for which the tasks are intended. This is especially true because of their familiarity and experiences with certain procedures or algorithms. It is the role of the instructor to draw out the idea that getting only the correct answer is not sufficient and that it is more important to examine the mathematical structure and the ideas behind the tasks. Teacher educators can accomplish this goal by focusing on questioning strategies built around challenging student assertions and creating generalization rather than simply steering and rephrasing student responses (Knott et al., 2008). By emphasizing this ideal, preservice teachers in Instructor 2's class were involved in more active engagement with mathematical ideas (Walshaw & Anthony, 2008) and participation in a communicative reasoning process (Lerman, 2001). Instructor 2's questioning strategies led to preservice teacher discourse that maintained the high level of cognitive demand of the surface area task. Preservice teachers' own educational backgrounds often lack sufficient mathematical understanding, therefore it is critical that the cognitive demand of tasks remain high so that they can provide effective instruction in the future. Questioning strategies that elicit strong discourse among preservice teachers is one way to ensure rigorous learning of mathematics content and specialized content knowledge in preservice elementary teacher classrooms.

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THE EVOCATION AND ENACTMENT OF CONCEPTUAL SCHEMES: UNDERSTANDING THE MICROGENESIS OF MATHEMATICAL COGNITION THROUGH EMBODIED, ARTIFACT-MEDIATED ACTIVITY

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When instructional designers develop content-targeted pedagogical situations, their practice can be theorized as engineering students' development of conceptual schemes. To account for the contributions of students' prior schemes and situated experiences towards their development of conceptual schemes, I suggest a distinction between the evocation of schemes and enactment of situations. The former suggests that the design of an instructional situation can activate students' prior schemes. The latter suggests that the structure of students' activity in an instructional situation determines the nature of newly constructed schemes. I contextualize my views using empirical data from a case study of early fraction instruction.

Background and Objectives

An established principle of reform mathematics education is to foster students' understanding of mathematical concepts and reasoning skills through problem-solving activities (e.g., NCTM, 2000). These activities are often designed as scaffolded interactions between teacher and student, whereby the teacher presents students with an instructional situation and attempts to guide them to realize a particular instructional goal (e.g., Newman, Griffin & Cole, 1989). In many cases, instructional artifacts are also introduced to support students' appropriation of a particular mathematical concept/scheme (e.g., Dienes, 1964).

For better or worse, the measure of a successful instructional activity is often defined by whether or not the interaction leads students to successfully appropriate the targeted learning objective. While this statement should appear self-evident, various national performance indicators (e.g. National Mathematics Advisory Panel, 2008), as well as the research literature on students' 'misconceptions' (e.g., Smith, diSessa, & Roschelle, 1993) can be interpreted as indirect evidence that our collective understanding of design practice remains underdeveloped.

Indeed, the challenges that students have learning fraction concepts have been well documented (see Lamon, 2007), and a major concern in early rational-number instruction is that many students cope with the topic's inherent challenge by progressively learning to rely primarily on their ability to perform procedural algorithm, enacted as symbol manipulation, which they apply as a means of demonstrating their competence in this mathematical subfield (e.g., Freudenthal, 1983, Chapter 5). The pitfalls and limitations of relying on purely procedural fraction knowledge are poignantly demonstrated in the case of Liping Ma's (1999) elementary school *teachers*: While perfectly capable of performing the appropriate arithmetic procedures for solving fraction-division problems, these teachers were unable to explicate their solution procedures in the form of mathematically accurate scenarios.

Consequentially, more work is needed to explicate the rationales underlying the otherwise tacit design choices that inform instructional practice, as well as to understand the effects that these choices can have upon students' cognition and learning outcomes. A primary objective of this paper will be to explore how the design of an instructional situation can influence students' mathematical cognition, so as to inform a general framework for both fostering and analyzing students' interactions with instructional artifacts and/or activities.

The claim to be advanced is that an instructional activity may be designed so as to purposefully and productively *evoke* students' existing funds of knowledge, and/or to *enact* situations that support students' development of targeted concepts/schemes.

This thesis emerged from, and will be supported by empirical data from a number of case studies involving the implementation of an experimental unit that leverages multimodal activity to support students' learning of fraction concepts (Charoenying, 2010). The specific pedagogical content area involves fundamental notions of rational numbers (specifically, part-to-whole conceptualizations of the *a/b* fraction form). My analysis will explore how the material embodiment of a mathematical task (in the form of an instructional activity) contributes to students' subjective cognitive schema. The functional unit of analysis guiding this examination is the interplay between students' *schemes*, and the instructional *situations* that both influence and give rise to them (Vergnaud, 2009).

Theoretical Framework

Expanding our Notions of Inter-Subjectivity

A socio-cultural perspective of learning is that instructors mediate the speech, actions and perceptions of learners (Vygotsky, 1978; see also Newman, Griffin, & Cole, 1989). Through this process, the learner appropriates and internalizes the socially established meanings of cultural forms such as mathematical signs and concepts.

Bartolini Bussi and Mariotti (2008) have elaborated upon the notion of mediation by characterizing the artifact mediated interactions between instructor and learner as a 'didactic cycle,' whereby the instructor progressively guides the learner to understand the link between the artifact used to complete a task and the formal mathematical meanings.

Although the inter-subjective dialectic between instructor and learner is clearly a vital component of teaching and learning, this framing essentially constrains any analyses along two dimensions—the knowledge, goals, and beliefs of the instructor (Schoenfeld, 1998; see also Ma, 1999); and the prior knowledge and cognition of the learner. Given the fact that not all instructional interactions are successful in supporting students' development of a targeted concept/scheme, a purely socio-cultural analyses would imply some failing on either the part of instructor or learner.

Rather than continue to problematize unsuccessful teaching and learning interactions in terms of the participants, I believe it may be more productive to problematize our existing theoretical frameworks for analysis (Smith, diSessa, Roschelle, 1993). Here, I suggest that a potential gap in socio-cultural accounts of learning is that they do not adequately address the contributions of learners' tacit multimodal activity underlying action and expression to cognition, or their prior knowledge to learning (cf., Abrahamson, 2009). A potentially more productive approach towards designing and understanding learning interactions, would be to incorporate a more nuanced examination of how the *intra*-subjective contributions of students' multi-modal perceptions and prior-funds of knowledge interact within a designed, instructional situation.

Evoking Schemes and Enacting Situations

I would begin by highlighting that whereas learners may experience some activities as conceptually "meaningless" and others as "meaningful," what can be common to these activities are the material objects students encounter, the operations they conduct upon and with them, and the discernable results of these actions.

Here, I propose utilizing the verb "to *evoke*" to refer to the activation of students' prior schemes. It is widely accepted that familiar artifacts, symbols, and signs can evoke an individual's pre-existing schemes. Similarly, students' schemes may be evoked upon their discovery of familiar features present in an instructional situation. Understanding and anticipating the schemes that are likely to be evoked by particular design features would position designers and educators to better interpret and address students' responses as they guide them towards the desired learning objective.

In tandem, I propose utilizing the verb "to *enact*" to describe how students' experience of a situation can be purposefully designed so as to support a particular learning objective. Bruner (1966) had originally used the term "enactive representation" to elucidate how children's actions and activities in the world contribute to their development of iconic and/or symbolic forms of representation. Bruner had suggested that only after "something" is first acted upon and experienced in the world, can it be referenced; first as an object of thought, and later as an icon and/or symbol (see also Hutchins, 2006). Following this reasoning, I argue that the educator/designer can guide students to enact a situation— to act upon, observe, and/or produce situated phenomena —the experience of which can form the basis from which students can subsequently "reverse-engineer" the targeted instructional schemes/concepts the educator wishes for them to develop.

In summary, given the prominent and established role that artifacts play in supporting early fraction instruction, it is important to understand how the design and utilization of these objects can influence students' mathematical cognition. I have proposed the constructs of evocation and enactment in order to further elucidate the relationship between the schemes that are activated when students' notice particular features of a design, the forms of interactions that the design supports, and ultimately, the contributions of situated activity towards mathematical learning. Finally, the instructional design presented herein is influenced in part by theoretical claims that cognition in general (Barsalou, 1999), and mathematical reasoning in particular (Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999), are grounded and tacitly instantiated in real-world experiences. Therefore, an explicit design consideration was to provide students with activities that leveraged their multi-modal perceptions and embodied actions in situ to help mediate and modulate the appropriation of mathematical forms and modes of reasoning (Abrahamson, 2009).

Methods

Given the dual objectives of this project: to better understand student learning, and improve instructional materials, design-based research (Brown, 1992; Confrey, 2005) was selected as an appropriate investigative approach.

This work draws on a series of case studies conducted during an ongoing design-based research study examining how students learn through interacting with tangible mathematical objects. I acted as teacher–researcher-designer. Students from two grade general-education classes (n=54) and from a Grade 3-5 self-contained special-education class (n=7) participated in a series of small group and one-on-one, video-taped teaching interactions.

My design rationale for building the instructional artifacts was as follows: First, I sought a familiar physical object rather than a diagrammatic or computational illustration. I felt it would be important for students to be able to interact with the objects so as to experimentally create, confirm, and reverse mathematical conjectures. A second criterion aimed to encourage broad dissemination was that the design be inexpensive, easily procured, and simple to use.

The resulting instructional design is a tutorial session that engages students in problem solving activities involving water and standard kitchen measuring cups (see Figure 2a).

Through a premeditated sequence of problem posing and facilitated hands-on inquiry, students are guided to perform and report perceptual judgments, measure quantitative aspects of the situation, uncover patterns, and so doing reformulate their initial judgments of part-to whole relationships as aligned with mathematical algorithms and vocabulary. Along the way, the teacher challenges students to warrant/evaluate their claims using the available materials.

As with traditional instructional representations, such as area models and number lines, a key affordance of measuring cups is that they enable students to harness their visual modality to support their mathematical judgments. Thus, to compare the unit fractions 1/3 and 1/4, one could place the respective measuring cups side-by-side (see Figure 2b). More importantly, the measuring cup activities provide a tangible means for teachers to guide students to physically enact situations they believe are analogous to the mathematical schemes they wish students to develop. For example, to help students develop a part-to-whole scheme, students could be guided to enact a situation in which they iterated the same volume from a unit-fraction cup into a one-whole cup measurer until it was filled (see Figure 2c). These activities also provide opportunities for teachers to observe and assess student reasoning.

Figure 2a. Standard kitchen measuring cups.

Figure 2b. A side by side visual comparison of two unit fraction measures.

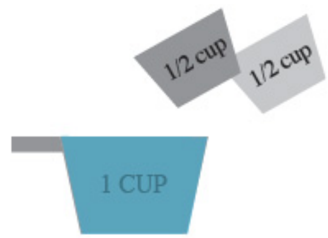

Figure 2c. Enacting a part-to whole scheme by iterating ½ cup measures into 1 cup.

The specific case study presented in this paper is from a one-on-one tutorial session with a Grade-3 special education student, hereafter referred to by the pseudonym "Manny." Pre-assessment data and consultation with the classroom teacher revealed that Manny was performing far below grade level in mathematics.

Results and Discussions

The pedagogical objective of the tutorial session was to guide Manny (age=8) to formulate a *qualitative* understanding of individual unit fractions and the part-to-whole scheme by using the measuring cups to physically enact mathematically analogous situations.

Enacting Mathematically Analogous Situations

After allowing him to familiarize himself with the measuring cups, I ask him how many scoops from the 1/2 cup measure are needed to fill the 1 cup measure. Manny correctly guesses two scoops. I then instruct him to enact his conjecture by iterating 1/2 cup measures of water into a one cup measure. While this instructional sequence would seemingly suggest some understanding of the part-to-whole scheme, his haphazard response to a similar question posed with the 1/4 cup measure reveals otherwise.

R: How many scoops do you need with this one [the 1/4 cup measure]?

M: Three, No four.
R: You need four of these? Why?
M: [Manny looks at the symbol for 1/4] It's one-half

His utterances suggest he has yet to understand the mathematical meaning signified by the fractional notations inscribed on the measures. I write 1/4 onto a slate.

R: No. Can you read one over four. Or you can say one fourth.

I then instruct Manny to iterate four scoops of water from the 1/4 measure directly into the one whole cup measure. He then reverses the operation by scooping water directly out of the one whole cup measure (into a separate reservoir of water) using the 1/4 measure.

R: How many of these guys? [I show him the 1/3 measure]
M: Three.
R: What do you call this one [the one third measure]?
M: I don't know.
R: One over three. Wait. Before you scoop, you have to write it. [Manny writes 1/3, and iterates the 1/3 cup measure into the one whole cup (figure 3)].

Figure 3. Manny iterating water iterating from the 1/3 measure into one cup

A Conflict Between Prior Knowledge and Perceptions

Having guided Manny to enact two situations I believe will support his understanding of the part-to-whole relationship of unit fractions, I purposefully attempt to dis-equilibrate his existing number scheme. I write 1/3 and 1/4 onto a slate.

R: Alright, question Manny. Which fraction is bigger? One third or one fourth?
M: That one [indicating the 1/3 cup measure].
R: You are correct, but why is this one bigger? Can you explain it to me. This is a one third cup [I hold the 1/3 cup next to its numerical inscription] this is a one fourth cup [I hold up the 1/4 cup next to its numerical inscription]. Why is it bigger? [I point again to the 1/3 cup] Why is one-third bigger?
M: Wait. [He matches the cups to the inscription. So this one's bigger! [M points back to the 1/4th cup measure]! No. This is!? [points again to the 1/3 cup measure] I don't know!
R: What do you mean you don't know? You can see with your eyes!

Interestingly, the prior number schemes that the numerical inscriptions evoke for Manny overrule his perceptual judgment. He picks up the 1/4 cup measure, physically nests it into the 1/3 cup, and again tentatively concludes that 1/3 is "bigger."

Evoking a Recently Enacted Situation

To reinforce Manny's intuitions and help him to appropriate a part-to-whole scheme, I guide Manny to reflect upon his earlier enactment of the mathematically analogous situations.

R: Okay. How many times do you need to scoop this one [How many times must the 1/3 measure be iterated to fill 1 cup?]

M: Three

R: And how many times do you need to scoop this one to fill this one.[How many times must the 1/4 measure be iterated to fill 1 cup?]

M: Four.

R: Do you understand why one third is bigger than one fourth yet? Let's see if you can explain why this fraction [inscription for 1/3 written on a whiteboard] is bigger than this one [inscription for 1/4 written on a whiteboard]

M: Because this one doesn't put too much water in it. [simulates scooping of water using the 1/4 measure into the 1 cup measure] And this one [points to the 1/3] can scoop a lot of water [more than the 1/4 measure].

Manny then examines each measurer in turn again, including the 1/2 measure. He physically simulates the pouring action with each into the one whole cup measure. I remove the measures and inscribe the fractional notation for 1/5 and 1/10 onto a board.

R: Which one would be bigger? [1/5 versus 1/10] Which one is going to be a bigger fraction? If these were cups, which would be bigger? [Manny attempts to locate a 1/5 cup and 1/10 cup measure] We don't have these cups...

M: Oh this one [he points to the 1/5 inscription]

R: Why? If this were a scooper…

Manny is initially unable to articulate an answer. He looks to the inscription of 1/10 and utters 11, which could suggest that the arrangement of numbers evoked his addition scheme. He looks again to the inscription of 1/5 before suddenly exclaiming:

M: Wait now I know! This one [the inscription for 1/5] only needs 5 cups, the small one [the inscription for 1/10] needs 10 cups!

Strikingly, the prior enactment of the mathematically analogous situation—which is arguably a function of the concrete, physical properties of the instructional artifact— provided an embodied point of reference for Manny's generalization (see also Pratt & Noss, 2002 on 'situated abstraction'). Manny's statement suggests that he has begun to build an experiential resource commensurate with the mathematical law that $a * 1/a = 1$. A generalized scheme has been abstracted from the situation. One is left to infer that the utility of an instructional situation is contingent on its capacity to support students' enactment of situations that are analogous to the scheme to be learned.

In many respects, Manny's initial interpretation of the 1/*a* form for fractions is typical of students his age, his learning disabilities notwithstanding. What differentiates this particular instructional activity from more traditional approaches involving drawn representations such as area models or number lines for example, is that Manny is able to physically enact both the composition and decomposition of fractional parts into wholes, and so doing, construct an understanding of fractions that is directly grounded in experience. Additionally, the physical medium allows me as teacher to design an activity I believe corresponds to the mathematical concepts/schemes I wish for him to appropriate. Just as importantly, it allows me to indirectly infer the changes in his conceptual understanding by directly observing his actions.

Summary and Conclusions

It is widely accepted that mathematical concepts can emerge through guided acts of representation, as students attempt to articulate emerging schemes in language and any other available means of objectification. While the contributions of a more knowledgeable other are instrumental to this process, I would argue that an analysis of an instructional interaction is incomplete unless one also accounts for the interplay between the instructional situation, and the cognitive and multi-modal perceptual resources that learner student brings to bear.

I have proposed two mechanisms for characterizing learners' interactions with an instructional artifact or activity. First, I highlight the fact that instructional activities are situations that can evoke pre-existing schemes in the mind of the learner. Awareness of this mechanism is arguably of vital importance for instructional designers. As a design heuristic, this notion of evocation leads the designer to consider how students might perceive and experience an instructional situation. Vitally, it prepares designers to better account for the otherwise unexpected ways that a student might make sense of a novel instructional situation. Knowing a priori that a particular design feature is likely to evoke a particular scheme for students (e.g., how the discrete units of an area model evoke counting schemes), would help to inform how an educator/designer chooses to mediate students' learning activities.

Second, I use the term "enact" to describe how students' situated activity can be structured so as to help facilitate their construction of a given scheme. Schemes are not constructed *ex nihilo,* but from students' encounters with, and assimilation of, new situations.

It stands to reason then that the manner in which students' situated activity is organized and orchestrated is central to their conceptual development. Instructional designers can selectively determine the experiences students have, and thereby influence the trajectory of students' cognitive development. Students' enactment of an instructional situation furnishes them with a set of experiences that may form the cognitive bases for the instructor's targeted learning objective.

Although evidence from only a single, brief case study episode is provided, the full corpus of data suggest that the Water Works design appeared to support students' articulation of their mathematical ideas by allowing them to physically enact a variety of representative situations. On the other hand, situations involving fractions for which there was no corresponding measuring cup such as 1/5, 1/100 etc., could not be directly modeled with the set of measurers. The students in the study did not appear to immediately transfer their newly constructed schemes (e.g., four scoops from the 1/4 measure to fill one cup) to the purely symbolic representations for which there had been no situation-specific analog. This is a clear limitation of the present design. In such cases, an abstract, drawn or computationally modeled representation of the scheme/situation might have been much more practical.

In conclusion, the argument advanced by introducing the constructs of evocation and enactment is this: Educator/designers effectively determine the actions and outcomes that arise from students' interactions with a particular situation. Therefore they influence in large part the schemes students construct. Conceptualizing instructional practice in terms of structuring learning situations may provide educators, designers, and researchers alike with new productive insights for anticipating and evaluating the effectiveness of a proposed instructional design.

Finally, the instructional design presented in this study is a low-cost, low-tech example of an embodied representational context that can be adapted by teachers in *any* classroom.

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ENGENDERING MULTIPLICATIVE REASONING IN STUDENTS WITH LEARNING DISABILITIES IN MATHEMATICS[1]: SAM'S COMPUTER-ASSISTED TRANSITION TO ANTICIPATORY UNIT DIFFERENTIATION-AND-SELECTION

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We examined how a student with learning disabilities (SLD) in mathematics constructed a scheme for differentiating, selecting, and properly operating on/with units that constitute a multiplicative situation, namely, singletons ('1s') and composite units (abbreviated UDS). Conducted as part of a larger teaching experiment in a learning environment that synergizes human and computer-assisted teaching, this study included 12 videotaped teaching episodes with a 5th grader (pseudonym-Sam), analyzed qualitatively. Our data provide a window onto the conceptual transformation involved in advancing from absence, through a participatory, to an anticipatory stage of a UDS scheme—a cognitive root for the distributive property. We postulate this scheme as a fundamental step in SLDs' learning to reason multiplicatively, and highlight the transfer-empowering nature of constructing it at the anticipatory stage.

Introduction

This study examined how students with learning disabilities (SLD) may construct a scheme for reasoning about multiplicative situations. Learning to reason multiplicatively is a major feat for elementary age children (Harel & Confrey, 1994; Kamii & Clark, 1996; Sowder, et al., 1998;

Steffe, 1994), whether or not they are identified as having learning disabilities/difficulties in math. Fostering it in all children is critical, because it provides a basis for understanding and properly using not only multiplication and division, but also fractional, proportional, and algebraic reasoning (NCTM, 2000, 2006). Lacking such a basis seems a key hurdle in SLDs' progress toward the latter, advanced concepts (Xin, 2008; Xin, Wiles, & Lin, 2008) and a cause for being and feeling 'stuck' in mathematics.

This study, a part of the NSF-funded *Nurturing Multiplicative Reasoning in Students with Learning Disabilities*[1] (NMRSD) project, examined SLDs' construction of a scheme for operating on, and coordinating, not just one but two sets of composite units (Steffe & Cobb, 1998) at a stage conducive to solving novel tasks ('transfer'). Such tasks may call for finding the sum or difference between two quantities (e.g., 'You have 7 boxes, 8 crayons each; I have 3 boxes, 8 crayons each; How many more crayons do you have?'). To solve such tasks in the absence of tangible 1s, a solver must identify and coordinate the units involved additively and multiplicatively. She either first multiplies (7x8=56, 3x8=24) to find the total of 1s in each set and then subtracts (56-24=32)—a *Totals-First* method, or first subtracts to find the difference in *composite units* and then multiplies the resulting, new set of composite units by the number of 1s in each (7-3=4; 4x8=32)—a *Difference-First* method. Our research question was how SLD may learn to differentiate, select, and operate on those quantities while forming a cognitive basis for what adults refer to as the distributive property (e.g., 8x(7-3) = 8x7 - 8x3).

Conceptual Framework of this Study

The NMRSD project is developing a software that draws on three research-based frameworks: a constructivist view of learning from mathematics education, generalization of word-problem underlying structures ('story-grammar') from special education, and machine (or statistical) learning from computer sciences. For this study the constructivist scheme theory (Piaget, 1970, 1985; von Glasersfeld, 1995) and its recent extension into the reflection on activity-effect relationship (*Ref*AER*) account (Simon et al., 2004; Simon & Tzur, 2004; 2004; Tzur, 2007, 2008) provided the cognitive lens. In particular, we used Tzur & Simon's (2004) distinction between the participatory and anticipatory stage in the construction of a new mathematical conception for designing instructional tasks and for assessing Sam's ways of operating. This stage distinction drew on von Glasersfeld's 3-part notion of scheme: (a) recognition of a situation, which sets the student's goal, (b) a mental activity associated with that situation and goal, and (c) an expected result. At the participatory stage, a learner forms a novel anticipation—an invariant relationship between the activity and a newly noticed/linked effect it brings forth. However, this *AER* is yet to be linked to a scheme's first part, and the learner can only access it if prompted for the activity, which regenerates the link (see Roig & Llinares, 2009). At the anticipatory stage, the learner links the novel *AER* with a host of 'structurally similar' situations, thus abstracting spontaneous (prompt-less) access to it across contexts (see Woodward, et al., 2009). We built on Tzur and Lambert's (in press) recent articulation of prompt features to guide decisions about sequencing tasks that could foster, and help assessing, transition to the participatory and then anticipatory stage of UDS.

The content-specific constructs of our framework draw on the *Initial*, *Tacitly Nested*, and *Explicitly Nested Number Sequences* (Steffe & Cobb, 1988)—three schemes a child uses in assimilating and operating on abstract composite units (CU). In the latter, the child conceives of smaller CUs as embedded within larger CU. For example, 3 CU of 8 singletons (3 boxes of crayons) and 4 CU of 8 singletons (4 boxes) are embedded within 7 CU of 8. Coordinating such same-size CU is key for early multiplicative reasoning in general and for children's construction of a UDS scheme in particular.

Methodology

Within the NMRSD project, this case study consisted of 12 videotaped teaching episodes in which Sam, a 5th grader, was jointly taught by the software and the second author. Sam was a strategic case, because RTI-like, problem-based pre-intervention assessment by the software and researchers indicated his 'readiness' for UDS. He had formed anticipatory stages of operating on CU multiplicatively to find the total of 1s (e.g., How many cubes are in 6 towers of cubes, 3 cubes each? Denoted $6T_3$ here) and additively to find CU sums/differences (e.g., You have $7T_3$; I have $4T_3$; how many more T_3 do you have?).

The software was designed based on results of previous (Tzur et al., 2009; Woodward et al., 2009; Xin et al., 2009; Zhang et al., 2009) and on-going teaching experiments with SLD. It presents students with problems in an interactive environment, and collects and analyzes data from the student's work (e.g., prompts needed, solution processes used) to select and present the next problem. This includes assessing if a student is ready to start the next level of problems. During our work with Sam, human teaching played a significant role to gain insight into software improvement—studying its impact (or lack thereof) on SLD learning of the intended mathematics.

Typically, software tasks commence within a context of producing, actually and/or mentally, a CU in the form of a tower made of several cubes (e.g., $7T_8$). One type of UDS problems asks the child to compare two different sets of same-size towers (e.g., $7T_8$ and $3T_8$). For example, the software may initially present those towers, then cover them, and pose the tasks: "*Jacqueline has a collection of 7 towers with 8 cubes in each. Mercedes has a collection of 3 towers with 8 cubes in each. How are these collections similar? How are they different? Who has more cubes and how many more?* A second type of problems switches the quantities, so two equal sets are made of different-size CU (e.g., $7T_8$ and $7T_3$). In teaching sessions in which the software was the primary source of UDS tasks, we often suggested to Sam to use one of the mini-tools available in the software (e.g., a calculator, a simulation of fingers that depict 'double-counting'). We also frequently asked Sam about the referent units of his solutions and how he figured these out.

We also used two interventions outside the software. First, Sam's work indicated that he did not have a conventional meaning for "*more*". For him, this term referred not to the difference between two quantities, but to the total of units in the larger quantity. Thus, we asked Sam to place two groups of Lego cubes (12 and 9), and then pair-off (1-to-1) as many cubes as possible. The researcher then clarified that (a) the number of non-paired cubes is the difference between the two sets (which Sam knew already) and (b) "more" refers to this number. This intervention seemed to create a shared, proper meaning for "more". The second intervention, which preceded Excerpt 2 (see Results), took place during the third UDS episode, as the software continued posing the first type of UDS problems. Sam seemed to struggle with operating on the various units, so we asked him to draw the two sets of towers on a paper. We said that a drawn tower could show each individual cube or just a schematic line/bar that Sam would label with the number of cubes it signified. Importantly, Sam felt the need to draw each cube in every tower (i.e., producing 1s), a need that was later diminished.

Data analysis occurred throughout the study. In our on-going analysis we discussed and recorded significant events after each teaching episode. We paid close attention to possible indicators (prompts needed) that Sam was in the participatory or anticipatory stage of UDS, and used our observations to plan for the next episode(s). In our retrospective analysis we began with those teaching episodes in which we previously identified segments when Sam was working in either stage of UDS. We transcribed and analyzed those episodic "snapshots" line-by-line to articulate advancements in Sam's mathematical thinking.

Results

In this section, we first briefly summarize data pertaining to Sam's lack of unit differentiation-and-selection (UDS) scheme. We then present and analyze data in support of claiming his transition to (a) the participatory stage and (b) the anticipatory stage of UDS. This includes his learning of a conventional meaning for 'more' or 'less', and a task in which he applied ('transferred') anticipatory UDS to a novel context.

Absence of a UDS Scheme

Once the NMRSD software determined that Sam has constructed an anticipatory stage of multiplication, and of *additively* operating on the *same* composite units (SUC scheme, e.g., you have $7T_8$ and I have $5T_8$; how many T_8 do we have altogether?), it moved to a UDS task: "Jacqueline has a collection of 12 towers with 8 cubes in each. Mercedes has a collection of 7 towers with 8 cubes in each. How are these collections similar?" He selected 'They both have 8 cubes in each tower'. Then, asked about difference, he selected 'Jacqueline (J) has more towers'. The software asked, "Altogether, who has more cubes?" Sam responded not with a name, but with an attempt to compute how many cubes J has (calculator). First, he multiplied 7x7=49, followed by 49x8=392, and noted "That's not right." The researcher (R) prompted, "Why did you do 49 times 8?" Sam calculated 7x8=56 (his *available* multiplication scheme) and stated that M has 56 cubes *in each tower* (unit conflation). Then, R said that the task asks *who* has more cubes, J or M. Sam responded by finding (calculator) 12x7=96 and stating, "Jacqueline." R asked if this makes sense; Sam nodded (yes), "Because J has more towers".

The software proceeded, "How many more cubes does Jacqueline have?" and R said that Sam could use the calculator. Sam re-read the task, stated he does *not* see how the calculator could be used, and after some thinking said, while inserting the numbers into the computer: "96 divided by fifty … (looks at R) Was it 58 or 56?" R, after saying "56," asked Sam why dividing. Sam said that this will give him the number he's looking for (clarifying this was "how many more cubes," and that multiplication would not work as it gives too high-a-number). R noted that Sam could have also used addition or subtraction. After a short pause to think of these options Sam keyed 96÷56, saw the answer (1.7142…) and responded in big surprise: "Whoa!" He noted that multiplication and division would not work, contemplated "96 minus … No," and added (calculator) while saying: 96+56=152. R asked him again why adding would be proper and explained to Sam the importance of his *reasoning* about the operation. In response, Sam stated that multiplication, addition, and division (!) gave him too large a number, calculated 96-56=40, and claimed this seemed correct.

The entire exchange indicated to us that Sam was not using an operation based on a reasoned choice (about difference). Rather, he tried one operation after another and judged the answer's plausibility on the basis of size/form. He also did not indicate any consideration of finding the difference in CU (towers) as a first step to be followed by multiplication (e.g., $12T_8$-$7T_8$=$5T_8$, 5x8=40 cubes). Thus, R began working with Sam on such a method and its link to the transposed one (e.g., 12x8–7x8), that is, on constructing UDS.

Participatory Stage of UDS

To foster Sam's construction of UDS, we realized he would first need to attribute a proper meaning—*difference*—to the terms 'more' and 'less'. An alternative meaning he seemed to attribute was evident in his responses to SUC tasks. For example, in the above situation (J has $12T_8$, M has $7T_8$), if asked how many more *towers* did J have, Sam would most often respond '12'. We briefly note that (a) such a meaning was found in other SLDs during the larger study and (b) it made sense to him because J has more *and* she has 12 so '12-is-the-more'. Sam quickly adopted the conventional meaning that R introduced through tasks about Lego singletons (1s). Thereafter, he consistently applied the terms to the difference between two quantities, either singletons or composite units, and strategically selected subtraction to find it. He seemed ready to construct UDS, which was fostered via tasks with 'easy' numbers.

The software stated: "John has a collection of 12 towers with 3 cubes in each, Sarah has a collection of 8 towers with 3 cubes in each." Sam playfully built the $12T_3$; he was asked and properly responded that both John and Sarah have 3 cubes in each of their towers. Asked how many towers each child had, he responded 12 and 8 respectively, responded that John and Sarah do not have the same number of towers, and that John has more cubes-in-all than Sarah. Excerpt 1 below provides what transpired next (C=computer, S=Sam, R=researcher).

Excerpt 1 (September 22, 2010)

S: [Reads the task in C out loud] "So, how many more cubes does John have than Sarah has?" and continues: Okay, so, take [away] 8. That's going to leave me with 4.

R: Four what?

S: Four towers; [and] 4 times 3 is 12.

R: You know that one, right?

[A little later, after R asked S to open the software Toolbox and use the calculator.]

R: If you do 12 x 3 and 8 x 3 … just to check the other way.

S: (Inserts 12 x 3 = while saying) 12 times 3 equals 36.

R: That's how many cubes - who has? [Pause] That's John, right?

S: (Inserts 8 x 3 = while saying) 8 times 3 equals 24. (Appears to think what's next) Twen … Twenty … Twenty-four (Inserts 24 and the *symbol for multiplication*, '*', while saying) times … [no] subtract, minus … [Clears calculator screen.]

[At this point, after R suggested to subtract the smaller number from the larger, S found 36-24=12 and, with a 'high-five' from R, they noted that it is, and must be, the same answer as Sam obtained by first subtracting the towers and then multiplying.]

Excerpt 1 indicates Sam's learning to operate in a UDS situation. He (a) differentiated the 1s (cubes) and unit-rate (3 cubes/tower) from the CU (towers), (b) selected the latter and operated on it to find the difference (via subtraction), (c) selected the unit-rate and used it as operand for (d) multiplying by the difference (4 towers) to figure out the number of cubes (12) that John had more than Sarah. Once prompted for solving the task in the alternative method, Sam's attempt to use multiplication indicated he was yet to establish UDS.

We argue that Sam's construction of UDS was at the participatory stage for a threefold reason. First, his execution of the *Difference-First* method, while proper and independent, followed R's teaching of using such a method earlier in that session. Thus, theoretically, the claim about anticipatory stage is not possible. Second, a fully established (anticipatory) UDS would consist of adeptly using either method. Excerpt 1 provides evidence that, initially, Sam thought of multiplying the totals in each collection (36x24) instead of subtracting. Third, data from the following two sessions showed that, when the software opened an episode with a UDS task Sam was unable to solve it via the *Difference-First* method without being prompted by R, although Sam himself stated it was easier. Excerpt 2 shows his choice of initial method (*Totals-First*) and incorrect operation (multiplying totals) after properly answering what's similar/different about J's and A's collections ($13T_9$ and $3T_9$ respectively). Because Sam's *single*, wrong method/answer was sufficient to *self-prompt* him for shifting to the alternative, *Difference-First* method, we consider his work as indication of high participatory stage.

Excerpt 2 (October 20, 2010)
C: Altogether who has more cubes?
S: (Selects 'Jacqueline'.)
C: How many more cubes does Jacqueline have?
S: (Pulls up the calculator tool. Inputs: 13 x 9 = 117; 3 x 9 = 27; 117 x 27 = 3159. He then clears the calculator screen and inputs: 13 – 3 = 10; 10 x 9 = 90.) Ninety!

Anticipatory Stage of UDS

We culminate with data that show Sam's UDS at the anticipatory stage. As *Ref*AER* stresses, a participatory and anticipatory stage differ not in the nature a scheme's parts (situation/goal, activity, effect), but rather in the learner's access to it—with or without prompt. Excerpt 3 shows Sam's prompt-less solution to a task presented outside the computer at the start of an episode 3 weeks after he indicated the high participatory stage (Excerpt 2). In the previous episode, he solved a tower/cubes UDS task without a prompt; here, we show how he applied ('transferred') it to a novel, realistic word problem with larger numbers.

Excerpt 3 (November 10, 2010)
S: (Reads the problem): "Evan's daughter (Sarah) needs to prepare 18 birthday bags. In each bag she plans to put 8 candies. Ron's daughter (Lihi) needs to prepare 7 birthday bags. She also plans to put 8 candies in each bag. Who needs more candies, Sarah or Lihi? How many more candies does that daughter need?" (He writes, 'Sarah needs')
R: Do you need a calculator?
S: (Nods yes): Hm-hmm. (Opens the calculator, inserts 18x8=144, comments it is the same as 12x12, clears the calculator and uses it for 18-7=11.) Eleven? Hmm. They are candies (clears screen). So what do I need to do here? I found the difference, which is 11. (Pauses, then shifts method again) Seven times … (calculates 7x8=56, then 18x8=144, and finally 144-56=88, so he exclaims): Eighty-eight!!
R: [After asking Sam to explain why he solved it this way, what did he find in each calculation, and Sam's writing 'Sarah needs 88 more candies' and stating he subtracted to find the difference in candies] Is there a different way to solve the problem?
S: [Appears to be thinking] Hmm …
R: I think you almost started one … Anything in your mind that you could have done?
S: Well, I did 18 minus 7, which is 11, and then 8 times 11 is 88! (Looks at R proudly.)

Excerpt 3 provides evidence that Sam has established the UDS scheme as an invariant way of operating in multiplicative situations where two quantities of the same unit rate (e.g., 8 candies per bag) are compared to find the total difference in singletons (1s). Establishing UDS at the anticipatory stage allowed assimilation of and operation on quantities given in a realistic situation. Interestingly, he considered solving the novel situation via the *Difference-First* method. However, he seemed to lose track of the second step (as he later explained to R's question). Yet, the anticipatory scheme empowered his resourceful, independent shift to and successful completion of the *Total-First* method. In turn, he could return to and successfully complete the alternative method, with proud awareness of the answer identity.

Discussion

This study makes a twofold contribution. First, it portrays cognitive changes in forming a unit differentiation-and-selection (UDS) scheme. UDS was articulated while studying how multiplicative reasoning evolves in students with learning disabilities/difficulties (SLD) in mathematics. For these students, and likely for their normal achieving peers, UDS serves as an intermediate cognitive step. It builds on a student's assimilatory scheme of multiplicative double counting, used for the goal of quantifying a total of 1s via distributing the given numerosity (unit-rate) of a composite unit over a number of such units (Steffe, 1994; Steffe & Cobb, 1998). It is constructed through and applied to solving problems in which the learner's goal is to *compare two such quantities*, each consisting of so many composite units (e.g., How many more marbles are in 11 bags with 7 each than in 6 bags with 7 each?).

Articulating UDS shows the need to foster SLDs' intentional identification of how two quantities are similar/different in terms of 3 types of units of a multiplicative situation. Sam, and often other SLD, could initially operate on just one level of units, that is, 1s. However, to strategically and effectively employ mental activities for finding the difference in total, one must distinguish singletons from composite units and operate on the latter while using two or three levels of units (Steffe & Cobb, 1988). The UDS scheme involves anticipating and coordinating *Difference-First* and *Total-First* methods. In the former, the child first selects and compares the two quantities in terms of the *numerical difference between composite units* (e.g., 5 more bags in the situation above). To this end, as Sam *taught us*, a child may have to re-learn a proper meaning for 'more' and 'less'. Then, the child has to re-select the numerosity of each composite unit (unit-rate) and anticipate multiplicatively distributing it over the difference found (e.g., 7 marbles/bag x 5 bags of the difference only = 35 singletons in the difference). This process requires simultaneous operation on at least two levels of units. In the latter, *Total-First* method, the child first selects and multiplicatively computes the total of 1s in each quantity—a step that *may* involve two levels of units (e.g., 7x11=77 marbles, 7x6=42 marbles). This turns the situation into an additive comparison, for which subtraction is called upon as a second step (e.g., 77-42=35 marbles). Our study shows that the anticipatory, well-coordinated stage of a UDS scheme empowers its application (transfer) to novel situations. We postulate it as a conceptual root of what knowledgeable adults call 'the distributive property of multiplication over addition' (e.g., 7x(11-6) = 7x11-7x6).

Second, our study shows that computer-assisted learning opportunities can be designed by using ongoing analysis of student understandings as a basis for task selection. In the NMRSD software, tailoring tasks to student assimilatory schemes is achieved by operationalizing constructs of the *Ref*AER* framework (Simon, et al., 2004; Tzur & Simon, 2004). In particular, software programming draws on Tzur and Lambert's (in press) work of linking the participatory stage to Vygotsky's notion of ZPD and identifying three parameters of a prompt: (a) its *locus*—whether self-generated within a learner's mental processes or by an outside entity, (b) its *focus*/*essence*—ranging along the continuum from generic (e.g., could you solve the problem in a different way) to specific (e.g., could you begin by finding how many more towers does J have?), and (c) *number*—ranging from one, to a few, to many prompts. Using these parameters, as well as the *time* it takes a student to solve each problem and the *computer actions* she or he is using (e.g., typing 11 and then hovering with the mouse over '*' before typing '-' and '6' followed by '='), the NMRSD software successfully determined and fostered Sam's progress. He started when seemingly not yet having constructed a basic scheme of multiplicative double counting. The software taught him that scheme and determined he had constructed it at the anticipatory stage, as well as a scheme for additively coordinating same-size composite units (SUC). It then shifted to UDS, gradually changing the prompts given to Sam to engender his progress to participatory and then anticipatory stage (including situations with different unit-rates, such as comparing $7T_3$ with $7T_5$, or all units differing, such as $7T_3$ and $9T_2$). While the software 'progressed' based on assessing Sam's thinking, its work was synergized with human teaching (e.g., detect non-conventional meaning for 'more', introduce novel, realistic tasks, provide social-emotional support). This synergy empowered not only transfer to novel situations, but also Sam's quite effortless learning of a more advanced, pre-algebraic scheme (e.g., "You have $9T_7$; you receive 28 more cubes; how many towers will you have once all cubes are put into T_7?").

Endnotes

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SOCIAL JUSTICE IN A MATHEMATICS COACHING PROGRAM: COACH GROWTH PROGRESSIONS

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In this paper we report on a project integrating social justice pedagogy in a professional development program training mathematics coaches to work in grades kindergarten through eight. The goal of our research was to study the coaches' growth in understanding of and commitment to social justice pedagogy in the mathematics classroom after participation in the social justice component of the three-year coaching training program. Findings reveal six progressions of coach growth.

Goals

Research, theory and practice around equity and diversity in mathematics education has grown to include more work in recent years that addresses liberatory education and social justice from both US and international perspectives (Burton, 2003; de Freitas, 2008; Frankenstein, 1987; Gutstein, 2003, 2006, 2008; Gutstein & Peterson, 2006, Sriraman, 2008). Within the scholarship we also find a body of work addressing the concept of resistance in social justice work (Gutstein 2006; Satterthwaite, Atkinson, and Martin, 2004; Filax, 1997). In this framework, action against the inequities, injustices, and oppressions in the world in which one lives is the resistance of social justice.

In this paper we report on a project integrating social justice pedagogy in a professional development program training mathematics coaches to work in grades kindergarten through eight. The coaches are trained to provide support and professional development opportunities for the mathematics teachers in their buildings. Most of the schools in the project are low-achieving schools representing a variety of groups underrepresented in mathematics education. Social justice is a component of the conceptual framework for the project, and is therefore an appropriate topic to be integrated into the training and support of the coaches. The goal of our research was to study the coaches' growth in understanding of and commitment to social justice pedagogy in the mathematics classroom after participation in the social justice component of a three-year mathematics coaching training program.

In the following pages we put forth the theoretical framework for our study; explain the research project, its methods and findings; and discuss the findings and their implications for mathematics education professional development around social justice.

Theoretical Framework

> The goal of social justice education is to enable people to develop the critical analytical tools necessary to understand oppression and their own socialization within oppressive systems, and to develop a sense of agency and capacity to interrupt and change oppressive patterns and behaviors in themselves and in the institutions and communities of which they are a part. (Adams, Bell, & Griffin, 2007, p. 2).

The first topic we have to acknowledge is that there is such a phenomenon as social justice and conversely that social injustice exists. A claim that we are teaching for social justice positions us at a point of recognizing sites of social injustice and teaching toward a goal of social justice. Given society's dynamic nature and the obscure manifestations of injustice, a condition of social justice is not easily obtainable; nor does working for social justice have a fixed end. Thus our goal is not necessarily to reach a state of social justice. Rather it is to develop the ability to identify social injustice, address it within school and community settings, and take actions to create change leading toward social justice. When we talk about developing ourselves as socially conscious catalysts for change, we acknowledge that we have a responsibility to play an active role in working toward social justice (DeVries & Zan, 1996; Green, 1971; Tom, 1984).

The second acknowledgement we make is that the nature of injustice in our society results in school-age students facing "persistent and profound barriers to educational opportunity" (Darling-Hammond, 1995, p. 465). Social injustice means that children are denied opportunities to learn and grow. When we work toward social justice, we acknowledge these barriers and make a commitment to transform the educational fabric toward a more fair and inclusive educational setting. "Without acknowledgment that students experience very different educational realities, policies will continue to be based on the presumption that it is the students, not their schools or classroom circumstances, that are the sources of unequal educational attainment" (p. 465).

According to Bell (2007), "The goal of social justice is full and equal participation of all groups in a society that is mutually shaped to meet their needs" (p. 1). Teaching for social justice means that in addition to using "good teaching strategies," we use these strategies within a context of working toward social justice. When we talk about socially just teaching the focus is on pedagogical practices to help all students succeed.

In mathematics education, indentifying issues of social injustice with links to curriculum topics is not necessarily difficult for the informed professional. However, for the teacher in development, moving social justice perspectives into practice can be a challenge. In the content disciplines, mathematics among them, content traditionally takes center stage and becomes the focus of pedagogy and the unit of analysis in assessing student learning. Thus, for teachers, attention to social justice is often interpreted as a shift away from the content, an uncomfortable position when one is accountable for student mathematics learning. These challenges for teachers to identify, explore and understand social injustices, to apply the practices of socially just teaching, and to see themselves as agents of change to address injustices in their world(s) was an influence in the development and implementation of the coaching program.

Methods

Participants, Context, and Data Sources

The participants in our study were twenty mathematics coaches who participated in the study across three years of their involvement in a MSP-funded mathematics coaching training program. All are licensed or certified teachers who are hired by school districts to serve as full time mathematics coaches, one per building, for the duration of the coaching training. Schools enrolled in the coaching program may be primary, elementary, intermediate or middle schools, and the project supports coaches across all of grades kindergarten through grade eight. Not all coaches continued in the program for all three years of available training, with funding issues being the primary reason schools dropped out of the program.

During the three years of the study, in addition to the mathematics education sessions in the coaching program, the coaches also engaged in nine professional development sessions focused on equity, diversity and social justice. The conceptual framework for the project includes pedagogical elements, mathematics content elements, and contextual elements; it is in the contextual elements that the equity, diversity and social justice component of the project rests. The goal of the social justice curriculum in the project is to teach the coaches about social justice, and to motivate them to bring social justice perspectives to bear on their daily work with teachers.

It is important to note three particular factors of this project in terms their impact in on the reliability of our data. Throughout the research project, the coaches were open and honest in revealing the limits of their knowledge of social justice in mathematics education. Given the potentially sensitive nature of the discussions, discomforts of revelations of non-awareness and acknowledgements of racism, we might have questioned the degree of honesty. But we trust the honesty and credit their openness to three factors:

- The nature of the professionals in the coaching program,
- The nature of the program as a whole, and
- The style of pedagogy practiced in the social justice lessons.

The professionals who participate in the coaching program are teachers who have chosen to enter a new field in mathematics education in its early stages of development. Taking a risk in leaving their classroom positions to embark on this professional growth opportunity suggests a strength of mind that distinguishes them from many of their peers. Additionally, the coaching program in which they are being trained is an intensive internship model where the coaches are coaching from the first day of their training. Participation includes a three-year expectation of training, involves whole group meetings for two days monthly, and small group meetings for two additional days monthly. The coaches come to know each other well, and bond to each other early in the program over the challenges of the work and the learning curve they experience. Finally, the social justice pedagogy responded at all times to the participants' needs and emerging growth, never judging, always understanding of the lack of awareness, the fears, and the resistances. Together these three factors made for group dynamics that resulted in trustworthy data.

Curriculum	Work products	Feedback/Survey Tools
Introduction to equity and diversity		Equity and diversity in the schools
Introduction to Social Justice	What does social justice mean to you? Documentation of group discussion.	Pre-SJ: Does Social Justice belong in the coaching program conceptual framework?
Video: The House We Life In	Reflective Writing on The House We live In.	Post-SJ: Does Social Justice belong in the coaching program conceptual framework?
We are the World and We Are Hungry lesson.	We Are the World And We Are Hungry activity sheet.	Final year descriptive narrative on growth in terms of growth regarding social justice in mathematics education.
Video: The Color of fear		
Color of Fear Follow-up Discussion	Assistant's notes from Color of Fear discussion	

Table 1. Data Sources

Our data sources included open response survey instruments administered prior to the project, part way through the project, and after the last lesson. Coach work products from sessions and researcher reflections on curriculum and teaching served as data sources as well. One particular discussion session was documented with verbatim notes because there was no other product from the session to capture the coaches' understandings. See Table 1. for all data sources. In this discussion of data sources, in order to distinguish this work from the work of the rest of the coaching program, we use the word project to refer to the social justice in mathematics research part of the training program, and program to represent the coaching training overall.

Near the end of year one in the program, coaches participated in a lesson introducing equity and diversity in mathematics education and were asked at that time for their feedback on equity and diversity efforts in their schools. Once the group started in year two of the project, they began a six-session series intended to teach them about social justice, and to motivate them to integrate social justice into their work with teachers and students in their schools. In year 3 of the project, the coaches participated in two follow-up lessons addressing their on own struggles and reflections upon their growth.

Data Analysis

Our data analysis focused on document analysis in reviews of the curriculum, coach responses on multiple surveys, observation captured in a transcript of a discussion, coach work products, and researcher field notes that included our reflections, notes from the sessions, and planning discussions. We should note here how every session included from four-six observers from the program, providing a kind of reliability check on our interpretations of the events. Follow-up discussion with these observers became a part of our researcher reflections.

To begin analysis we progressed through all documents that were coach work products, survey responses, and transcripts and researcher notes. We reviewed each coach's documents chronologically through all lessons. We compared coach perspectives, language, and quality of work products progressively coach-by-coach, looking for indications of the coaches' understanding of and commitment to social justice pedagogy. As is customary in qualitative research, successive readings across the data allowed the coaches' growth patterns to emerge.

Results

Our analysis allowed us to identify the initial positions of the twenty participants and six different progressions in the coaches' growth. We found that the coaches were clear about what they did not know from the first days of the project. Reflecting back to the methodological note about the reliability of the data, we believe coach revelations to be honest. As a whole, the group started with varying degrees of naiveté, some believing they were attentive to equity and diversity in their work, only to realize later how far they needed to come. However, there were three particular cases that were distinctive within the group. One coach, Marjani*, who we discuss later, was the most informed at the outset, and credited that to her lived experience of injustices. Another coach, Margaret, began the project as a disconnected spectator and was one of only two coaches from whom we saw no growth. A second from whom we saw no growth was Mitch, who also started with an articulation of a connection to mathematics that suggested a potential and particular barrier to accepting social justice pedagogy. Mitch wrote early in the project "Honestly, I don't know how it will impact me as a math coach. I teach all children who make up a classroom, so I am not consciously aware of social justice per se, while teaching mathematics or working with a fellow teachers." At the end of his experience with us, he wrote that the social justice work did not belong in the coaching program because it "took a lot of our time away from the [coaching] material." Nadine, whose growth is clear in the data we discuss below, started with the following perspective:

> I don't believe the mathematics classroom is the place to have this debate. The study of mathematics supersedes socio-political, cultural conditions. For me it explains and uncovers the wonders of the universe, the responses of humankind, the nature of mankind and the predictions of the future. It confirms the idea of intelligent design.

The overall results in terms of coach growth were encouraging to say the least, as all but two of the coaches demonstrated growth. That growth was individualized per coach, both in terms of the amount of growth and the nature of it. Through our analysis, we identified the following six growth progressions, where data reflected movement as follows:

- From mathematics to students and teachers to consideration of the context
- As a pivoting center (reference) from self to student to student within the context
- From spectator to participant
- From naiveté to deeper understanding
- From the self within a social justice context to expanding boundaries of social justice
- From validation to expansion into mathematics

A Shift in Paradigm

The first three examples of movement listed above collectively represent a shift in paradigm for each participant. For one coach, Jessie, the shift was one that moved her focus from mathematics to students and teachers and eventually to consideration of the context. She wrote,

> To be honest, social justice and mathematics education wasn't even on my radar when we first began this discussion… I was more concerned about treating all students with respect and trying to meet their academic and social needs… I now look at students and teachers and try to understand where they have come from and where they are now….Although I knew about social justice or thought about it in the past, I didn't think about how it plays a part in mathematics education.

Valarie's case is an example of paradigm shift representing a change in her pedagogical center from herself as a person and teacher to the student in context. "I always thought I was helping children be all that they can be, but it was in my eyes, not in each child's eyes….I am … more aware of children's backgrounds and what they bring socially and educationally… I believe I got here by our discussions and being able to see others' point of view."

Susan examples a shift from spectator to participant. She revealed her spectator status when she wrote "I have always thought that everyone regardless of background, race, economics etc. should have an equal opportunity to education. I never really thought (I know this now) how that was going to happen." She then wrote that she also did not know "how to make it make it happen. It just should" suggesting a shift to her role as a participant. Finally, she writes "Where I am now, there are so many things to learn about with regards to where people come from, background, economics, and to learn and know how to help provide this equal opportunity."

Deeper/Broader Understanding

The last three examples of movement listed above collectively represent movement to a deeper or broader understanding on the part of each participant. In her writing, Nadine characterizes a change from naiveté to deeper understanding, progressing from a starting with a naïve belief that she was aware to knowing how she has much to learn. She wrote:

> My definition for equity on the first day was 'equity means every child gets what he or she needs.' I was so proud thinking 'I really nailed <u>this</u>' and I was surprised when I shared my answer with the group that Cynthia didn't jump right up and say, 'Yes yes. That is a great answer.' Instead, she said 'Hmmmm' and without another word moved on to the next person… It was the movie about the group of men who came together to confront the issue of race that had the most significant impact on me. I was embarrassed for the Caucasian man and ashamed to see a little of myself in his naiveté about the disparity that exists even today between races, and how that disparity continues to live today in part because of ignorance that it exists at all. Today, at least I am aware that it exists to a greater degree than I fully understood and it is something I need to work on.

Rita's data revealed a case of broader understanding, moving from the self within a social justice context to expanded boundaries that now include a broader world of social justice understanding. She wrote,

> When we first began this discussion of social justice I considered myself to have already begun a self-reflective process prior to our start. Even knowing at that time I still had much growing to do, I had no idea how much growth that would entail… Discussions we have had … have overflowed to discussions with Nadine and Marjani outside of [the coaching program]… I appreciate the discussions greatly and feel I have gained a broader sense of humanity and equity, as my current beliefs are challenged by new information. I appreciate the fact that awareness of social justice has been heightened as I feel it has impacted my own perceptions of my self.

Rita's major growth was from a place of comfort in her definitions and the process. She became "much more analytical of actions, beliefs, etc., not only in the educational setting but in all areas of my life."

Finally, Marjani was a special case of a coach whose broadening and deepening change included an expansion into mathematics as a context. Additionally, she was the only coach who actually was comfortable taking her learning into her role of a coach. She found the social justice readings and pedagogy validating from the start.

As an African American woman teaching in a racially diverse urban school district, Marjani found the films, readings, and discussion validating. She wrote:

> The film was very informative and served to validate my experiences and the experiences of other people of color. I have been in conversation about racism and its effects for as long can remember. My career has been impacted by my choices to work with inner city youth and their instructors…Raising 3 African American young men with the help of my husband, and experiencing the difficulties of racism as it tried to hinder our success, makes me very sensitive to the issues… I thought and reflected on the film when I read chapter 3 of Eric Gutstein… As I work with my teachers I encourage them to reflect on their treatment/reaction to various students. I also try model respect and relationship-building with them.

Discussion

Each mathematics education professional development provider designs and delivers programs to meet the needs of their constituents. Those programs often include combinations of experiences with mathematics, viewing films, and engaging in activities, readings and discussions. Our project was no different in that regard. But the growth found in this project does suggest the value of three contextual elements. One contextual element was the on-going review of the data, and the revised curriculum that resulted from that. Significant to the changing process was the risk-taking that was necessary on the part of the project, to push discussions in very uncomfortable directions, especially in the context of mathematics, which many believed to be value free and socioculturally neutral.

A second contextual element contributing to the growth in this project was the intensive, prolonged, and community-like engagement of the coaches described earlier. That context provided us the opportunity to integrate our project over a longer period of time than we would have had in typical professional development projects or university coursework. With the growth we found taking all of the three years to become realized by the coaches, it is clear that summer and holiday breaks way from their work, time between social justice lessons, and a project that is sustained and coherent provided allowed room and time for reflection and growth in nearly all of our coaches.

The coaches had assignments to talk to someone outside of our sessions about some element of our work together. One group of coaches took that a little further, suggesting a third contextual element contributing to growth. Three coaches, Rita, Nadine and Marjani worked in the same urban school district, spent time together as colleagues outside of their schools, and became friends. On the long, monthly drives to and from the coaching program trainings, and in their additional sessions two more days each month, they talked to and challenged each other regarding social justice pedagogy. They became critical friends (Nieto, 2000; Zeichner and Hoeft, 1996), referencing in written reflections those drives and the friendship that developed over their growth around social justice pedagogy in mathematics.

We close with comment on one final growth element that relates to the social justice work to the coaches' role as professional development leaders in their schools. As the coaches entered the third year of the project, Marjani was the only one showing any evidence of taking her social justice learning into her role as a coach. Since that application of their learning was a goal in our teaching, and because we had so little time left in the project, we made an assignment to the group to push them to think more about that aspect of their coaching work. The coaches stopped us; they could not do what we were asking them to do, and clearly articulated the ways in which they were not ready for it, and what they needed to do before they could apply their understanding to their coaching work. They revealed self-regulatory and self-directed behaviors, an aspect of growth we had hoped for, had not seen, and had not planned for. They needed from us only the space to reflect upon, synthesize, and name what they found to be incredible personal growth. They believe that only then could they – and would they – take their growth into their work as coaches.

Endnotes

* All names used in this paper are pseudonyms.

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zKnowledge Foundations for Teaching Algebra for Equity

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This paper presents our initial work to develop trajectories of preservice teachers' knowledge of teaching algebra for equity. We have developed two overlapping hypothetical learning trajectories (HLTs) for teachers' learning to engage and motivate diverse students to solve algebra problems. These HLTs are being used to guide the development of assessments, interviews, and learning activities for a pilot group of preservice teachers. The HLTs also serve as learning materials to help the preservice teachers understand how their future middle grade students learn to solve algebra word problems.

This paper describes work in the first phase of an NSF funded project whose goal is to enhance preservice teachers' knowledge for teaching algebra to diverse students. Specifically, the project focus is to design, develop, and test technology-enriched teacher preparation strategies to address equity in algebra learning for all students. The foundation for this work is a set of Hypothetical Learning Trajectories (HLTs) for teaching algebra for equity, which will be described in the paper.

Moses and Cobb (2002) stated that "the most urgent social issue affecting poor people and people of color is economic access. In today's world, economic access and full citizenship depend crucially on math and science literacy" (p. 5). Ladson-Billings (2009), citing Haberman (1996), notes that "a serious effort toward preparing teachers to teach in a culturally relevant manner requires a rethinking of the teacher preparation process" (p. 143). We agree with Moses and Ladson-Billings and we concur with researchers who conclude that traditional teaching methods are one source of difficulties that students of color have in school-based learning. Weiner (2005) found that success in the classroom only came after a shift in teachers' attitudes about teaching, learning, and culture. Research (Irvine, 1990; Lewis, 2009) has found that White teachers are sometimes not sensitive to the cultural needs of African-American students in the classroom. Both verbal and nonverbal interactions between the student and teacher are often misinterpreted and can lead to negative consequences for the African-American student (Lewis, 2009).

The gap in mathematics achievement between White, Hispanic, and African-American students has been documented since 1973. Data from the past 30 years show that White students have consistently outperformed students of color for each of the testing points (Cooper & Schleser, 2006). Although there has been recent improvement among Hispanic and African-American students, the achievement gap is still an issue (The Nation's Report Card, 2007). Furthermore, several "national and international comparisons of student achievement indicate that it is between fourth and eighth grade when U.S. students in general, and minority students in particular, fall rapidly behind desired levels of achievement" (Balfanz & Byrnes, 2006, p. 144). There are several probable causes for the continuation of the achievement gap in mathematics, including, “weak and unfocused curricula, shortage of skilled, trained, and knowledgeable mathematics teachers, unequal opportunities to learn challenging mathematics, and unmotivated students” (p. 144).

The importance of teacher quality has been identified as a key element to closing the mathematics achievement gap [now known as the opportunity gap] and increasing achievement of all students (Ladson-Billings & Tate, 2006, The Education Alliance, 2006). Teachers who are highly qualified and certified to teach mathematics have stronger pedagogical and mathematics knowledge, and therefore are more likely to have a better understanding of how students best learn mathematics. Highly qualified teachers will have better instructional practices than less qualified teachers (Darling-Hammond & Sykes, 2003). Teachers who are highly qualified and who have the ability and knowledge to effectively teach mathematics can “produce as much as six times the learning gains produced by less-effective teachers” (Singham, 2003, p. 589). Preservice education is the starting point for developing high quality teachers for all students. In a review of research on preparing teachers for diversity, Sleeter (2001) noted that while many White preservice students expect to work with children of another cultural background, most have little knowledge of or experience in cross-cultural settings. Preservice teachers acknowledge family support as a factor in their own previous success in mathematics education, but also hold the beliefs that mathematics ability is inherited from the parents, and that socio-cultural factors can restrict or promote learning (de Freitas, 2008). Sleeter stated that "the great bulk of the research has examined how to help young White preservice students (mainly women) develop the awareness, insights, and skills for effective teaching in multicultural contexts. Reading the research, one gains a sense of the immense struggle that involves" (p. 101). She concludes that many of the programs and courses aimed at general multicultural awareness and knowledge have had limited success, partly due to the limited opportunities for direct and relevant experience that preservice teachers have in confronting real issues of diversity in the classroom.

Teachers in Ladson-Billings’ (2009) Dreamkeepers project offered these suggestions for teacher preparation programs: recruit teacher candidates who have expressed an interest and a desire to work with African-American students; provide educational experiences that help teachers understand the central role of culture; provide teacher candidates with opportunities to critique the system in ways that will help them choose a role as either an agent of change or defender of the status quo; systematically require teacher candidates to have prolonged immersion in African-American culture; provide opportunities for observation of culturally relevant teaching; and conduct student teaching over a longer period of time and in a more controlled environment (Ladson-Billings, 2009).

In designing the HLTs, we reviewed the research literature to examine effective strategies for equity both for preservice teachers and for middle grades learners. Cognitive, affective, and cultural factors were included as primary criteria for identifying strategies that might be effective. An important underlying goal is to explore and develop activities and strategies that are effective both for preservice teachers and the students they will eventually teach in the classroom.

Teaching and Learning Algebra

Specific learning models for algebraic thinking are being identified and used to guide the development and design of HLTs. We began by identifying research-based learning trajectories for specific emergent algebraic thinking at the middle grades (Chazan & Yerushalmy, 2002).

Although there is considerable research on teaching algebra, the nature of preservice teachers' learning trajectories for developing deep algebraic thinking is less well known. Our research focuses on identifying, describing, and documenting these trajectories so that knowledge of algebra, often found to be deficient in preservice teachers (You & Kulm, 2008), can be addressed during preservice work. Learning activities will be constructed around the strategies for providing access and encouraging student engagement in challenging mathematical problems.

There are many studies on teachers' knowledge for teaching function concept. For example, Norman (1992) found that the secondary teachers tended to have inflexible images of the concept of function that restricted their abilities to identify functions in unusual contexts and to shift among representations of functions. The sampled teachers were able to give formal definitions of function, were able to distinguish functions from relations, and were able to correctly identify whether or not a given situation was functional but did not show strong connections between their informal notions of function and formal definitions. Consistent with Norman's findings, Chinnappan and Thomas (2001) found that all four pre-service secondary teachers in their study had a preference for thinking about function graphically, weak understanding of representational connections, and limited ability to describe applications of functions. Difficulty in constructing functions was also observed by Hitt (1994), who found that teachers had difficulty in constructing functions that were not continuous or were defined by different algebraic rules on different parts of the domain. Even (1993) found that many prospective secondary teachers did not hold a modern conception of a function as a univalent correspondence between two sets. These teachers tended to believe that functions are always represented by equations and that their graphs are well-behaved. None of the teachers had a reasonable explanation of the need for functions to be univalent and over-emphasized the procedure of the "vertical line test" without concern for understanding. On the other hand, Chazan, Yerushalmy, and Leikin (2008) found that teachers who were provided an opportunity to discuss and use alternative approaches were able to rethink their ideas about functions and equations.

Description and Specifications of Learning Trajectories.

The first phase of the project focuses on two interrelated research questions: What are the trajectories of middle grade students' knowledge and skill in algebra? and What are the trajectories of preservice teachers' knowledge of teaching algebra for equity? We believe that the basis of preservice teachers' knowledge for teaching algebra for equity rests on their knowledge of students' learning and motivation. We have developed two separate but overlapping hypothetical learning trajectories (HLTs) for solving algebra problems. One HLT characterizes the learning development of middle grades students. The second HLT characterizes teachers' use of strategies to engage and motivate diverse students in learning algebra. The trajectories follow the model of Simon and Tzur (2004), who provided the following set of assumptions about the characteristics and use of a hypothetical learning trajectory (HLT):

1. Generation of an HLT is based on understanding of the current knowledge of the students involved.
2. An HLT is a vehicle for planning learning of particular mathematical concepts.
3. Mathematical tasks provide tools for promoting learning of particular mathematical tasks and are, therefore, a key part of the instructional process.
4. Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of the HLT (p. 93).

The HLTs facilitate building on existing knowledge and developing deeper knowledge of the topics. In order to describe how this knowledge is built, we began with the model of Lamberg and Middleton (2009) in constructing HLTs. This model contains (a) descriptions of the conceptual scheme at each level of learning, (b) summaries of the cause/effect mechanisms that characterize students' current knowledge, (c) cognitive interpretations of current knowledge, including possible misconceptions, and (d) intermediary understandings that are necessary for bridging to the next level of the learning trajectory (p. 237). Figure 1 provides a first version of the HLT for solving algebra problems. The trajectory, based on findings from research and best practice, begins with a direct translation scheme and provides rationales and cognitive interpretations of reasons for students using this approach. The last column provides possible steps to bridge to a more sophisticated scheme for solving problems. The intermediary schemes represent increasingly sophisticated problem solving strategies. The highest level scheme in this hypothetical model is a heuristic approach based on Polya's (1945) work and subsequent research that has applied that scheme.

Conceptual Schemes	**Cause/Effect**	**Cognitive Interpretation**	**Bridging Steps**
Direct Translation Scheme: Translate directly from words into expressions or equations.	Students have been taught that certain words stand for operations or equivalence. (Koedinger & Nathan, 2004; National Council of Teachers of Mathematics, 2000). Textbook word problems often to support this strategy.	Students who can read and comprehend text may be successful using this strategy for some word problems. (Chapman, 2006) However, errors in translation are common, and many problems do not lend themselves to the approach. (Pape, 2004)	Practice using heuristics to understand problems, such as draw picture or diagram, make a table, identify the data and unknowns (Johanning, 2004, 2007). Delay writing equations, then write the equations in words.
Textbook Four-Step Scheme: Read, identify data, identify unknowns, write the equation.	In an attempt to reflect Polya's four step approach Polya (1945), yet keep a simple procedure, textbooks often use this strategy with simple word problems that provide reasonable success in applying this approach.	Students can use the strategy with simple word problems, as an aid in analyzing the data and assigning variables to the unknown(s) (Johanning, 2004, 2007). The approach does little to support finding relationships needed to write an equation (Greer 1993, Johanning, 2004).	Practice using heuristics to understand problems, such as draw picture or diagram, think of a similar problem, use numbers to find examples.
Generalized Pattern Scheme: Generate numbers and generalize the pattern.	In early grades, students learn to "plug in" numbers to verify statements. Since numbers are concrete and familiar, they can be used to generate data that satisfy the conditions in a word problem.	Students are familiar with numbers and can use them to satisfy problem conditions. They can use inductive reasoning to generalize familiar patterns or arithmetic sequences. (Zazkis & Liljedahl, 2002)	Practice using heuristics to understand problems, such as draw picture or diagram. Use differences to identify the type of pattern (linear, etc.)
Heuristic Scheme: Apply heuristics to understand, plan, carry out, look back (Polya)	An heuristic approach addresses the need to focus on understanding the relationships and conditions in a problem and the need to plan how to solve it before attempting to write equations (Johanning, 2007, Polya, 1945).	Students can begin to use heuristics for identifying the data and unknown, drawing diagrams, and tables. (Bednarz & Janvier, 1996; Kieran, Boileau, & Garancon, 1996). Then can use subproblems, special cases and develop strategies such as remembering similar problems (Kieran, 1996).	Reflect on the solution to find another one, or a better or generalized solution. Think about other similar problems previously solved.

Figure 1. HLT for writing symbolic representations for verbal or non-routine problems.

The HLT presented in Figure 2 addresses the issue of equity through the use of conceptual schemes that have potential to engage diverse students in learning algebra and problem solving. A central focus of our current research is to collect and analyze data in order to further understand and revise the proposed schemes.

Conceptual Schemes	**Cause/Effect**	**Cognitive Interpretation**	**Bridging Steps**
Situated learning scheme: Provide an instructional context that allows students to have concrete and hands-on experiences with math knowledge and skills. Build math learning on realistic, open-ended, culturally relevant problems that students solve using a variety of skills, concepts, and tools.	Traditional math instruction focuses on symbolic representations exclusively, or moves too quickly from concrete to abstract lessons. This kind of transition results in poor skill development and limited conceptual understanding. (Hollar & Norwood, 1999; Karsenty, 2002; O'Callaghan, 1998)	Students develop math understanding by constructing their own culturally relevant knowledge (Ladson-Billings, 1994), building from more concrete to abstract ideas. Activities that apply math in contexts support effective learning (Pellegrino, Chudowsky, & Glaser, 2001). Student engagement in planning and carrying out the activity builds "ownership" and understanding (Silva & Moses, 1990).	Use concrete materials such as balances, algebra tiles, and everyday objects to provide concrete (Their, 2001) and hands-on experiences, before introducing formal symbols, definitions and rules. Problem solving activities can be used that gradually are more open-ended and provide opportunities for students to devise solution strategies.
Culturally relevant context scheme: Use contexts for activities that are based in and relevant to students' cultures and lives.	Many students do not see the relevance of math and have low self-expectations for learning math. Providing interesting and relevant contexts can be motivational. (Ladson-Billings, 1997)	Learning does not take place unless students are engaged in the lesson (Their, 2001). A "zone of proximal development" is necessary in which students can learn which enhances motivation for learning math.	Adapt math activities and problems that have relevant contexts, individualized to the interests of a particular class, group, or individual student (Ladson-Billings, 1995).
Critical pedagogy scheme: Provide learning activities in which students investigate the sources of mathematical knowledge, identify social problems and plausible solutions, and react to social injustices.	Many students do not see the social relevance of math. Providing interesting and relevant social contexts can be motivational. (Ladson-Billings, 1995)	Problem-based learning engages students in using math to address and solve problems that are drawn directly from possible social or community issues. The context can motivate and engage students (Boaler, 2000).	Adapt math activities and problems that have social contexts, individualized to the interests of a particular ethnic or interest group, or individual student (McLaughlin, Shepard, & O'Day, 1995; Stinson, 2004).

Figure 2. HLT for engagement in meaningful, interesting, and culturally relevant algebra problems.

These HLTs are being used by the project in two ways. They serve as frameworks to guide the development of assessments, interviews, and learning activities for a pilot group of preservice teachers. These data will inform the revision and further development of the HLTs and the learning activities aimed at developing preservice teacher knowledge of teaching algebra for equity (KATE). The HLTs also serve as learning materials, along with problem solving activities, to help the preservice teachers understand how their future middle grades students learn to solve algebra word problems. The teachers develop activities and lessons that address the bridging steps necessary to progress in the trajectories. Finally, they present lessons in Second Life to a simulated class of diverse middle grades students.

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EXPLORING PRESERVICE TEACHERS' BELIEFS ABOUT THE MATHEMATICS EDUCATION OF ENGLISH LANGUAGE LEARNERS

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This study explores preservice teachers' beliefs about the mathematics education of English language learners. In all, 164 preservice teachers responded to an online survey designed by the researchers. Through the use of descriptive statistics, the data shows that preservice teachers are open to the use of native language; believed that mathematics was devoid of culture; and were more empathetic to the experiences of English language learners if they themselves had experienced trying to learn a second language.

This pilot study sought to expand on research that examined teachers' beliefs regarding the instruction of English Language Learners (ELLs[1]) (e.g. Reeves, 2006; Torok & Aguilar, 2000). Our goal was to understand preservice teachers' (PSTs) beliefs about the mathematics education of ELLs; specifically we considered the following research question: What are elementary preservice teachers' beliefs toward (a) the use of language in the school context; (b) the interconnection of language and mathematics; and (c) teaching mathematics to ELLs?

With the increased number of ELLs that are currently mainstreamed into classrooms (Costa, McPhail, Smith, & Brisk, 2005), along with the projected 40% of school students that will speak a language other than English at home by 2030 (AACTE, 2002, as cited in Lucas & Grinberg, 2008), it is vital to prepare teachers to work with this specific population. Within mathematics education, this need is magnified when one considers how ELLs have fared on national high stakes tests. Historically a large percentage of ELLs have performed below the basic[2] achievement level on the National Assessment of Educational Progress (NAEP) mathematics exam. This trend continued in the latest iteration of the assessment. In 2009, 72% of the eighth-grade ELLs tested were below the basic level compared to just 25% for non-ELLs. At the twelfth-grade, the numbers were 80% and 35% respectively, and at the fourth-grade the numbers were 43% and 16% respectively.

While these percentages indicate that the assessments should be analyzed with a focus on accessibility for ELLs (Kieffer, Lesaux, Rivera, & Francis, 2009), they also support the argument that teachers at all levels need to understand and be willing to incorporate into their teaching practice specific methods of helping ELLs learn mathematics deeply. The National Center for Educational Statistics (2002) reported though that of the 41% of teachers working with ELLs in their classroom, only 13% received adequate preparation. While there are many avenues in which to strengthen this preparation, one important component is for teacher preparation programs to consider the beliefs that PSTs have regarding the mathematics education of ELLs. As Philipp (2007) defined, beliefs can be thought of as "lenses that affect one's view of some aspect of the world or as dispositions toward action" (p. 259). Thus, how a PST views the mathematics education of ELLs will affect how they design and implement mathematical experiences for ELLs in the classroom. By understanding what PSTs believe, teacher preparation programs can design meaningful experiences that will promote a critical analysis of their beliefs.

Theoretical Perspective

We used an on-line survey to ascertain PSTs' beliefs about the mathematics education of ELLs. While admittedly there are downsides to using surveys (as discussed in Ambrose, Clement, Philipp, & Chauvot, 2004), our reasoning for the use of a survey was two-fold. First, we were concerned that PSTs might answer in a manner they thought was expected of them if we used an interview setting. As Sapsford (1999) noted, "a straightforward question can all too easily evoke a rhetorical or ideological response" (p. 106). We hypothesized that through an anonymous, on-line survey, PSTs would be more honest about their beliefs. Second, a more qualitative approach (such as interviewing or observations) would have been more time-consuming. As such, fewer PSTs would have been able to participate in the study. This was an important consideration since we were interested in understanding the beliefs of a population of PSTs and not just a select few.

The survey instrument was constructed by the researchers due to the lack of appropriate instruments found. At the beginning of another study (Fernandes, Anhalt, & Civil, 2009), which sought to understand how PSTs' beliefs were challenged as a result of conducting two task-based interviews with ELLs and reflecting on the interviews in a structured report, the second author could not find any surveys that would be appropriate to use in the context of mathematics. While survey instruments that focused on teachers' beliefs about working with ELLs or teaching diverse students were found (e.g. Reeves, 2006; Tatto, 1996; Torok & Aguilar, 2000), none focused specifically on the beliefs of PSTs regarding the mathematics education of ELLs. This lack of direct focus was a point of contention for us as the context of mathematics was just as important as the context of working with ELLs. As Cooney, Shealy, and Arvold (1998) noted, beliefs that are central in one domain are not necessarily central in another. Thus we conjectured that framing the statements in the context of the mathematics classroom would encourage the PSTs to consider the nuances of working with ELLs in the domain of mathematics as opposed to in general.

The survey consisted of 14 questions. Each of the statements measured the strength of agreement or disagreement of PSTs' beliefs related to the mathematics education of ELLs with a 5-point Likert-type scale. Each respondent was asked to choose from one of the following choices: Strongly Disagree; Disagree; Undecided; Agree; Strongly Agree. Each of the statements used in the survey was based on research and/or adaptations from items found in other surveys[3]. Respondents also answered questions relevant to their demographic information such as gender, years of teaching experience, years of experience working with ELLs, spoken languages, and ethnicity.

Methods

We piloted the survey with elementary PSTs from a university in the south east of the country in spring 2010. All of the PSTs were working toward their teaching certification through an accredited teaching certification program and were enrolled in one of the two mathematics methods courses offered that semester. All students enrolled in both methods courses were invited to participate in the study. The second author discussed the study with the two methods teachers and asked each of them to announce the survey to their students. The PSTs were provided a web link that directed them to the survey that was hosted on http://surveyshare.com. At the beginning of the survey, individuals were asked to read a consent form detailing the specifics of the study. If the individual indicated consent, they were then provided access to the survey questions. Participants were not asked any identifying information such as name or address. Overall, 164 PSTs participated, a return rate of almost 100%.

To ensure the validity of the survey, the participants were asked at the end of the survey to answer questions pertaining to the readability and clarity of the survey. In particular, participants were asked the following questions:

1. Were there any ambiguous questions?
2. Was there anything that you did not understand?
3. Was there anything that we did not ask that we should have asked?

Comments from the PSTs about what they did not understand indicated that three were unsure about questions related to their knowledge of teaching and three wanted an option of student teaching/clinical experience to be included in the "Years of experience teaching ELL students." Related to the former, one respondent commented, "I'm not in the classroom full time, so how am I supposed to know about teaching [ELLs]?" This type of comment was echoed in answers to question 1. Here respondents discussed less the ambiguity of the wording of the questions and more that they were unsure of how to answer the questions based on their experience. All of the PSTs noted that in these cases, they clicked the choice 'undecided.' These responses from the PSTs did not reveal any obstacles to their understanding or ability to answer any of the survey items.

The survey results were imported into an MS Excel file that was used to do the analysis. We calculated the percentage of PSTs who agreed with an item by adding the percentage of those who either strongly agreed or agreed with a statement. Similarly, we calculated the percentage of PSTs who disagreed with an item. These percentages are shown in Table 1 for each item. To further understand the alignment of the PSTs' beliefs with the literature, we obtained a total score for each PST by attaching a numerical value for a response in the Likert-type scale: 1 for strongly disagree; 2 for disagree; 3 for undecided; 4 for agree; and 5 for strongly agree, while reverse coding some items if needed. For us, a score of 1 represented a response that was least aligned with the research literature regarding ELLs and 5 was the most aligned. For example, the statement "Students in the US should be taught in English only" was reverse coded with a PST strongly agreeing scoring 1 and strongly disagreeing scoring 5. Since beliefs can only be inferred, we conjectured that a PST who believed that all students should be taught in English only would be less open to the use of native language as a resource in the classroom. Items 1, 2, 5, 6, 7, 8, and 12 from Table 1 were reverse coded. In this survey, a PST could have obtained a minimum of 14 and a maximum of 70. Our next step was to order the scores and use quartiles to partition the 164 students into four groups, where G1 represented the lowest quartile and G4 represented the highest. Finally we compared the means of the scores from the PSTs in the upper quartile (G1) to those in the lower quartile (G4)

Results

We examined the results of the survey descriptively as we sought to answer our research questions and interpret the results in light of the existing literature. First we report on the demographic information collected about the participants to contextualize the population who responded to the survey. We then report on the percentages of participants that agreed, were undecided or disagreed with the individual statements by considering three main categories of belief statements related to our research questions: (a) beliefs about language in the school context; (b) beliefs about the interconnection of language and mathematics; and (c) beliefs about teaching mathematics to ELLs. These percentages are summarized in Table 1. Finally we report the findings from the comparison of different groups based on the quartile calculations.

Demographic Information

Of the 164 participants, 153 were female and 11 were male. Of the choices 'European American,' 'African American,' 'Hispanic,' 'Asian,' or 'Other,' 85% of the PSTs reported being European American/Caucasian/White. (Note that some PSTs chose the 'other' category and filled in Caucasian or White.) Of the other 15% of respondents, 7.32% self-reported as African American, 1.22% as Hispanic and 3.05% as Asian. Most of the PSTs either indicated no formal teaching experience or reported classroom observations and clinical teaching required in coursework as their only teaching experience. Finally, over 85% had limited to no experience to working with ELLs prior to the study.

Belief	% Agree	% Undecided	% Disagree
Language in the School Context			
1. Students in the US should be taught in English only.	21.37	13.4	65.2
2. It is more important for immigrants to learn English than to maintain their native language.	41.4	19.5	39
3. In a mathematics classroom, ELLs should be allowed to discuss course material with each other in their native language.	76.83	12.8	10.37
4. ELLs should be assessed in their native language on the State tests.	35.98	37.80	26.22
Interconnection of Language and Mathematics			
5. Mathematics uses symbols and ideas that are not associated with any language or culture.	52.44	23.17	24.39
6. Mathematics is an ideal subject for transitioning recent immigrant students into English.	55.49	33.54	10.98
7. If ELLs can hold a conversation in English, they should have no more of a problem, than non-ELLs, learning mathematics in an English only classroom.	29.88	21.34	48.78
8. Mathematics is not language intensive.	25	15.85	59.15
Teaching Mathematics to ELLs			
9. Whenever possible, ELLs should receive instruction in their native language until they are proficient enough to learn via English instruction.	67.07	15.24	17.68
10. When working with ELLs in a mainstream classroom, teacher lesson plans should incorporate content objects as well as language objectives.	69.52	33.54	5.49
11. Teacher should incorporate explicit language learning strategies in the mathematics lesson if they have ELLs in the class.	66.47	25	8.54
12. Teachers should use the same standards in evaluating the work of all students in the class.	48.18	18.29	33.54
13. It is important to incorporate aspects of the students' culture into mathematics teaching.	84.15	9.15	6.71
14. Culture plays a significant role in the learning and teaching of mathematics.	56.1	28.05	15.85

Table 1. PSTs' beliefs about the mathematics education of ELLs

Beliefs about Language in the School Context

A majority of the PSTs (65.2%) disagreed or strongly disagreed with the statement "Students in the US should be taught in English only." Moreover, 76.83% agreed or strongly agreed with allowing ELLs to use their native language when discussing mathematics content in the classroom. Given that most of the PSTs surveyed were monolingual in English (83%[4]), these results point to the PSTs being open to languages other than English being used in the mathematics classroom. However, even though PSTs seemed open to the use of native language for teaching and small group discussions in the class, they seem split on the importance ELLs should give towards learning English rather than maintaining their native language. Namely, 41.4% of the respondents agreed that it is more important for ELLs to learn English while 39% felt that learning English should not come at the expense of the ELL's native language. This agrees with the literature (e.g. Walker, Shafer, & Iiams, 2004; Reeves, 2006) where teachers may assume that the use of native language may interfere with ELLs learning English and thus, more importance should be given to learning English. Finally, the PSTs were about evenly divided when asked about the use of native language on exams, with 35.98% agreeing, 26.22% disagreeing, and 37.8% undecided when responding to the statement "ELLs should be assessed in their native language on the State tests." There are a number of possibilities for the large percentage of students who are undecided. For example, it is possible that PSTs had difficulty deciding between whether it was a beginning ELL or a more advanced ELL that was allowed to take the tests in their native language.

Beliefs about the Interconnection of Language and Mathematics

A little more than half (52.44%) of the PSTs agreed with the statement, "Mathematics uses symbols and ideas that are not associated with any language or culture," with 23.17% being undecided and 24.39% disagreeing. This is in line with beliefs about mathematics being a universal language that is structured and the same for everyone regardless of country or region (Kloosterman, 1999). Indeed, both authors have regularly had informal interactions with PSTs who claim that "2+2=4" is true in any language or culture. This view of universality may account for why a majority of PSTs (55.49%) agreed that mathematics would be ideal to transition recent immigrant students into English. About a third (33.54%) of the PSTs, however, were undecided. It is possible that the PSTs that comprised this percentage see mathematics as inherently difficult and challenging and thus felt that mathematics would not be ideal for any student, regardless of language ability. When asked to consider if ELLs who could hold a conversation in English would have no more of a problem that non-ELLs learning mathematics in an English only classroom, a little less than half of the PSTS (48.78%) disagreed. Moreover, almost 60% of the PSTs disagreed with the statement "Mathematics is not language intensive."

Beliefs about Teaching Mathematics to ELLs

More than two-thirds of the PSTs agreed that ELLs should receive instruction in their native language until they are proficient enough to learn via English. This view is in-line with research (e.g. Moschkovich, 2000) that suggests that educators should draw upon resources ELLs bring with them, such as their native language. Just about or above two-thirds of the PSTs also agreed that teachers should focus on language when working with ELLs be it through focusing on objectives related to language or enacting explicit language learning strategies. Of the PSTs that did not agree with these ideas, a majority (85.9% for objectives and 74.5% for enacting) were undecided suggesting perhaps that the PSTs might be unsure of how or why a mathematics teacher should focus on language. Less than half of the PSTs (48.18%) believed that teachers should use the same standards in evaluating the work of all students, with about a third disagreeing and 18.29% undecided. This distribution points to the fact that while PSTs want to be fair in assessing what their students know and can do, a majority at least question how that fairness should play out in the classroom. In regards to how culture should impact the teaching of mathematics, a large number (84.15%) of PSTs felt that it is important to incorporate aspects of the students' culture into mathematics teaching. While we cannot know for certain from this statement alone, it seems that PSTs did not consider the depth to which a student's culture should be incorporated. Instead, they may have felt that, for example, using a context with which an ELL may be familiar in a story problem constitutes incorporating a student's culture. Indeed, when asked if culture should play a *significant* role in the learning and teaching of mathematics, the percentage of respondents who agreed dropped to 56.1, a 33% decrease.

Comparison of Groups

The total scores of the PSTs were divided into quartiles, with Min=33, Quartile 1= 51, Median=55, Quartile 3=58, and Max=62. A box-plot of the scores is shown in Figure 1. Comparing PSTs in G1 (or the scores in upper quartile) to PSTs in G4 (or the scores from the lower quartile), we found the biggest differences in the means of items 1 (G1=3.93, G4=2.71), 7 (G1=3.66, G4=2.49) and 2 (G1=3.32, G4=2.27). This indicates that PSTs in G4 tended to put more emphasis on English only education and were also more likely to mistake conversational proficiency for academic language fluency than PSTs in G1. Items 6 (G1=2.63, G4=2.37), 9 (G1=3.78, G4=3.22) and 5 (G1=2.95, G4=2.32) had the smallest differences of means between G1 and G4. The means in item 6 and 5 were closer to 1 indicating that the PSTs may have felt that mathematics is a difficult subject for all students, independent of language and culture. Item 9 showed that PSTs from G1 and G4 believed that ELLs should receive mathematics instruction in their native language until they are proficient enough to learn via English instruction. Given the beliefs of PSTs from G1 towards language (as seen in items 1, 7 and 2), it is possible that these PSTs view the teaching of ELLs as the exclusive responsibility of the ESL teachers in the school and may be less open to making accommodations in their own classrooms. Demographically, PSTs from G1 and G4 were similar in terms of gender, ethnicity, and previous experience with ELLs, however, the two groups differed in their exposure to a second language. There were 18 PSTs in G1 that spoke English only, compared to 28 in G4; and 23 that spoke an additional language at varying levels of fluency in G1, compared to 13 in G4. Considering this difference, the alignment of the PSTs' beliefs seen in items 1, 7, and 2 to the research could be attributed to their experience in learning a second language.

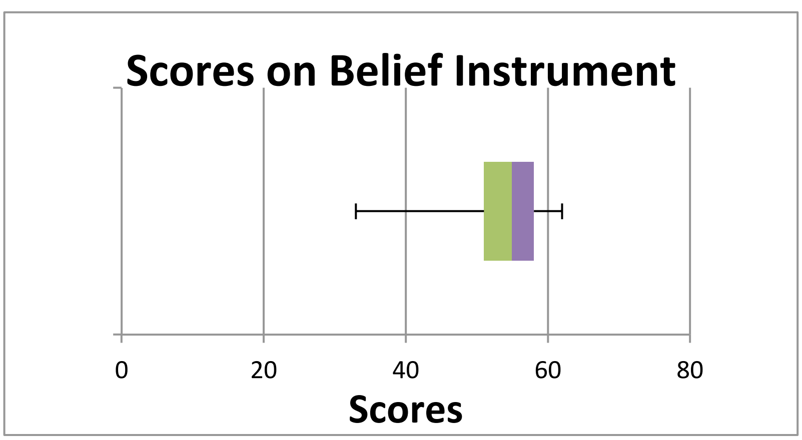

Figure 1. PSTs' overall scores on belief survey

Discussion

The calculation of percentages of PSTs who agreed or disagreed with a particular statement revealed some interesting connections among the PSTs' beliefs. One in particular concerns how PSTs may see culture within mathematics and mathematics teaching. An overwhelming 84% of PSTs agreed that a student's culture should be incorporated into mathematics teaching, with about 56% stating that culture should play a significant role in the teaching of mathematics. However, about 75% of the PSTs either agreed or were undecided that the symbols and ideas used in mathematics are not associated with any language or culture. These numbers indicate a possible disconnect for PSTs in the teaching of mathematics. One strategy for teaching ELLs suggested from research (e.g. Short & Echevarria, 2004) is to contextualize mathematics through the use of realia and the culture of ELLs in meaningful ways. The possible belief structure of the PSTs based on the percentage calculations suggests that even though PSTs may agree with the strategy suggested in the research, they may not understand how it is feasible to do so in the area of mathematics given that mathematics itself is devoid of context. If this is indeed the case, it sheds light on why teachers (both at the preservice and inservice levels) incorporate culture sometimes in seemingly superficial ways (e.g. changing the names of a story problem).

Finally, as seen in the comparison of groups, the PSTs who had beliefs that fell in the upper quartile (meaning that these PSTs had beliefs which were most consistent with findings from research) reported more experiences learning a language other than English than those PSTs who fell into the lowest quartile. It seems then that by experiencing the learning of another language, PSTs may have empathy for the experience that ELLs go through on a daily basis. However, this empathy may not translate into their teaching practice, as it is possible that they do not see it as their role as a classroom teacher to provide ELLs the type of support called for in research. If this empathy is cultivated, challenged, and nurtured though, PSTs that display the beliefs seen in this survey may develop the habit of mind and teaching practices that are consistent with best practices for teaching ELLs.

Closing Thoughts

While we acknowledge that beliefs are not static and in fact are affected by many factors at any given point in time, it is important that teacher preparation programs take into account what a PST is thinking regarding the mathematics education of ELLs. As can be seen through the findings and discussion from this pilot study, the use of a survey can provide a window into how PSTs may view the inclusion of language in the school context, the interconnection of language and mathematics and the teaching of ELLs. Mathematics teacher educators can use the valuable insight gained from the survey to design meaningful, relevant, and engaging experiences that challenge and confront PSTs' beliefs in an intellectual and respectful manner.

Endnotes

1. We consider English Language Learners (ELLs) as those students who are still developing a proficiency in English and consist of students who speak a language other than English at home.
2. The basic level is the lowest of the three levels a student can obtain on a NAEP assessment and is defined as "mastery of prerequisite knowledge and skills that are fundamental for proficient work" (NCES, 2009).
3. Proper credit was given in all cases where items from other surveys were adapted.

4. This percentage includes those respondents who either reported 'English' or 'English' plus an elementary/limited/brief amount of a different language when asked, "What languages can you speak? Please explain."

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HIGH-STAKES STANDARDIZED MATHEMATICS TESTING IN URBAN HIGH SCHOOLS: AN EQUITY PERSPECTIVE

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This article examines mathematics opportunities at two urban high schools in the context of high-stakes standardized testing. Case studies were conducted at two urban high schools in low-income neighborhoods. Data includes opportunities in mathematics offered to students at the school level, in terms of mathematics course sequencing, and at the classroom level in terms of task cognitive demand and participation modality. In particular, this article focuses on two related questions: 1) Do high-stakes standardized testing set high standards for all students? and 2) Do mathematics teachers "teach to the test" and if so, how?

Educational equity is often framed as a significant rationale for high-stakes standardized testing (Diamond & Spillane, 2004). Proponents of high-stakes standardized testing argue that the tests hold schools and teachers accountable to providing all students with a high quality education (No Child Left Behind Act, 2008). Others (e.g., Au, 2007; Koretz, 2008) contend that an emphasis on testing prompts schools and teachers to adapt their curricula to "teach to the test." The primary goal of this paper is to explore the two sides of this argument, in the context of mathematics education at two urban high schools that serve Black and Latino/a students from low-income families. While other studies explore the impact of standardized testing on teachers' instructional practice by strictly using a methodology of surveying or interviewing teachers (e.g., Barksdale-Ladd & Thomas, 2000), this study examines the potential relationship between high-stakes standardized testing and equity in mathematics education empirically, with data from classroom observations.

The paper begins with a presentation of the research context and description of data sources. It continues with a presentation of results organized around the dual themes of standardized testing as promoting high standards for all students and teachers "teaching to the test." The discussion section summarizes the study's findings and poses questions for further research.

Methods

Research Context

Urban school districts, because of their size, typically serve a diversity of students, along racial, linguistic, religious, and socioeconomic dimensions. However, while a city may be diverse across many dimensions, it is typically an aggregate of smaller neighborhood units, which are often homogenous in terms of race and socio-economic class. African American and Latino/a students from the lowest income families tend to be clustered in schools in particular urban neighborhoods (Lipman, 2004). The research described in this paper is conducted in two such neighborhoods in New York City, each with about 130,000 residents: Bushwick is primarily Latino/a (split among Puerto Ricans, Dominicans, Mexicans, and Ecuadorans), and Brownsville, about a mile away, is primarily African American. These neighborhoods are among the ten lowest income neighborhoods citywide; more than half of families in Bushwick and more than 2/3 of families in Brownsville are in the bottom two quintiles of city income levels (Furman Center, 2008).

Harwood and Carver are pseudonymously named high schools located in these two neighborhoods. At Harwood, in 2009-2010, just over two-thirds of students self-identified as "Hispanic or Latino/a," and the other nearly one-third self-identified as "Black or African American." About 90% of the school's families were recipients of public assistance. Carver also serves students from low-income families; about 81% of its families were recognized as recipients of public assistance. In 2009-2010, about 90% of Carver's students self-identified as "Black or African American," and the remainder of students self-identified as "Hispanic or Latino/a" (statistics cited in this paragraph are from New York State Education Department, 2010).

Harwood and Carver were the two partner schools for the Centering the Teaching of Mathematics on Urban Youth (CTMUY) professional development project in 2009-2011. CTMUY is an NSF-funded, integrated research and professional development project, focused on improving classroom-level opportunities to learn mathematics provided to students at high schools in underserved urban neighborhoods (for more details about the professional development project, see Rubel, in press and Rubel & Chu, accepted for publication).

Data Sources

Data for this analysis includes school-level data, teacher-level data, and classroom-level data, described in more detail below.

School data. Ethnographic school visits were conducted at Harwood and Carver in 2008-2009 and 2009-2010 as part of the schools' participation in CTMUY. Interviews were conducted with assistant principals and principals, as part of the data collection for that larger project about each school's course offerings in mathematics and rationale for those offerings. In addition, student standardized testing data was gathered from annual state report card.

Teacher data. Three focal teachers at Carver and four focal teachers at Harwood completed a Likert-scale survey which included items, among others, about the degree to which they emphasize a variety of objectives, such as preparing for standardized tests, increasing students' interest in mathematics, or preparing students for further study in mathematics.

Classroom observations. As part of the teachers' participation in CTMUY, each of three focal teachers at Carver and four focal teachers at Harwood were visited for ten classroom observations across the 2009-2010 school year. Observations were conducted in four clustered rounds across the school year. Each teacher's set of ten observations was conducted in the same class period, with the same group of students. Fieldnotes were taken during the classroom observations and were then expanded into detailed narrative descriptions. Two teachers each had single observations that were highly atypical, either because of teacher illness or student attendance, and those two observations were dropped from the data set.

Each lesson's main mathematical task was identified and then classified in terms of its cognitive demand, using categories from Henningsen & Stein (1997). Cognitive demand can be low-level, if the task presented to students relies strictly on memorization or if it is a strictly procedural task, that does not offer connections to concepts, understanding or meaning. Tasks that have a high level of cognitive demand might be complex and non-algorithmic ("doing mathematics) or procedural in nature, however, the task's procedures connect explicitly to concepts, understanding or meaning.

A second quantitative measure of each observed lesson pertains to the various participation modalities offered to students. I utilized a set of categories adapted from Weiss, Pasley, Smith, Banilower & Heck (2003): listening; investigating or problem solving; discussing; reading, writing, or reflecting; using technology; or practicing skills. Each lesson was subdivided according to the various participation modalities offered to students, and those modailities were quantified in terms of relative minutes of instruction.

Results

Results are presented in terms of two themes. First, I explore the notion of high-stakes testing as a mechanism that promotes high standards for teachers and students, using the examples offered by Carver and Harwood. Next, I analyze project data to investigate if and how teachers at Carver and Harwood "teach to the test" in the context of high-stakes standardized testing.

High Standards and/or Gatekeeper?

In New York, to graduate from high school, students must pass a state standardized exam in mathematics that corresponds with state courses in Algebra, Geometry, or Algebra 2/Trigonometry. However, because of the actual, or perceived, nature of mathematics as a strictly cumulative discipline, in practice, the state's entry-level algebra exam plays the dual role of functioning as the 'high standard for all students' and as the gatekeeper of high school graduation. Table 1 contains the percentage passing rates on the state's entry-level algebra exam, for the classes of 2009 and 2010 at Carver and Harwood at the end of their four years of high school. As shown, at Harwood, in both the classes of 2008 and 2009, only about half of its students passed the entry-level algebra exam by the end of high school. As a result, Harwood did not meet the Adequate Yearly Progress (AYP) thresholds determined by the No Child Left Behind Act in either year. After two years of not meeting AYP, Harwood was then labeled a "school in need of improvement (SINI)" and was subject to a state quality review of their mathematics program. In 2010, their results improved significantly and exceeded the AYP threshold, with 72% of the graduating class passing the state's entry-level algebra exam.

Carver High School, on the other hand, had relatively stable frequency of passing the algebra exam: 65% of students in the class of 2008, 72% of students in the class of 2009, and 66% of students in the class of 2010 passed the algebra exam by the end of high school. Carver's results exceeded the AYP threshold in 2008 and 2009. Although a higher percentage of students in the class of 2010 passed a mathematics exam than the class of 2008, in 2010, because of the changing nature of the AYP thresholds, Carver did not make AYP in mathematics (all data is from New York State Education Department, 2008, 2009, 2010).

	Class of 2008	Class of 2009	Class of 2010
Carver	65% (98)	72% (93)	66% (92)
Harwood	48% (105)	52% (102)	72% (97)

Table 1. Percentage of students passing entry-level algebra exam

As we see in Table 1, the percentage of seniors at Carver who passed the state algebra exam remained roughly constant, with a slight increase in 2009 and a corresponding decrease in 2010. Harwood, on the other hand, showed dramatic improvement in 2010. One interpretation of these results is that, after two years of low performance on the mathematics exams, the sanctions imposed on Harwood pushed the school and teachers to focus their efforts on preparation for the algebra examination. In fact, at Harwood, in 2009-2010, the school structured their staffing resources so that the eleventh and twelfth graders who had not yet passed the algebra exam had mathematics class two periods each day. This evidence suggests that the mechanism of the state standardized test, and the sanctions that come with poor performance, pushed Harwood to focus its resources on its struggling students, by offering them double the amount of instructional minutes in mathematics.

A second interpretation of this data focuses on the mathematics opportunities that result from an institutional emphasis on a *minimal* requirement. In other words, the accountability system does not rate schools in terms of the learning opportunities they provide to all students. Instead, the accountability system sanctions schools whose students do not meet a minimum requirement, in this case, passing an entry-level algebra exam. So at Harwood, in the 2008-2009 and 2009-2010 school years, students who did not pass the algebra exam at the end of their ninth grade year were placed in remedial “repeater classes” to retake the algebra course. Many students continued to cycle through the remedial classes all the way through their four years of high school. For instance, in the fall semester of 2009-2010, nearly 67% of all Harwood general education students were taking or re-taking the New York State entry-level Integrated Algebra course. So by the end of their four years of high school, while 72% of Harwood’s students from the class of 2010 passed the state’s algebra exam, many of these students (and of course, the 28% of the class who did not pass the algebra exam) were not given the opportunity to study geometry or any other mathematics as part of their high school education.

The case of Carver High School strengthens the interpretation of high stakes standardized testing functioning as gatekeeper. At Carver, during the 2008-2009 and 2009-2010 school years, students progressed through a sequence of Algebra, Geometry, Algebra 2/Trigonometry, irrespective of whether they passed the entry-level algebra exam. Those students who did not pass the exam were assigned to before school, after school, or Saturday sessions to continue to practice for that exam. However, once Carver did not make AYP in mathematics in 2010, Carver changed their course sequencing policy for their students in 2010-2011. While all students progressed from 9th grade algebra to 10th grade geometry, any eleventh or twelfth grade student who had not passed the entry-level algebra exam was removed from the mathematics course progression and tracked into a designated remedial “test-prep” mathematics class.

The analysis in this paper has, thus far, focused on school-level processes of course sequencing in the context of high-stakes standardized testing. In the next section, the analysis zooms in to the classroom level to examine the issue of if and how teachers at Carver and Harwood “teach to the test.”

“Teaching to the Test”

One way to examine the potential relationship between high stakes standardized testing and teachers’ pedagogical practices is to survey teachers. At the start of the CTMUY project, for example, all seven focal teachers at Carver and Harwood reported that they place moderate or heavy emphasis on preparing students for standardized tests. On the same survey, only three of the seven focal teachers indicated that they place moderate or heavy emphasis on preparing students for further studies in mathematics. Similarly, only three of the seven focal teachers indicated that they place moderate or heavy emphasis on increasing students’ interest in mathematics. Even though this is an extremely small sample of teachers, these results suggest that these teachers take the accountability system of high-stakes standardized tests seriously, mirroring findings of Barksdale-Ladd & Thomas (2000). I expand upon these findings to illuminate how a teacher’s emphasis on preparing students for standardized tests might be expressed in mathematics instruction. I build upon the definition of curriculum as presented by Au(2007) as including a) subject matter content knowledge; in this case, the mathematics content that is included in a curriculum, 2) structure or form of curricular knowledge; how mathematics is structured and presented within a curriculum, and 3) pedagogy; how that mathematical knowledge is communicated.

Mathematics content of curriculum. The high stakes standardized tests and the mathematics learning standards are both created and endorsed by the state and the school district, and they are therefore, assumedly, in direct correspondence. In the case of New York, high schools and their teachers are accountable to aligning the mathematical content of their 9th grade curriculum to the state’s algebra standards. Proponents of high-stakes testing argue that this standardizes the opportunities offered to students. So even in low-income neighborhoods like Bushwick and Brownsville, students are studying the same set of algebraic concepts and skills as everywhere else in the state. Seen in this way, "teaching to the test" pushes teachers to maintain high mathematics standards for their students by using the state's mathematics standards to guide the mathematical content of their curriculum.

Form of curriculum. In schools like Harwood and Carver, where students enter high school with a history of poor test scores, teachers are under enormous pressure for their students to achieve good results on high-stakes tests. Another interpretation of "teaching to the test," especially in contexts like these, is, therefore, that teachers will pattern the actual mathematical tasks they pose to students in the classroom, or the structure or form of mathematical knowledge, to precisely reflect the tasks that are offered on the standardized exams. By their very nature, standardized test tasks are limited to the types of questions that can be expressed in multiple-choice or short-answer form, have a single correct answer, and perhaps most importantly, can be completed in a very short time. These tasks typically require low-level forms of knowledge in that they tend to either require recalling a term or a executing a well-defined procedure (Williams, 2010).

I analyze this issue of "teaching to the test" first by examining the cognitive demand level of the mathematical tasks in the 68 observed classes at Carver and Harwood. Table 2 contains the distribution of cognitive demand level of these mathematical tasks in the 29 observed lessons at Carver and the 39 observed lessons at Harwood.

	Low-Level			High-Level	
	Memorization	Procedures without connections	Other	Procedures with connections	Doing mathematics
Carver	10	10		8	
Harwood	1	27	2	9	1
Total	11	37	2	17	1

Table 2. Task Cognitive Demand

As shown in Table 2, 50 of the 68 tasks were rated as low-level tasks, indicating that they were either memorization tasks (11), focused on procedures without connection to concepts, meaning or understanding (37), or were otherwise low-level (2). Only 18 of the 70 tasks were rated as high-level tasks, indicating primarily that they were tasks that focused on procedures with connection to concepts, meaning or understanding (17). In addition, 17 of these 18 high-level tasks were limited to only two of the seven focal teachers. In other words, the students in the classes of four focal teachers were offered only low-level tasks, in the 40 observed lessons of those teachers. These results strengthen the claim that teachers "teach to the test" by offering students low-level mathematical tasks, which correspond to the types of tasks found on standardized tests.

Pedagogy of curriculum. Another means of analysis as to if and how teachers are "teaching to the test" is to consider the forms of participation opportunities they make available to students in their mathematics classes. Table 3 contains the relative frequency of each participation modality offered to students in the 29 observed mathematics classes at Carver and the 39 observed mathematics classes at Harwood.

Teacher-centered modes of participation, such as listening to the teacher, practicing skills and “housekeeping” (taking attendance, collecting homework, distributing textbooks) dominated the observed lessons at both Carver and Harwood, as shown in Table 3. In fact, in the observed classes of four of the seven focal teachers, these teacher-centered modes of participation occupied more than 80% of the instructional time (not shown in Table 3). Notably, teachers organized classes, on average, for students to participate by “practicing skills” 37% of the mathematics time at Carver and 32% of the time at Harwood. At both schools, the “practicing” consisted of teachers distributing worksheets with tasks modeled specifically on previous state test items. Teachers created these worksheets using a free, Internet-based application, which consists of a search engine that generates such worksheets in printable form. Student-centered modes of participation, like whole-class discussion, listening to other students, problem solving or investigating, using technology, or writing were offered much more infrequently in the observed lessons, on average: only 26% of the time at Carver and 42% of the mathematics time at Harwood.

Forms of Participation	Carver 29 classes 60 min each	Harwood 39 classes 48 min each
Teacher-centered		
Listening to teacher	29%	22%
Practicing	37%	33%
“Housekeeping”	8%	3%
Total	74%	58%
Student-centered		
Discussing	10.%	12%
Listening to students	2%	3%
Investigating	9%	22%
Writing	4%	4%
Using technology	1%	1%
Total	26%	42%

Table 3. Frequency of Participation Modalities

There is a relationship between the cognitive demand of a lesson’s task and the participation modalities offered to students to complete that task. Low-level tasks, which typically require memorization of facts or execution of steps of a well-defined procedure, lend themselves to listening and practicing. The teachers in this study who exclusively offered low-level tasks to their students also nearly exclusively offered the teacher-centered forms of participation of listening to the teacher and practicing. In contrast, tasks with higher-level cognitive demands, in which students are challenged to detect patterns or make and communicate connections across mathematical representations, afford more varied opportunities for students. The teachers in this study who offered high-level tasks to their students also offered student-centered participation modalities like investigating or discussing.

Discussion

In this paper, I have presented dual themes related to mathematics education in a context of high-stakes standardized testing. In some ways, the current accountability system in New York forces high schools to maintain high standards for all students. While in the past, high school students could earn “local diplomas” without passing any state mathematics exams, current state mandates require that students pass a state mathematics exam in order to graduate from high school. In the context of schools in low-income neighborhoods, this mandate has had an impact in that schools are now required to place their 9th grade students in algebra courses, as opposed to primarily placing students in remedial level courses in "pre-algebra" or "consumer mathematics."

Although most of the students at Carver and Harwood pass their schools' 9th grade mathematics courses, most of the 9th grade students do not pass the corresponding state exam. This presents the schools with a dilemma: should the students retake a course that they have already taken and passed, should the students proceed through the high school mathematics curriculum, or should the school restructure its entry-level course to stretch over multiple years? The accountability system does not reward schools for having students take more or more challenging mathematics courses, and instead, sanctions schools for having students not meet the minimum requirement. Therefore, I argue that the accountability system encourages schools to follow the first option, i.e. cycle students through remedial courses, with a test-preparation emphasis, until they pass the test. The effect that this has had at Carver and Harwood is that this entry-level mathematics examination is positioned as a "finish-line" towards graduation, instead of a starting point for continued success. Students need knowledge in geometry and algebra 2/trigonometry to succeed in college placement tests or in college courses, so this inversion of a minimum requirement to function as an end goal ultimately does students a great disservice.

Others have claimed that "the weight of the high stakes testing environment falls heaviest on the shoulders of low income students and students of color" (Au, 2009, p.3). More specifically, Jones, Jones & Hargrove (2003, p.115) posit that low income students "are hit doubly hard (by high-stakes standardized testing) – not only do they tend to have lower scores on high stakes tests that may block them from subsequent opportunities, but the instruction that they receive might actually be worse than the instruction that they received before the testing policy was implemented." This claim is substantiated, perhaps, by this data and analysis.

While it remains hypothetical, of course, as to what the mathematics instruction at Carver and Harwood might resemble without high-stakes standardized testing, the observed structure and pedagogy of the curriculum at both Carver and Harwood are in alignment with standardized testing in several ways. Focal teachers most often presented students with tasks of low-level cognitive demand whose form corresponds to the low-level form of the state exams, typically involving recalling a vocabulary term or short, well-defined procedure. In addition, these low-level tasks were presented in a classroom pedagogical environment that privileges teacher-centered forms of participation. Students were typically expected to participate in mathematics, throughout the school year, by listening (to the teacher) and practicing, using worksheets created from a bank of test items readily available to teachers on the Internet. Opportunities for students to investigate patterns, to solve non-routine problems, to use technology to discover relationships, or to write or reflect about a mathematical concept or process were far more limited in the observed classes.

This preliminary analysis suggests that the context of high-stakes standardized testing is complex and merits analysis with an equity lens. A common perspective is that high-stakes standardized testing promotes equity by setting and enforcing high standards in mathematics. This analysis has demonstrated that we also need to consider the ways in which high-stakes standardized testing "exacerbates inequities" (Diamond & Spillane, 2004). This study has demonstrated the particular ways that high-stakes standardized testing also functions as a gatekeeper in urban high schools. In addition, this study provides description and nuance to how teachers "teach to the test" by providing students with low-level mathematical tasks using teacher-centered forms of participation.

Notes

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THE GENDERING OF MATHEMATICS AMONG FACEBOOK USERS IN ENGLISH SPEAKING COUNTRIES

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Using an innovative recruitment tool, the social network site Facebook, survey data were gathered from samples of the Australian general public and from around the world. Views on the gendering of mathematics, science, and ICT were gathered. In this paper we report the findings from six of the 15 questions on the survey, and only from respondents in predominantly English-speaking countries. The findings reveal that the majority was not gender-stereotyped about mathematics and related careers. However, if a gendered view was held, it was overwhelming to endorse the male stereotype. Male respondents' views were more strongly gendered than were females'.

Prologue

The new focus on nature seems to be encouraging parents to indulge in sex differences even more avidly…. From girls' preschool ballet lessons and makeovers to boys' peewee football… the more we parents hear about hardwiring and biological programming, the less we bother tempering our pink and blue fantasies. (Freeman-Greene, 2009, p.11)

Providing a Context

Publication of student achievement data from large scale testings ensures that gender differences in mathematics learning continue to attract sustained attention from both the research and broader communities. Results from the Organisation for Economic Co-operation and Development's [OECD] Programme for International Student Assessment [PISA], for example, receive considerable media attention on their release. Such media reports on performance may include comments on gender differences. Often, however, simplified summaries of complex data are presented (Forgasz & Leder, 2011). This is not altogether surprising, given reporters' time and space constraints. That such media accounts often shape and sway public opinion, including views on gender issues, is well documented (e.g., Barnett, 2007; Jacobs & Eccles, 1985).

Possible Explanations for Gender Differences in Mathematics Achievements

Multiple explanations have been put forward for the persisting patterns of gender difference in mathematics achievement. After a detailed review of relevant literature, Halpern et al. (2007) concluded that the reasons for the overlap and differences in the performance of males and females were multifaceted, could not be explained by a single factor, and that "[e]arly experience, biological constraints, educational policy, and cultural context" (p. 41) could all play a part. Geist and King (2008) referred to pervasive societal beliefs about gender linked capabilities and their impact:

Many assumptions are made about differing abilities of girls and boys when it comes to mathematics. While on the 2005 NAEP girls lag only about 3 points behind boys, this is only a recent phenomenon. In the 1970's, girls actually outperformed boys in all but the 12th grade *test…. assumptions about differing levels of ability pervade not just the classroom, but home. (pp. 43-44)*

In their detailed model of achievement motivation, and implicitly of academic success, Wigfield and Eccles (2000) highlighted the influence on students' learning and behaviours not only of learner-related variables but also of the overall context in which learning occurs, that is the attitudes, actual and perceived, of critical "others" in the students' home, those at school, and societal expectations more generally.

Societal Expectations - Public Views about Mathematics: Gauging Public Opinion

Attempts to measure directly the general public's views about mathematics, science, or ICT, or the teaching of these subjects, or their impact on careers, are rare. With respect to mathematics, for example, more than two decades have passed since a genuine attempt was made by the Victorian (Australia) state government to gauge parents' attitudes towards their daughters' education and career (McAnalley, 1991). This exercise was linked to the state-wide media campaign, *Maths Multiplies Your Choices,* a program introduced to encourage parents to think more broadly about the likely influence of mathematics on their daughters' careers. Since that time, in Australia there has been no similar concerted, large scale, and appropriately funded, attempt to measure the general public's views about school mathematics, its link with technology, and possible career options.

A decade ago in the UK, Sam and Ernest (1998, p. 7) noted that "there are relatively few systematic studies conducted on the subject of myths and images of mathematics. We need an answer to the question: What are the general public's images and opinions of mathematics?" Lucas and Fugitt (2007) similarly argued that the public's views on mathematics and mathematics education were rarely sought. Yet, they found that Mid-West USA residents responding to a 10-item survey were generally interested in, and often well informed about, the way mathematics was taught in schools. The respondents generally believed that a good mathematics education offered young people a better and successful future; schools failed to offer effective mathematics education because too much emphasis was placed on technology and not enough on the basics; teachers often exerted too much pressure and criticism to the detriment of their students' attitudes to mathematics; and teachers should make learning mathematics more enjoyable. Issues such as these were also explored in the study reported in this paper. The general public's views about aspects of mathematics were explored. Also examined was whether or not the views expressed were gender stereotyped.

The Study

In the present study, data from respondents in countries in which English is the dominant language – Australia, Canada, Ireland, New Zealand, UK, and USA – were analysed. While data were also gathered from participants living in many other countries around the world, in this paper we focus predominantly on countries where English is the most commonly spoken language for the following reasons:

- In broad terms, cultural differences are small among citizens across these countries
- Almost without exception, the mathematics achievements of males in these countries are found to be superior to females'

All six countries participated in PISA 2009. The gender differences in the mean scores for these countries are shown in Figure 1 – the data were drawn from Thomson, De Bortoli, Nicholas, Hillman, and Buckley (2010).

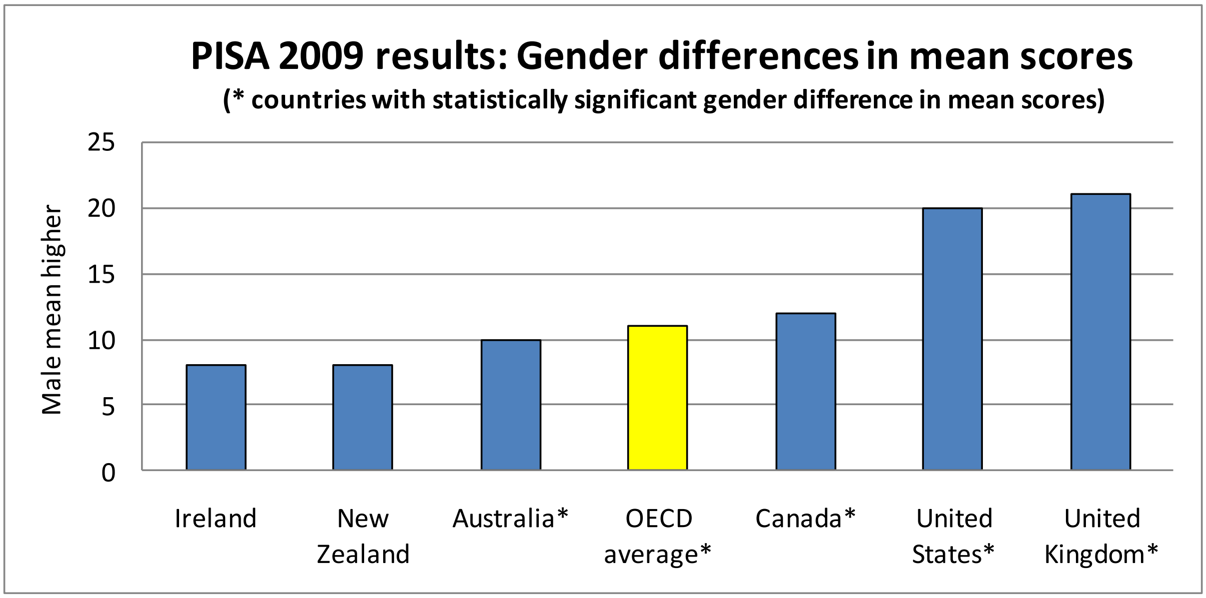

Figure 1. PISA 2009 results by country and gender difference in mean scores

From Figure 1 it can be inferred that there is much overlap in the performance of males and females. However, it is also apparent that males outperformed females in each country. As noted by Thomson et al. (2010), the differences in the mean scores were statistically significant for Australia, Canada, UK, and USA.

For many countries participating in the Trends in Mathematics and Science Study [TIMSS] in 2007, females' mean scores in mathematics were higher than males'. However, there were several countries, including all participating English-speaking countries, for which the reverse was true. Gender differences in favour of males were noted as follows at the Grade 4 and Grade 8 levels respectively: Australia (6 points, 16 points*), New Zealand (1 point, did not participate), England (1 point, 5 points*), Scotland (did not participate, 3 points) and USA (6 points*, 3 points) – see Thomson, Wernert, Underwood, and Nicholas (2008, pp. 59, 61). [NB. * indicates statistically significant gender differences.]

In this study, it was of particular interest to determine if members of the general public in English speaking countries held views that might support the hypothesis that performance differences are consistent with the perceptions of society at large with respect to the gendering of mathematics learning outcomes.

The data presented in this paper are a subset of a larger study which involved gathering data from 12 different heavy foot-traffic sites throughout Victoria (see Leder & Forgasz, 2010 for details) as well as from an 'advertisement' to recruit participants placed on the social network site, Facebook (http://www.facebook.com). Facebook was selected as the recruitment site to reach a more diverse international group of participants. The overall aim of the study is expressed concisely in an excerpt, provided below, of the "explanatory statement" made available to Victorian participants and required for obtaining ethics approval for the study.

We have stopped you in the street to invite you to be a participant in our research study. ...We are conducting this research, which has been funded by [our] University, to determine the views of the general public about girls and boys and the learning of mathematics. We believe that it is *as important to know the views of the public as well as knowing what government and educational authorities believe*.

A modified version of this statement was used in the online survey that was directly linked to the Facebook advertisement. That is, Facebook users who clicked on the advertisement were directed to the online survey.

The Instrument

To maximize cooperation and completion rates, the surveys used (in the street and on Facebook) were limited to the same 15 core items. These focused on personal background data; the learning of mathematics at school; perceived changes in the delivery of school mathematics; beliefs about boys and girls and mathematics, and their perceived facilities with calculators and computers; and careers. In this paper we focus on six questions related to the importance of mathematics and the gendering of mathematics. The six items analysed and discussed are:

- Should students study mathematics when it is no longer compulsory?
  Yes/No/Don’t know/Depends
- Who is better at mathematics, girls or boys?
  Girls/Boys/Don’t know/Depends
- Do you think studying mathematics is important for getting a job?
  Yes/No/Don’t know/Depends
- Is it more important for girls or boys to study mathematics?
  Girls/Boys/Don’t know/Depends
- Who are better at using calculators, girls or boys?
  Girls/Boys/Don’t know/Depends
- Who are better at using computers, girls or boys?
  Girls/Boys/Don’t know/Depends

Respondents had the option to explain their responses to each question. However, only quantitative data analysed are presented and discussed here. [Space constraints precluded the reporting of findings from the open-ended responses and from quantitative analyses by respondent gender.]

Participant Recruitment, Online Survey, and the Social Networking Site, Facebook

The social network site, Facebook (http://www.facebook.com), was the avenue adopted to recruit the participants from whom data are reported in this paper. With the rapid advancement of internet technology, online surveys have become a viable method for data-collection in research (e.g., Sue & Ritter, 2007). Social network sites (SNS) such as Facebook are rapidly gaining worldwide popularity. “As of March, 2010, Facebook is the second ranked site on the Internet traffic metrics on alexa.com, accounting for almost 5 percent of all global page views” (Hull, Lipford, & Latulipe, 2010, p.1).

Except for research in which SNS users’ profiles and SNS usage are investigated, studies using Social Network sites [SNS] as a method of recruiting participants are relatively rare. In one such study, Howell, Rodzon, Kurai & Sanchez (2010) administered a well-being and happiness survey to participants recruited from a college and from the SNS, Craigslist. Incentives were offered for both data gathering methods. Although the completion rate was lower for those recruited via SNS (68.5%) than from the college (93.6%), the quality of data obtained by the two methods was comparable.

The Sample

There were 314 participants who completed surveys via Facebook. The advertisement was designed through Facebook's commercial advertising campaign system. The system allows particular groups to be targeted. Countries targets were changed on a weekly basis, but at all times, only those over 18 were sought. The advertisement appeared randomly on individual Facebook users' homepages in the targeted countries. Clicking on the advertisement was voluntary, as was completion of the survey. Thus the respondents were considered to represent a random sample of Facebook users over the age of 18 from a range of countries. The respondents represented 57 different countries around the world. Six were countries in which English is the dominant language: Australia, Canada, Ireland, New Zealand, the United Kingdom, and the United States. The composition, by gender (M=male, F=female), of the sample from each of these six countries is summarised in Table 1.

Country	N (M, F)	Country	N
Australia	74 (39M, 35F)	New Zealand	1 (OM, 1F)
Canada	12 (5M, 7F)	United Kingdom	22 (12M, 10F)
Ireland	2 (0M, 2F)	United States	6 (4M, 2F)

Table 1. Sample size, by gender, for English speaking countries

Results and Discussion

Since the Australian sample was the largest, we decided to explore if the combined data from the five other countries differed significantly from the Australian data. Chi-square tests were conducted on the responses to the six questions and no statistically significant differences were found in the response distributions for any item. This allowed us to confidently combine the data from all six countries. The valid percentage frequency distributions of the 117 participants' responses to the six questions are illustrated in Figure 2.

The data in Figure 2 indicate that the respondents generally agreed that:

- Students should study mathematics when it is no longer compulsory (65%), and that
- Studying mathematics is important for getting a job (72.2%)

For each of the questions to which respondents were asked to indicate whether boys or girls were more proficient, the majority of respondents claimed that there was no difference. This indicates that the majority is not gender-stereotyped in their views on boys' and girls' mathematics capabilities or their proficiency with calculators and computers, and that mathematics is considered equally important for both boys and girls.

However, a less positive pattern is also discernible in the data. If respondents held a gendered view on mathematics capability, calculator or computer proficiency, or for whom mathematics was considered more important, they were much more likely to indicate that it was boys rather than girls:

- Better at mathematics: 31.1% said boys, 10% said girls
- Studying mathematics is more important: 2.2% said boys, none said girls
- Better at using calculators: 14% said boys, 1.2% said girls
- Better at using computers: 34.5% said boys, 3.4% said girls

The data re-inforce earlier reported research findings. That is, there appears to be little difference between the perceptions of the general public and those of stakeholders (e.g., students, teachers, or parents) that boys are more talented mathematically and more able with technology than are girls (see Leder, 1992 with respect to views on mathematical talent; see Forgasz, 2009 with respect to teachers' views on technology for mathematics learning). There is also no evidence of a change in these perceptions over time – disappointing and pessimistic outcomes.

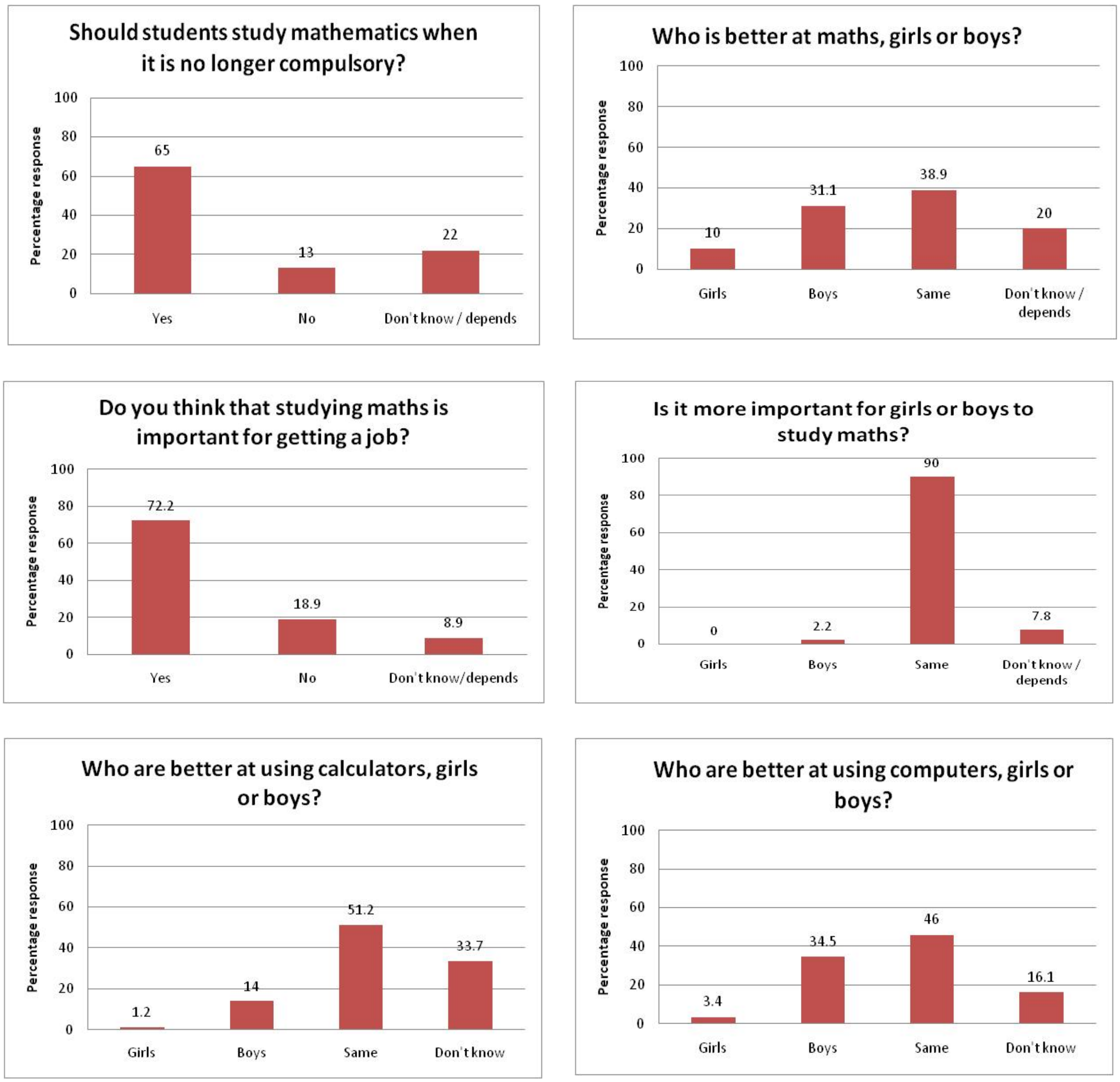

Figure 2. Valid percentage responses to the six survey items

Final Words

As noted above, gender differences favouring males in mathematics achievement are evident in large scale international testing results (PISA and TIMSS) in English speaking countries. In Australia, results from the National Assessment Program for Literacy and Numeracy [NAPLAN] reveal a similar pattern (see Leder & Forgasz, 2010). What is particularly alarming about the NAPLAN results is that the gender differences in favour of males are found at each grade level being tested: grades 3, 5, 7, and 9. This appears to be a retrograde trend since when Fennema was writing about gender differences in the 1970s (e.g., Fennema, 1974), there was little evidence of gender differences emerging prior to about grade 7. Gender differences in mathematics learning continue to attract media attention (see Forgasz & Leder, 2011). If the stories are not carefully crafted, the public will come away with views and beliefs shaped by the basic (simplistic) words they read (and interpret) and the images they see.

The data presented in this paper suggest that the general public from the targeted English speaking countries who engage with Facebook hold views that resonate with contemporary mathematics achievement data that they will have had thrust at them by way of the popular media and press. When the age profile of the respondent group is considered – no respondent was older than 60 and 85% was under 40 – the data become of even greater concern. It would appear that there is a critical mass of the younger generation who are reverting back to holding gender-stereotyped beliefs in favour of males with respect to mathematics and with the technology associated with mathematics learning. Their parent generation, mainly "Baby Boomers", lived through the era of feminist agitation that happened in the 1970s and 1980s. Their views were tapped by way of the street-based survey; many over 60s are in the streets during weekday hours. It was clear from the data that "[C]ompared to older respondents, the younger cohort was more likely to consider boys to be better than girls at mathematics and also better with calculators" (Leder & Forgasz, 2011).

The data presented here, although limited to a relatively small group of Facebook users in English speaking countries, lend weight to the contention that we may have come full circle and are now confronted with a young adult group holding traditionally gender-stereotyped beliefs about the domains of mathematics and the technology associated with mathematics.

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AN ELEMENTARY 3D VISUALIZATION LEARNING TRAJECTORY

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This research team has developed a learning trajectory on 3D visualization for elementary children. This paper focuses on the interchange among the three Spatial Operational Capacity framework representations, namely, 3D models, 2D conventional diagrams, and semiotic abstract representations, and the critical role of a dynamic computer interface that simulates the representations. The trajectory is described through an actions-on-objects (Connell, 2001) lens and the project's strong problem-solving approaches that are critical to its successful enactment.

The National Research Council's report, *Learning to Think Spatially* (2006), identifies spatial thinking as a significant gap in the K-12 curriculum, which, they claim, is presumed throughout but is formally and systematically taught nowhere. They believe that spatial thinking *is* the start of successful thinking and problem solving, an integral part of mathematical and scientific literacy. The importance of visual processing has been documented by researchers who have examined students' performance in higher-level mathematics. For example, Tall et al (2001) found that to be successful in abstract axiomatic mathematics, students should be proficient in both symbolic and visual cognition; Dreyfus (1991) calls for integration across algebraic, visual and verbal abilities; and, Presmeg (1992) believes that imagistic processing is an essential component in one's development of abstraction and generalization.

The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that in their early years of schooling, students should develop visualization skills through hands-on experiences with a variety of geometric objects and use technology to dynamically transform simulations of two- and three-dimensional objects. Later, they should analyze and draw perspective views, count component parts, and describe attributes that cannot be seen but can be inferred. Students need to learn to physically and mentally transform objects in systematic ways as they develop spatial knowledge.

Using design-research (Cobb, et al, 2003) principles this research team has developed a learning trajectory on spatial development for elementary children guided by the Spatial Operational Capacity (SOC) framework developed by van Niekerk (1997) based on Yakimanskaya's (1991) work. This paper focuses on the interchange among SOC representations and the critical role of a dynamic computer interface, through an actions-on-objects (Connell, 2001) lens. The project's strong problem-solving approaches make this possible. It is conducted in a dual-language urban elementary school within one of the largest public school districts in the mid-southwestern United States. More than 70% of its students are designated "At Risk" and at least 50% of its students are English Language Learners.

Theoretical Frameworks

The *spatial operation capacity* (SOC) framework (van Niekerk, 1997; Sack & van Niekerk, 2009) that guides this study exposes children to activities that require them to act on a variety of physical and mental objects and transformations, as prescribed by the National Council of Teachers of Mathematics (NCTM, 2000) to develop the skills necessary for solving spatial problems. The framework (see Figure 1) uses: full-scale figures, that, in this study, are created from loose cubes or Soma figures, made from 27 unit cubes glued together in different 3- or 4-

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

cube arrangements (see Figure 2); conventional 2D pictures that resemble the 3D figures; semiotic representations such as front, top and side views or numeric top-view codings that do not obviously resemble the 3D figures; and, verbal descriptions that may be accompanied by gestures using appropriate mathematical language (Sack & Vazquez, 2008).

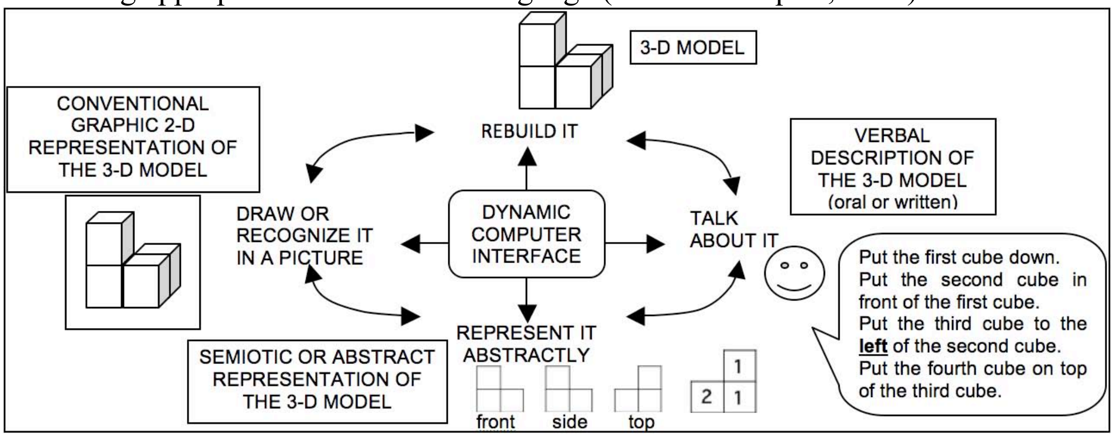


Figure 1. Multiple representations within 3-D visualization

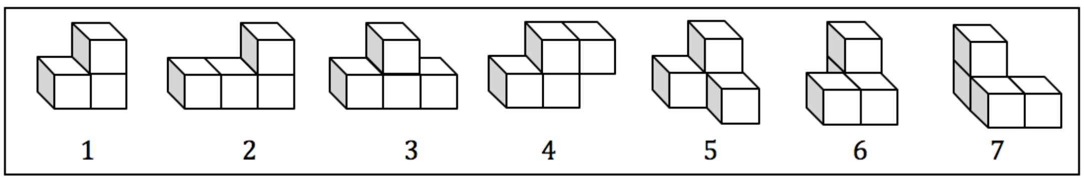


Figure 2. The Soma set can be made by gluing unit cubes together.

The project utilizes a dynamic computer interface, Geocadabra (Lecluse, 2005). Through its Construction Box module, complex, multi-cube structures can be viewed as 2-D conventional representations or as top, side and front views or numeric top-view grid codings (see Figure 3). These options can be (de)selected according to instructional goals. The Control-line-of-view option allows the user to move the figure dynamically using the mouse or by clicking on the arrows at the ends of the space's triaxial system.

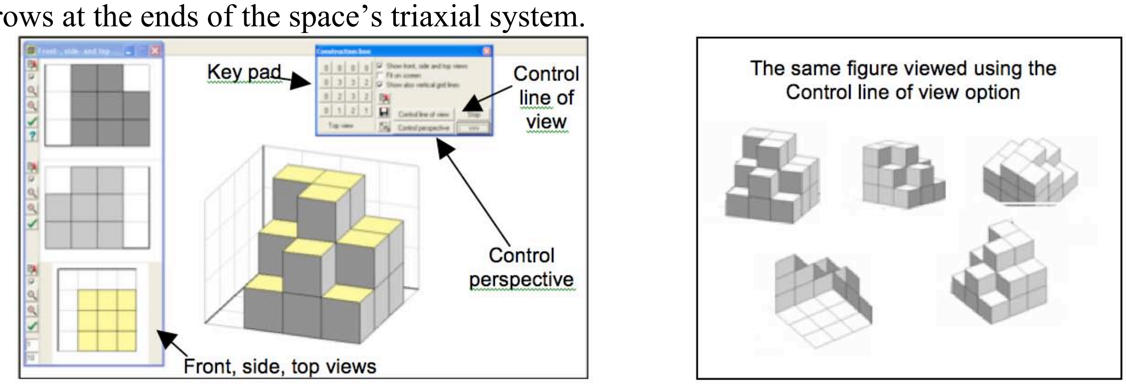


Figure 3. The Geocadabra Construction Box

Connell's (2001) *action on objects* metaphor guides the discussion section of this paper. Through carefully designed activities, children act strategically upon manipulative objects as they solve problems. Computer images that replicate the attributes of the physical objects then behave as real objects in the mind of the learner. The Geocadabra Construction Box interface integrates the SOC representations in the form of a dynamic image that can be moved to provide the same views as if moving about a 3-D object; a 2-D image when the figure remains static; and if selected, simultaneous semiotic top-view numeric, or face view representations. Follow up problems or questions require the child to relate to the newly instantiated and defined object of thought, which becomes the basis upon which later mathematical thinking occurs. This model extends as the child develops his or her own problems based upon the objects that were recently defined. "This ability, to pose one's own problems and to then successfully solve these problems, provides further opportunity for growth in mathematical thinking and problem solving" (p. 161).

Methodology and Context

Since the project's inception in 2007-2008, a university-based researcher and two teacher-researchers have formed the research team working with a group of children weekly for one hour in teacher-researcher, Vazquez' 3rd-grade classroom within the school's existing after-school program. English and Spanish parent/guardian and student consent-to-participate forms are sent home to parents of all 3rd grade children. All respondents are accepted into the program. The research team uses socially mediated instructional approaches to support a problem-solving environment that fosters students' creativity according to readiness and interest.

Design research methodology (Cobb, et al, 2003) guides this study's instructional decisions based on learning trajectories developed from an instrumentalist standpoint (Baroody, et al, 2004). This conceptual and problem-solving approach aims for "mastery of basic skills, conceptual learning, and mathematical thinking" using any "relatively efficient and effective procedure as opposed to a predetermined or standard one" (p. 228). Each lesson is part of a design experiment followed by a retrospective analysis in which the research team determines the actual outcomes and then plans the next lesson. This may be an iteration of the last lesson to improve the outcomes, a rejection of the last lesson if it failed to produce adequate progress toward the desired outcomes, or a change in direction if unexpected, but interesting, outcomes arose that are worthy of more attention. Data corpus consists of formal and informal interviews, video-recordings and transcriptions, field notes, student products and lesson notes.

Results and Discussion

During the learning trajectory's introductory lessons, children interact with loose cubes and the Soma figures initially solving problems with the 3D models and with 2D task cards (e.g., see Figure 4a). These illustrate a variety of assemblies of two Soma combinations in different orientations requiring figure identification and classification. Thus, learners become familiar with the SOC framework's 3D and 2D conventional graphic representations before they are introduced to the Geocadabra virtual interface. By the middle of the second month, children begin to digitally reproduce figures printed in a customized manual (e.g., see Figure 4b). These activities provide the children opportunities to coordinate numeric top-view codings with 2D pictures. In addition, there is a strong focus on enumeration of cubes in the manual's figures. Whereas beginning learners generally are able to determine the numbers of cubes in relatively simple figures (such as the left-most task card in Figure 4a), very few can do so with figures containing hidden cubes (as in Figure 4b). Battista (1999) has shown that many learners count visible faces when asked to find the number of unit cubes in 3D rectangular arrays, often double counting edge cubes and triple counting vertex cubes. This research team's pre-program interviews, September, 2010, using both 3D rectangular arrays and multi-level structures as in Figure 4b, yielded similar results. Some children recognize that cubes are hidden but lack the mental structuring capacity to determine precisely how many in a logical fashion.

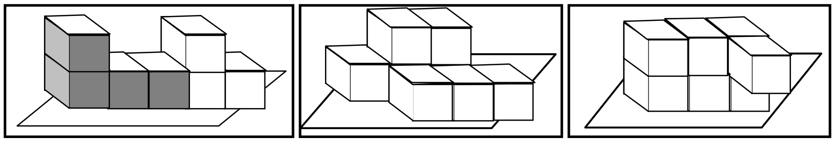


Figure 4a. Task cards increasing in level of difficulty

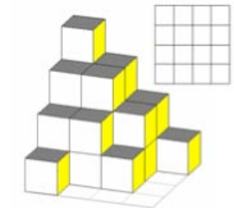


Figure 4b. Task 1*f*

By working systematically with increasingly more complex figures and by enabling the Geocadabra Construction Box's Control-line-of-view function has developed children's capacity to solve such enumeration problems. The learner's ability to see the digital figure in the same way as a 3D model enhances the digital experience. This way, since a stack of cubes grows by repetitive clicks on the corresponding position on the number grid, the children know that the computer figures can have no holes beneath any visible cubes. The ability to rotate the digital image allows them to see the sides, back and bottom views of the figure. Thus, movement from conventional 2D graphical images to virtual 3D models via the numeric top-view semiotic representation is realized.

After developing reasonable proficiency with the Geocadabra Construction Box through the manual's tasks, more open-ended problems are posed to further develop children's mental imaging capacity with respect to volume concepts. They are expected to correlate the numbers in the Geocadabra Construction Box grid with the height of each stack of cubes in each 2D picture; and, to verify that the number of cubes in the 2D picture is the same as the sum of the numbers in the Geocadabra Construction Box grid. Children create their own structures consisting of 24 unit cubes, using the Geocadabra Construction Box. Initially, they may choose to build the figure using 24 loose cubes, but most discard the loose cubes almost immediately. Conventional 2D images of these digital figures are converted by the researcher into new task cards essentially created by the children. The creator then draws the numeric top-view code and a peer decodes and re-creates it on the computer, first hiding the visual figure, and then showing it to check that it matches the task card figure. In addition, the peer checker ensures that the figure consists of exactly 24 cubes by adding the numbers in the top-view numeric grid on the computer. Examples of student-created task cards are shown in Figure 5. Enumeration of these figures using symmetry and slicing is encouraged and shared during whole class discussions when verbal language is developed. The research team has noted that children seem to have more difficulty enumerating 3D rectangular arrays than the types of figures shown in Figure 5. However, with practice, they are able to represent 3D rectangular arrays using numeric top-view coding grids. This is a critical step in their understanding of formula-based volume concepts (see Sack, & Vazquez, 2010).

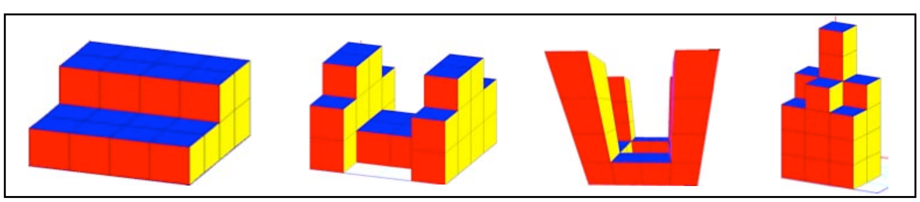
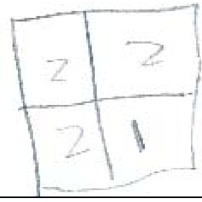
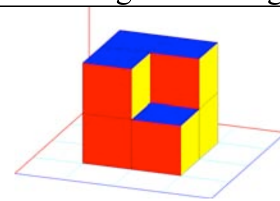
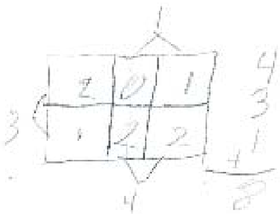
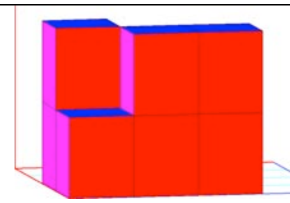
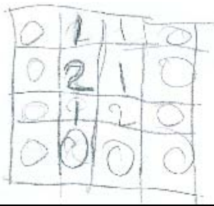
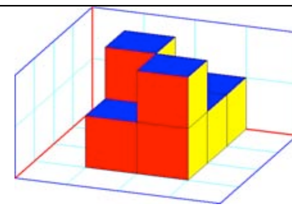
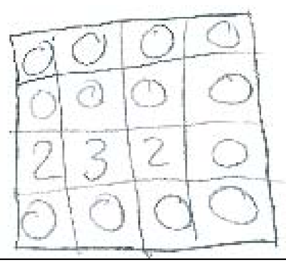
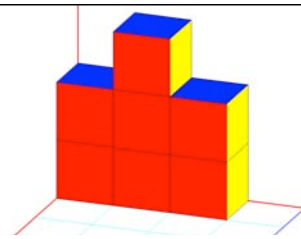


Figure 5. Students' 24-cube figures

Numeric top-view coding	Peer solutions	Original task figure
A 	Soma #1 and Soma #5 OR Soma #1 and Soma #6	
B 	Soma #6 and Soma #7	
C 	Soma #5 and Soma #6 OR Soma #2 and Soma #3	
D 	Soma #4 and Soma #1 OR Soma #2 and Soma #1	

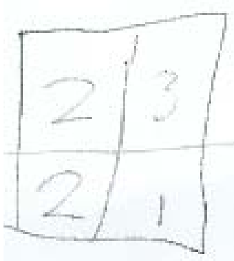
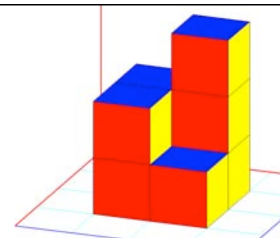
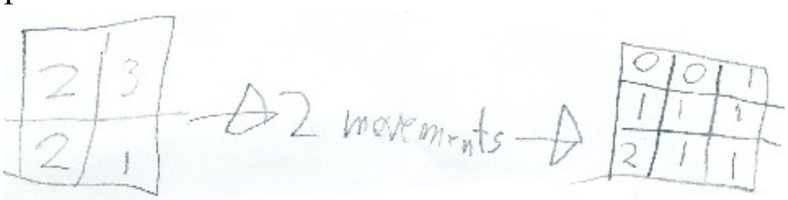
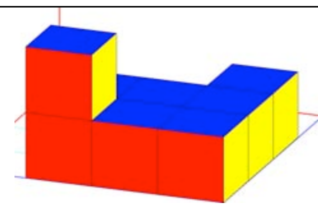
Numeric top-view coding	Peer solutions	Original task figure
E 	E1 Soma #7 and Soma #4 E2 Soma #3 and Soma #5 OR Soma #6 and Soma #2	
F 		

Table 1. Examples of student-made semiotic puzzles and peer solutions

The discussion now moves to Connell's (2001) notion of deepening mathematical understanding through self-developed problems that encompass all of the visual SOC representations. Using a complete set of seven Soma figures, children first select two to create a 3D assembly structure that can be reproduced on the Geocadabra Construction Box, unlike the rightmost structure in Figure 4a. The following week, without the aid of the computer interface, each child draws the numeric top-view coding from the picture of his or her own structure that the research team has formatted into a task card. Regardless of whether the children remember which Soma figures were used they know that the figures were assembled from two different ones since these pictures are their own creations. These semiotic codings become puzzles for their peers to decode using only the set of seven Soma figures. Examples are shown in Table 1. Some students drew 4-by-4 grids (as in C and D), which is a throwback to the grid provided by the Geocadabra Construction Box interface. Others (as in A, B and E) understood that rows or columns containing zeros were not needed. The child who created grid B enumerated the cubes in her structure, reflecting on work that had been done some weeks prior to this activity, to verify the volume of her structure.

To begin some puzzle solvers use enumeration strategies. A sum of 7 must mean that Soma #1, the only figure with 3 cubes, is used in combination with a 4-cube Soma figure, as in puzzles A and D. Some children are able to directly state which two Soma figures can be used to solve the puzzle. Others mentally visualize at least one Soma figure in the coding. The children quickly realize that selecting two random Soma figures is a time-consuming and ineffective strategy. A more effective scaffolding strategy discovered by some children is to use Soma #1 together with a loose cube. For example, in Puzzle B, the left side appears to be Soma #7 (see Figure 6). Then, the child places Soma #1 on the right side and holds a loose cube to its left as shown in Figure 6, to create Soma #6. Some children use one or two loose cubes in their interim solutions before they arrive at the assembly solution.

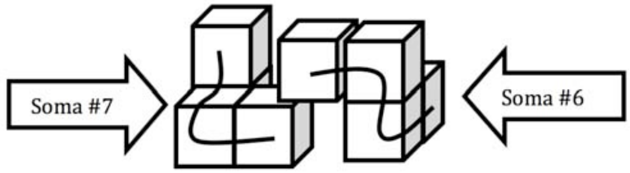


Figure 6. Using Soma #1 to solve puzzle B

While the 2D task cards were created with a 3D stimulus via the Geocadabra Construction Box computer interface, the reverse process, using the semiotic puzzle as a stimulus and the 3D model as a product, is performed without the aid of the computer interface. The capacity to visualize develops further when children are able to find more than one Soma figure combination to solve these particular problems as shown in puzzles A, C-E (Table 1). Two different children solved puzzle E using three different Soma figure combinations (solutions E1 and E2).

These examples are evidence of problem situations created and solved by the children based on the objects that were previously used as stimuli. Puzzle F is particularly interesting because this child created a unique extension of the original task using his two solutions, shown as E2. He realized that the two Soma figures were the same as the ones he had used for his own assembly shown in F to the far right. He stated that with "two movements" of the Soma figures he needed for puzzle E he could re-create his own assembly.

Following the puzzle posing activity, the teacher researcher displayed for the first time the formal SOC framework as shown in Figure 1 in order to elicit metacognitive interpretation of this graphic organizer from the children. She asked if anyone recognized any of the project's activities within the figure. One child pointed out that the puzzle creation activity (Table 1) started with the 3D model to create 2D pictures on the computer. Another recognized that the numeric top-view codings that they created from their 2D pictures belonged to the semiotic or abstract representation. She also stated that the class had moved from the semiotic to the 3D figures directly without the computer. When asked about the verbal description, one child said that they do it all the time when they think and share solutions together. The participating children have made sense of the SOC framework through the myriad of hands-on experiences over the past 5-6 months without prompting from more-knowledgeable adults. This is strong evidence that through carefully designed activities, using strategically chosen manipulatives, deep mathematical knowledge (Connell, 2001) and generalized abstraction (Presmeg, 1992) can develop in unprecedented ways. Furthermore, this occurrence demonstrates how the instructional team attends to and accommodates child-centered contributions to the development of the learning trajectory.

Conclusions

The capacity to move among the three SOC visual representations, namely, 3D models, 2D conventional pictures and semiotic representations, was developed through the Geocadabra computer interface. However, through extended problems created by the learners themselves, it is remarkable that they are able to move from the semiotic representation to the 3D model independently of the computer (Connell, M., personal communication, September, 2010). The research team considers this to be evidence of the children's growing ability to visualize as they move between semiotic and 3D assembly models. Through their different configurations the Soma figures provide a high degree of complexity to the instructional tasks and force the children to engage in mental transformations in ways that would not be possible if they used loose cubes.

The actions on objects (Connell, 2001) occur concretely with the 3D models, virtually through the Geocadabra interface and ultimately as mental imaging through the powerful problem-solving approaches developed by the research team. These evolve through the reflective practices enacted by the team immediately following each lesson (Sack, & Vazquez, 2011). The team has a very strong child-centered philosophy, coupled with the belief that children learn best when engaged in problems of their own creation (see also Connell, 2001).

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DEVELOPING ELEMENTARY TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF RATE-OF-CHANGE IN ENGINEERING THERMODYNAMICS: A DESIGN STUDY

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The developing pedagogical content knowledge of elementary-certified teachers was supported in a course on concepts in algebra and calculus. Engineering thermodynamics was used as a domain through which participants could model rate-of-change for a variety of functions. The design of the professional development course is examined, and teachers' struggles and triumphs with course content described. Results show teachers' backgrounds in elementary algebra proved a barrier to their ability to treat relationships among variables as mathematical objects. Once these barriers were overcome rate-of-change served as a powerful support to model complex thermal applications that utilized rate-of-change as an index of efficiency.

This paper describes a design-research project that focuses on developing the mathematical knowledge for teaching of elementary-certified teachers who are interested in becoming STEM specialists in the middle grades. The case described is of the design of a graduate-level course focusing on the mathematics of change and how it relates to engineering thermodynamics. We first describe our approach to teacher knowledge, and why pedagogical content knowledge is the key lever for practical change in the classroom. Then we address design-theory and why it is an appropriate paradigm for studying teachers' mathematical learning at the same time as we attempt to improve it. Next, we describe the course design, dwelling on modeling as the pedagogical approach to the course, the mathematical content and its sequencing, and engineering thermodynamics as an ideal context for the development of essential concepts of calculus. Lastly, we describe the ways in which teachers interacted with this content, and particularly how they grappled with their "fuzzy" prior knowledge of algebra, attempting to repair holes in their understanding, even while continuing to move forward on more advanced content.

Pedagogical Content Knowledge

Teachers' knowledge has long been promoted as a lynchpin variable, connecting standards and curricular innovation with classroom practice and student learning. Ostensibly, teachers' understanding of content, pedagogy, students, and curriculum enables them to better design instructional environments, tailor activities to the needs of their students, and to assess and improve their practice for the purpose of increasing student learning outcomes.

Lee Shulman and colleagues are generally credited with the initial promotion of teacher knowledge in this lynchpin role (e.g., Grossman et al., 1989; Shulman, 1986; Wilson, Shulman, & Richert, 1987). They generated a number of descriptions of teacher knowledge as an attempt to show teachers' knowledge as multi-faceted, drawing from a number of experiences in their early lives as well as university coursework, teacher education, and practice (Shulman, 1986, p. 9). In general, research has focused on three categories of teacher knowledge: (a) subject matter knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge. Content knowledge refers to both subject matter knowledge and its organization or lack thereof, while pedagogical content knowledge involves content knowledge as it is directly related to the teaching of specific subject matter. Finally, curricular knowledge attunes to knowledge of the programs of study and instructional materials and technology used by teachers and students.

There is some evidence from certification studies as well as large-scale surveys and quasi-experimental studies that indicates that teachers' knowledge has a significant impact on student gains in achievement over time (Darling-Hammond, 2000; Hill, Rowan, & Ball, 2005). This information is mixed, showing mathematical content knowledge as only moderately correlated with student achievement gains. Pedagogical Content Knowledge also shows a significant moderate correlation with student gains separable from mathematical content knowledge. This separability supports Shulman and colleagues' initial assumptions that several funds of knowledge interact to impact instruction and subsequent learning.

More recently Hill, Ball, & Schilling (2008) argue that the knowledge required for teaching mathematics, Mathematical Knowledge for Teaching (MKT), is multifaceted, incorporating aspects of: 1) common content knowledge held by all mathematically sophisticated occupations, 2) content knowledge specialized to the teaching of mathematics, 3) knowledge of student learning of mathematical content, 4) knowledge of the practices of teaching mathematics, and 5) knowledge of mathematics-related curriculum. Each of these knowledge components has some effect on teachers' ability to develop, select, and deliver tasks at an appropriate developmental level at the appropriate time in mathematical sequence and student learning. Moreover, each of these components supports and updates the other components as teachers' learning of mathematics and mathematics-relevant pedagogy grows. It is likely that, unless each of these aspects of MKT is addressed, and their interdependence emphasized, that professional development for mathematics improvement will result in only modest student gains over only a short term (Ball & Bass, 2000).

Of these facets of MKT, pedagogical content knowledge is of primary concern because it lies at the confluence of the content and the curriculum; it is the knowledge base of mathematics education: PCK comprises the knowledge and skills required for one to transform the other facets of MKT (like mathematical content, knowledge of curriculum, knowledge of students) into a set of experiences, activities, and environments that optimize the likelihood of students learning. In our program, we attempted to improve elementary teachers' content knowledge related to fundamental concepts in algebra and calculus utilizing Modeling Instruction as the pedagogical theory for both teacher learning and for student learning. It was by integrating Modeling Instruction with teachers' own content learning experiences that we hoped to impact their ability to understand and apply this content in an appropriate pedagogical manner with their own classrooms—hypothetically connecting new content with new knowledge of curriculum and pedagogy—addressing three of the five aspects of MKT through PCK.

Modeling as an Integrative Construct

One of the key goals for professional development is for teachers to develop new models of content, teaching, and curriculum. Models are conceptual structures—mental (and oftimes physical or inscriptional) representations of real things, real phenomena. *Modeling* as it refers to content knowledge, involves building, testing and applying conceptual models of natural phenomena and is a practice that is central to learning and doing science and mathematics (Hestenes, 1992). In fact, it can be accurately stated that Mathematics, Science, and Engineering are all fundamentally modeling enterprises. Educationally, modeling has been touted as a unifying theme across science and mathematics education as recommended by both the National Science Education Standards (1996) and the Common Core Standards for Mathematics (2010).

In sum, we think with models, we think through models, and we think about models. Our approach to instruction attempted to embody this fact and exploit this facet of human cognition.

Modeling Instruction as an instructional theory aims to correct many weaknesses of the traditional lecture/demonstration method in science and mathematics education, including the fragmentation of knowledge, student passivity, and the persistence of naive beliefs about the physical world. Modeling Instruction objectives begin with the goal to engage students in understanding the physical world by ***constructing and using scientific models*** to describe, to explain, to predict and to control physical phenomena. To do this, students must gain proficiency with ***basic conceptual tools*** for modeling real objects and processes, especially mathematical, graphical and diagrammatic representations. This is the underlying curricular model we employ at all levels in the reported program: Build and use models of important scientific content, utilizing important mathematical and representational tools.

It has been shown in a variety of studies of student learning that context is critical for coming to understand mathematical concepts and skills. Moreover, the capacity to create models of scientific phenomena, and to test those models is dependent on the development of mathematical ways of thinking about the phenomena, including the ability to make sense of patterns in data. For this reason, modeling truly is an integrative construct, connecting mathematics and scientific content through meaningful activity.

Unfortunately, teachers are not exposed to mathematical modeling of scientific phenomena during their undergraduate careers to any great extent. This is especially true for middle school teachers. With typically one course in mathematics at the College level, and only one non-calculus-based science course, middle school teachers trained as generalist elementary education majors are just not equipped to handle the traditional mathematics curriculum let alone Algebra I and Geometry which are currently common offerings in US schools. The same can be said for middle school science, where teachers may teach life science, physical science and earth and space science in any given year. It is therefore imperative that the curriculum and instruction that takes place for teachers closely resembles that which they are expected to provide for their students—yet not be so prescriptive as to squelch personal style, initiative and inspiration. We structured our classroom tasks and pedagogy based on the past 20 years of research on modeling (citations from Lesh, Hestenes here).

Design Research and Curriculum Development

Lamberg & Middleton (2009) describe how design theory can provide a rigorous paradigm for disciplined inquiry in learning and instruction, and a pragmatic framework for defensible change in curriculum and instruction. Briefly, this theory outlines seven "phases" through which a research program progresses, beginning with more grounded methods, where the researcher attempts to understand a little-researched area of inquiry, consolidating evidence about learning and practice, until a local theory of change can be generated that seems to explain and predict positive movement along a trajectory of change. This *theory*, then is used to design curricular tasks, which serve as operational definitions of that theory. Through iterative testing and revision, tasks are made more effective, while at the same time, the theory is refined, refuted, and made more useful and explanatory (Author, 2008). Design research, thus has two primary outcomes: A theory of change that describes how learning progresses across a defined set of topics, and a mechanism for at least partly driving that change—curriculum materials or other supports that have been refined over time to be optimally (or at least *reasonably*) effective.

Design theory is a kind of modeling theory that focuses on the development of systematic knowledge about teaching and learning. As such, it is consistent with the theory of learning and instruction we employed in the reported research: Modeling Instruction. The study reported here is situated in Phases 2 and 3: Development of the Artifact (in this case, the course curriculum and activities), and Feasibility Study (initial trialing and iterative improvement of the course).

Description of Modeling Course

Participants

The focus teachers in this study were 11 elementary certified teachers. Two additional students (masters' degree graduate students in mathematics education with no prior teacher training) participated as well. Of the teachers, two had a calculus course early in their collegiate experiences, but the rest had no mathematics beyond basic College Mathematics and Mathematics Methods for Elementary Teachers. All were female, and all were teaching elementary grades subjects. On a pre-Calculus assessment, none scored higher than 50%, and the majority scored around the 20% or chance level.

Course Foci

The modeling course for teachers focused on developing key mathematical content related to modeling scientific and engineering concepts. We emphasized discussing the most common linear, polynomial, exponential, and inverse square functions and their representations, focusing especially on the analysis of change in the context of applied problems. We focused on these concepts in the context of modeling situations of thermodynamics, which is the study of energy transformation. We chose thermodynamics because we hypothesized that most of the teachers had everyday experience with heat and temperature, energy, pressure and work, whether formally in a classroom or informally solving problems in their daily lives. We emphasized convection and conductance as key ideas in thermodynamics because they were the most accessible concepts in thermodynamics while also providing convenient ways to talk deeply about rate of change: Newton's law of cooling for convection and thermal exchange in conductance both are examples of exponential functions, while radiation follows the classic inverse-square law. Our goal was to formalize the in-service teachers' knowledge while exploring applications of thermodynamics in engineering contexts, environmental science, and biology such as passive solar heating and cooling, properties of insulators, and heat capacity of materials.

Thermodynamics is an area of science where several key variables, such as entropy, cannot be measured directly; one can only measure them by examining related variables and doing the algebra. So in this case the mathematics is absolutely critical for understanding the science—thermodynamics cannot be understood at even a basic level without it. In general, we wanted to emphasize the deep connections between mathematics and science, and help the teachers understand how one can better understand science by knowing the mathematics.

Structure of Course

We attempted to structure the course so there was a common substrate of activities in each homework assignment and weekly meeting of the course. Each week, teachers were engaged in the following tasks:

Book club write-ups. Students read 1 to 3 chapters from *Calculus Made Easy* (Thompson, 1914) each week. To get them to reflect deeply on the content of that book, students were required to write a review of the chapter(s). In their review, we asked them to focus on what they know, what they wanted to know, and what they had learned: 1) What you know: A description of the reading, as if you were describing the concepts to an 8th grade algebra student; 2) What you want to know: Analysis and insights you have about the study of change, and especially those things you feel confused about; and 3) What you have learned: Students listed those ideas and ways of thinking they had that they didn't have prior to reading the text.

Weekly Modeling Problem. In each class session, we presented the students with a new thermodynamics problem to model. Following class, they were required to do a formal lab write-up of their model. Their analyses addressed the following points: 1) A description of the relationship they attempted to model; 2) A description of the methods they used to model the relationship; 3) The problem solution presented in a general form (e.g., a rule, procedure, or symbolization, which was often a combination of formulae, graphs, and data tables); and 4) An analysis of why the problem solution was correct.

Problems emphasized convection rates, conductance/insulative properties of materials and radiation rates, which mathematics paralleled the content being developed in the Book Review assignments, but required application in messy, data-rich situations.

Weekly Mathematics Problems. Finally, based on the modeling problem and ideas introduced in the book club, we presented students with a set of algebra and calculus problems to attempt. Their response to each problem included all their work, including mistakes, and their answer, if they arrived at one; and a reflective statement describing the core mathematical idea the problem was addressing.

For the reported course, we organized the mathematics content into 7 modules:

- Proportional Reasoning: Direct, Inverse, and Joint Variation
- Average Rate of Change: $\Delta y/\Delta x$, Analyzing Functions Qualitatively, Rational Expresssions
- Families of Functions: Linear, Polynomial, Exponential/Logarithmic/Power
- Comparing Functions within Families: Coefficients, Exponents, Bases, Local Maxima/Minima, Inflection Points
- Derivatives as local average rate of change
- Derivatives at a point; Derivatives as Functions; Limits
- Accumulations; Area under a curve

Structure of Daily Sessions

Each 4.5-hour class session was broken into three 1.5 hour activities. One activity focused on discussion of the week's readings and calculus concepts. The second activity focused on discussing any difficulties participants had with understanding and completing homework activities. The third activity engaged teachers in modeling thermodynamics problems and applying the readings and problems.

Each modeling session was organized into ***modeling cycles*** which move students through all phases of model development, evaluation and application in concrete situations — attempting to promote an integrated understanding of modeling processes and acquisition of coordinated modeling skills (Lesh & Yoon, 2007). Typically, the instructor set the stage for classroom activities, ordinarily with a demonstration and class discussion to establish common understanding of a question to be asked of nature. Then, in small groups, students ***collaborated*** in planning and conducting experiments, collecting and analyzing data to answer or clarify the question. Teachers were required to present and justify their conclusions in oral and/or written form, including a ***formulation*** of their own models for the phenomena in question and an ***evaluation*** of the set of models by comparison with data. Technical terms and concepts were introduced by the instructor only as they were needed to sharpen models, facilitate modeling activities and improve the quality of discourse.

Additionally, in 3 of the 8 sessions, we carved out an hour to examine the Common Core Standards for Science and Mathematics for middle grades, and map the key content they were learning in the course to the key mathematical ideas emphasized by the Standards. Teachers developed maps of the content to show how it fit together in an integrated fashion in science and mathematics classes.

Analysis of Sessions and Course Revision

The research team consisted of three mathematics educators with engineering backgrounds, and an expert in teacher professional development. The research/teaching team met each week following the daily sessions. In the manner of Cobb et al (1997), we reviewed student work, planned subsequent sessions, and revised tasks and assignments based on the discourse engendered in the daily sessions. In particular, we reviewed the difficulties students had in their conceptual understanding of the algebra, and their facility with its manipulative skills, and developed new tasks to help them bridge this knowledge, while still forging ahead thinking about derivatives and rate-of-change as a function.

Sources of data included their homework, book club reflections, modeling problems, and their posts to the class discussion group, which we developed to be able to handle ad-hoc questions just-in-time.

Results and Discussion

We set out to focus on a mathematically rich course focused on thermodynamics, and eventually we were able to talk about difficult mathematical concepts to explain and model scientific phenomena. However, the level of resistance and frustration the teachers exhibited in the first three weeks of the course was palpable. Their homework reflections and midterm course evaluations suggested that they were not thinking about modeling phenomena. Instead, they were highly frustrated or confused about concepts like algebraic field properties and representation of proportional quantities using fractions. We used their class discussions, homework assignments, and book reviews to gain insight into what sense they were making of modeling thermodynamics ideas using mathematics.

We found that the majority of students' ways of thinking about the mathematics was not at a level where discussions about rate of change could be interesting or productive to the teachers. Given our data and feedback from the teachers, we concluded there were two major obstacles to their understanding. First, a chief barriers we faced was the confluence of factors associated with teachers' knowledge and the nature of disciplinary knowledge and how these factors could be made compatible and even complementary. Second, there was a real disconnect between the community of mathematics and that of the sciences regarding what was being modeled and what its conceptual substrate is. With motion and mechanics the connections between the particular functions being modeled is pretty close, but when there is no macroscopic motion to serve as a grounding metaphor (i.e., in thermodynamics), students must rely on more abstract understanding of variable quantity, function, rate, direct and indirect proportions, and the combinations of these ideas as they are manifest in the typical power functions and exponential functions we see in science applications not derived from motion examples.

Teachers' Content Knowledge

Most of our students had not taken a course in mathematics, or had significant content professional development since their 1st or second year of college. Most teachers, because they are elementary certified, took required courses up through college mathematics, and therefore have little experience with algebraic methods, functions and functional reasoning, the mathematics of change, or geometry and its relationship to algebraic relations. As a result, teachers' knowledge of and understanding of variable, function, field properties and their tie to the manipulative skills of algebra, basic concepts of exponents, factors and multiples, and proportional reasoning was generally poor and uneven across the cohort.

Nearly all teachers possessed poor mathematical self-efficacy. Reasons for this include poor prior performance resulting from poor prior learning experiences, knowledge that their coursework was not sufficient for a developed conceptual understanding and procedural skill, and the fact that much time has passed in which they have not continued mathematical learning nor practiced mathematical skills previously learned. As a result, teachers' confidence in their conjectures and insight appeared to be tentative; their ability to judge their learning and understanding (meta-cognition) was under-developed.

Given the breath and depth of these problems, we took steps taken to deal with these issues. We immediately stepped back from a pre-Calculus level of instruction to a high school algebra II level, to an 8th-grade pre Algebra/Algebra I level with some high school ideas integrated. We focused on scaling back on the quantity of homework to allow for students to focus on ideas discussed in class, and developed homework activities based on class discussion the day following, so that teachers can do relevant work the following week. We used 3 of 8 to focus on the most basic ideas of quantities, algebraic manipulative skills and their conceptual basis in the field properties, recognizing the form of familiar functions (linear, polynomial, exponential). Once we helped the teachers construct a foundation of basic mathematical concepts, we used rate of change as an integrative concept to help students look at different regions within the same curve, and the same regions across different curves, to examine the behavior of these functions, and their similarities and differences.

Teachers' Disciplinary Knowledge

In science, the mathematical models developed can be derived from specific circumstances to assist the student to be able to describe their behavior. In mathematics, mathematical models must have a generality across applicable situations—this implies that, if an idea is developed within a context, there must be other application contexts where the student is expected to transfer their knowledge, and some opportunity to make the mathematical abstractions explicit and their transferable structure overt.

Much of school science is done without any mathematics or any significant mathematics. This make students not lazy per se, but the teachers were disinclined to mathematize or to frame scientific questions or conjectures in mathematical terms other than just basic correspondence relations. Functional reasoning was not a habit of mind. Overreliance on solving for a number detracted the teachers from the examination and description of relationships. We believed that the general practice of making science "conceptual" typically eliminates the critical mathematical structure of the concepts in favor of more qualitative or general depictions of phenomena

The use of software for scientific data analysis was tremendously helpful to both the teachers and us. However, the learning curve for students to learn the software takes a lot of time and focus away from the mathematical content itself. Graphing Calculators are excellent tools, but a distinct effort had to be made to develop some proficiency with these tools. Excel was also excellent, but the teachers had little to no experience with spreadsheets or other data systems, so this also took much of their time and energy. The important aspects of these programs are to allow students to see the form of relationships, to create an explicit algebraic relationship by manipulating cell formulae (excel) and lists (graphing calculators). As the semester progressed, the teachers moved from viewing technology as a barrier to understanding to using it as a tool for thinking about modeling complex thermodynamic situations.

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TEACHING STUDENTS TO ESTIMATE PROBABILITIES: THE FREQUENTIST APPROACH AND ITS RELATIONSHIP WITH STATISTICAL UNDERSTANDING.

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An interview with a sixth-grade student illustrates how her number sense and understanding of variability relate to her ability and proclivity to apply a frequentist (statistical) approach to probability tasks. A general suggestion for teaching about mathematics of uncertainty through the gradual strengthening of estimation, as per the historical development, is also discussed.

Theoretical Background

The Three Views of Probability

There are three main views of probability: classical, frequentist, and subjective (Shaughnessy, 1992). Using the classical view, one first partitions a sample space into equally likely outcomes. The probability of an event is simply the ratio of the number of outcomes in which that event occurs to the total number of outcomes. In contrast, the frequentist approach to probability involves repeated trials. A person using a frequentist approach might conduct a simulation with a large number of trials, examine the data, and assert probabilities based on the observations. If the number of trials is large enough, and if the results are repeated in other contexts, then the probability is judged reliable. A classical approach examines *a priori* how different arrangements of events could happen in order to develop a uniform distribution model. The frequentist approach is mathematically more related to statistics, since it involves the search for a distribution and subsequent application of the distribution's properties. Thus the mathematics behind the frequentist approach tend more to the notions of limits and convergence, as relating to the law of large numbers (Shaughnessy, 1992). The third view of probability, the subjective view, also takes into account an individual's own knowledge, opinions, or feelings. Reliance upon subjective reasoning may signal misconception, lack of confidence, or uncertainty of the relevant mathematics, but its use is not necessarily irrational. A child might always express a favorite color to be most probable on a spinner, while some situations, such as the probability of a Mars landing in the next century, can only be estimated by a subjective approach (Jones, Langrall, & Mooney, 2007).

Development of the Mathematization of Statistics and Probability

The oldest examples of statistical thought each related to the concept of estimation (Bakker, 2003). Examples from Indian, Egyptian, and Greek stories contained phenomena similar to the *mode*, *mean*, and a measure called the *midrange.* For data that has a symmetric distribution, the mean, median, mode, and midrange all coincide, so there is no need for their distinction. Bakker (2003) found, using classroom teaching experiments, that modern-day students also benefited from beginning with estimation while learning measures of center. It was not until students were faced with the task of computing an average with non-symmetric data that they felt the need to develop and formalize other methods of average.

The parallel between the historical development of average and the historical development of probability is the original expectation of symmetry (or uniformity), the subsequent adjustment to increase the accuracy (or number of successes), and the reliance upon estimation. The oldest manuscript describing observed frequencies and non-uniform distribution was written in the 13th

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century, while the first known and solved probability problem (by Galileo) took place about 1620 (Batanero, Henry & Parzyz, 2005). The conflict between theoretical calculation and observed frequencies is what led to the development of more rigorous combinatorics methods (Batanero et.al., 2005). A teaching experiment using non-uniform dice by Nilsson (2009) showed that students went through a similar process when their previous conception of uniformity was challenged by observation. When the amount of variation was too great, the students reexamined their assumptions based on empirical data and modified their theoretical model.

Probability and Statistics Frameworks

Jones, Langrall, Thornton, and Mogill (1999) developed a framework for the development of probabilistic thinking through the middle grades by observing students' responses to tasks that could be solved using classical probability. The framework consists of four levels: subjective, transitional, informal quantitative, and numerical. At level one, a student might only be able to recognize certain or impossible events; at level two, most or least likely events. By level three, students' quantitative reasoning and measures are used to describe likelihood. By level four, students are able to assign numerical probabilities (Jones et al., 1999). As a student's level increases, the tendency to use subjective judgments decreases. Polaki (2002) validated this framework in a study in South Africa, and found that the highest levels of probability thinking require part-whole reasoning, while students without even part-part reasoning generally operate in the subjective level.

Watson, Collis, and Moritz (1997) describe a complementary framework of probabilistic thought. There is a hierarchy of four levels: prestructural, unistructural, multistructural, and relational, and students must pass through two cycles within the levels to achieve the highest realm of probabilistic thinking. The first cycle involves the development of probability as a measure, while the second cycle involves the development of that measure. Considering that the second cycle requires part-part reasoning for the multistructural level and part-whole for the relational level (Watson et al., 1997), this view is compatible with that of Jones et al. (1999). The additional levels of the framework more clearly describe the phenomena of students' demonstrating a relational level of reasoning for a single task, while a subjective level for a multiple-part task.

Both of these frameworks support the view that probabilistic instruction should begin with part-part comparisons instead of part-whole relationships. However, the classical view of probability necessitates part-whole comparisons in order to declare a uniform distribution (at least implicitly). Since comparing frequencies of outcomes is essentially performing a statistical task, perhaps further insight into the requisite knowledge for learning probability can be obtained by studying existing frameworks for the learning of statistics. According to Shaughnessy, "there are important connections between probability and statistics, particularly when repeated trials of probability experiments generate a distribution of possible outcomes" (2007, p. 981).

A relevant connection is the interaction between students' understanding of expectation and variability. Watson, Callingham, and Kelly (2007) developed a framework for the understanding of expectation and variation with six levels ranging from idiosyncratic, with little or no appreciation of either variation or expectation, to comparative distributional, in which links between variation and expectation are established in comparative settings with proportional reasoning. To compute simple probabilities with the frequentist approach, only the fifth level of statistical reasoning is needed: understanding of the relationship between variation and expectation within a single context. It is at this level that students can articulate an expectation; at levels four and below, they are able to articulate only comparative aspects such as more or less likely. Thus the higher levels of this framework are distinguished by the use of part-whole reasoning, as was the case with the probability frameworks. In order to use the frequentist approach to *develop* a theoretical distribution, one must determine a permissible amount of variance between observed frequencies and a theoretical distribution, i.e., a search for a signal of variability (Shaughnessy 2007). This requires the highest level of statistical reasoning, for it necessitates the comparison of distributions across two groups: the observed frequency and the expected frequency (Watson et al., 2007).

Purpose

The purpose of this study was to examine the relationships between students' understanding of probability and statistics. The frequentist approach to computing probabilities relies on an understanding of what constitutes an acceptable level of variance. On the other hand, students' expectations for variance may be influenced by an assumption of uniform probability. Thus we sought to ascertain whether the students' level of statistical understanding could help explain their level of probabilistic understanding, and consequently further understand the role of the teaching of statistics and variability in teaching probability.

Methods

An extended clinical interview was conducted with one sixth-grade female from Virginia. The interview was divided into two sessions, each lasting approximately thirty minutes, involving a total of nine tasks. The first session, consisting of the first seven tasks, was designed to gauge the student's level of probabilistic thinking as per the frameworks of Jones et al. (1999) and Watson et al. (1997); questions and activities similar to their released items were used to assess the student's probability thinking. In order to ascertain the student's level of statistical thinking, tasks similar to those described by Watson et al. (2007) and Bakker (2003) were used. Each task was presented orally by the first author who served as the teacher researcher throughout the study. Manipulatives available to the student included physical dice, pencil and paper, and virtual spinners. Following are the tasks presented to the student in the first session.

1. If you were to roll this die (student is presented with a six-sided die), do you think it's easier to roll a one or a six?
2. In mathematics class, there are 13 boys and 16 girls. If a teacher were to write the names of the students on slips of paper and draw one out of a hat, would it be more likely that the name would be that of a boy, that of a girl, or is it equally likely?
3. There are two boxes, Box A and Box B. Box A has six red marbles in it. Box B has 60 red marbles. Box A has four blue marbles, while Box B has 40 blue marbles. If you want to pick a blue marble, from which box should you choose?
4. Create a possible graph depicting the monthly high temperatures in your hometown, given that the average yearly high temperature was 69 degrees.
5. A game is played where two spinners, each 50% black and 50% white, are spun. If they are both black, then person A wins; if they are different colors, person B wins. Which person would be more likely to win?
6. Estimate the number of penguins in a photograph in which the penguins are not of uniform size.
7. Ten Twizzlers of different colors are placed in a bag, and the student is asked to guess which color is most likely to be drawn out.

Each session was videotaped, and the authors planned additional tasks after discussing their individual interpretations of the video. The second session consisted of an extension of Task 4 and a new task that merged two dice throwing tasks, the first by Watson and Kelly (2007) and the second by Nilsson (2009). These were designed to explore the role of variance in the student's perception of the likelihood of outcomes.

The focus of the first session was that of a clinical interview: both the questions and the sequence of tasks were chosen in advance, and no attempt to teach the student was made. The second session was more of a teaching experiment (Steffe, 1991), as the student was encouraged to examine her notions of variance and expectation in the following tasks.

8. Create and critique possible graphs depicting the average yearly temperature.
9. Predict the results of rolling a six-sided die 60 times. The dice in question were (a) fair, (b) had two ones and no sixes, (c) were "loaded" in favor of ones.

Results

Probability Tasks Results

The student response to the first task was that it was equally likely to roll a one or a six: "because there are the same number of sides, so I don't think it really matters the number."

The second task response was "a girl, because there's more girls than boys." This correct response using part-part reasoning indicates achievement of the multi-structural level of Watson's framework (Watson et al., 1997) and the transitional level of Jones' framework (Jones et al., 1999).

According to Watson et al. (1997), a correct response to the third task is associated with the relational level of reasoning. The conversation associated with this task proceeded as follows:

Student: I would think they would be the same, but, ... maybe Box B just because it has a bigger number?

Interviewer: What makes you think they might be the same?

Student: Because that has 10 and that has 100, so out of, like, 100 percent would be 60 percent for both of them, to 40 percent.

Interviewer: What made you think that maybe Box B would be the one to pick?

Student: Ummm...maybe because it has the bigger number. I don't know.

The student's initial correct response to the task was based on proportional reasoning, which is requisite for the higher levels of probabilistic thinking in both Jones' and Watson's frameworks. The fact that she thought that Box B having a bigger number might make it the one to pick suggested that there might be some confusion over the law of large numbers. It was suspected that the student might believe that having "large numbers" is desirable when computing probabilities, but that she had not yet conceptualized a justification.

The teacher researcher then presented her with the option of simulating the task using Probability Explorer (Lee, 2005) because he wanted her to become familiar with the program for a future task and also wanted to see her reaction to the result of an experiment. Specifically, he wanted to see if the result of the experiment would help her make a decision. The conversation continued:

Interviewer: Which do you think is more likely to come up, red or blue?

Student: Red.

Interviewer: Why?

Student: Because there are more reds than blues.

Interviewer: [Clicking the button to simulate a grab, which was blue.] Why do you think it came up blue instead of red?

Student: I don't know, I'm not sure.

Interviewer: What if I click it again, do you think it will come up blue again, or do you think it will come up red?

Student: I'm gonna guess red. I guess it's a fifty-fifty chance because there's two colors. I don't know.

The inconsistency in the responses here is indicative of the transitional level in Jones' framework. The simulation didn't help her decide an answer - in fact, it encouraged her to even question the part-whole reasoning with which she earlier seemed comfortable. Here the term “fifty-fifty” seemed to mean that it could be either of two outcomes rather than a rejection of the part-part reasoning earlier displayed. This language pattern has often been noted by researchers (e.g., Jones et al, 1999; Shaugnessy & Ciancetta, 2002). The uncertainty of her response seems to indicate a lack of knowledge or confidence in the relationship between her theoretical model and empirical trials.

The student again said that there was a “fifty-fifty” chance that each person would win when faced with the fifth task on spinners. This was not surprising, since it involves a compound event and hence would require the higher levels of probabilistic thinking (Jones et al., 1999). When presented with a simulation in which the spinners came up differently seven times out of ten, the student decided that choosing different colors probably had an advantage. This aligns with Shaughnessy and Ciancetta’s (2002) results which indicated that playing this game and seeing variation could help students reject their equiprobable hypothesis.

The seventh task was also aimed at understanding the student’s level of probabilistic thinking. She estimated that blue would be most likely to be drawn out since there were more blues than any other color. A blue was drawn and not replaced, leaving 2 green, 2 blue, and 2 yellow. She was asked again what color was most likely to be drawn, and she said "blue, green, or yellow" because they had the highest number. When the blues were replaced, she said that the chance of a blue would increase because the number of blues increased. This is evident of a relational understanding (comparing the possibility across two sample spaces).

Overall, the student's level of probabilistic reasoning appeared to be “informal quantitative” in the framework of Jones et al. (1999). She displayed use of part-part reasoning throughout and at times exhibited part-whole reasoning such as percentages. She was able to compare sample spaces and make relational judgments such as “more or less likely,” while at the same time used subjective judgments for compound events. This indicated that she was in the first-cycle relational stage or second-cycle idiosyncratic stage in the framework of Watson et. al. (1997).

Statistics Tasks Results

When creating a scale for her graph in Task 4, the student made the lowest value 30 degrees and the highest 80 degrees “because I think the coldest it would get is around thirty” and “eighty is probably about the highest it gets around here.” Bakker (2003) observed similar behavior when asking students to explain their understanding of “average,” in which several students gave a response indicative of a mid-range concept.

The student seemed to display an expectation for variation in temperatures, as the temperature trends cooler in the winter months and warmer in the summer, but the overall mean of the temperatures she displayed was much lower than 69 degrees. It appeared that she was focused on the variation and the range but not the mean. Based on these initial findings, it was decided to follow up in the next session with other graphs to compare to hers in order to determine her level of understanding of variability and expectation via the framework of Watson et al (2007).

When judging the number of penguins in the sixth task, she originally did not want to guess, but said "two million" upon encouragement. I believe that the hesitation toward calculation was due to the fact that the penguins were not uniform in size and were in rows of non-uniform width. She was offered a ruler, and encouraged to create a way to estimate using proportional reasoning. She counted the number of penguins in the bottom row, estimated visually the number of rows, and multiplied to get an estimate of 1,200 penguins. This was similar to the method that Bakker (2003) found children using to estimate the number of elephants in a picture, although the students in his study first found an "average" block. She made no mention of the fact that the penguins in the row she chose were larger than the penguins in the other rows, even after that error led her to a much smaller estimate than she had originally guessed. Her behavior in this task was opposite of her behavior in the temperature-graphing in which she initially focused on the mean and subsequently on the variation. In the penguin counting she was initially focused on the variation (causing the difficulty in counting) and subsequently ignored the variation when performing the calculation.

In preparation for Task 8, the student was introduced to four graphs of supposed student work and asked to critique them as possible graphs for the average high temperature. The first graph depicted a uniform temperature of 69 degrees, which she decreed unlikely since "February was too warm…and August is too cold." The second graph was also unlikely, for although the temperatures varied, they did so linearly, which was "too perfect." However, she did express that the first two graphs were possibilities. The third and fourth graphs both had varying temperatures, and neither graph increased or decreased linearly. However, by putting a pencil across the graph at 69 degrees, the researcher showed her that graph 3 had substantially more data below the line than above the line (this would technically be comparing to the median, but the data is nearly symmetric and hence the mean and median are close). It was thought that this would have led her to believe that it was not a possible graph of the data with mean 69 degrees, and she did make the observation that "it was too low." She drew a similar line using the fourth graph, and found about the same number of temperatures above and below the line. But when asked whether the third or fourth graphs were more likely, she chose the third because in the fourth graph the change in temperatures was closer to linear. She seemed to value the observed variance and high probability of "randomness" from month to month more than the presence or absence of the desired overall mean.

When faced with the die rolling tasks, the student was first asked to write down her prediction of how many of each number would come up. She said that she expected an equal number, 10, of each of the six values on the dice, which is indicative of level 2 on the scale of variation used by Watson and Kelly (2007), a "strict probabilistic prediction"(p. 3). When she rolled a standard die, six came up 17 times, which she attributed to chance. When asked if she would like to revise her prediction, she declined and indicated that she still thought each number would come up ten times if she rolled again.

The results of sixty throws of the second die (biased in favor of one) included a total of 21 ones and zero sixes. It was expected that the student would think that having no sixes was very unusual, and that she might wonder about the fairness of the die. She said that she thought it was "weird," but once again attributed the outcome to "chance" and declined to revise her prediction.

The roll of the third die (weighted) resulted in 29 ones, this time with two sixes. Once again, the student was not surprised by the result, attributing it to chance. It was expected that she would immediately question the die, for Nilsson (2009) found that students noticed unexpected frequencies with non-uniform dice and resolved the conflict between theoretical and empirical probability by developing an empirical model.

The teacher researcher felt that it might not occur to the student that some of the dice might be unfair, so she was encouraged to examine the second die more closely. After she noticed that there were two ones and zero sixes, she explained that she was no longer surprised about the presence of 21 ones. “The ten from the sixes went to the ones and 21 is near 20.” She was then asked what might explain the 29 ones in the third trial. She ruled out the possibility that there were any numbers missing with the third die, because she had rolled at least one of each number. The teacher researcher encouraged her to watch the die as it spun, and she noticed that it was “loaded” by the way it landed. When asked to predict the way it was loaded and determine how many ones would be expected, she first suggested that we should expect to get 20 ones, just as in the second problem, but was unable to articulate a reason for this prediction. She was then asked, “If I were to roll this die a million times, how could you predict how many ones would come up?” She suggested that we had rolled “almost 50 percent ones,” in the first 60 rolls, so we could “roll it another 100 times, and see if it came up about half ones.” When asked why she chose 100 times instead of more or less than 100, she said “maybe do more, like 300.” She had suggested the use of a frequentist approach to compute probability.

Discussion

The student's performance on the probability tasks showed initial understanding and preference for part-part comparisons and the occasional use of part-whole reasoning when faced with classical probability tasks. However, when engaged with actual events, she showed a lack of confidence in the application of such comparisons to make predictions by rejecting the results of her comparisons and relying instead on subjective judgments. While her performance fell into the “informal quantitative” stage (Jones et al. 1999), the reliance on subjective probability in simulation showed the value of also considering the student's statistical understanding. Her initial demonstration of a “purely probabilistic” approach to variance shows her view of the calculation of probability as a deterministic exercise. Without further statistical understanding of variance, average, and the law of large numbers, she was inconsistent with both the application of probabilities to make predictions beyond a single event and the reconciliation of her prediction with a contrary outcome.

The interactions with this student led us to believe that after the second session, she became more likely to consider a frequentist approach to calculating probabilities of events that can be simulated. This mirrors the historical development of probability, in which theoretical notions of expectation are strengthened or rejected based on the observation of the frequency of outcomes. Because the student had been exposed to classical probability, she originally focused on the mathematical task of comparing outcomes in a uniform sample space. The performance on the tasks showed consistently that her probabilistic reasoning was in transition between focusing on part-part relationships and focusing on part-whole relationships. The statistical tasks also showed that she had difficulty gauging what was a reasonable level of variation, resulting in her subjectivity in and hesitation toward the rejection of a theoretical model based on trial outcomes. When tasks involved calculations, she ignored variation, and when tasks did not involve calculations, she focused on the variation.

Conclusions

Statistics and probability understanding are connected, for the evaluation of probabilistic claims with the frequentist approach requires one to both understand and expect variance in situations of uncertainty. Students who are taught to calculate probabilities using the classical approach may have difficulty reconciling empirical evidence that differs from their calculations. By exposing children to probabilistic situations in which their intuitions and theories are challenged, teachers can encourage them to evaluate their own and then others' claims. Future research may explore in more detail the relationship between the transitions that students make between levels of probabilistic understanding and understanding of variation and expectation.

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AN INFORMAL FALLACY IN TEACHERS' REASONING ABOUT PROBABILITY

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The main objective of this article is to contribute to the limited research on teachers' knowledge of probability. In order to meet this objective, we presented prospective mathematics teachers with a variation of a well known task and asked them to determine which of five possible coin flip sequences was least likely to occur. To analyze particular normatively incorrect responses we utilized a brand new lens – the composition fallacy – instead of the traditional lenses and models associated with heuristic and informal reasoning about probability. In our application of the new lens we were able to determine that fallacious reasoning, not just heuristic reasoning, can account for normatively incorrect responses to the task. Given the success of the new lens, we contend that logical fallacies are a potential avenue for future investigations in comparisons of relative likelihood and research in probability in general.

The general purpose of this article is to contribute to the paucity of research on (prospective) teachers' knowledge of probability (Jones, Langrall & Mooney, 2007; Stohl, 2005). More specifically, the purpose of this article is to merge the established thread of investigations into comparisons of relative likelihood (e.g., Borovcnik & Bentz, 1991; Cox & Mouw, 1992; Hirsch & O'Donnell, 2001; Kahneman & Tversky, 1972; Konold, Pollatsek, Well, & Lohmeier, & Lipson, 1993; Rubel, 2006; Shaughnessy, 1977; Tversky & Kahneman, 1974; Watson, Collis, & Moritz, 1997) with a developing thread of investigations into prospective teachers' comparisons of relative likelihood (e.g., Chernoff, 2009, 2009a, 2009b).

In order to achieve the general and specific goals detailed above, prospective teachers, as has been the case in past research, were presented with five different sequence of heads and tails – derived from flipping a fair coin five times – and were asked to declare which sequence was least likely to occur. However, unlike previous research, we utilize a brand new lens to account for certain responses; we demonstrate that certain responses fall prey to the fallacy of composition (i.e., because parts of a whole have a certain property, it is argued that the whole has that property). Further, we contend that informal fallacies, in general, create a new research opportunity for those investigating comparisons of relative likelihood.

A Review of the Literature

In mathematics education (and psychology) research, comparative likelihood responses are categorized, in a broad sense, into two particular categories: correct responses and incorrect responses. While correct responses are, for the most part, associated with normative reasoning, different individuals account for incorrect responses differently. In particular, two types of reasoning – heuristic (Tversky & Kahneman, 1974, LeCoutre, 1992) and informal (Konold, 1989) – have dominated the research literature.

Heuristic reasoning

Psychologists Daniel Kahneman and Amos Tversky (1972) asked a group of individuals whether there would be more families with a birth order sequence (using B for boys and G for girls) of BGBBBB or GBGBBG. Further, the same individuals were asked whether there would be more families with a birth order sequence of BBBGGG or GBGBBG. Kahneman and Tversky argued that individuals who declared one sequence as less likely were reasoning according to the *representativeness heuristic*, where one "evaluates the probability of an uncertain event, or a sample, by the degree to which it is: (i) similar in essential properties to its parent population; and (ii) reflects the salient features of the process by which it is generated" (p. 431). In other words, the BGBBBB sequence of births was seen as less likely than the sequence GBGBBG because the ratio of boys to girls (in the parent population) is one to one and the BBBGGG sequence of births was seen as less likely than the sequence GBGBBG because BBBGGG did not appear random. The representativeness heuristic, in essence, provided a new interpretation of normatively incorrect responses to comparisons of relative likelihood.

Informal reasoning

Building upon certain task developments introduced by Shaughnessy (1977) (i.e., including the equally likely option and providing a response justification), Konold et al. (1993) introduced a different version of the relative likelihood task than had been seen in the past. For example, Konold et al. asked individuals "which of the following is the most likely result of five flips of a fair coin?" and provided them with the following options, "a) HHHTT b) THHTH c) THTTT d) HTHTH e) all four sequences are equally likely" (p. 395). Further, the researchers gave students a most likely version of the task followed by a least likely version. They found, for the most likely version, certain participants answered using *the outcome approach* – "a model of informal reasoning under conditions of uncertainty" (Konold, 1989, p. 59) – and for the least likely version subjects answered using the representativeness heuristic. The outcome approach represented, as had the representativeness heuristic earlier, a new interpretation of normatively incorrect responses to comparisons of relative likelihood.

Theoretical Framework

With a few exceptions (e.g., Abrahamson, 2009), there has been a lull in the past number of years in the creation and development of fresh perspectives and interpretations of normatively incorrect responses to comparisons of relative likelihood. Inspired by the notion that novel perspectives and interpretations to normatively incorrect responses to comparisons of relative likelihood have, in the past, established new domains of research (e.g., Konold's most likely version leading to the outcome approach), we introduce the fallacy of composition to account for certain responses to comparisons of relative likelihood. In doing so, contend that informal fallacies, in general, may provide a original research domain; however, given pagination limitations associated with the present article, we decided to limit our scope and, further, our theoretical framework. As such, our impending analysis of results will consist of one particular fallacy: the fallacy of composition.

Put simply, *the fallacy of composition* occurs when an individual infers something to be true about the *whole* based upon truths associated with *parts* of the *whole*. For example: Bricks (i.e., the parts) are sturdy; buildings (i.e., the whole) are made of bricks; therefore, buildings are sturdy (which is not necessarily true). Applying the fallacy of composition framework to existing research (for example, and more specifically, Kahneman and Tversky's (1972) research involving the birth of boys and girls where participants deemed the sequence BGBBBB less likely than the sequence GBGBBG): the birth of a boy or a girl (i.e., the parts) occurs at a ratio of 1 to 1; birth order sequences (i.e., the whole) are made of births of boys and girls; therefore, birth order sequences should have a 1 to 1 ratio of boys to girls, which is not necessarily true. Beyond the application of the fallacy of composition framework to existing research, as presented, we will also demonstrate, in the analysis of results, certain participants in our research associated certain properties of individual coin flips (e.g., the ratio of heads to tails) to be true for sequences derived from individual coin flips – demonstrating our new perspective for normatively incorrect responses to relative likelihood comparisons.

The Task

As presented in Figure 1 below, the task given to participants was both similar and different to the task utilized by Konold et al. (1993), which we will comment on, in turn.

Which of the following sequences is the *least likely* to occur from flipping a fair coin five times:

a) HHTTH
b) HHHHT
c) THHHT
d) HTHTH
e) THHTH
f) all five sequences are equally likely to occur

Justify your response...

Figure 1. The relative likelihood task

Fundamentally, and is the case with all subsequent research since Konold et al.'s version of the task (e.g., Cox & Mouw, 1992; Chernoff, 2009; Hirsch & O'Donnell, 2001; Rubel, 2006), the structure of our task is similar Konold et al.'s. For example, participants are asked to pick out a particular sequence as more or less likely or to declare that all sequences are equally likely to occur and, subsequently, are asked to justify their response. The minor differences, however, between our current version of the task and Konold et al.'s version lie in the number of sequences available to choose from and certain sequences have been replaced with others.

Participants

Participants in our research were (n =) 147 prospective mathematics teachers enrolled in a methods course designed for teaching elementary school mathematics. More specifically, the 147 participants were comprised of five classes, each containing approximately 30 students, which were all taught by the same instructor. Of note, the topic of probability had not been covered in the course they were taking part in at the time of the research and, further, it was determined, afterwards, that none of the individuals involved in the research had ever answered an alternative version of the relative likelihood task. Individuals were given as much time as necessary to complete the task.

Results and Analysis

As presented in Figure 2 below, the majority of participants' responses fell into three different categories.

Students	***HHTTH***	***HHHHT***	***THHHT***	***HTHTH***	***THHTH***	***equally likely***
Total (147)	***1***	***38 (26%)***	***1***	***15 (10%)***	***1***	***91 (62%)***

Figure 2. Breakdown of responses from the 147 participants

Most participants, that is, 91 of the 147 (or 62%), responded correctly, that each of the five sequences were equally likely to occur. (However, not all 91 participants who answered correctly provided proper or, for that matter, normatively correct response justifications.) Fifteen (or 10% of the) participants responded incorrectly that the sequence HTHTH was least likely to occur and 38 (or 26% of the) participants declared that HHHHT was least likely to occur. For the analysis of results and in order to bolster the claim that the fallacy of composition can be used to account for particular responses to comparisons of relative likelihood, certain response justifications from the 38 individuals that responded HHHHT as least likely to occur are analyzed in detail.

The Ratio of Heads to Tails

As seen in the responses of Dave and Jerry, presented below, they both reference the ratio of heads to tails in the sequences they are presented and, further, declare that the sequence B (i.e., HHHHT) is least likely to occur because it has a heads to tails ratio of 4 to 1, whereas all other sequences have a ratio of 3 to 2.

Dave: I think B is least likely to occur because the others are all 2 only B is 1.

Jerry: B, it's the only one with one T and four Hs, the rest have 3Hs and 2Ts.

However, neither Dave nor Jerry elaborate beyond "that" one sequence has a different ratio of heads to tails, which leaves one to infer as to "why" they would think a 4 to 1 ratio of heads to tails is less likely. Based upon previous research (e.g., Tversky & Kahneman, 1974), one may argue that Dave and Jerry find the equally likely sequences to not be equally representative and, further, that the sequence with a heads to tails ratio of 4 to 1 is less representative, because the parent population of coin flips would have a ratio of 1 to 1, and, thus, less likely. However, the eight responses presented below tell a slightly different story.

Emboldened in the response justifications below, three individuals mention, in one form or another, that the 4 to 1 ratio is less likely because of the notion of equiprobability.

Gustav: It's more unlikely to have 4 heads and one tail **because there is a 50% chance**.

Igor: **Because the likeliness of both is the same**. It's unlikely that out of 5 the ratio would be 4:1 instead of something like 2:3.

Farhan: **because it's 50/50** so it will probably be 3H2T or 3T2H so B has the least likelihood of happening (4H1T).

However, none of the three individuals explicitly reference where the equiprobability they are using is coming from, which is not the case with the justifications of Hermine, Keanu, and Ferdinand.

Hermine: **Because the chance for H and T are both ½**. So it will be least likely to have H, H, H, H, T.

Keanu: **Both heads and tails have an equal chance for getting flipped**. I think B as in B H had more chance.

Ferdinand: I think that HHHHT is the least sequence that is more likely to occur, **because the chances are when you flip a coin once, the possibility of getting a tail is ½ because there are 2 sides on a coin.**

From the responses above, the notion of 50-50, 1/2, 1 to 1 ratio or equiprobability used in their justifications is derived from an individual coin, which, as Keanu declares, "ha[s] an equal chance of getting flipped."

Taking their justifications one step further, Duncan and Andrew make reference to individual coin flips, which each have a 50-50 chance of landing heads and tails.

> *Duncan:* I chose B because there are more H's than T's and they are not spread out. **A likely guess would be a 50/50 chance for each, eg: H, T, H, T,...**
>
> *Andrew:* b is least likely because **if each flip gives a 50/50 chance of resulting in heads/tails**, it is more likely that in five flips more tails would result than one. I mean it is not as likely that one result would dominate over the other (but possible).

For both individuals, since each individual coin flip has a 50-50 chance of heads or tails, the sequence with a 4 to 1 ratio of heads to tails is least likely to occur because the ratio is furthest away from 50-50 or 1 to 1.

As found in past research, responses reference the ratio of heads to tails as the reason why the coin flip sequence with a 4 to 1 ratio (i.e., HHHHT) is least likely to occur; however, we break tradition from previous research and demonstrate that the fallacy of composition – not just Tversky and Kahneman's (1974) sample to parent population determinant of representativeness – can account for the above ten responses. For example, all responses declare that the ratio of heads to tails for flips of a fair coin is 1:1 (i.e., the bricks); further, they note that the sequence (i.e., the building) is comprised of five flips of a fair coin; therefore, the sequence (i.e., the building) should also have a heads to tails ratio of 1:1 (i.e., the bricks), which, simply, is not true. As such, the expectation of a 1:1 ratio of heads to tails for individual flips of a fair coin (not the sample to parent population) leads the individuals to declare that the sequence with a head to tails ratio of 4:1 (the furthest "away" from 1 to 1) is least likely. A similar situation is revealed when looking at those responses, which traditionally would have been analyzed according to Tversky and Kahneman's reflection of randomness determinant.

The Appearance of Randomness

Aaron, Candy, Zoni, and Geena, declare that the sequence HHHHT is least likely to occur because of the "long" run of heads. However, the responses do not get into detail as to why the run of 4 heads in a row make the sequence less likely.

> *Aaron:* The chances of getting the same one four times is least.
>
> *Candy:* because the chances of getting 4 same sides are small.
>
> *Zoni:* B. It's more unlikely to flip four of the same sides 4 times in a row.
>
> *Geena:* B. Because that has the most of one side of the coin, which makes it less likely.

Peter, on the other hand, provides some insight, in his response below, as to why the sequence with a run of 4 heads is less likely.

> *Peter:* There is less of a chance of getting the same answer four times in a row. It's more likely to get a variety of answers.

Peter, and perhaps Aaron, Candy, Zoni, and Geena are expecting some mixture of heads and tails and not a run of 4 heads in a row. Presented in terms of previous research, Peter and the other individuals are focusing on the sequences appearance of randomness (Tverksy & Kahneman, 1974). More specifically, the run of 4 heads in a row is not locally representative, which demotes the sequences appearance of randomness and, further, demotes its likelihood.

A large number of responses, however, provide a very particular reason for the why the run of 4 heads in a row in less likely to occur than the other sequences. Chiefly, the sequence HHHHT is not likely to occur because the fair coin, which is used for individual flips that make up each of the sequences, is equally likely to result in heads or tails. References to the equiprobability (i.e., the 1 to 1 ratio of heads to tails) associated with the coin (and the flip of the coin) are emboldened in the response justifications below. Further, the response justifications are categorized in two different (sub) groups.

The first group, made up of the responses from Velma, Wally, Nina, and Carol, all reference that the four heads appear in the sequence. Further, they argue that it would be less likely to have four heads in the sequence because, as Wally mentions, "each side has an equal chance."

> *Velma:* **Because there are 2 sides the percentage is 50-50** so that means it is more likely to have 3 tails or 3 heads. 4 heads is possible, but if you flip a coin, I think 4 heads isn't going to happen a lot.
>
> *Wally:* B **because each side has an equal chance**. Therefore, it's very hard to flip a coin and get the same side 4 times.
>
> *Nina:* I first thought the answer was F, but my gut feeling was telling me the correct answer was B. **To consistently flip heads when you have equal opportunity to flip tails** has made me chose B.
>
> *Carol:* B is least likely because it would be hard to flip a coin 5 times in a row and have it land on heads **because coins are pretty much equally weighted throughout and there is just as much chance of it landing on tails as heads** and heads is probably heavier since the design has got more metal on it. 50/50 chance.

The above four responses, we contend, fall prey to the fallacy composition. For example, all four responses declare that the coin, which has two equally likely sides, has a 50-50 chance of landing on heads or tails (i.e., the bricks); further, all participants note that the sequence (i.e., the building) is made of flips of that fair coin; therefore, the sequence (i.e., the building) should also have a heads to tails ratio of 1 to 1. As was also seen in the ratio of heads to tails responses earlier, the equiprobability of the coin and coin flips (i.e., the bricks) correlates with an expectation of "heads to tails equiprobability" for the sequence, which is not found in the sequence with four heads and, as such, is deemed less likely.

The second group, made up of the response justifications from Quinn, Reno, Susan, Terry, and Uma, also reference the number of heads, that is, four, found in the sequence. They do, however, also reference that the four heads occur in a row. Further, all five individuals declare that the sequence with four heads in a row is less likely to occur because, as Quinn declares, "there is a fifty-fifty chance it will land on heads or tails."

> *Quinn:* **Because there is a fifty-fifty chance it will land on heads or tails**. So for it to land on head four times straight is possible, but the least likely to happen.
>
> *Reno:* I think HHHHT is least likely to occur **because there is a 50% chance to land on H or T** so you probably wouldn't get H 4 times in a row.
>
> *Susan:* B is least likely to occur because there is a very low chance to get 4 head in a row. **It is a 50% chance to get H or T** so it is the most unlikely sequence.
>
> *Terry:* The reason why is **because a coin has 2 sides, so there is a 50/50 chance of one or the other**, so it is unlikely for so many heads to appear in a row. It would be more likely to have 3 or 2 heads (or tails).
>
> *Uma:* I believe this is **because there is a fifty-fifty percent chance that it will land heads or tails** and 4 head in a row is very unlikely to happen.

All five responses above – which, in the past, would have been accounted for with the reflection of randomness determinant of the representativeness heuristic – we contend, are also falling prey to the fallacy composition. All responses declare that the coin, which has two equally likely sides, has a 50-50 chance of landing on heads or tails or has a heads to tails ratio of 1 to 1 (i.e., the bricks); further, the sequence (i.e., the building) is derived from flips of a fair coin (i.e., with a heads to tails ratio of 1 to 1); therefore, the sequence (i.e., the building) should also have a heads to tails ratio of 1 to 1 and, as such, the sequence with too many heads is deemed least likely to occur. As was seen in the ratio of heads to tails and in the "four of one side responses," the heads to tails ratio of the coin (i.e., the bricks) correlates with an expectation of the same heads to tails ratio for the sequence, which is not found in the sequence with four heads (in a row).

Concluding Remarks

Demonstrated in the analysis of results, the fallacy of composition accounts for certain responses (i.e., certain normatively incorrect responses) to the relative likelihood task (which we have introduced in this article). Also as demonstrated in the analysis of results above, all responses presented can be framed within the fallacy of composition. In particular, participants: make note of the 1 to 1 ratio of heads to tails of the coin (alternatively expressed throughout as: "fifty-fifty," "equiprobable," "the possibility of getting a tail is 1/2"); note that sequence is made up of flips of said fair coin; and, as such, determine (fallaciously) that the sequence of coin flips should also have a heads to tails ratio of 1 to 1. In other words, the properties associated with the fair coin (i.e., the brick), which make up the sequence (i.e., the building), are expected in the sequence. Subsequently, when looking at potential responses, the one sequence with a heads to tails ratio furthest away from 1 to 1 (i.e., the only sequence with a heads to tails ratio of 4 to 1 when compared to all others sequences with a heads to tails ratio of 3 to 2) is deemed the least likely to occur. The respondents are taking their knowledge of equiprobability in one context and transferring it to another without validation of equiprobability in the new and expanded context. It is in this reasoning that the fallacy of composition lays.

Discussion

Research involving comparisons of relative likelihood has, historically, been focused on accounting for individuals' responses – both correct and incorrect. Through investigating normatively incorrect responses, research has developed a variety of theoretical models (e.g., the representativeness heuristic, the outcome approach, and the equiprobability bias) to account for incorrect, sometimes incomprehensible, responses. In a more general sense, the incorrect responses have, in the past, been accounted for by contending that individuals were employing heuristic or informal reasoning in their justifications. In more recent years, there has been a lack of developments and new perspectives to response justifications associated with comparisons of relative likelihood. We have, however, in this article, presented a fresh perspective, the composition fallacy, to account for certain response justifications. This novel perspective, we contend, opens a new area of investigation for future research on comparisons of relative likelihood: the use of logical fallacies. While, in this article, we have demonstrated the descriptive power of the fallacy of composition, more research in the new domain will lead researchers to determine to what extent informal logical fallacies can describe response justifications to comparisons of relative likelihood and, in doing so, determine to what extent logical fallacies are a part of teachers' knowledge of probability. Through the identification of these logical fallacies within teachers' probabilistic knowledge, teacher educators can assess the origins of teachers' non-normative (e.g., heuristic, informal, fallacious) probabilistic reasoning. Once assessed, educators can then address the teachers' knowledge and model effective strategies for assessing and responding to their future students' non-normative probabilistic reasoning.

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DATA MODELING IN ELEMENTARY AND MIDDLE SCHOOL CLASSES: A SHARED EXPERIENCE

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This paper argues for a renewed focus on statistical reasoning in the elementary school years, with opportunities for children to engage in data modeling. Data modeling involves investigations of meaningful phenomena, deciding what is worthy of attention, and then progressing to organizing, structuring, visualizing, and representing data. Reported here are some findings from a two-part activity (Baxter Brown's Picnic and Planning a Picnic) implemented at the end of the second year of a current three-year longitudinal study (grade levels 1-3). Planning a Picnic was also implemented in a grade 7 class to provide an opportunity for the different age groups to share their products. Addressed here are the grade 2 children's predictions for missing data in Baxter Brown's Picnic, the questions posed and representations created by both grade levels in Planning a Picnic, and the metarepresentational competence displayed in the grade levels' sharing of their products for Planning a Picnic.

Introduction

The need to understand and apply statistical reasoning is paramount across all walks of life, evident in the variety of graphs, tables, diagrams, and other data representations that need to be interpreted. Elementary school children are immersed in our data-driven society, with early access to computer technology and daily exposure to the mass media. With the rate of data proliferation has come increased calls for advancing children's statistical reasoning abilities, commencing with the earliest years of schooling (e.g., Franklin & Garfield, 2006; Langrall, Mooney, Nisbet, & Jones, 2008; Lehrer & Schauble, 2005; National Council of Teachers of Mathematics [NCTM], 2006; Shaughnessy, 2010). We need to rethink the nature of young children's statistical experiences and consider how we can best develop the important mathematical and scientific ideas and processes that underlie statistical reasoning (Franklin & Garfield, 2006; Langrall et al., 2008; Watson, 2006). One approach in the beginning school years is through data modeling (English, 2010; Lehrer & Romberg, 1996; Lehrer & Schauble, 2007; Lehrer & Schauble, 2000).

Data modeling is a developmental process, beginning with young children's inquiries and investigations of meaningful phenomena, progressing to identifying various attributes of the phenomena, and then moving towards organising, structuring, visualising, and representing data (Lehrer & Lesh, 2003). As one of the major thematic "big ideas" in mathematics and science (Lehrer & Schauble, 2000, 2005), data modeling should be a fundamental component of early childhood curricula. Limited research exists, however, on such modeling and how it can be fostered in the early school years. Indeed, the majority of the research on mathematical modeling has been confined to the secondary and tertiary levels, with the assumption that elementary school children are not able to develop their own models and sense-making systems for dealing with complex situations (Greer, Verschaffel, & Mukhopadhyay, 2007).

In this paper, I first consider briefly the core components of data modeling relevant to the present activity, namely, structuring and representing data, and informal inference (specifically, making predictions). I also consider the role of task context in data modeling. Specifically, I address the following questions:

- What was the nature of the three grade 2 classes' predictions for the missing values in a table of data for *Baxter Brown's Picnic*?
- What questions were posed and representations created by one second-grade and one seventh-grade class in *Planning a Picnic*?
- How was metarepresentational competence displayed in the sharing of products between the second-grade class and the seventh-grade class?

Structuring and Representing Data

Models are typically conveyed as systems of representation, where structuring and displaying data are fundamental—"Structure is constructed, not inherent" (Lehrer & Schauble, 2007, p. 157). However, as Lehrer and Schauble indicated, children frequently have difficulties in imposing structure consistently and often overlook important information that needs to be included in their displays or alternatively, they include redundant information. Providing opportunities for young children to structure and display data in ways they choose, and to analyze and assess their representations is important in addressing these early difficulties.

Constructing and displaying data models involves children in creating their own forms of inscription. By the first grade, children already have developed a wide repertoire of inscriptions, including common drawings, letters, numerical symbols, and other referents. As children invent and use their own inscriptions they also develop an "emerging meta-knowledge about inscriptions" (Lehrer & Lesh, 2003). Children's developing inscriptional capacities provide a basis for their mathematical activity. Indeed, inscriptions are mediators of mathematical learning and reasoning; they not only communicate children's mathematical thinking but they also shape it (Lehrer & Lesh, 2003; Olson, 1994). As Lehrer and Schauble (2006) emphasized, developing a repertoire of inscriptions, appreciating their qualities and use, revising and manipulating invented inscriptions and representations, and using these to explain or persuade others, are essential for data modeling. In a similar vein, diSessa has argued for the development of students' metarepresentational competence, which includes students' abilities to invent or design new representations, explain their creations, and understand the role they play (e.g., diSessa, Hammer, Sherin, & Kolpakowski, 1991).

Informal Inference: Making Predictions

There has been limited research on young children's abilities to make predictions based on data, an important component of beginning, informal inference. Although young children obviously do not have the mathematical background to undertake formal statistical tests, they nevertheless are able to draw informal inferences based on various types of data (Watson, 2007). Predictions can be based on aspects of the problem scenario and context, and children's understanding of the data presented. As pointed out by Watson (2006), one of the aims of statistics education is to help students make predictions that have a high probability of being correct. Yet in the real world, decisions are required where there is uncertainty and where several alternatives might be reasonable. Hence, young children's exposure to informal inference involving uncertainty is an important learning foundation if a meaningful introduction to formal statistical tests is to take place in secondary school.

The Role of Context

The nature of task design, including the task context, is a key feature of data modeling activities. Children need to appreciate that data are numbers in context (Langrall, Nisbet, Mooney, & Jansem, 2011; Moore, 1990), while at the same time abstract the data from the context (Konold & Higgins, 2003). Moore emphasised that a data problem should engage students' knowledge of context so that they can understand and interpret the data rather than just perform arithmetical procedures to solve the problem.

The need to carefully consider task design is further highlighted in research showing that the data presentation and context of a task itself have a bearing on the ways students approach problem solution; presentation and context can create both obstacles and supports in developing students' statistical reasoning (Cooper & Dunne, 2000; Pfannkuch, 2011). In designing the present activities, literature was used as a basis for the problem context. It is well documented that storytelling provides an effective context for mathematical learning, with children being more motivated to engage in mathematical activities and displaying gains in achievement (van den Heuvel-Panhuizen & van den Boogaard, 2008).

Methodology

Participants

The participants were from an inner-city Australian school, situated in a middle socio-economic area, with an enrolment of approximately 500 students from Prep (K) -7. The three first-grade classes (2009, mean age of 6 years 8 months) continued into the second year of the study, the focus of this paper (2010, mean age of 7 years 10 months, n=68). The grade 7 class (n=24), who participated in the *Planning a Picnic* activity, described below, had an age range from 12 to 13 years.

Research Design

A teaching experiment involving multilevel collaboration (English, 2003; Lesh & Kelly, 2000) was adopted here. This approach focuses on the developing knowledge of participants at different levels of learning (student, teacher, researcher) and is concerned with the design and implementation of experiences that maximise learning at each level. The teachers' involvement in the research was vital; hence regular professional development meetings were conducted. This paper addresses aspects of the student level.

Activities and Procedures

The final activity implemented in the second year of the study continued the story context (purposely created) from the first year of activities. The context involved the adventures of Baxter Brown (a "westipoo"—West Highlander X toy poodle). The children requested more stories about Baxter Brown in the second year of the study; hence a story about Baxter Brown's picnic was created as the context for the first part of the activity (*Baxter Brown's Picnic*). For the *Planning a Picnic Activity*, two story picture books about foods and picnics were read to the grade 2 classes prior to their planning their own picnic.

For the *Baxter Brown's Picnic Activity*, the children (as a whole class) were presented with a table of six different items that he and each of his five canine friends chose to take on their picnic. The final column of the table was left blank, as indicated in Table 1 below. After discussing what they noticed about the values and variation in values across the table, the children were invited to predict the number of Oinkers that Baxter Brown and each of his friends might take on the picnic.

Table 1. Items Taken to Baxter Brown's Picnic

	Liver Straps	Beef Discz	Dentastix	My Dog Gourmet Beef	Bones	Oinkers
Baxter B.	3	5	2	1	3	
Monty	2	7	1	2	1	
Fleur	4	0	3	4	5	
Daisy	3	1	4	3	2	
Lilly	5	3	0	2	4	
Pierre	7	5	2	6	10	

For *Planning a Picnic* (grades 2 and 7), an initial whole-class discussion focused on questions the children might ask about planning a class picnic. In their groups, the children then listed five items they would like to take on the picnic, which were recorded by the teacher in a table on an interactive whiteboard. The children were subsequently asked what might be done with the data and what questions they might ask about the data. Each group's question was recorded on the board, with brief discussion on how some of the questions might be refined. In their groups, the children proceeded to answer their question and were to display their findings using whatever representation they liked. They were provided with a range of recording material including blank chart paper, grid paper, and chart paper displaying a circle shape. The children could use whatever of these materials they liked; no encouragement was given to use any specific recording material. On completion of the activity, the groups reported back to their class peers on how they answered their question. The grade 2 children were subsequently asked how their responses might compare with those of the other grade two classes, and were then asked to consider how the grade 7 classes in their school might respond to the activity. On the suggestion of one of the second-grade teachers, we administered the *Planning a Picnic* activity in one seventh-grade class. We then brought together the teachers and students from the second-grade class and the seventh-grade class for a sharing of how they worked the activity.

Data Collection and Analysis

In each of the second-grade classrooms, all whole-class discussions were videotaped and audiotaped; likewise, in each class, two focus groups (of mixed achievement levels and chosen by the teachers), were videotaped and audiotaped, with all tapes subsequently transcribed. There were 17 groups of second-grade children (3-4 per group), five in one class and six in each of the remaining two classes. For the seventh-grade class, the teacher chose mostly two-member groups, making 11 groups in total. The sharing of products between the two grade levels was videotaped and transcribed. All artifacts were collected and analyzed along with the transcripts. Where appropriate, iterative refinement cycles for analysis of children's learning (Lesh & Lehrer, 2000) were used, together with constant comparative strategies (Strauss & Corbin, 1990) in which data were coded and examined for patterns and trends.

Selection of Findings

Grade Two Children's Predictions for Baxter Brown's Picnic

In contrast to the children's use of informal inference in the first year of the study (English, in press), where they used the variation and range of values in a table of data to predict unknown values, the context of the present activity appeared to inhibit the children's ability to abstract the data from the context (Konold & Higgins, 2003). Each class initially identified the blank column as the first feature they noticed, with one child explaining, "Nobody wants Oinkers." In predicting how many Oinkers each of the dogs might take to the picnic, the children predicted small values less than 10, with their reasoning mainly based on the total number of other items each dog was bringing and the fact that if a larger number of Oinkers were brought to the picnic, the dogs "might get sick," "get a tummy ache," or "get fat." One child suggested zero, "because there has to be something that he doesn't like." There were some responses however, indicating an awareness of the need to consider the nature of the existing values, such as, "Because he (Monty) doesn't eat that much of anything else so he mustn't eat that much." In response to a child who predicted that Baxter Brown would take zero Oinkers, because he already has many other items, the teacher accepted the response as a reasonable prediction. Another student, however, disagreed, stating, "I don't think it's reasonable because he's pretty of a greedy guts so I think he would have more" (basing her decision on the existing item values for Baxter Brown).

On asking each class to consider the scenario of Baxter Brown taking 26 Oinkers, Monty 33, Fleur 50 etc., the majority of children used the task context to decide that these values were inappropriate. Comments such as, "They're um too big, the dogs would probably get a tummy ache and get sick" and "It's too heavy for them to carry to the picnic," were common. On the other hand, other responses suggested that some children were aware of the need to focus on the data itself, for example, "They would be bigger than all the numbers," "Ten is the highest number you can go up to," There's only one two-digit number," and "Because there would be too much."

Children's Questions and Representations for Planning a Picnic

The findings reported in this section focus on the responses of the selected second-grade class and the seventh-grade class. The table created by the grade 2 class appears below; a comparable table was developed by the grade 7 students.

<u>Table 2</u>. Picnic Items Chosen by the Grade 2 Class

Group1	Group 2	Group 3	Group 4	Group 5
choc chip cookies	sandwiches	blanket	food	cup cakes
fruit	fizzy drinks	fruit	picnic basket	cake
sausage rolls	cookies	cake	sunscreen	juice
cordial	fruit	esky	drinks	fruit soft drink
sandwiches	fruit pudding	soft drinks	chairs	carrots

The questions posed by each class are listed below. These questions addressed the table of data displaying the items each group would take on their picnic.

<u>Grade 2</u>

- Did everybody choose healthy items?
- Is there a most popular food?
- What are the different types of items?
- Did everybody choose the same items?
- Is there a most popular item?

<u>Grade 7</u>

- What percentage of foods are in each food group?
- How many different picnics brought 2 or more healthy foods?
- What percentage of foods are unhealthy?
- What is the most popular item on the list, soft drink or sandwich?
- How many items are processed foods in each picnic?
- What percentage of groups brought fruit on their picnic?

- What was the most popular food?
- What percentage of groups chose sandwiches compared to groups who chose fruit to bring on their picnic?
- Is there more healthy than unhealthy food?
- What food group does the majority of food from all of the picnics come from?

Not surprisingly, the grade 2 students' questions were less sophisticated than their older counterparts, resulting in a few difficulties in answering their question and representing their findings. Nevertheless, the younger children displayed a wider range of representations, albeit less sophisticated than their older counterparts. Each grade 2 group made use of the table of chosen items given to them (Table 1) and displayed a range of inscriptions in analyzing their data. For example, one group who addressed their question, "Did everybody choose healthy items?" placed a X on what they considered to be unhealthy items, a * on healthy items, a 0 on "things that aren't food," and a created symbol of mixed shapes for "fruit/sugar." This group also drew a food pyramid, with a focus on healthy and unhealthy items, and followed this with a third representation, a circle divided into halves displaying drawings and labels of "junk food" and "healthy foods." Four of the grade 2 groups made a list of selected items, before constructing a bar graph (3 groups) or a pie graph (cut into thirds; 1 group). One of the groups explained how their construction of a bar graph made them change their initial answer to their question:

> *Our question is, "Is there a most popular food?" There is, there is, the answer was, there is not any popular food because there were, there's 3. We, um, our finding things out was that all the things, we've made all the things that go together on the graph here and then we found out, we recorded how many different stuff there was and on one square it means that um, it means that there was one thing, on two squares it means that's there's two things and it keeps on going up to 6. And then we found out that there was no most popular food. There were 3 tying, drinks, cakes and picnic stuff…We wrote first that there was a popular thing but then when we ended up doing the graph, it ended up that there was, um, three populars.*

In contrast to the grade 2 children, all but one of the 11 grade 7 groups chose only one representation, with vertical bar graphs and pie graphs being equally popular (each chosen by 5 groups, with the display of percentages prominent). One group who created a pie graph also made a tally chart first. The remaining group created a line graph. When asked why they selected a line graph in preference to a bar graph, this group explained, "Well we thought because there are so many foods, drawing bars to make them seeable would be quite squishy; we just thought it would be easier to read if it was a line graph."

The children's foregoing explanations indicate a metarepresentational competence where they were able to explain and justify the representations they generated and also understand the role these played. Further evidence of such competence was evident in the sharing of products between the two grade levels.

Children's Metarepresentational Competence in Sharing Products for Planning a Picnic

As indicated in the methodology section, one grade 2 class and one grade 7 class came together to share their products for the *Planning a Picnic* activity. The grade 2 teacher initially asked her class to recall how they predicted the grade 7 students might work the activity. The children responded that "They won't have the same ideas," and "We said that they might be better because they'd had more years."

As the grade 7 class presented their products to their younger counterparts, there were several displays of metarepresentational competence at both grade levels. One grade 7 group reported that they solved their question using a bar graph that showed percentages of the particular items targeted in their question. When asked why they chose this representation, the group explained, "We tried a pie graph but we couldn't like split it into the right amount of groups." When invited to define a pie graph for the grade 2 children, one group member explained, "A pie graph is a circle that you put lines into and then color sections which is what, yeah, is what you chose." When the grade 2 children were asked to compare their bar graph representations with the grade 7 group, they responded that theirs was easier to read as "They (grade 7) used percentages and we don't know about percentages yet." However, the younger children were able to interpret the grade 7 representation when asked what the most popular and least popular item was:"Cause it's got the names at the bottom (labels under X axis). I was looking at the fruit one and I knew that it was the most…cause it's got the highest thing (bar) that goes up." In answering their question, "How many different picnics brought two or more healthy foods?" another grade 7 group justified their selection of a bar graph in preference to a pie graph by explaining, "Cause if you did like a pie graph… you wouldn't really show each group and how many items each individual group brought."

A follow-up grade 2 class discussion on how their working of the activity compared with the grade 7 students included comments such as: "We took more healthy food than they did;" "They were really bad choices;" "They did pie graphs and we didn't know like how to;" and (they did) "The line graph." In a follow-up question, the grade 2 children commented that 100% means "all of it" (circle) and "to understand the pie, we can look at it and see if it adds up to 100%."

Discussion and Concluding Points

Three main issues arising from the children's responses are worth highlighting here—the role of task context in the grade 2 children's predictions, the nature of the questions and representations created by the grade 2 and 7 classes, and the metarepresentational competence displayed in their sharing of products.

As previously noted, children need to appreciate that data are numbers in context, while at the same time abstract the data from the task context. Although context provides meaning in statistics (Garfield & Ben-Zvi, 2008), it can create both obstacles and supports in student's statistical reasoning (Pfannkuch, 2011). The purposefully created context of Baxter Brown and his canine friends organizing a picnic appeared to hinder the children's analysis of the table of data (Table 1). Only a few children justified their predictions by considering the nature (range and/or variation) of the values displayed, with the majority making contextual inferences such as the need to consider the dogs' health. The role and impact of task context require careful consideration in designing statistical activities; clearly a good deal more research is needed here to guide the development of data modeling in the early years.

Posing questions about the class selection of picnic items was a comparatively new learning experience for the second-grade children and did present some difficulties, resulting in discussion on how some of the questions might be refined. Such difficulties can be expected—transforming initial questions into more specific statistical questions is not an easy step, especially for young children (Konold & Higgins, 2003). Not surprisingly, the grade 7 students generated more sophisticated questions, applying mathematical understandings they had developed during their additional years of schooling. Nevertheless, both grade levels displayed metarepresentational competence in generating, describing, explaining, and justifying their representations. Interestingly, most of the grade 2 children, in contrast to their grade 7 counterparts, created more than one representation and could identify the links between their representations. The sharing of products was a rich learning experience for both grade levels, providing opportunities for appreciating different approaches to dealing with data and for questioning, explaining, and interpreting the data models of others. Consideration should be given to creating such sharing opportunities across grade levels.

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PROBABILISTIC MISCONCEPTIONS OF PRESERVICE TEACHERS

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Probabilistic misconceptions are common among students and teachers alike. This study involved 18 preservice secondary mathematics teachers and documents some of the common probabilistic misconceptions held by these preservice teachers. Data was collected through a survey involving eight probability problems and two semi-structured interviews. Findings indicate that the most common misconception held by these preservice teachers related to time axis, compound events, and availability.

Many people struggle with probabilistic misconceptions in situations of uncertainty. A misconception is more than just a mistake. Mistakes are easy to make and can be made for a number of different reasons: e.g., carelessness or distractions from another classmate. A misconception, however, is an incorrect concept held by a person that leads to a systematic pattern of errors (Khazanov, 2008). Teachers' practices and beliefs play an extremely important role in student learning (Beswick, 2006, 2008; Thompson, 1984), therefore identifying and addressing probabilistic misconceptions held by preservice teachers is important because they may pass those misconceptions on to their future students. This study examines and documents some of the probabilistic misconceptions held by preservice teachers.

Theoretical Framework

Shaughnessy (1992) argued that people need to "learn about probability and statistics just to be able to make reasonable decisions (as consumers or voters or even in choosing a career) on the basis of the mounds of data and probabilistic statements that confront them" (p. 95). Any well-informed citizen or consumer should have a good understanding of probabilistic statements – free from misconceptions. However, probabilistic misconceptions are common among students of all ages – elementary students, secondary students, college students, and even adults (Shaughnessy, 1992; Fischbein & Schnarch, 1997).

Fischbein and Schnarch (1997) explored common probabilistic misconceptions of students at various ages. A seven-item questionnaire was developed and administered to five groups of students: 20 students in fifth-grade, 20 students in seventh-grade, 20 students in ninth-grade, 20 students in eleventh-grade, and 18 preservice mathematics teachers. The seven items on the questionnaire were each related to seven common probabilistic misconceptions. Each of these is briefly discussed below (for further discussion see Fischbein & Schnarch, 1997; Kahneman, Slovic, & Tversky, 1982). Misconceptions related to the use of the representativeness heuristic involve an individual estimating the likelihood of an event based on how similar that event is to the population that it comes from. Misconceptions associated with negative and positive recency effects involve a person relating the probability of an event to the most recent similar events (e.g., flipping a coin a certain number of times). Misconceptions related to compound and simple events involve a misunderstanding of all possible outcomes for two events occurring simultaneously (e.g., rolling two dice). Misconceptions related to the conjunction fallacy involve a person believing that, under certain conditions, the probability of an event appears to be higher than the probability of the intersection of the same event with another. Misconceptions related to sample size involve individuals neglecting the importance of the size of a sample when estimating probabilities. Misconceptions related to the availability heuristic involve a person estimating the likelihood of an event based on how easy it is to think of an example or recall a particular instance of the event. Finally, the time-axis fallacy involves a person inverting the time axis of cause implying effect.

Fischbein and Schnarch (1997) hypothesized that all of the probabilistic misconceptions would decrease with age, and specifically, that they would stabilize during what Piaget labeled as the formal operational period (ages 12 and above). However, the actual findings of the study were extremely varied. Some of the misconceptions decreased with age, some remained constant, some increased with age, and some were absent altogether. Misconceptions dealing with the representativeness heuristic, negative recency, and the conjunction fallacy all decreased with age. Misconceptions dealing with compound and simple events were frequent among the students in the study and stable across all ages. Misconceptions dealing with the effect of sample size, the availability heuristic, and the effect of the time axis all increased with age. Only one of the misconceptions, positive recency, seemed to be absent from the students.

In summary, probabilistic misconceptions are common among students, teachers, and citizens (Fischbein & Schnarch, 1997; Wilkins, 2007; Shaughnessy, 1992). The purpose of this study is to identify and discuss probabilistic misconceptions of secondary mathematics preservice teachers.

Methods

Participants

Eighteen preservice secondary mathematics teachers participated in the study. All of the preservice teachers were graduating seniors in a five-year teacher preparation program. The preservice teachers graduate with an undergraduate degree in mathematics after four years and a master's degree in education after one additional year. The preservice teachers' undergraduate degree in mathematics included at least one course in probability or statistics, which they had all already taken. At the time of the study all of the preservice teachers were enrolled in a senior capstone course in which one of the researchers was the instructor.

One limitation of the study is the dual relationship that one of the researchers had with the participants in the study. The researcher personally knew all of the participants because they were students in her class. However, the researcher and the participants had never before discussed probabilistic misconceptions. Participation in the study was voluntary and was in no way linked to course credit.

Data Collection

The study consisted of an initial survey and two follow-up semi-structured interviews. The initial survey was designed to identify probabilistic misconceptions held by the preservice teachers. All eighteen of the preservice teachers in the study completed the initial survey. The initial survey included eight problems, each of which focused on one of the seven common misconceptions outlined previously by Fischbein & Schnarch (1997) with two of the problems focused on the effect of sample size.

Preservice teachers completed the eight problems by selecting the correct answer from three or four answer choices. In addition, preservice teachers were asked to explain their thinking for each of the eight problems. It was extremely important to have the preservice teachers explain their problem solving process for each of the eight problems. In this way the researchers were able to more accurately determine whether or not the preservice teachers actually held a particular probabilistic misconception.

From the results of the initial survey, six preservice teachers who demonstrated probabilistic misconceptions were selected. Of these six preservice teachers, only two of them participated in the follow-up interview sessions. The semi-structured interviews were especially beneficial because they allowed the researcher to "experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267). Both of the follow-up interview sessions were video recorded for future analysis.

This first semi-structured interview consisted of a short background interview and a time for the preservice teacher to work on two probability problems. The background interview focused on the preservice teachers' prior experiences with mathematics in general and with probability and statistics in particular. The background interview also focused on the preservice teachers' attitudes about teaching probability and statistics in the future. The preservice teacher then engaged with two probability problems that related directly with two common probabilistic misconceptions, the availability heuristic and compound and simple events.

The second semi-structured interview consisted of a probability activity that dealt specifically with the representativeness heuristic. The activity combined a hands-on experiment in which the preservice teacher flipped six fair coins 50 times in order to find experimental probabilities and also created a theoretical model for the same situation. While completing the probability activity, the interviewer probed the mathematical and probabilistic understanding of the preservice teacher by asking questions such as: "What is your reasoning for this?" and "Why did you choose this approach?"

Data Analysis

The analysis of the data included both quantitative and qualitative methods. From the initial screening survey, the percentage of preservice teachers who demonstrated the various probabilistic misconceptions was calculated. The video recording from the two semi-structured interviews was transcribed and analyzed for rich evidence of probabilistic misconceptions held by the preservice teachers.

Results

The percentage of teachers who held the various probabilistic misconceptions is presented in Table 1. The most common misconception held among these preservice teachers is related to the effect of the time axis (72.2%). Interestingly, after all eighteen participants completed the initial survey they animatedly debated their answers to this particular problem. The second most common misconception held among these preservice teachers is related to compound and simple events (44.4%). The third most common misconception held among preservice teachers is related to the availability heuristic (33%). The specific problems dealing with these three probabilistic misconceptions from the initial survey can be found in Table 2 below.

Effect of time axis	72.2%
Compound and simple events	44.4%
Availability	33.3%
Effect of sample size	27.7%
Effect of sample size	22.2%
Conjunction fallacy	11.1%
Negative recency	5.5%
Positive recency	0%
Representativeness	0%

Table 1. Percent of preservice teachers who demonstrated misconceptions (N = 18)

Effect of the time axis
Jack and Jill each receive a box containing two white marbles and two black marbles. a) Jack extracts a marble from his box and finds out that it is a white one. Without replacing the first marble, he extracts a second marble. Is the likelihood that his second marble is also white smaller than, equal to, or greater than the likelihood that it is a black marble? b) Jill extracts a marble from her box and puts it aside without looking at it. She then extracts a second marble and sees that it is white. Is the likelihood that the first marble she extracted is white smaller than, equal to, or greater than the likelihood that it is black? Explain your thinking.
Compound and simple events
Suppose Cathy rolls two dice simultaneously. What of the following has a greater chance of happening? a) Obtaining two sixes. b) Obtaining a five and a six. c) Both have the same chance. Explain your thinking.
Availability
When choosing a committee comprised of 2 members from among 10 candidates, the number of possibilities is: a) Smaller than… b) Equal to… c) Greater than… the number of possibilities when choosing a committee of 8 members from among 10 candidates. Explain your thinking.

Table 2. Initial survey problems

Semi-Structured Interviews

For the purposes of this paper, the results from the semi-structured interviews will focus primarily on Amanda, one of the two preservice teachers who participated in the follow-up interviews. During the first interview, Amanda was given two tasks. These tasks were related to availability and compound and simple events. The misconceptions associated with these two types of problems were among the most common in the initial survey. On the initial survey, Amanda did not seem to hold a misconception related to availability, but she did seem to hold a misconception related to compound and simple events.

The first task related to availability (see Figure 1). Amanda's initial approach was to draw all the possible paths in each of the grids. This turned out to be a daunting task and led Amanda to some confusion. She eventually took a step back and tried to relate the paths in each of the grids to a more concrete example. She exclaimed,

> Trying to figure out the number of paths in the grid is like trying to pick out a pizza. If you have to choose a certain kind of crust, a certain kind of cheese, a certain kind of

> meat, and a certain kind of vegetable to put on the pizza, then you would just multiply the number of each to get the total number of possible pizzas.

Amanda related this back to the number of possible grids by saying,

> For the first *x*, you've got one, two, three, four, five, six, seven, eight possibilities to go to the next row. And then from whatever *x* you pick there, you've got eight more ways to go to the next row… yeah, that would give you all the paths because it would be just like making a tree.

This reasoning convinced Amanda that she had developed a correct method to find the possible paths in each grid by multiplying the number of rows in the grid by the number of symbols in each row. Based on Amanda's final answer to the problem and her reasoning supporting that answer, Amanda did not seem to demonstrate a misconception related to the availability heuristic.

Consider the grid below. A path is a polygonal chain of line segments, starting at the top row and proceeding to the bottom row and meeting one and only one symbol in each row.

Grid A	Grid B
xxxxxxxx	xx
xxxxxxxx	xx
xxxxxxxx	xx
	xx
	xx
	xx
	xx
	xx
	xx

a) Are there more paths possible in Grid A?
b) Are there more paths possible in Grid B?
c) Are there about the same number of possible paths in each grid?

Figure 1. Availability task

The second task related to compound and simple events (see Figure 2). For the first question in this task, Amanda's response was,

> It's one-half…it doesn't matter…this reminds me of genetics…and it doesn't matter what your first child is, I mean, we know that his first child is a son, but that has no bearing on what the [other child is]. Every time you have a child, you have a one-half probability of it being a boy or a girl. So it's one-half because they are completely independent events.

Amanda's response was especially interesting because she is a double major in both mathematics and biology. Because of her focus on biology and her past experience in genetics classes, Amanda was very passionate to say that no matter what we know about a situation (such as the gender of one child) the probability of the other child being a boy will *always* be one-half. Amanda was confident that the probability of the second child being a boy was one-half. In response to the second question in this task, she said that it still did not matter what the gender of the eldest child was. She said, "even if you ask the probability that…the other child is a girl, it's still one-half." Based on her response to this task and her reasoning for her response, it seems as though Amanda does hold a probabilistic misconception related to compound and simple events. More specifically, Amanda's misconception hinges on the idea that the events are "independent" of each other. Amanda's misconception also suggests that she does not take into consideration the fact that there are four different possibilities for Mr. Smith's two children; the two children could either be a boy and a boy, a boy and a girl, a girl and a boy, or a girl and a girl.

1) Mr. Smith is the father of two. We meet him walking along the street with a young boy whom he proudly introduces as his son. What is the probability that Mr. Smith's other child is also a boy?
2) We meet Mr. Smith (who we know to be the father of two) in the street with a boy. He is very elaborate with his introduction, presenting the boy as his eldest child. What is the probability that Mr. Smith's other child is also a boy?

Figure 2. Effect of time axis task

During the second interview, Amanda engaged with a task that had both an experimental and a theoretical component (see Figure 3). The hands-on experiment required that she flip six coins 50 times and record the number of heads (out of six) for each of the 50 flips. Before starting the experiment, Amanda hypothesized that "three heads would come up the most…there's just more ways to get three heads than six heads." This shows that Amanda had a notion of which combinations of heads and tails would be the most common. Also, when flipping the coins, Amanda observed, "one head is coming up more than one tail." This surprised her. She thought that she should have about the same number of flips when there was only one head showing as when there was only one tail showing. This observation shows that Amanda also understood the notion of symmetry in the distribution of the number of heads out of six.

Based on the experiment, Amanda developed an experimental probability model. This model described the probability of getting x number of heads when six coins were flipped at one time. These beginning observations seem to indicate that Amanda has a good understanding of the distribution and sample space she is dealing with and that she does not hold a misconceptions related to the representativeness heuristic.

<table>
<tr><td colspan="9">1) Flip 6 coins 50 times and record the number of heads in the chart below.</td></tr>
<tr><td rowspan="2"></td><td>6 H</td><td>5 H</td><td>4 H</td><td>3 H</td><td>2 H</td><td>1 H</td><td>0 H</td><td rowspan="2"></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="9">2) Based on your data, what is the probability of getting 6 heads? 5 heads? 4 heads? 3 heads? 2 heads? 1 head? 0 heads?</td></tr>
<tr><td colspan="9">3) Make a list of all possible outcomes for flipping 6 coins.</td></tr>
<tr><td colspan="9">3) Based on your list, develop a mathematical model to find the theoretical probability for the outcomes of flipping 6 coins.</td></tr>
</table>

Figure 3. Experimental and theoretical probability task

The theoretical component of the task required that Amanda develop a theoretical probability model, again describing the probability of getting *x* number of heads when six coins were flipped at once. Amanda initially used a tree diagram to list out all the possible outcomes for flipping six coins. From her tree diagram, Amanda quickly articulated that a flip of TTTTTH is very different from a flip of HTTTTT. She then explained that you could "move around" the one head to get all the possible ways that the flip of six coins could have one head. Because Amanda had an intuitive notion of symmetry, she knew that this was similar to one tail (which she also explained was the same thing as looking at five heads). Amanda also explained that there would be a lot more ways to "move around" three heads than just one head. Because it would have been difficult to count all the theoretical ways of getting *x* number of heads on a flip of six coins based on her tree diagram, Amanda switched to using her understanding of combinations to develop the theoretical model.

Throughout this activity, Amanda did not seem to demonstrate that she held any probabilistic misconceptions associated with the representativeness heuristic. She did demonstrate that she held intuitive notions of symmetry and combinations related to the activity based on her observations during the experiment and the ease in which she calculated her theoretical model.

Discussion and Conclusions

The purpose of this study was to explore and document some of the common probabilistic misconceptions of preservice teachers. Although all of the preservice teachers in this study are undergraduate mathematics majors who have taken at least one course in probability and statistics, these preservice teachers still demonstrate probabilistic misconceptions. The most common probabilistic misconceptions of the eighteen preservice teachers were related to the effect of the time axis, compound and simple events, and availability.

Most secondary mathematics preservice teachers are required to take at least one probability and statistics course, similar to the preservice teachers in this study. However, because data analysis and probability is one of the National Council of Teachers of Mathematics' five content standards for pre-kindergarten through grade 12 mathematics education (NCTM, 2000), it might be necessary to examine both the probability and statistics courses and methods courses of secondary mathematics preservice teachers in order to discuss any potential probabilistic misconceptions of preservice teachers.

Based on Amanda's responses during the second interview, it was clear that she valued the theoretical model over the experimental model. Amanda developed both an experimental and a theoretical model for finding the probability of *x* number of heads when flipping six coins. Both models accurately described the probabilities. However, Amanda trusted her theoretical model over her experimental model. She compared the experimental model to the theoretical model and found that the experimental model was not "off by that much." This is most likely due to the fact that Amanda had little experience with experimental probabilities; she was much more familiar with theoretical probabilities. This is typical for many preservice teachers like Amanda; they do not often have meaningful experiences with activities in which they find both the experimental and the theoretical probabilities and compare the two models.

One benefit of a task like this – developing both an experimental and theoretical model for finding probabilities and then comparing the two models – is to show preservice teachers that *both* models are valuable. It is important to explicitly explore the connections between both models to enforce the validity, usefulness, and power of the experimental model.

Identifying and discussing probabilistic misconceptions of preservice teachers should be an important part of the teachers' methods courses as preservice teachers will have an influence on their future students' understanding of probability. Further, having preservice teachers engage in hands-on experiments involving well-designed tasks may be one way to help them address their own misconceptions (Shaughnessy, 1982; Wilkins, 2007, Khazanov, 2008) and better prepare them to help their future students develop sound probabilistic conceptions.

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PROSPECTIVE TEACHERS' USE OF REPRESENTATIONS IN SOLVING STATISTICAL TASKS WITH DYNAMIC STATISTICAL SOFTWARE

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This study examined a random stratified sample (n=62) of prospective teachers' work across eight institutions on three tasks that utilized dynamic statistical software. Our work was guided by considering how teachers may utilize their statistical knowledge and technological statistical knowledge to engage in cycles of investigation. Although teachers did not tend to take full advantage of dynamic linking capabilities, they utilized a large variety of graphical representations and often added statistical measures or other augmentations to graphs as part of their analysis.

Purpose of Study

Dynamic statistical software tools have become more common in schools in the past decade, and mathematics teacher educators are beginning to use these tools in courses for prospective mathematics teachers. Although the tools are available, teachers' effective use of these tools in classrooms is influenced by their own understanding of how to use the tools to explore statistical ideas. In this paper we examine how prospective teachers use representations of data when solving statistical tasks using Fathom (Finzer, 2002) or TinkerPlots (Konold & Miller, 2005).

Theoretical Framework and Background

Lee and Hollebrands (2008, in press) proposed a framework that characterizes the important aspects of knowledge needed to teach statistics with technology (see Figure 1). In this framework, three components consisting of Statistical Knowledge (SK), Technological Statistical Knowledge (TSK), and Technological Pedagogical Statistical Knowledge (TPSK) are envisioned as layered circles with the inner most layer representing TPSK, a subset of SK and TSK. Thus, Lee and Hollebrands propose that one's TPSK is founded on and developed with teachers' technological statistical knowledge (TSK) and statistical knowledge (SK).

Within Statistical Knowledge is prospective teachers' ability to engage in transnumeration (Wild & Pfannkuch, 1999) as a process of transforming a representation between a real system (real-world phenomena) and a statistical system (ways of modeling the phenomena statistically) with an intention of engendering understanding (Pfannkuch & Wild, 2004). Thus, teachers should be able to collect data, represent them meaningfully with graphs and computed statistical measures, and translate their understandings of the data back to the context. Often times, transnumeration occurs when data is represented in some way that highlights a certain aspect related to the context that can afford new insights into the data.

Within TSK, our focus is on how prospective teachers can take advantage of technology's capabilities to automate calculations of measures and generate graphical displays, and the ways they use these graphs and measures to explore data and visualize abstract ideas (Chance, Ben-Zvi, Garfield, & Medina, 2007). For example, how do prospective teachers visualize measures (e.g., mean) and graphical augmentations (e.g., shade a region of data, showing squares on a least squares line), and how do they take advantage of ways to link multiple representations?

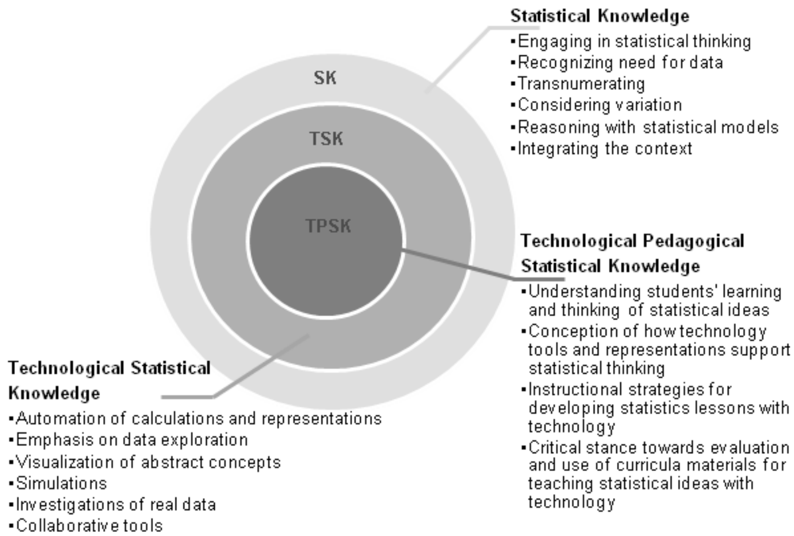

Figure 1. Components of Technological Pedagogical Statistical Knowledge (adapted from Lee & Hollebrands, in press).

Several researchers have studied teachers' use of dynamic statistical tools, usually focused within a single professional development experience or course at a specific university (e.g., Doerr & Jacob, 2011; Hammerman & Rubin, 2004; Makar & Confrey, 2008; Meletiou-Mavrotheris, Paparistodemou, & Stylianou, 2009). Overall these studies have shown that dynamic tools can provide opportunities for teachers to increase their approaches to statistical problem solving, moving beyond traditional computational based techniques and utilizing more graphical based analysis. Teachers using TinkerPlots and Fathom in these studies often combined graphical and statistical measures by either adding a measure to a graph or using a graph to make sense of a statistical measure computed separately. Such analysis by teachers often affords opportunities to consider an aggregate view of a distribution that incorporates reasoning about centers and spreads (Konold & Higgins, 2003). Although prior research discussed how teachers analyzed data where a link among representations can be inferred, researchers often did not focus their analysis on how teachers utilized linked representations.

Method

The overarching research question for the study was: *When teachers use technology tools to solve data analysis tasks, in what ways do they use representations (dynamic and static) to investigate these problems?* To further research teachers' use of dynamic statistical tools beyond small samples and limited contexts, data were collected from eight different institutions in which faculty were using materials (Lee, Hollebrands, & Wilson, 2010) developed by the PTMT project[1]. What is reported here is based on analysis of teachers' work on three tasks that use similar statistical concepts and tools in either TinkerPlots or Fathom (see Table 1).

The faculty implementing the materials attended a week-long summer institute to become familiar with technologies, specific tasks and data sets, and pedagogical issues. Across institutions, materials were implemented in a variety of courses, some focused on using technology to teach middle or secondary mathematics, and others on statistics for elementary or middle school teachers. The courses predominately enrolled prospective teachers, with a few practicing teachers or graduate students. Each teacher worked individually and created a document that described his or her work, including illustrative screenshots of ways they used technology in solving the task. A total of 247 documents were collected across institutions and blinded to protect teacher, faculty, and institutional identity.

Table 1. Research tasks

	Task as Posed in Materials
Ch 1 Task TinkerPlots	[Note: Faculty had a choice of two similar data sets to use of state level school data from two regions of the US. Data included attributes such as average teacher salary, student per teacher ratios, expenditures per student] Explore the attributes in this data set and compare the distributions for the South and West [Northeast and Midwest]. Based on the data you have examined, in which region would you prefer to teach and why? Provide a detailed description of your comparisons. Include copies of plots and calculations as necessary.
Ch 3 Task Fathom	Explore several of the attributes in the 2006Vehicle data set. a) Generate a question that involves a comparison of distributions that you would like your future students to investigate. b) Use Fathom to investigate your question. Provide a detailed description of your comparisons and your response to the question posed. Include copies of plots and calculations as necessary.
Ch 4 Task Fathom	Explore several of the attributes in the 2006Vehicle data set. a) Generate a question that involves examining relationships among attributes that you would like your future students to investigate. b) Use Fathom to investigate your question. Provide a detailed description of your work and your response to the question posed in part a. Include copies of plots and calculations as necessary.

To begin analysis, four documents were randomly selected from each chapter. Through iterative discussions by the research team[2], examining documents and making sense of prospective teachers' work, both top-down methods of Miles and Huberman (1994), and grounded theory (Strauss & Corbin, 1990) were used to develop and apply a coding instrument. The coding instrument that emerged was based on theory from research on statistical problem solving, particularly cycles of exploratory data analysis and typical phases (ask a question, represent data, analyze/interpret, and make a decision, e.g., Wild & Pfannkuch, 1999) and use of static and dynamic representations in statistics and other domains of mathematics; and on categories and codes that emerged from analyzing an initial random sample of teachers' work. In the coding procedures, each teacher's work was chunked into smaller cycles of investigation that included four phases (Choose Focus, Represent Data, Analyze/Interpret, and Make Decision). Within each phase, several categories were used to characterize the work (e.g., number of attributes, type of representations, what was noticed, interpretations, type of claim).

In the second phase of analysis, we examined a larger randomly chosen subset of documents. From an initial review of the 247 documents it was obvious that some responses were more detailed than others, contained more statistical investigation cycles, and used more representations. Thus, each document was classified as either short or long. Short responses were typically 1 page and included 1-2 screenshots with minimal explanation. Others were classified as a long response. We then conducted a stratified random sample to have proportional representation of short and long responses, and to select about 25% of our documents (Table 2).

Table 2. Design of stratified random sample

	Chapter 1			Chapter 3			Chapter 4		
	Short	**Long**	**Total**	**Short**	**Long**	**Total**	**Short**	**Long**	**Total**
Total Documents Collected	52	50	102	12	29	41	38	66	104
Stratified Random Sample	13	12	25	4	8	12	9	16	25

Pairs of coders were assigned 8-12 documents to code. All documents were initially coded individually, then the pair met to discuss, compare, record inter-rater reliability, and come to consensus for each document. During early stages, several coding categories were clarified and new categories emerged that were then adopted into the coding procedures by all coders. For several categories of codes discussed in this paper, the overall IRR across documents was 0.923 for coding representations used, and 0.94 for coding measures added to representations. There was low initial agreement about coding an augmentation to a graph (0.523), which led to discussions to establish agreed upon coding and a better definition of a graphical augmentation.

Results

A major advantage of dynamic statistical software is the ability to link representations and view different representations and statistical measures or graphical augmentations that can perhaps lead to interpretive insights. Thus, to organize our cross-document and cross-chapter analysis, we first focused on whether or not a teacher provided evidence of linking representations (see Table 3). We considered a teacher was dynamically linking two or more representations if there was evidence they had to interact (e.g., click on or drag a data point, select a region of data) with data in one representation and use information linked (often highlighted) about data in another representation. A teacher was coded as using static representations if he coordinated two or more representations without direct interactions. With differences noted across chapters, we carefully examined use of representations within chapters.

Table 3. Number of responses providing evidence of linking representations

	No Evidence of Linking	Dynamic Linking	Static Coordination
*Chapter 1 (TinkerPlots) n= 25	13 (52%)	11 (42%)	3 (12%)
Chapter 3 (Fathom) n=12	8 (66.67%)	2 (16.67%)	2 (16.67%)
*Chapter 4 (Fathom) n=25	14 (56%)	7(28%)	6 (24%)
*Totals (n=62)	35 (56.5%)	20 (32.3%)	11 (17.7%)

*Percents do not sum to 100 since 2 cases in Ch1 and 2 cases in Ch4 were coded as both static and dynamic.

Representation Use in Chapter 1 Task

In coding Chapter 1 documents, we did not specifically list a data card of a collection as a representation of that data, as it was automatically available in TinkerPlots. Data cards look like a stack of index cards, with each card representing a case (e.g., the state of Virginia) and containing the values for that case for each attribute (e.g., Average salary, Census region) in the dataset. Thus, we considered this as a representation of data readily available. Data Tables and Plots, however, were considered representations as teachers needed to drag down a Data Table to view a different numerical representation, or use a Plot and construct a visual representation. The most common Plots created were dotplots and box plots. When teachers engaged in dynamically linking or statically coordinating information in more than one representation, they could have been explicitly using or attending to a data card as a representation in that process. This focus was typically evident with those who used dynamic linking. The most common purpose for dynamic linking was to identify a particular value for a specific case of interest (e.g., clicking on a particular case icon in the graph and using the data card to determine the value of an attribute) or to click on a different attribute in the data card that would augment the graph by recoloring the data icons for a new attribute of interest. In the first type of linking, teachers were often focused on special cases such as those that appeared to be outliers and often situated these cases in comparison to the aggregate (see Figure 2 as an example of this type of linking). It was also inferred that teachers linked the data card and graphical displays in order to report specific values (e.g., data point at Q1) or compute measures such as the range. In the latter type of dynamic linking, the addition of a second or third attribute in the plot often increased the complexity of analysis as teachers considered relationships among attributes.

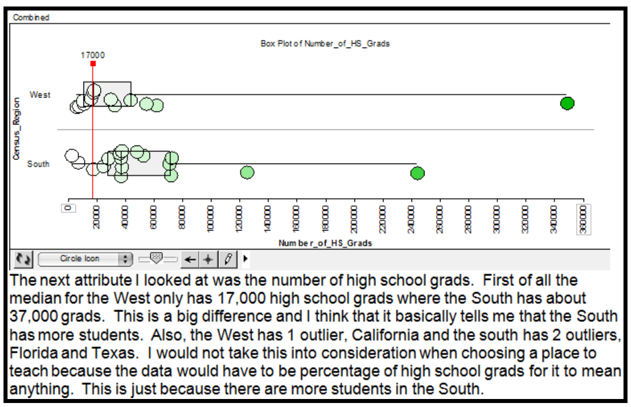

Figure 2. Using dynamic linking to identify state names of data points considered outliers.

In considering ways teachers may have used their TSK, we further examined relationships among their actions of augmenting graphs, adding statistical measures to graphs, and whether they linked representations. All teachers who dynamically or statically linked representations also augmented their graph. Teachers who interacted more with a graph through augmenting tended to also engage in linking representations (66.6% of teachers who augmented also linked). This pattern was not as strong when considering a relationship between adding a statistical measure to a graph and linking representations. Only 50% of teachers who added statistical measures to a graph also linked representations, and 3 of the 7 teachers who did not add any statistical measures to a graph still engaged in linking representations.

Representation Use in Chapter 3 Task

In Chapter 3, teachers were asked to explore several attributes in the data and to determine questions for their future students to compare distributions. More than half of the examined documents were done and submitted within Fathom, rather than in a Word document, so they could leave many of their representations viewable in Fathom and write their responses in a text box. Those that did their response in a Word document interspersed their text responses with purposeful screenshots of their work in Fathom. The data table was not specifically listed as a representation in Chapter 3 documents as teachers used this as a primary view of the data in Fathom and for grabbing the label for attributes to graph, similar to how teachers used the TinkerPlots data cards in Chapter 1. In one case a teacher used the data table as a dynamically linked representation; it was thus coded as a representation. Chapter 3 documents tended to be long (71%) yet had the smallest percentage of teachers across all chapters who showed evidence of linking representations. The two teachers that linked dynamically did so to examine particular cases and clicked on data points in a graph to locate specific values for attributes in a data table. The two teachers that demonstrated evidence of static linking did so near the end of their problem solving to examine trends across cycles. Both purposes are the same as in Chapter 1.

Teachers working on the Chapter 3 task used a wide variety of representations, and often 2-3 representations per cycle of investigation. They seemed to take advantage of the ability to generate multiple graph types (box plots, dot plots, scatterplots), and used Summary Tables extensively (many had more than one). Several teachers also added statistical measures (e.g., mean, median) to the graphs in Fathom; however in two cases, it seemed they were mainly using a graph window as a place to compute a statistical measure (e.g., IQR, StDev) whose location in the distribution did not add a way to reason about the measure in relationship to the aggregate.

Representation Use in Chapter 4 Task

In coding representations in Chapter 4 documents, again we did not specifically list the data table as a representation, as it was used as a primary view of data in Fathom. Six of the 25 documents sampled from Chapter 4 were completed entirely within Fathom rather than as a Word document. Of the 25 teachers, only 44% (n=11) linked their representations statically or dynamically. Overall, the most common purpose for linking was to compare the position of groups of cases across graphical representations and to make statements about their relationships. Of the four teachers that statically linked representations, three linked between two or more graphs. This linking typically occurred as a teacher noted the shape of a graph and commented whether or not a second graph using a different attribute was similarly shaped. Furthermore, of the four teachers that statically linked representations, two linked numerical measures with a graphical representation. One of these teachers linked the standard deviation to a histogram in an attempt to make sense of magnitude of the standard deviation. There was no evidence that the other teacher attempted to make sense of, or interpret the slope of the least squares line that was explicitly linked. Five teachers either implicitly or explicitly linked their representations dynamically. Three of these teachers linked two graphs, one linked a graph with the data table, and another linked the numerical values of sliders (controlling values of coefficients in a model) to characteristics of a graph of the model. Two teachers that linked dynamically and statically did so by linking several univariate graphs, and one teacher also linked a boxplot to a scatterplot.

In considering the ways teachers may have used their TSK, we further examined relationships among their actions of augmenting graphs, adding statistical measures to graphs, and the ways in which they did or did not link representations. Six out of 11 teachers who dynamically or statically linked representations also augmented their graph; whereas, 40% of teachers neither augmented their graph, nor engaged in any type of linking among representations, and 66% of teachers who did not augment their graph also did not link representations. Five of the 11 teachers who linked representations also added statistical measures to their graphs. Out of 16 teachers who only displayed data graphically, two-thirds (11 of 16) did not link their graphs. In contrast, all teachers who either statically or dynamically linked representations (n=11) used graphical displays.

Cross-chapter Representation Use

Teachers in Chapter 3 tended to use several more and unique types of representations of data in their response than those in Chapter 1 or 4. This may be an artifact of more than 50% of teachers in the Chapter 3 sample submitting their work in a Fathom document. Using Fathom as an analysis and reporting environment may give better insight into all the representations teachers used. When reporting work in a separate document and asked to supply screenshots of their work, teachers may not report all their representations used. The lack of evidence of linking representations may be an artifact of the task posed for Chapter 3 or may be due to the difficulty of illustrating linking when many representations are in a single Fathom document.

The graphs teachers created in Chapter 1 and 3 (tasks about comparing distributions) were typically double box plots or dot plots, with a few bar charts and scatterplots in Chapter 3. While Chapter 1 teachers most often used one graph per investigation cycle, teachers in Chapters 3 and 4 typically used more than one graph per cycle. A much wider variety of representations (simple box plots and dot plots, double box plots and dot plots, histograms, and scatterplots) were used in Chapter 4 (questions about relationships among attributes), and almost always there was more than one representation per cycle. In Chapters 1 and 3, the most common use of dynamic linking was to coordinate a single graph with either a data card or data table to find out details about a specific case of interest. Only occasionally in solving the Chapters 1 and 3 tasks did teachers use dynamic linking to look at an interval of data. However, teachers who linked representations in a dynamic way in Chapter 4 often were comparing the position of groups of cases across graphical representations and using such noticing to make statements about relationships. Static linking was done across chapters to compare trends in graphs of different attributes across cycles. This was often done towards the end as teachers reflected on their work and made a final decision.

Teachers who dynamically linked representations tended to also augment a graph (75% across all chapters). Most of the teachers (71%) who either did not link representations or only statically linked representations did not augment a graph. Chapter 3 teachers added more summary statistics to provide detail to fully answer the questions they explored. In Chapter 3, 75% of teachers added statistical measures to summary tables as compared to 28% in Chapter 4. A similar percentage of teachers from Chapter 3 and 4 (58% and 56% respectively) added statistical measures to graphical representations as compared to 72% of the teachers in Chapter 1. This may be attributed to features of TinkerPlots and Fathom. Whereas, Fathom offers the ability to summarize statistical measures in a summary table, TinkerPlots affords being able to easily incorporate summary statistics, such as measures of center, on graphical representations.

Discussion

Given the dynamic nature of Fathom and TinkerPlots, and emphasis in the materials on using dynamic linked representations, it was somewhat surprising how few teachers' responses provided evidence of either dynamic or static linking among representations. We recognize, however, that lack of evidence of linking does not mean that teachers did not engage in this activity, just that they did not report their work or findings in a way that we could infer that linking had occurred. In addition, it was apparent that those who linked in the Chapter 1 and 3 tasks did so for similar purposes, often focused on specific individual cases, whereas the Chapter 4 teachers who linked often did so to examine group propensities. We wonder if this difference is related to the nature of the tasks (comparing distributions versus relationships among attributes). In the next phase of analysis, it will be important to have a better way of capturing how a teachers' use of representations is connected with the complexity of their statistical problem solving (i.e., what is a relationship between their SK and TSK). Based on our cross-institutional sample, teachers educators may need to provide many opportunities to engage teachers in tasks that explicitly encourage dynamic linking. This may facilitate teachers understanding and using dynamic capabilities in their statistical work, and hopefully their work with students.

Endnotes

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THE TEMPORAL CONCEPTION: STUDENT DIFFICULTIES DEFINING PROBABILISTIC INDEPENDENCE

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This article discusses results from interviews investigating students' understanding of probabilistic independence and mutual exclusivity. Three students compared several sets of events in various sample spaces. Data collected from these interviews gives evidence of a temporal conception wherein students think of independence as reliant on a chronological sequence of events and conditioning. With this approach, students may see some sample spaces (e.g. spinners) as having pair-wise independence for all events. In other sample spaces, (e.g. decks of cards) students see pairs of events as both independent and not independent, this being determined by whether replacement occurs.

Introduction

Probability and Statistics have been increasingly emphasized in elementary and high school education over the past two decades (National Council of Teachers of Mathematics [NCTM], 1989; NCTM, 2000). This emphasis has also extended into post-secondary education. Mutual exclusivity and independence have gained such emphasis, making it important to accurately gauge students' conceptualizations of these two ideas. This article discusses results from interviews aimed to better understand how students think about independence and mutual exclusivity in turn contributing to the theoretical framework of how we approach learning in this area.

Manage and Scariano (2010) found that an alarmingly high percentage of undergraduate students who were enrolled in a course in probability and statistics had fundamental misunderstandings about the relationship between the ideas of independent events and mutually exclusive events. They found this by directly assessing students' understanding of this relationship through a non-scientific, multiple-choice survey of 217 students.

The researchers explored student responses to two questions. In each question, two events, A and B, are assumed to have nonzero probabilities in the same sample space. The first question presented a Venn diagram of the sample space with non-intersecting areas, labeled "*A*" and "*B*," within a rectangular sample space. The question stated that *A* and *B* are mutually exclusive and gave four responses: "*A* and *B* are independent events," "*A* and *B* are not independent events," "*A* and *B* may or may not be independent events," and "I really don't know how to do this problem" (Manage & Scariano, 2010, p. 18). The second question presented the fact that *A* and *B* are independent, have nonzero probabilities, included no diagram, and gave four responses: "*A* and *B* are mutually exclusive events," "*A* and *B* are not mutually exclusive events," "*A* and *B* may or may not be mutually exclusive events," and "I really don't know how to do this problem" (Manage & Scariano, 2010, p. 19).

In the first question, 68.3% of students incorrectly chose the first answer choice, namely "*A* and *B* are independent events." In total, 88% of students incorrectly answered this question. With respect to the second question, 36% of students gave the incorrect first response, "*A* and *B* are mutually exclusive events" whereas 23.3% of students responded correctly. By the results of the first question, students seem to think that these two ideas have a direct relationship, that mutual exclusivity implies independence. This misconception seems less prevalent in the second question, since the responses were more evenly distributed than they were in the first (Manage & Scariano, 2010).

D'Amelio (2009) found that most participants could not correctly identify a method for calculating the probability of the union of mutually exclusive events. Most students mistook the product, rather than the sum, of the two events' respective probabilities for the proper calculation. These results suggest students' confusion about the use of such a product when calculating certain probabilities. They also point to a similar misconception to that found in Manage and Scariano (2010), particularly a misunderstanding of the distinction between independent and mutually exclusive events. Confusion between calculating the probability of the intersection of two events rather than their union provides an alternative explanation for this mistake.

Shaughnessey (1992) identifies two equivalent definitions of independent events given that the two events *A* and *B* are in the same sample space and have nonzero probabilities:

Definition 1- $P(A\|B) = P(A)$	**Definition 2-** $P(A \cap B) = P(A) \times P(B)$.

Table 1. Definitions of Independence

Both D'Amelio (2009) and Manage and Scariano (2010) use Definition 2 in their research. Each of these researchers also define two events *A* and *B* as "*mutually exclusive* if and only if (A and B) = $A \cap B = \varnothing$" (p. 15). From this definition, if *A* and *B* are mutually exclusive then $P(A \cap B) = 0$. So, by the zero product property, it cannot be the case that $P(A \cap B) = P(A) \times P(B)$ and $P(A \cap B) = 0$ when *A* and *B* have nonzero probabilities.

Of the research deliberated, the explicit relationship between independence and mutual exclusivity was found in only three articles (Keeler & Steinhorst, 2001; Kelly & Zwiers, 1988; Manage & Scariano, 2010). Manage and Scariano use the reasoning discussed above whereas Kelly and Zwiers address this relationship in the context of student misunderstanding. They provide several examples of how each of these ideas can be explored separately in a classroom. Kelly and Zwiers contend that, "most of the confusion arises because we, as instructors, do not take the time to relate the two concepts." They then blatantly state the relationship between the two ideas- "mutually exclusive events are (almost) never independent." They attribute the "almost" in this last quote to the "pathological cases" when one or both events considered have zero probability (Kelly & Zwiers, 1988). Keeler et al., however, acknowledge the relationship as a common misunderstanding among students.

In considering pedagogical implications for this research, we find that students have many difficulties with both conditional probability and independence (Shaughnessey, 1992). This research goes on to say that students' "misconceptions of conditional probability may be closely related to students' understanding of independent events and of randomness in general" (Shaughnessey, 1992, p. 475). He points out that many researchers "advocate introducing the concept of independence via the conditional probability definition (Definition 1), as they believe this is more intuitive for students" (1992, p. 475). This intuition comes in the context of without-replacement problems. If the sample space remains unchanged, then the first experiment bears no affect on the second experiment.

Other pedagogical research in this area discusses students' misconceptions related to independence almost exclusively with respect to conditional probability (Tarr & Lannin, 2005). Tarr and Lannin (2005) justify their concentration on these types of misconceptions, citing Shaughnessey (1992), and focus on replacement and non-replacement situations because of the prevalence of these types of problems in the typical curriculum. Tarr and Lannin state that "within this context [that of *with-replacement* situations], an 'understanding of independence' is demonstrated by students' ability to recognize and correctly explain when the occurrence of one event does not influence the probability of another event" (2005, p. 216). There is also an emphasis that students understand the change of an event's probability in non-replacement conditional probability problems is due to the change of the sample space.

While Shaughnessey (1992), Tarr and Lannin (2005), and Keeler and Steinhorst (2001) suggest conditional probability as a context for independence D'Amelio (2009), Kelly and Zwiers (1988), and Manage and Scariano (2010) each explore student misconceptions outside of the conditional probability setting. This could provide some reasoning into why D'Amelio (2009) and Manage and Scariano (2010) had such disturbingly low numbers of correct responses. Supposing that the students' previous curricula addressed independence in the context of conditional probability, the students may not have been able to correctly reason about the relationship between independence and mutual exclusivity without such a context. Perhaps this is what Kelly and Zwiers are arguing when they say that, "we, as instructors, do not take the time to relate the two concepts," referring to mutual exclusivity and independence (1988, p. 100).

Since independence can be defined outside of a conditional probability setting, it is not limited to an order of experiments. This can be seen in a standard deck of cards. The event of drawing a spade is independent of the event of drawing an ace. We can see this since P(spade) = ¼, P(ace) = 1/13, and P(ace ∩ spade) = 1/52 = 1/4 × 1/13. This example demonstrates that independence can be accessibly thought of outside the context of conditional probability. This can prove useful since mutual exclusivity is also defined without respect to time. For instance, in the above case, it can quickly be demonstrated that P(heart∩spade) = 0 since this intersection is empty; so these two events are mutually exclusive.

Altogether, we see that different researchers emphasize two different contexts for the independence of two events. Some focus on conditional probability for the definition of independence (Definition 1), while others focus on the product definition (Definition 2). Shaughnessey (1992) provides an explanation of why educators and some researchers focus on conditional probability when dealing with independence in that it is more intuitive for students to explore independence in the context of conditional probability. Regardless, students' misconceptions about the relationship between independence and mutual exclusivity prevail.

Methods

Interviews were conducted with three undergraduate students who were enrolled in a Junior-level Proofs course (Alex, Betty, and Caroline). Betty had recently taken an undergraduate course in probability and statistics, whereas Alex and Caroline had not. In these interviews, the students were asked to give definitions of independent events and mutually exclusive events as well as provide examples of each. The students were then given various sample spaces and events within those sample spaces and asked to determine whether pairs of events were independent. The purpose of these interviews as part of broader research was to gauge the students' understanding of these concepts and the relationship between the two. Two initial sample spaces were discussed and then others were explored as students and the interviewer responded to situations throughout the interviews.

The first sample space consisted of a standard deck of cards and a fair six-sided die. The events discussed were the simultaneous drawing of a card and rolling of the die. For example, event A was "drawing a spade and rolling any number on the die." Event B was "drawing any card and rolling a three on the die." This sample space is made up of two smaller sample spaces (let's call them "subsample spaces") that are often considered individually. Each of these subsample spaces is independent of the other, as the card drawn would have no effect on the die and vice versa. The second sample space was similar to the first in that it consisted of two subsample spaces. In this sample space, an event consisted of tossing a fair coin (heads and tails) and spinning a spinner with five colors (blue, red, green, orange, and yellow) with given probabilities (.3, .2, .2, .2, and .1, respectively). So, for example, event A was tossing a head and spinning blue.

These sample spaces were chosen to observe how readily students identified the subsample spaces as independent and how they thought of the events in these sample spaces with respect to independence and mutual exclusivity. The complexity of each sample space also provided somewhat familiar situations, the combination of which prevented the participants from simply remembering probabilities from previous experience. After several pairs of events in each sample space were discussed, the interviewer and participant each introduced different sample spaces in discussion. For instance, Betty discussed a deck of cards without a die and the interviewer brought up the sample space of a die without the deck of cards when interviewing Caroline. This allowed the interviewer and participant to discuss caveats and nuances of the concepts of independence and mutual exclusivity in their own terms.

Results

Alex defined independence as, "[when] the outcome of one event does not affect the outcome of a subsequent event." This definition implies an emphasis on a sequence of events, where one of the events being considered must occur prior to the other. With regard to mutual exclusivity, however, Alex was less certain of a formal definition- changing his phrasing twice throughout the interview and eventually declaring, "Performing an event or series of events causes a subsequent event to have zero probability of happening." Again, Alex implies that this relationship is defined over a period of time. When prompted for an example of independent events, Alex gave the example of a die. He stated that rolling a six on the first roll of a die does not affect rolling a six on the second roll of a die. This example is consistent with his definition, implying that the two events in consideration take place at separate times.

Betty stated that, "Two events are independent if the probability of A occurring does not affect the probability of B occurring." Betty also described the independence of events A and B using the equation $P(A) = P(A|B)$. In comparison to Alex, this definition of independence does not necessarily imply that one event must occur before the other. But, when prompted for an example of independent events, Betty described the act of picking a card from a deck of fifty-two cards, and putting it back so that the probability of picking a second card is not affected. Similarly, when asked for an example of events *not* being independent, Betty provided the case of picking a card and not replacing it. These examples are consistent with the notion of independence in the context of a "with replacement" and "without replacement" conditioning event. In contrast, Betty defined mutually exclusive events with the statement, "you can't have both at the same time." This definition explicitly states that the events can be compared instantaneously. Here, Betty gave the example that the queen of hearts and jack of diamonds are mutually exclusive, since they cannot both occur when one card is drawn.

Caroline's definition for independence was similar to the other two participants, stating, "Two events are independent if they do not affect each other." Caroline's example of independent events was different from both Alex's and Betty's in that Caroline described "everyday events" rather than "artificial" events (such as dice or cards) that are typically investigated in the classroom setting. Caroline described how "the probability of someone wearing a red shirt is independent of their age." Similarly, when prompted for an example of events that are not independent, Caroline provided the example of someone who is forgetful is less likely to win a student lottery for a football ticket, since they are less likely to enter the lottery. This example implies a directly causal relationship, where the lower probability of the first event (entering the lottery) decreases the probability of a later event (being selected in said lottery).

It should be noted that all three participants showed initial difficulty in differentiating between the concepts of independence and mutual exclusivity, although each did eventually distinguish between the concepts. Alex, for instance, initially stated that the two concepts are "more or less the same thing" and "pretty much synonymous." Betty's first definition for mutually exclusive events, which she quickly changed, was $P(A \cap B) = P(A)*P(B)$, the mathematical definition of independence. Caroline had greater difficulty defining mutual exclusivity, stating, "It sounds like they would be independent of one another." From this, we see at least initial difficulty distinguishing between these two concepts among all the participants, consistent with the literature (Manage & Scariano, 2010).

Independence in the Sample Spaces

In the first sample space, the participants explored two pairs of events (Table 2). All three participants reacted in similar ways to the question, "Are these events independent?" In each case, all three students established various cases, beginning with some form of the question, "Do you put the card back?" In the first case, the participant described successfully performing one event, putting the card back, reshuffling (or resetting) the deck, and then successfully performing the other event in question. In each interview, the participant claimed that the events would be independent in this case. In the second case, the sequence of events was identical, except that the card was not replaced and the deck was not reshuffled.

Answers were slightly more varied in this second case. For instance, Alex thought that neither pair of events was independent, since you "change the context," but only considered event A before event B and event C before event D. Betty considered whether A came before B or vice versa and similarly for events C and D. In the first pair, Betty responded that A before B gave independence, but B before A did not. With the second pair, each event would change the probability of the other, causing C to always be not independent of D. Caroline generalized all pairs of events under this case, explaining that without replacement you affect the probability of the second event, so that no two events will be independent without replacement.

		Card	Die
Pair 1	Event A	Spade	"Any Number"
	Event B	"Any Card"	3
Pair 2	Event C	Spade	3
	Event D	Heart	4

Table 2. Events in Sample Space 1

Event B produced the most diverse explanations. Alex seemed to think of event B as impossible after drawing any card without replacing it. He attributed this to the fact that the first card would be a member of "any card," so that drawing any card would not be possible, since you will have removed one of them. Betty thought of Event B as always successful and independent. Caroline thought of event B as always successful as long as a card was remaining, but independent only when the first card was replaced. It should be noted also, that explanations of whether events on the die were independent were almost nonexistent. All three participants focused on whether the card was replaced. Alex even said, "With the die, it really doesn't matter what you do… If you're just rolling dice over and over, those are always independent."

The second sample space provided much more homogeneous responses among the participants. Most of the events discussed were single outcomes from each subsample space (i.e. one color from the spinner and one side of the coin). A few events discussed were similar to events A and B from the first sample space, where a compound event from one of the subsample spaces was considered. When asked to compare any two events in the sample space, all three participants thought of every pair of events as independent from each other. This reflects the lack of emphasis on the die in the first sample space. Successfully completing an event on both a coin and a spinner does not physically remove an object from the sample space. This fact has strong implications about the students' understanding of the relationship between independence and mutual exclusivity.

Mutual Exclusivity in the Sample Spaces

The participants seemed to struggle much more with determining two events' mutual exclusivity than with their independence. Alex changed his definition of mutually exclusive events twice, each time after encountering examples that challenged his definition. Initially, Alex's definition seemed consistent with the mathematical definition. He explained a scenario of it raining or not raining, saying, "It couldn't be raining and not raining at the same time." This phrasing alludes to an empty intersection. Throughout the interview, Alex changed his definition to more closely align with how he discussed independence, so that one event would cause a subsequent event to be impossible. Interestingly, as a corollary to his last definition of mutual exclusivity, Alex pointed out that two mutually exclusive events couldn't be independent. He made this connection before the interviewer discussed any relationship between the two terms.

Because he changed his definition, Alex's responses in the beginning of the interview convey a different conceptualization of mutual exclusivity than do his later responses. When comparing events C and D he determined that these two events are not mutually exclusive since drawing a spade does not cause drawing a heart to have zero probability. In his exploration of the second sample space, Alex concludes that no two events are mutually exclusive. This relies on his responses regarding independence, since no event in the second sample space changed the probability of a subsequent event, no event could cause a subsequent event to have zero probability.

Betty's responses to questions of mutual exclusivity were consistent with the correct mathematical definition in both sample spaces. This is because her definition of the concept was a typical translation into "everyday" language, "You can't have them both at the same time." This is not too surprising since Betty was the only participant to have had recent instruction in probability, even though Betty showed early confusion about the distinction between independence and mutual exclusivity.

Caroline also had difficulty establishing a definition for mutual exclusivity. Her final definition, "Two events that do not include parts of each other," is very similar to the mathematical definition. Most of Caroline's responses in each sample space were consistent with correct responses, except two responses in the second sample space. Event A in this sample space was "spinning blue on the spinner and tossing heads on the coin" and event E was "spinning blue or yellow and tossing heads." Caroline concluded that event A is not mutually exclusive of event E, since it was a subset of event E. But Caroline claimed that event E is mutually exclusive of event A, since you could successfully perform event E and not event A by spinning a yellow and tossing heads.

The Temporal Conception

We see from these results that students think about independence of two events by assuming the occurrence of one event and then considering whether this changes the probability of a second event that occurs chronologically after the first. The tendency for students to think of independence in this way causes a temporal conception. With all three participants, questions of events' independence were answered with students considering sequences of events. With Alex, though not in the case of Betty or Caroline, this temporal conception was also evident in his definition of mutually exclusive events.

It should be noted that, had the interviewer asked the participants to determine if the first event in a sequence of trials had changed the probability of a second event, the responses that the participants gave would have been generally correct. The misconception is that examples of chronological dependence and independence generalize to the standard definition of probabilistic independence. For instance, in the second sample space, Alex viewed all events as independent and not mutually exclusive. Meanwhile, Betty and Caroline viewed all events in the second sample space as independent, with some pairs being mutually exclusive and some not. In reality, the majority of the events discussed were mutually exclusive of each other (e.g. blue/heads, red/tails, orange/heads, etc.) and therefore not independent since most had nonzero probabilities. Furthermore, in the first sample space all participants found at least some pairs of events to be both independent (with replacement) and not independent (without replacement). This allows students to think of independence as a consequence of time and therefore allows the misconception that the same two events can be both independent and not independent.

Discussion

Understanding this temporal conception could provide insight into how students confuse the concepts of independence and mutual exclusivity. For instance, under this conception, a student could conclude that mutually exclusive events can be both independent and not independent, as was seen in the first sample space with events C and D. One may suggest that the phrase "mutually exclusive" and the word "independent" are somewhat synonymous in their everyday context, as hinted in the literature (Manage & Scariano, 2010). This was evident in the initial responses by each participant. This explanation, however, only helps to explain the responses of those students who mistakenly thought that one term implied the other. The temporal conception could potentially help explain how students thought events could be "mutually exclusive and maybe or maybe not independent" or "independent and maybe or maybe not mutually exclusive."

Kelly and Zwiers (1988) touched on the notion that students had temporal conceptions about independence. The authors discuss statement, "events A and B are independent if knowledge about whether A has occurred provides us with no knowledge about whether has occurred," (1988, p. 98) noting that, "In a very subtle way an element of time is hinted at in such a statement and it often confuses students." (1988, p. 98) They then emphasize the importance that students understand that independent events are independent regardless of time. While I agree with the importance of such a concept, it is not entirely obvious that understanding this will help students avoid the temporal conception. It is equally important that students understand that knowledge of whether events are independent is calculable without any element of time, assuming that one knows the theoretical probability of each event and of their intersection.

It can be argued that this subtle element of time is also present when discussing conditional probability. This is important since much of the research suggests that independence be taught in the context of conditional probability (Shaughnessey, 1992; Tarr & Lannin, 2005; Keeler & Steinhorst, 2001). For instance, the phrase "given that the other event has occurred" implies that perhaps the first event's occurrence was chronologically prior to the second. Further research could investigate what relationships exist between these concepts with respect to this element of time. It is also important to understand how we as educators can overcome this temporal conception. Analysis of current school mathematics curricula could allow insight into how students build notions of independence. For instance, repeatedly seeing without replacement scenarios of conditional probability as a context for how two events can be "not independent" could enable students to develop the temporal conception with independence.

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IMPULSIVE-ANALYTIC DISPOSITION: INSTRUMENT PILOT-TESTING

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The likelihood-to-act (LtA) survey measures impulsive and analytic dispositions in solving mathematics problems. The current version has 16 impulsive and 16 analytic items. Its validity was assessed using a sample of 27 in-service and 92 pre-service teachers. Both the impulsive and analytic subscales were found to have internal consistency reliability, but they were not correlated with one another. The impulsive subscale was predictive of correctness in classifying the LtA items. The analytic subscale was predictive of how well a participant would perform in Part 2 of a math test after taking Part 1 and being warned that some items could be tricky.

Many pre-service teachers for elementary and middle grades appear to have an *impulsive disposition*—a proclivity to spontaneously proceed with an action that comes to mind without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation (Lim, 2006). For example, Lim (2009) found that 22 out of 28 pre-service middle-school teachers used the same strategy to solve two superficially-similar but structurally-different problems.

As mathematics educators, we want to help students advance from impulsive disposition to *analytic disposition*—where one analyzes the problem situation (Lim, 2006). An instrument that can identify problem-solving disposition will be valuable to both students and teachers. With this purpose in mind, we developed the likelihood-to-act survey (LtA). Based on our analyses of the data collected using the original 18-item version (Lim, Morera, & Tchoshanov, 2009), we increased the number of items from 18 to 32 and revised some of them. To establish the validity of this enhanced version, we developed and administered an open-ended questionnaire, a classification exercise, and a multiple-choice mathematics test.

Constructs Related to Impulsive-Analytic Disposition

Impulsive disposition can be viewed from various perspectives. From a cognitive perspective, impulsive disposition is related to the *Einstellung effect*—the phenomenon of solving a given problem in a fixated manner even when a better approach exists (Luchins, 1942). Ben-Zeev and Star (2001) used the term *spurious correlation* to account for students' association-based behavior: "when a student perceives a correlation between an irrelevant feature in a problem and the algorithm used for solving that problem and then proceeds to execute the algorithm when detecting the feature in a different problem" (p. 253). Einstellung effect, spurious correlation, and impulsive disposition emphasize different aspects of the same phenomenon: (a) Einstellung effect refers to a mental fixation, (b) spurious correlation refers to a feature-algorithm association, and (c) impulsive disposition refers to a cognitive tendency.

From a problem-solving perspective, problem-solving disposition is related to metacognition. Actively monitoring one's progress in relation to a goal is an indicator of analytic disposition. Impulsive disposition, on the other hand, is inferred when students "read, make a decision quickly, and pursue that direction come hell or high water" (Schoenfeld, 1992, p. 356). We consider impulsive disposition to be an externalization of certain beliefs such as "there is only one correct way to solve any mathematics problem—usually the rule the teacher has most recently demonstrated to the class" (p. 359).

From a psychological perspective, impulsive disposition can be viewed as a personality trait. Kagan et al. (1964) regard a child as (a) *impulsive* if the child responds to a question with an inaccurate answer but in a short response time, and (b) *reflective* if the child has an accurate answer but long response time. Nietfeld and Bosma (2003) found that college students' impulsive-reflective style is consistent across three types of tasks: verbal, mathematical, and spatial.

From a teaching-learning perspective, impulsive-analytic disposition can be viewed as *ways of thinking* (Harel, 2008) or *habits of mind* (Cuoco, Goldenberg, & Mark, 1996). In addition to helping students develop mathematical understanding, teachers can help students to develop desirable habits of mind such as being analytic in solving a problem instead of being impulsive by applying the first idea that comes to mind. Self-awareness is a crucial first step towards transforming habits of mind from undesirable to desirable. A way to create awareness is to have an instrument that can accurately assess one's own impulsive-analytic disposition.

Means to Assess Impulsive-Analytic Disposition

A reliable way to investigate students' problem solving behaviors is through task-based interviews (Clement, 2000; Goldin, 1998) and think-aloud protocols (Ericsson & Simon, 1993). Ways of thinking such as impulsive anticipation and analytic anticipation can be identified from the analysis of students' responses to interview tasks (see Lim, 2006). Although well-suited for uncovering problem-solving disposition in individual students, this mode of data collection is not practical for large-scale assessment of students' problem-solving disposition.

Well-designed mathematical problems can be an effective and efficient means to assess impulsive-analytic disposition. Frederick (2005) designed a three-item test for assessing cognitive reflection. One of the items is: "A bat and a ball cost $1.10 in total. The bat costs $1.00 more than the ball. How much does the ball cost?" (p. 27). The most common wrong answer, 10 cents, is considered impulsive. A reflective person, on the other hand, is likely to realize that the difference between $1.00 and 10 cents is not $1.00 but 90 cents. Other problems, such as missing-value problems involving non-proportional situations, can also be used to elicit impulsive behaviors.

An efficient way to measure cognitive and psychological constructs is through the use of questionnaire. For example, the *Need for Cognition Scale* (NfCS; Cacioppo & Petty, 1982) measures one's desire to engage in a complex thought. In the context of mathematics problem solving, Lim et al. (2009) developed the LtA survey to measure problem-solving disposition along the impulsive-analytic dimension. The LtA survey consists of six-point Likert items where participants are asked to indicate how likely they are to respond to a given mathematical problem in the described manner. The LtA survey, like most Likert-scale questionnaires, can be used for self-assessment.

The purpose of our study was two-fold. First, we sought to improve internal consistency reliability by lengthening the LtA survey to a 32-item measure. We then examined the validity of the LtA survey by determining whether the LtA scores were related to accuracy in classifying the LtA items and performance on a two-part mathematics test.

Method

Participants

There were three groups of participants: (a) 27 in-service teachers and 10 pre-service teachers in a program for improving mathematics and science education in El Paso; (b) two mathematics classes for pre-service EC-8 (Early Childhood to Grade 8) teachers with 33 and 22 students respectively; and (c) one class of 27 pre-service EC-8 teachers. Because this round of data collection was designed for testing and piloting the instruments, a convenience sample of in-service teachers was used. Data collection involving the first group was integrated into a 2.5-hour lesson on problem-solving disposition. The participants first took the LtA survey and an open-ended questionnaire. They then experienced their own problem-solving disposition via a three-problem activity involving clickers which are remote units for a personal response system that records student responses. They were introduced to impulsive and analytic dispositions via a PowerPoint Presentation and were then asked to classify each of the 32 LtA items based on whether they considered the act described in the item analytic or impulsive. The activities for the second group of participants differed slightly from that for the first group in that the three-problem clicker activity was excluded because of shorter class time. The third group of participants took the LtA survey and the two-part math test.

Instruments

Likelihood-to-Act survey. The current version of the LtA survey has four categories: algebra, fraction, word problem, and non-mathematically-specific description. In each category there are four impulsive items and four analytic items. A pair of items for each category is presented below.

aA3 $(x-7)(x-4)=0$. When asked to solve for x, how likely are you to study the equation and predict the solution? [*analytic, algebra*]

iA3 $(x-5)(x-8)=0$. When asked to solve for x, how likely are you to multiply out the terms (i.e., FOIL) and then solve $x^2-13x+40=0$ using the quadratic formula? [*impulsive, algebra*]

aF1 $\frac{3}{5}+\frac{11}{12}+\frac{1}{12}\frac{3}{5}+\frac{11}{12}+\frac{1}{12}$. When asked to find the answer for the above arithmetic expression without using a calculator, how likely are you to begin by studying the fractions to see if you can predict the answer? [*analytic, fraction*]

iF1 $\frac{3}{4}+\frac{1}{10}+\frac{9}{10}\frac{3}{4}+\frac{1}{10}+\frac{9}{10}$. When asked to find the answer for the above arithmetic expression without using a calculator, how likely are you to begin by finding the common denominator? [*impulsive, fraction*]

aW3 Paula is cycling from home to school. At 8 o'clock she has already cycled 2.4 miles. When asked to find her rate of cycling, how likely are you to analyze the problem situation instead of dividing 2.4 by 8? [*analytic, word problem*]

iW3 Jimmy is walking from home to school. At 7 o'clock he has already walked 1.4 km. When asked to find his rate of walking, how likely are you to use the $d = rt$ or $r = d/t$ relationship and obtain 0.2 km/hour? [*impulsive, word problem*]

aG1 In solving a problem in mathematics, how likely are you to interpret and understand the problem thoroughly before deciding what to do? [*analytic, general*]

iG1 In solving a problem in mathematics, how likely are you to use the first idea that comes to mind? [*impulsive, general*]

Three measures can be derived from the LtA survey: (a) the analytic subscale is based on the 16 analytic LtA items, (b) the impulsive subscale is based on the 16 impulsive items, and (c) the analytic-impulsive difference is computed based on the difference between the analytic score and the impulsive score for each pair of items.

Open-ended Questionnaire. In this questionnaire, six open-ended questions were posed to uncover participants' initial approaches for solving selected impulsive items in the LtA survey.

The following question, for example, is associated with Item iA3: “What are the first few actions that you would take when asked to solve $(x - 5)(x - 8) = 0$ for x?” Two versions were created to cover the 12 non-general impulsive items (i.e., iA1-iA4, iF1-iF4, and iW1-iW4). The first author and two first-year graduate students coded all the responses in one version. Another team of a full-time research assistant and a final-year doctoral student coded all the responses in the second version. Members from both teams met to analyze the responses in a training set and to discuss general principles for analyzing. The inter-rater reliabilities for the two teams were 0.91 and 0.97 respectively. For each written response, two codes were assigned: (a) disposition code using a five-point scale to indicate whether the response has a strong or weak indication of analytic (A+ or A-) or impulsive (I+, I-) or is inconclusive (U); and (b) correctness code (1 = correct or no indication of misunderstanding, and 0 = presence of misconceptions or non-trivial errors). The analysis of students’ written responses was reported in the 2010 PMENA conference (Lim & Mendoza, 2010).

Classification test. The classification test was designed to assess the accuracy in determining whether the act described in each LtA item is analytic or impulsive. For each of the 32 LtA items, participants were asked to label the item as analytic or impulsive, and then to rate whether they were confident or not confident in their answer.

Belief survey. On a 10-point scale (1 = Impulsive; 10 = Analytic), participants were asked which point on the scale best describes their own problem-solving and that of their peers, high-school (HS) math students, HS science students, HS math teachers, and HS science teachers. On a five-point scale (1 = strongly disagree; 5 = strongly agree), participants were asked whether they agree with statements about their problem-solving disposition, the lesson/presentation on impulsive-analytic disposition, and the U.S. mathematics education with regards to impulsive disposition. Listed below are representatives of the statements in the survey.

- I consider myself more impulsive than analytic.
- Today’s lesson has helped me appreciate the importance of differentiating between impulsive disposition and analytic disposition.
- U.S. high school mathematics curricula promote impulsive disposition.

Two-part mathematics test. The test was divided into two parts to test the effect of warning about trickiness of problems on performance. The test items in both parts are structurally equivalent. Each part has 12 problems: 6 involving non-proportional situations and 6 involving fractions, ratios, or percents. The problems were designed such that students with an impulsive disposition would be more likely to choose a wrong answer choice. Below are two test items:

- When a candle has burned 23 mm of its original length, its height is 82 mm.
  What is the candle’s height when it has burned 46 mm of its original length?
  (a) 41 mm (b) 59 mm (c) 164 mm (d) None of the above
- Benito needs to increase the money he has now by 20% so that he can buy a $540 laptop. How much more money does he need?
  (a) $90 (b) $108 (c) $432 (d) $2700.

After completing the first part and before beginning the second part, an example (see Figure 1) was presented to caution participants that “some of the items in this test may be considered ‘tricky’ for some students.”

Consider the following problem:

> John bought a new car this year. If the car depreciates by 20% each year, the car will have depreciated by _______ in two years.
> (a) 30% (b) 36% (c) 40% (d) 64%

Many students choose "c" because 20% plus 20% is 40%, but the correct answer is "b".
The actual depreciation in dollar amount in the second year is less than that in the first year because the cost of the car after 1 year is less. For example, suppose the new car costs $10,000.
After 1 year, the car's value drops by $2000 (20% of $10000), and it is now worth $8000.
In the second year, its value drops by $1600 (20% of $8000). So after 2 years, its value drops by $3600, which is 36% of $10000.

Figure 1: An example to warn students to be cautious prior to working on Part 2

Results and Discussion

Reliability of the Various Measures

The internal consistency reliability estimates for the various measures in each instrument were calculated using Cronbach's alpha. Table 1 presents for each measure the number of items in the measure, the mean value, the standard deviation, the α-value, and the 95% confidence interval. The reliability values for most of the measures are relatively high, except for the Coded-correctness score from the open-ended questionnaire and Part 1 of the mathematics test. The reliabilities for the analytic and impulsive subscales, 0.81 and 0.74 respectively, were greater for this version of the LtA survey than those in the previous version, 0.63 for the 7-item analytic subscale and 0.64 for the 7-item impulsive subscale (Lim et al., 2009).

Table 1: Descriptive and Reliability Measures for the Various Instruments

	No. of Items	Mean	Standard Deviation	Cronbach α	95% Conf. Interval for α
LtA Survey (N = 119)					
Analytic Subscale	16	3.96	0.71	0.81	0.76 , 0.86
Impulsive Subscale	16	4.45	0.63	0.74	0.67 , 0.81
Analytic-Impulsive Difference	16	-0.49	0.83	0.77	0.71 , 0.83
Open-ended Questionnaire (N = 92)					
Coded-disposition Score	6	1.86	0.31	0.71	0.61 , 0.79
Coded-correctness Score	6	0.77	0.26	0.29	0.38 , 0.49
LtA Classification Test (N = 92)					
Classification-accuracy Score	32	0.59	0.12	0.76	0.69 , 0.83
Classification-confidence Score	32	0.75	0.09	0.86	0.81 , 0.90
Belief Questionnaire (N = 92)					
Self-disposition Belief[a]	2	2.88	0.38	0.84	0.75 , 0.89
Lesson-on-disposition Opinion[b]	4	4.43	0.05	0.93	0.90 , 0.95
Two-part Math Test (N = 27)					
Math-Part1 Score[c]	10	0.28	0.22	0.57	0.28 , 0.76
Math-Part2 Score	12	0.32	0.22	0.67	0.46 , 0.83

[a]The 10-point scale item was transformed into a 5-point scale. The 5-point item was reverse coded.
[b]All the four items about the lesson used a 5-point scale.
[c]Two items had zero variance (all 27 students got them wrong) and were removed from the scale.

A mean value of -0.49 for analytic-impulsive difference indicates that the participants in this study chose lower values for analytic items than for impulsive items in the LtA survey. A mean value of 1.86 for the coded-disposition score on a 5-point scale (1 = impulsive; 5 = analytic) also indicates that the 92 participants were generally more impulsive than analytic in their open-ended responses. A mean value of 2.88 for the Self-disposition Belief score on a five-point scale, on the other hand, indicates that the participants view themselves as more analytic than impulsive. Mean values of 0.23 (12 items; 0.28 is Table 1 was based on 10 items) and 0.32 (12 items) for the two parts of the math test indicate that the 27 pre-service teachers on average responded correctly to 23% and 32% of the 12 items. These findings suggest that participants are generally more impulsive than analytic although they might view themselves as more analytic than impulsive.

A mean value of 4.43 on a 5-point scale indicates that the participants have a favorable opinion about the lesson on impulsive and analytic dispositions. A mean value of 0.59 for the classification-accuracy score indicates that the 92 participants (Groups 1 and 2) have a 59% accuracy in classifying the 32 LtA items. The 27 participants (Group 3) showed a 9% gain from Part 1 to Part 2, suggesting that students performed better when they were warned to be cautious.

Correlations among the Various Measures

The correlation between the analytic subscale and impulsive subscale was -0.014, not statistically different from zero. In other words, the analytic and impulsive subscales were independent of one another. Interestingly, the difference score between the two subscale scores for each pair of items had a relatively high reliability of 0.77 (see Table 1).

Table 2 presents the correlation between the measures from the LtA survey and other measures that might be related to impulsive-analytic disposition. The disposition scores derived from coding of the six open-ended responses were positively correlated ($r = 0.373$) to the analytic subscale and negatively correlated ($r = -.0488$) to the impulsive subscale. These significant correlations were probably due to the similarity between the items in the open-ended questionnaire and the LtA items. In addition, their initial exposure to the LtA survey first might have influenced their subsequent written responses.

Table 2: Correlations between LtA Subscales and Other Measures

	Analytic Subscale	Impulsive Subscale
Analytic Subscale	1	
Impulsive Subscale	-0.014	1
Analytic-Impulsive Difference	0.745**	-0.678**
Coded-disposition Score	0.373**	-0.488**
Coded-correctness Score	0.256*	-0.205*
Classification-accuracy Score	0.107	-0.286**
Classification-confidence Score	0.183	-0.048
Self-disposition Belief Score	0.418**	-0.275*
Lesson-on-disposition Score	0.011	-0.041
Math-Part1 Score[a]	0.075	-0.205
Math-Part2 Score	0.595**	-0.140

$^{*}p < .05$, $^{**}p < .01$.

Participant accuracy in classifying items was negatively correlated to impulsive subscale ($r = -0.286$) but not correlated to analytic subscale. This finding suggests that students who have an impulsive disposition tended to make more mistakes in classifying items. Participant confidence in their classification was not correlated to either of the LtA subscales.

Participant self-reported disposition score was correlated to both the LtA subscales. The lesson-on-disposition score was not correlated to either of the LtA subscales.

Participant performance in Part 1 of the mathematics test, prior to the warning, was not correlated to the LtA subscales. On the other hand, student performance in Part 2 was correlated significantly to the analytic subscale but not to the impulsive subscale. This suggests that the warning about problems being tricky had an impact on students with an analytic disposition but not those with an impulsive disposition.

In summary, the two LtA subscales were reliable and uncorrelated. Other measures in the study were reliable and were appropriately associated with the LtA constructs.

Concurrent Validity

In addition to the above analyses, we were also interested in determining whether the LtA subscales could be used to predict the following variables: (a) self-reported GPA, (b) classification-accuracy score, (c) lesson-on-disposition score, (d) Math-Part1 score, and (e) Math-Part2 score. To perform these analyses, a hierarchical multiple regression was performed. In the first step of the multiple regression, variables representing participant characteristics (years of teaching experience, sex, grade band and subject area currently teaching or planning to teach) were entered into the regression model. In the second step, we typically entered the two LtA subscales (except for predicting Math-Part2 score). None of the variables mentioned above could statistically explain variability in self-reported GPA, the lesson-on-disposition score, and the Math-Part1 score. In other words, the LtA scores could not predict these three variables, but the LtA scores did predict classification-accuracy score and the Math-Part2 score.

Predicting Classification-accuracy Score. The variables entered in the first step of the regression model explained 3.3% of variability in the classification-accuracy score. None of these predictors was statistically significant. When the LtA subscales were added, the fraction of variance explained, R^2, increased by 10.3% to 13.6%. The impulsive subscale ($\beta = -.316$, $p = 0.011$) accounted for a significant portion of the variance. In other words, increased impulsivity was associated with worse item classification.

Predicting Math-Part2 score. This regression consisted of four steps instead of two. In the first step, participant sex and indicator variables representing grade band were entered and accounted for 3.5% of variability in the Math-Part2 scores. None of these predictors was statistically significant. In the second step, Math-Part1 score was entered and accounted for an additional 27.8% of variability. Increased Math-Part1 scores were associated with increased Math-Part2 scores ($\beta = .556$, $p = .010$). In the third step, the LtA subscale scores were entered and accounted for an additional 37.8% of variability. Increased analytic disposition ($\beta = .652$, $p = .000$) was associated with increased Math-Part2 scores. In the final step, values representing the interaction between Math-Part1 and the LtA subscales were entered into the regression model. These interactions explained an additional 13.1% of variability. The interaction between Math-Part1 and analytic disposition was statistically significant ($\beta = 1.75$, $p = .004$) in that individuals who had both high Math-Part1 scores and high analytic scores had higher Math-Part2 scores.

Concluding Remarks

The results obtained in this research phase indicate that a measure of problem-solving disposition along the analytic-impulsive dimension could be created using the LtA survey. The LtA scales demonstrated adequate internal consistency reliability. In addition, we found that the analytic and impulsive subscales were independent of one another, replicating the result of an earlier study in the behavioral decision theory on types of decision making styles (Morera et al., 2006).

In this round of data collection, participants were also asked to complete a series of other measures. The analytic and impulsive LtA scores were found to be correlated to certain scores derived from the open-ended questionnaire, the belief survey, and the two-part mathematics test. In particular, the impulsive score was found to be predictive of correctness in classifying the LtA items, and the analytic score was predictive of how well a participant would perform in Part 2 of the math test after the warning, while controlling for the effects of Math-Part 1 scores. In summary, these findings represent a first step in demonstrating the utility of the LtA measures.

The data we obtained were used to refine the LtA survey (see Lim & Mendoza, 2010) and the two-part mathematics test. The revised versions of the two instruments have been administered to more than 450 pre-service teachers. The Barratt Impulsiveness Scale Version 11 questionnaire (Patton et al., 1995), the NfCS questionnaire (Cacioppo & Petty, 1992), and the Cognitive Reflection Test (Frederick, 2005) were administered concurrently to determine how well the LtA survey correlates with these three instruments and how well the LtA scores predict students' performance in the two-part mathematics test.

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MATHEMATICAL PROBLEM SOLVING PRACTICES OF HIGH SCHOOL CHILDREN: FACTORS INFLUENCING CHOICES OF STRATEGIES

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In this work we investigated the problem solving behaviors of 3 highschool students as each solved four common non-routine problems with the goal to trace performance constancy across different subject areas and problem types. Additionally, we aimed to identify possible factors that influenced children's choices of heuristics in different problem contexts. The results suggested the individual's confidence and preference for the use of certain strategies. Inconsistency in the same individual's mathematics problem solving behaviors across different subject areas was revealed.

Introduction

The development of problem-solving ability among school children has been a persistent goal of mathematics education community for over a century; however, the issue of how to develop problem solving skills among learners continues to be a major dilemma. This is, in part, due to lack of specific knowledge about mathematical problem solving practices of children and factors that influence their choices and actions (English, 2010). Indeed, previous research studies on problem solving have primarily focused on effective implementation of problem solving instruction by examining students' problem solving performance on tasks (Anderson & White, 2004). These studies have identified some key factors for the success or failure of implementation of problem solving approaches in mathematics teaching; however they do not provide detailed accounts of individuals' problem solving behaviors. Muir, Beswick, and Willamson (2008) suggested that researchers must focus on understanding what successful problem solvers do and use that knowledge to help individuals develop their problem solving skills. They further argued that instead of focusing on whether particular strategies should be taught or not and how, greater attention must be devoted to understanding processes that individuals use when engaged in problem solving. In support of this suggestion we argue that knowledge about children's problem solving behaviors and factors that influence their mathematical practices while solving problems can better assist teachers in helping nurture mature problem solvers. Such knowledge is currently not well developed. The goal of the research we report here was to address this need.

The purpose of this study was to examine mathematical problems solving practices of three students in an attempt to determine whether the individuals' performances were consistent across different subject areas and problem types. Moreover, we were interested in identifying those factors that influenced children's choices of heuristics used in different problem contexts. Lastly, we intended to isolate common and unique patterns of behaviors that children exhibited as well as those factors that seemingly contributed to the institution of those patterns.

Context and Background Literature

Nearly two decades ago Lester (1994) summarized the research community's perspectives on qualities that distinguish successful problem solvers from those characterized as poor problem solvers, and concluded that good problem solvers: know more and their knowledge is well connected and composed of rich schemata, focus more on structural features instead of literal features of problems, are more aware of their own strengths and weakness in terms of problem solving, monitor and regulate their problem-solving efforts more routinely and, are more concerned about obtaining best solutions to problems. More recently however, English and Sriraman (2010) argued for a reconsideration of this list, indicating that since previous research had focused mainly on solving word problems emphasized in school textbooks, primarily routine and procedural then results concerning quality of problem solving and nature of problem solving performance of children should be more critically examined. The authors attributed the lack of success of school based practices for fostering problem solving skills among children to community's inadequate knowledge about *how* individuals come to make decisions about when, where, why, and how to use heuristics and strategies when faced with novel problem contexts. Naturally, focusing on applying these strategies, without understanding how and why individuals make decisions about pathways for solving problems is non-productive (English, Lesh, & Fennewald, 2008). Despite these criticisms several fundamental factors regarding effective mathematical problem solving have been identified. First, knowledge of heuristics and their appropriate use are recognized as fundamental to mathematical problem solving (Schoenfeld, 1992). There is evidence indicating that students' use of heuristic strategies is positively related to success in problem solving, although the effect may not always be significant (Kantowski, 1977). Yet a number of studies have shown the deficiencies that students exhibit when applying heuristics and metacognitive strategies to their problem solving processes (Schoenfeld, 1992).

Flexibility in strategy use has also been referenced as a key aspect of successful problem solving. Flexibility refers to the quantity of variations that can be introduced by an individual in the concepts and mental operations one already possess (Demetriou, 2004). Elia, Heuvel-Panhuizen, and Kolovou (2009) discussed two methods for studying strategy flexibility usage: inter-task flexibility (changing strategies across problems) and intra-task flexibility (changing strategies within problems). They used three non-routine problems to study the strategy use and strategy flexibility by 4th grade high achievers. An implicative statistical method was performed to determine whether the strategies used by students to solve the three problems were successful or not. Guess-and-check strategy was found to be the most crucial strategy that led to the success of the three pattern/algebra problems. An important finding was that higher inter-task strategy flexibility was displayed by more successful problem solvers, while intra-task strategy flexibility did not support the problem solvers in reaching a correct answer. An intra-task strategy flexibility study showed that the understanding to the problem influenced the correctness of the answer, instead of the flexibility of the strategies.

A study of four 6th grade students' problem solving behaviors was conducted by Muir, Beswick, and Williamson (2008). The strategies students used in solving 6 problems were analyzed and three categories of performance were proposed to associate with the levels of problem solving behaviors students exhibited including, naive, routine and sophisticated. The consistency of approaches across problems for each individual was also studied, and the conclusion was that most individuals consistently exhibited behaviors characteristic in one category. Since all 6 problems used in this research concerned number and number sense, the consistency in performance across different content areas was not revealed. Our goal in the current study was to contribute to the existing literature on mathematical problem solving by examining the mathematical problem solving practices of 3 children on 4 tasks selected from different content areas in order to identify ways in which children's choices of heuristics and orientation may have influenced their mathematical problem solving performance.

Methodology

A task-based interview methodology was used to closely observe and study three students as they worked on non-routine mathematical tasks. A case study report was developed for each student, describing, in detail, their actions during interview sessions. These case study reports were used, first, to identify and analyze the processes students used and patterns of problem solving behaviors they exhibited while solving problems; second, to describe and analyze the problem solving strategies children utilized, their choice of representations and metacognitive behaviors they accessed and used.

Although the major research project from which the data for the study was selected involved interviews with nearly 60 children, only three participants, Jazzy in 8^{th} grade, Liza and Yoni in 9^{th} grade (all pseudo names), were selected to serve as case subjects for the current study. This selection was due to several important considerations as described below.

First, all three participants had signed consent forms to participate in the longitudinal research project. Second, all three had worked on the same four tasks used as data collection sources. This would allow us to draw inferences regarding comparisons among their thinking and orientations, processes they used during problem solving episodes and metacognitive behaviors they exhibited. Additionally, the three participants offered a wide range of backgrounds and habits that would strengthen the potential for generalizability of the results. Despite their differences, the participants shared similar attributes; they were characterized as "successful" students of mathematics as measured by grades they had secured in their mathematics courses and results of the State mandated standardized exams. Lastly, each of the subjects displayed distinct behaviors when interacting with problems during the enrichment sessions: Liza exhibited flexible explorations when tackling problems, Jazzy showed strong reliance on calculator and numbers, and Yoni possessed the most sophisticated mathematics technique learned in school curriculum. Diverse performances were expected among the three participants.

Data Sources

The data sources consisted of two interviews with each of the participants. Each interview consisted of approximately 35-40 minutes. Each interview was tape–recorded and used in analysis. The first interview consisted of two parts: the first part was assessing participants' mathematics background information, their beliefs about mathematics, and their views on value of mathematics for their lives. The second part of the first interview the children contained problem solving episodes. During the second interview it was reassured that the participants solved the remaining problem selected for data analysis.

The participants were interviewed individually. The protocol for problem solving interviews suggested the least interruption from interviewers during students' problem solving work. The protocol also suggested that interventions be made only when a clear understanding what children were doing was not evident or if reasoning and justification was not shared by them. The children were not restricted by a specific amount of time or their representational systems they could use.

Four non-routine problems were selected to access participants' problem solving performances in patterns, functions, and geometry (Table 1 illustrates two of the interview questions). The problems were selected so to elicit different heuristics. The diversity of subject areas and heuristics served the aim of studying the consistency of individual problem solving behaviors/performances across problem types and subject matter contexts.

1. Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. If at the end each has 8 dollars, how much money did each have at the beginning?
4. What relationships exist among the areas of triangles and rectangle?

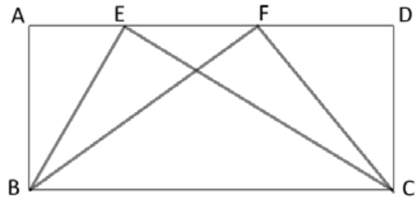

Table 1. Description of problems

Data Analysis

Using Mason's mathematical problem solving model, mathematical practices of each child were analyzed and modeled. Mason (1985) divided the process of problem solving into three phases: entry, attack, and review. The entry phase includes thinking about "what do I know," "what do I want," and "what can I introduce." The review phase further contains "checking," "reflecting," and "extending." However, a typical problem solving activity is seldom linear; an individual always goes back and forth when proceeding to the desired outcome. Also it is possible that the attack phase is difficult to observe or the review phase is missing. Generally speaking, this perspective could provide an overview of the entire problem solving process so that a clearer relationship between steps could be identified and studied. Mason's model, however, does not indicate the internal or external forces that impact the movement from one phase of process to next. Between each process, distinct motives/stimuli might exist, initiated by the individual or outside information that can either facilitate or prevent making progress towards a more general understanding and ultimately, more efficient problem solving performance. Mason's model as used in this research was modified in order to demonstrate certain internal (self-initiated) and external (interviewer-initiated) forces influencing mathematical work of the children.

Each interview episode was videotaped. We first reviewed each episode repetitively to track the distinct problem solving phases of each of the children on each of the tasks during each problem solving episode. A problem by problem performance model was developed for each child and then a cross problem performance analysis was completed.

A detailed description of the processes each child used, strategies they employed and metacognitive behaviors they exhibited was completed. Every strategy and representation used by the participants was identified and recorded. Following the completion of each case study analysis, the three cases served as data sources for the overall analysis.

Results

Table 2 provides an overview of particular behaviors and performances of children as they relate to average amount of time they spent on tasks, average number of instances of self-initiated questions, average number of times they switched strategies, average number of self-initiated testing and justifying episodes. As the data illustrate, although the three participants varied greatly in the average amount of time they spent on tasks, their performance along the self-initiated constitutive elements of the problem solving process was consistent, suggesting potential patterns of thinking and actions. The average numbers of self-initiated questions and times shifts in strategy usage had the least variety of items, indicating a less diverse use of these two behaviors. The average numbers of justifying answers, which included self-initiated justification and interviewer-initiated justification, varied more than the previous two items. The average number of scaffolding questions that the interviewers asked was the item with the most variety, which was also the least natural item for participants. A detailed cross case analysis of subjects' mathematical practices is offered in the following section.

	Average length of PS episode	Average number of self initiated questions	Average number of shifts in strategy usage	Average number of justifying episodes	Average number of interviewers' scaffolding questions
Liza	10'54"	0	1	1.5	1.75
Jazzy	9'57"	0.5	1	0.75	5.25
Yoni	7'57"	0.5	0.5	0.25	0.5

Table 2. Overview of participants' particular behaviors and performances

Patterns of Mathematical Problem Solving Performance among Children

Based on each student's problem solving episodes, Liza, Jazzy, and Yoni's general problem solving practices (despite content and heuristics) are illustrated in Figures 1, 2, and 3 respectively.

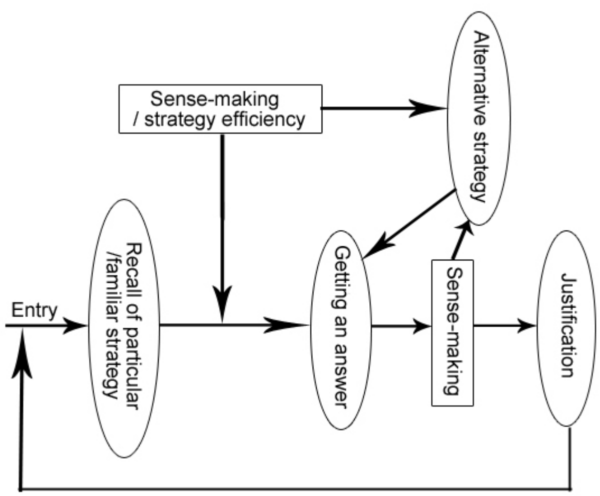

Figure 1: Liza's general problem solving orientation

Compared to the other two students, Liza's general problem solving process had a unique feature: sense-making. Sense-making was her way of self-monitoring, which was presented throughout all problem solving episodes. After she reached an answer, sense-making was her premier way to justify her response. It was one of the factors that could influence her switch of strategy. Liza had a belief that she could solve most problems eventually, thus she was more likely to deliberately re–enter the problem in order to gain a better understanding of the problem.

Jazzy's problem entry phase was different from other students. She always started her activity with manipulating numbers in order to get a sense of the problem. Her intra-task strategy usage depended largely on whether the strategy she used was by numerical or non-numerical. She tended to switch strategy instead of modifying information when she was not using a numerical strategy.

Yoni's general problem solving orientation was to stick with one method regardless of success or failure. He seldom justified his answers and assumed a problem was solved when he arrived as an answer (either right or wrong). His low intra-task strategy flexibility could be due to his self-confidence regarding his mathematical ability.

All three participants showed a high degree of involvement in solving the assigned problems during the interviews. They took ownership of the tasks and attempted to attack, review and solve them. The intensity of their involvement in tasks was reduced or diminished if they failed to see patterns, fully understand the information given, or make connections between what they knew and the contexts under study. An additional inhibitive factor included their ability to define mechanisms for gauging their own success in solving problems when using strategies most familiar to them.

The children's ability to identify relevant from irrelevant data, either embedded in the problem or deduced as the result of their own work, was a pivotal influence on their successful problem solving as evidenced by their willingness to reflect on options or reconsider approaches. Preoccupation with using familiar techniques learned in school curriculum served as a primary motive for participants' reluctance to focus on extending their understanding of the problems, reflecting systemically on what was given, or to even test and justify their answers.

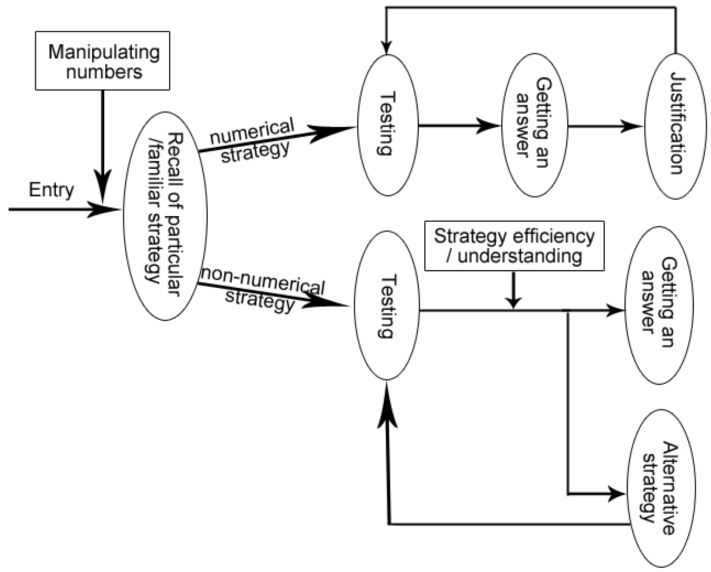

Figure 2: Jazzy's general problem solving orientation

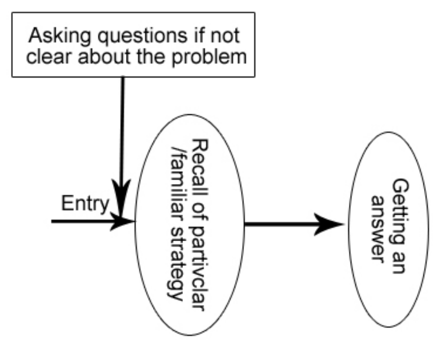

Figure 3: Yoni's general problem solving orientation

The children's particular orientation and their personalities influenced how they entered the problems during the initial phase, the degree of persistence they showed in solving them, and whether they tried to access additional strategies or engaged in metacognitive actions. The one subject with the least amount of interest in school mathematics and its content seemed most flexible in changing strategies. Her need for understanding and sense-making, as articulated during both interview sessions, may have been the primary force behind her natural desire to constantly examine the context at hand and to switch her approaches. On the other hand, the most academically successful student among the three, and the one with most sophisticated mathematical tools, appeared least flexible in his thinking and choices. Indeed, his attempts to use procedures he had learned in school prohibited him from taking the initiative to justify or verify his own answers or monitor his progress reflectively.

All three participants showed the tendency to enter problems using the technique of testing numerical values. They made, or refused to make modifications to their initial choices of numerical values to understand the problem better. Once, and if a deeper understanding of the problem was achieved, they were more willing to switch strategies. Most notably, they were also more successful in the use of newly adopted approaches. Despite this, those with greater control over numerical manipulation tended to remain loyal to the use of this approach. A shift from one strategy to another was the result of either a significant change in their level of understanding of the problem, or provoked by interviewers' questions.

All three subjects showed the tendency to use concepts and procedures most recently addressed in school at the time of data collection, regardless of whether these concepts were relevant to the problem under study. This was most notable when they worked on the geometry task. With the exception of Jazzy, the two other participants experienced difficulty when abstractions of specific knowledge became a focus of work.

The children's ability to access different representational modes was also driven by the contexts they had previously experienced in school experiences. The use of drawing a picture for illustrating the problem became only natural for two of the participants (Jazzy and Liza) when their initial attempt at using numerical data for answering questions seemed too cumbersome to be practical. Even when they were successful in use of the strategy they remained skeptical of the accuracy of their own responses. Formalizing and authenticating the final answer derived using this approach was endorsed to an outside authority (the interviewer), as opposed to self conviction.

Discussion

The results of this work provide additional evidence suggesting that self-monitoring is positively correlated with success in performance on certain mathematical activities (Cohors-Fressenborg, Sjuts, & Sommer, 2004; Cohors–Fressenborg et al., 2010). Liza used sense-making as a way of self-monitoring her progress on tasks and towards gauging her problem solving process accordingly. Jazzy used numerical computation as a way to self-regulating her actions and increasing her control over tasks, performed better. Yoni, who did not exhibit self-monitoring/regulating consistently during his problem solving processes, was not always successful in solving problems. Hence, we highlight that self-monitoring/regulating could be a significant influence on successful problem solving on both routine and non-routine tasks, consistent with findings of previous.

Our results also indicate that intra-task strategy flexibility usage does not imply success at reaching correct answers on tasks (Elia, Heuvel-Panhuizen, & Kolovou, 2009). However, we posit further that the level of intra-task strategy flexibility usage might depend largely on the individual's level of confidence and preference for the use of certain strategies. These constructs may not ensure that correct answers across different subject areas and heuristics might be reached. Instead, they may prevent the individuals from moving forward in securing an enhanced level of understanding of the problem.

Our data also revealed inconsistency in the same individual's mathematics problem solving behaviors across different subject areas and/or heuristics usage. This result is distinct from the conclusion of previous research that indicates most individuals exhibit consistent problem solving behaviors (Muir, Beswick, & Williamson, 2008). The factors that may impact the consistency in behaviors include the preference for the use of specific approaches and orientations (visual, graphical, pictorial, etc.), experience with specific subject area (number theory, algebra, geometry), familiarity with the heuristic needed to solve the problem, and personal belief about one's own mathematical ability and confidence in problem solving.

Personal orientation could largely impact one's problem solving behaviors throughout the entire process (i.e. Jazzy's numerical orientation). Personal belief about one's own knowledge and ability could impact one's confidence: Liza believed she could eventually solve all problems and Yoni believed he was good at mathematics; both of them exhibited noticeable confidence during their problem solving episodes. On the other hand, Jazzy's belief about problem solving ("the only right answer") influenced her attitude during the problem solving process: she asked for the right answer even when she was convinced by visual evidence.

Lastly, perhaps a puzzling finding of the study is the relationship between children's claimed level of confidence with mathematics and their problem solving performance. In virtually all past literature focused on the connections between affect and problem solving performance of children, the conclusion had been drawn that confidence and success in problem solving are directly related: the more confident an individual was in his/her mathematical ability, the better performance on problem solving was observed. Our research provides conflicting results. As described earlier, the most mathematically confident individual in this study, Yoni, showed the lowest degree of flexibility in thinking or control over tasks. It is not quite clear at this point what measures were used in previous work for determining the problem solvers' levels of confidence and whether claims were carefully studied. This issue merits further study.

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REFERENTIAL COMMUTATIVITY: PRESERVICE K-8 TEACHERS' VISUALIZATION OF FRACTION OPERATIONS USING PATTERN BLOCK

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This paper examines ten K-8 preservice teachers' visual representations of fraction operations using the four main pattern blocks. Data consist of figures made using the pattern blocks, drawn colored representations, and detailed written comments and algebraic formalism. The theoretical framework is drawn from representational theories and analyses of fraction operations, and work on coordination of different levels of units. The main result is that only those teachers meaningfully coordinating the different referent units in the fraction situations, were the ones consistent in their representations and reasoning, and in successfully establishing referential commutativity for multiplication of fractions.

Introduction

The usual way of representing a fraction numerically, is the expression $\frac{a}{b}$ where *a* is a whole number and *b* is a nonzero whole number. This representation has several interpretations such as part-whole, quotient, operator, and measurement (Kieren, 1980; Skemp, 1986; Olive, 1999; Olive & Steffe, 2002). In the *part-whole* meaning, the referent unit 1 is defined, the denominator indicates the number of congruent pieces into which the unit 1 is partitioned, and the numerator indicates how many of those congruent parts are selected. In the *sharing equally (partitive) division* model, the fraction is interpreted as the equi-partitioning (Olive & Steffe, 2002) of the quantity, *a,* into *b* congruent parts (shares), with the fraction, $\frac{a}{b}$ being the share of one person, relative to the referent unit for quantity, *a*. For example, if we share 3 chocolate bars among 5 friends, each friend gets 3/5 of ONE chocolate bar.

The repeated subtraction (*measurement* or *quotitive division*) interpretation attends to the instruction "How much (or how many) of quantity *b* is (or are) there in quantity *a*?" or "What is the measure of quantity *a* in units of size *b*?"

Children need to be aware of how the same quantity can be represented by many fractions (i.e. *fraction equivalence*) before the exploration of +, −, ×, and ÷ operations with fractions. Children should be able to recognize and create fractions equivalent to a given fraction, because they will frequently need to determine an equivalent fraction in order to add, subtract, multiply, or divide, in a way that makes sense to them (Sowder et al., 1998). For example, in adding or subtracting fractions, in the process of obtaining fractions of equal denominator, students must be able to refer to their knowledge about fraction equivalence.

In dealing with multiplication of two fractions, the understanding of what the multiplier, the multiplicand, and the product refer to is of paramount importance. The referent units for the multiplier, the multiplicand, and the product respectively are the multiplicand, the whole unit, and the whole unit. The algorithm $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ is effortless to memorize and to perform; however, to render fraction multiplication meaningful, children must be aware of the referent units for these fractions and what the product really indicates. Moreover, while both $\frac{a}{b} \times \frac{c}{d}$ and $\frac{c}{d} \times \frac{a}{b}$ yield the same numerical answer, the order matters if we want to conceptualize the referent units involved, namely the fact that the referent units are being swapped between the multiplier and the multiplicand. As will be explained in the results section of this study, although fraction multiplication is *algebraically* commutative, that commutativity is definitely not so obvious to construct referentially. Construction of referential commutativity requires proficiency in simultaneously coordinating various fraction relations and different levels of units meaningfully.

As for the division of fractions, the understanding that the referent unit for both the dividend and the divisor is the same unit whole, is necessary in order to make sense of the division operation. Moreover, it is equally important to realize that the quotient has no reference to the original unit whole; the quotient must be seen as a relation between the dividend and the divisor in order to develop an in-depth understanding of fraction division. For example, when dividing ¾ by ½ in order to find out how many ½ lb-bags of coffee we could make from ¾ lb of coffee, the answer is one and a half *bags* (not pounds of coffee). While fraction division has traditionally been related with the crude invert and multiply algorithm, most children and adults do not make sense of how this algorithm works (NCTM, 2000). Awareness of referent units for fraction division is crucial in order to develop any meaning for the algorithm.

This study investigates the above assertions (with special emphasis on referential commutativity of fraction multiplication) by analyzing the work of ten pre-service K-8 teachers.

Theoretical Framework

The theoretical framework of this study is drawn from the work by Steffe and Olive involving representations of operations with fractions and referent unit coordination (Olive & Steffe, 2002; Steffe & Olive, 2010). These researchers postulated a series of fractional schemes as a foundation for the construction of fraction operations (Olive & Steffe, 2002, p. 436). They also reported facility with whole-number sense as one of the main prerequisites of fractional thinking and reasoning (the *Reorganization Hypothesis*, Steffe, 2010). Their work with children in grades three through five involved children's representations and actions in fractional situations using electronic manipulatives called TIMA (Tools for Interactive Mathematical Activity)(Olive, 2000).

The idea of a representational system is also relevant to the research presented in this paper. This construct comprises written symbols, thinking aloud, physical manipulatives, and drawn representations (Behr et al., 1983). In what follows, we focus on K-8 pre-service teachers' drawn and physical representations of five main fraction operations modeled with pattern blocks, and on their abilities to connect their visual and written formalism.

Context and Methodology

This study investigates pre-service K-8 teachers' construction of fraction operation problems (equivalence, addition, subtraction, multiplication, division) using physical manipulatives (the four main pattern blocks). Ten pre-service K-8 teachers, whom the first author met weekly for two weeks in two-hour sessions, were selected from his "algebra for teachers" class to participate in this study. They demonstrated their solutions for each problem with both the actual pattern blocks and colorful drawings on a triangular grid (Figure 1). They also explained their reasoning in detail for each task with reference to their physical and drawn representations.

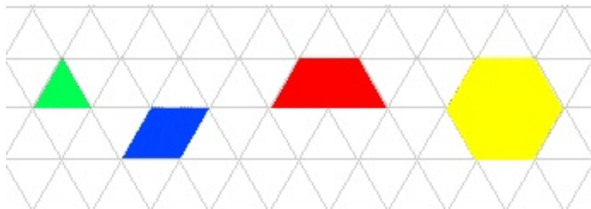
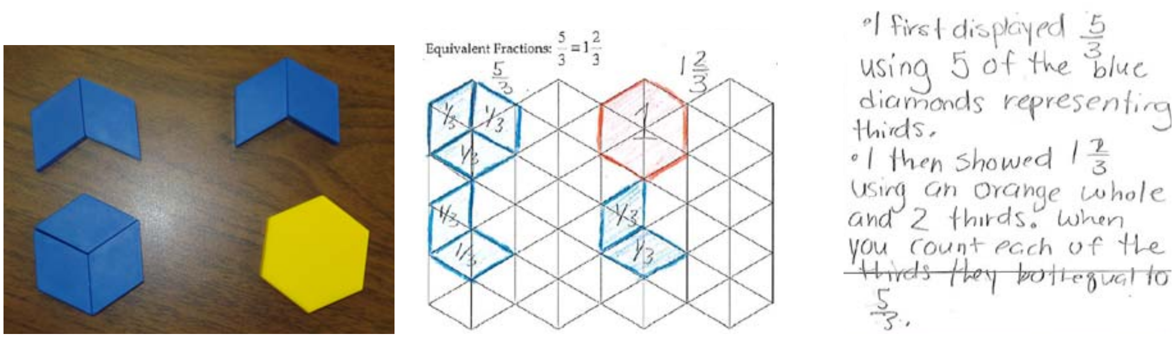


Figure 1 - The Four Main Pattern Blocks on Isometric Grid

Our data consists of photographed physical representations, and scanned drawn representations along with written arithmetic formalism and detailed comments. The purpose of creating these scanned versions was to conduct a retrospective, preliminary thematic analysis in order to find possible themes for a detailed analysis. The dataset was then revisited multiple times in order to generate a thematic analysis (using constant comparison methodology) from which the following results emerged.

Results & Analysis

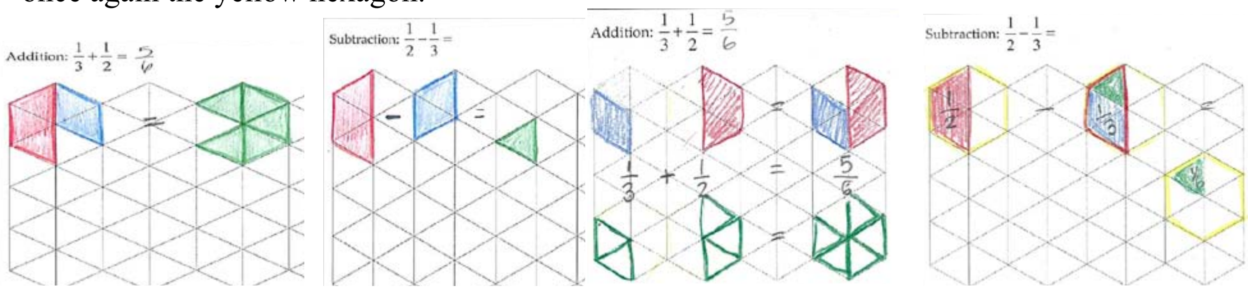
For the equivalence problem $\frac{5}{3} \equiv 1\frac{2}{3}$, all ten teachers pretty much came up with the same physical and drawn representations. Several of them even clarified their construction using a *counting the thirds* strategy (see Figure 2). We can infer that Edie relied on this strategy by explicitly labeling the thirds that are being counted and also by referring to the whole unit, the yellow (or orange) hexagon, at each step of her counting. Lauren used a similar reasoning by writing *3 "one thirds" equal one, so 5 thirds will have 2 "one thirds" left over with one yellow whole*.



2a 2b 2c

Figure 2: Edie's Physical (2a) & Drawn (2b) Representations & Written Work (2c)

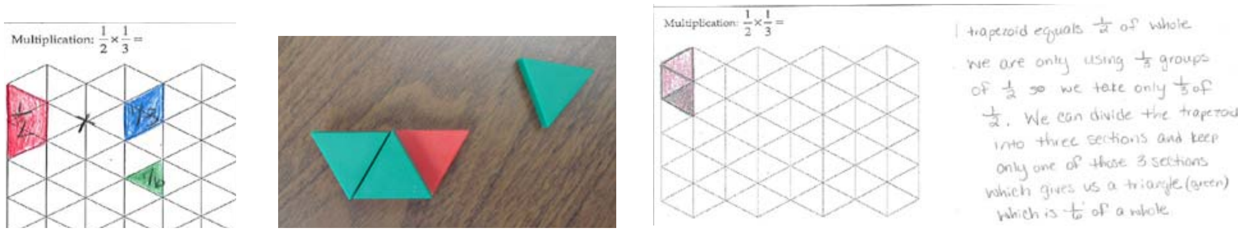
For the addition and subtraction tasks, not all teachers were as explicit with reference to the referent unit, the yellow hexagon, at each step of their construction. Moreover, although the instructions required explanation for each step, not all teachers, for instance, thought about using the idea of common denominator and involving that in their physical and drawn representations. As depicted in Figure 3, both Lauren and April arrive at the correct final answer; however, April's thinking seems to be more sophisticated than Lauren's in that she not only refers to the referent unit yellow hexagon (for the addends in the addition task, and for the minuend and subtrahend in the subtraction task), but she also decomposes each fraction into smaller pieces (green triangles representing sixths), thus making sense of the situation. She also clarifies both in her writing and her physical and drawn representations that the referent unit for those sixths is once again the yellow hexagon.



3a 3b 3c 3d

Figure 3: Lauren's (a b) and April's (c d) Drawn Representations for the + and − Tasks

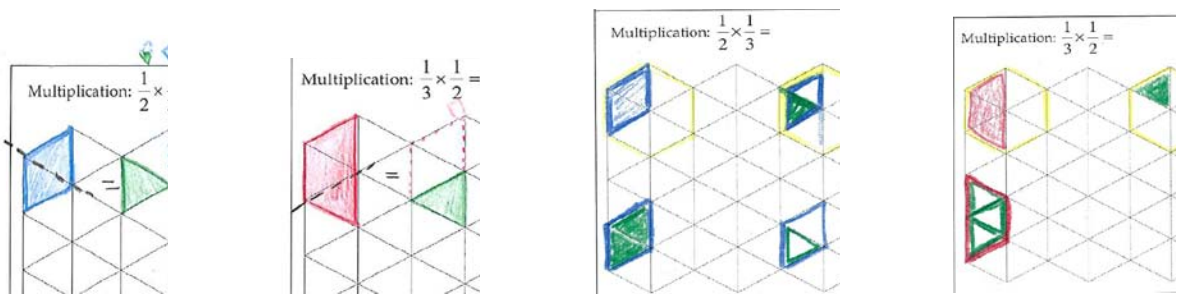
All teachers succeeded in establishing the referential commutativity of addition in dealing with the problems $\frac{1}{3}+\frac{1}{2}$ and $\frac{1}{2}+\frac{1}{3}$ in constant comparison of their physical and drawn representations. They also invalidated the commutativity of subtraction with very creative constructions. Multiplication and division tasks, on the other hand, were rather cumbersome for many of the participants. There were some who overcame this difficulty by appropriately relating the multiplier, the multiplicand, and the product (the dividend, the divisor, and the quotient in the division tasks) to their referent units. Only two participants were able to induce a referential commutativity for fraction multiplication, the most sophisticated behavior resulting from this research study. For the multiplication problems $\frac{1}{3}\times\frac{1}{2}$ and $\frac{1}{2}\times\frac{1}{3}$, many teachers constructed and drew both the multiplier and the multiplicand, the former being irrelevant, a non-operative interpretation of the problem situation. Several teachers swapped the role of multiplier and multiplicand. We begin the discussion on multiplication with these problematic representations. Emma modeled the first multiplication problem as the product of a red trapezoid and of a blue rhombus (Figure 4a). Kristie not only included both $\frac{1}{2}$ and $\frac{1}{3}$ in her constructions, but also swapped their roles as well (Figure 4b-c). Her referent units for $\frac{1}{2}$ and $\frac{1}{3}$ respectively were the whole unit (yellow hexagon) and $\frac{1}{2}$ (the red trapezoid). Her interpretation of $\frac{1}{2}\times\frac{1}{3}$ is actually $\frac{1}{3}\times\frac{1}{2}$, and vice versa. While she is very clear in establishing the referent unit of the product $\frac{1}{6}$ (green triangle) as the yellow hexagon, we postulate that Kristie failed to establish referential commutativity of multiplication due to the interchange of referent unit roles. Both Kristie and Emma used their same approaches for the other multiplication problem as well.



4a 4b 4c 4d

Figure 4: Emma's (4a) and Kristie's Constructions for $\frac{1}{2}\times\frac{1}{3}$

Lauren was one of the few who established the referential commutativity of multiplication in a meaningful and appropriate manner. For the half of a third problem, she started by constructing the third as the blue rhombus. She then bisected this third using a dashed line, as depicted in Figure 5a. She was aware of the fact that constructing a half (red trapezoid) was irrelevant in the problem situation. She also understood that the referent unit for the multiplier (the half) was the multiplicand (the third). It is also worth noting that she did not specify whether the product (the sixth) has the whole unit as the referent unit. She followed a similar approach for the third of a half problem, as depicted in Figure 5b.



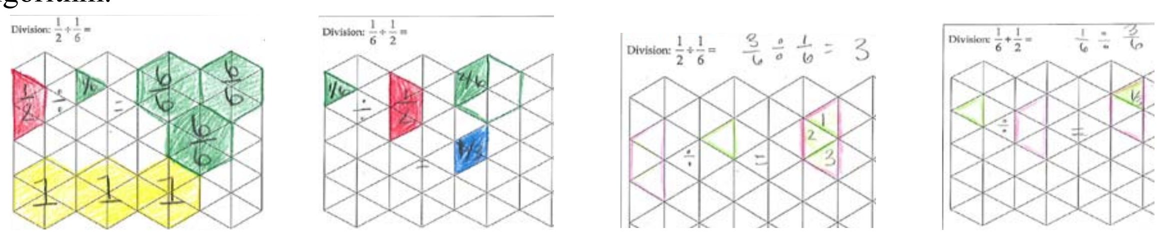
5a ***5b*** ***5c*** ***5d***

Figure 5: Lauren's (a, b) and April's Constructions for Fraction Multiplication Tasks

We can say that April followed a reasoning pattern similar to Lauren's, in that she was aware of the fact that constructing a half (red trapezoid) was irrelevant in the problem situation (Figure 5c). She also successfully interpreted the referent units of the multiplier and the multiplicand. As can be detected in Figures 5c-d, April took the whole thing one step further by also specifying the referent unit of the product "the sixth" as the yellow hexagon unit. Both Lauren and April established referential commutativity, but April's representations can be considered to be more sophisticated than Lauren's.

Teachers overall seem to have successfully applied the "How many of … are there in …" (or "how much of … is there in …") view in their representations of fraction division problems. Some teachers were very explicit in their reference to the yellow hexagon as the referent unit for the dividend and the divisor. Some others perhaps over-generalized this reference for the quotient by attempting to construct the quotient using the physical or drawn representations. What was the quotient's referent unit then? Was it the yellow hexagon, the dividend, or the divisor? Or something else? We begin our discussion on fraction division with Emma's drawn representations for the problems $\frac{1}{2}\div\frac{1}{6}$ and $\frac{1}{6}\div\frac{1}{2}$. Emma not only constructed both the dividend and the divisor, but she also drew the quotient as well (Figures 6a-b). She basically arrived at this construction with reference to her purely algebraic formalism using the invert-and-multiply method. In her interpretation, the quotient of the first problem, 3, is referred to the 3 yellow hexagon whole units. In fact, this corroborates our theory about Emma's view of fraction multiplication in which she constructed both the multiplier and the multiplicand (Figure 4a). For Emma, all these elements have to be represented using a pattern block.

Edie, on the other hand, seemed to have meaningfully constructed all the constituents in fraction division task. She explained "*How many sixths are there in a half? Equivalently, how many green triangles are in a red trapezoid? There are three sixths in a half.*" In her drawing, she also illustrated her way of counting those three sixths by labeling them as 1, 2, and 3, respectively (Figure 6c). Edie followed a similar reasoning for the other division problem by stating ""*How many halves are in a sixth? There is* $\frac{1}{3}$ *of a half in a sixth*." In her drawing, Edie used the idea of labeling, in an attempt to count, for which this time she used a fraction (Figure 6d). And that fraction, 1/3, the quotient, has nothing to do with the blue rhombus. It is true that the blue rhombus represents one third of the yellow hexagon whole unit, but in the context, as constructed by Edie, it refers to the quotient 1/3 with referent unit 1/2 (the red trapezoid), which also happens to be the divisor. We also observe that Edie meaningfully divides both the half by the sixth, and the sixth by the half algebraically, without reference to the invert-and-multiply algorithm.

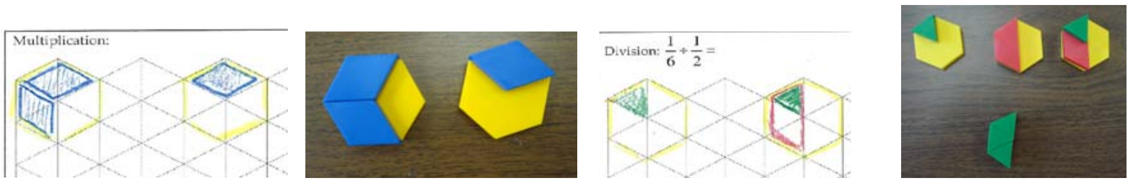


6a 6b 6c 6d

Figure 6: Emma's (a, b) and Edie's (c, d) Constructions

We conclude our findings with April's performance on fraction division tasks. We consider April's reasoning as the most sophisticated one in that she makes use of a variety of meaningfully connected fraction ideas. April was one of the few making sure to refer to the yellow hexagon whole unit and including it in her drawn and physical representations, whenever relevant (Figures 3d, 5c-d). For the division (of the half by the sixth, and the sixth by the half) problems, she proceeds in a manner very similar to Edie's (Figure 6c-d). The only difference is that April also includes the yellow hexagon in her drawing, which is an indication that she is aware that the dividend's referent unit is the yellow hexagon whole unit (She also uses the idea of labeling the sixths the same way Edie does). This stacking approach, which can be thought of coordination of referent units at different levels, is a powerful tool in making sense of fraction division. We look at April's coordination of referent units through her physical and drawn representations for the multiplication task $\frac{1}{2}\times\frac{2}{3}$ and the division task $\frac{1}{6}\div\frac{1}{2}$ simultaneously. For the multiplication task, she explains "*There are two 1/3 portions in 2/3; 1/2 of that is one 1/3 portion*." She also relates the multiplier, namely the 1/2, to its referent unit 2/3 (two blue rhombi); the multiplicand, namely the 2/3, to its referent unit yellow hexagon; and the product, namely the 1/3, to its referent unit yellow hexagon (Figures 7a-b). For the division task, she explains "*Division is how many or how much of a given will go into another given portion. 1/3 of*

1/2 will divide into 1/6 of a whole. Therefore 1/6 ÷ 1/2 = 1/3." She also relates the dividend, namely the 1/6, to its referent unit yellow hexagon; the divisor, namely the 1/2, to its referent unit yellow hexagon; and the quotient 1/3 to its referent unit red trapezoid (Figures 7c-d).



7a 7b 7c 7d

Figure 7: April's Constructions

April's referent unit coordination scheme for the fraction multiplication and fraction division tasks can be tabulated as follows:

Fraction Multiplication		**Fraction Division**	
Components	*Referent Units*	*Components*	*Referent Units*
Multiplier	Multiplicand	Dividend	Whole
Multiplicand	Whole	Divisor	Whole
Product	Whole	Quotient	Divisor

Table 1. April's Referent Unit Coordination Scheme

Conclusions and Discussion

This study aimed to investigate the visual representations of five main fraction operations (equivalence, addition, subtraction, multiplication, and division) created by K-8 pre-service teachers using pattern blocks. Analysis of these participants' physical and drawn representations, accompanied by their algebraic formalism and verbal reasoning, helps us to determine important insights into their sense-making of the mathematics they are exploring. These insights have direct implications for the teaching of fractions in a hands-on-activity based environment. Mathematics teachers should be more conscious and explicit in modeling problems because their models may lead to a misinterpretation of the problem situation, or even the solution to the problem, as depicted in this present study. For example, although some students were confident with their algebraic solutions for the multiplication tasks (using the multiply the numerators and denominators algorithm) and division tasks (using the invert-and-multiply algorithm), their interpretation of the processes differed considerably, when they were asked to represent these tasks using the drawn and physical representations. In particular, Emma's representations of fraction multiplication and fraction division indicate that she may not have a meaningful concept for these operations with fractions. She appears to lack the necessary three levels of units (Olive & Steffe, 2010) to mentally coordinate the relations among multiplier, multiplicand and product, with respect to their roles in the situation and their respective referent units. This lack of coordination is even more apparent in her representation of fraction division (see Figure 6a-b). These necessary relations are depicted in April's referent unit coordination (Table 1 above).

Although fraction multiplication is algebraically commutative, the representation of that commutativity requires sophisticated reasoning. Construction of referential commutativity requires proficiency in simultaneously coordinating various fraction relations meaningfully. Awareness of the referent units for each component (multiplier, multiplicand, multiplier), ability to recognize which fractions are operators and which are quantities, and ability to connect the representations of these to the written explanations and algebraic formalism are essential in establishing referential commutativity of multiplication within a representational system (Behr et al., 1983).

Research indicates that the multiplicative conceptual field is very complex and includes many concepts of mathematics, other than multiplication itself (Behr, Harel, Post, & Lesh, 1992; Harel & Behr, 1989; Harel, Behr, Post, & Lesh, 1992). "*Additive reasoning develops quite naturally and intuitively through encounters with many situations that are primarily additive in nature*" (Sowder, Armstrong, Lamon, Simon, Sowder, & Thompson, 1998, p. 128). Building up multiplicative reasoning skills, on the other hand, is not obvious; schooling and teacher guidance are essential to acquire a profound understanding and familiarization with multiplicative situations, especially with respect to fractions (Hiebert & Behr, 1988; Resnick & Singer, 1993). This present study indicates the importance for teachers (and students) to develop three levels of units structures and the skill to coordinate those units (Olive & Steffe, 2010).

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STUDENTS' DISTRIBUTIVE REASONING WITH FRACTIONS AND UNKNOWNS

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To understand relationships between students' quantitative reasoning with fractions and their algebraic reasoning, a clinical interview study was conducted with 18 middle and high school students. The study targeted a balanced mix of students with 3 different multiplicative concepts, which are based on how students coordinate composite units (units of units). Students participated in two 45-minute semi-structured interviews and completed a written fractions assessment. This paper reports on how students with the second and third multiplicative concepts demonstrated the use of a distributive operation in fraction and algebraic problem solving.

Fractional knowledge is regarded as important for algebraic reasoning (Kilpatrick & Izsak, 2008; National Mathematics Advisory Panel [NMAP], 2008; Wu, 2001), in part because such knowledge is a basis for typical algebra topics such as ratios and slopes of lines. More generally, fractional knowledge is important for learning algebra because it helps students advance their multiplicative reasoning (cf. Thompson & Saldanha, 2003). For example, generating strategies for multiplying fractions can help students develop a distributive operation (Hackenberg & Tillema, 2009): to determine 1/5 of ¾ of a yard a student can take 1/5 of each of the three one-fourths of the yard. So, 1/5 of ¾ is 1/5 of (¼ + ¼ + ¼). Thinking of 1/5 of ¾ in this way might be considered algebraic because it highlights how the distributive property emerges from reasoning and is powerful for solving problems. However, little research has focused on how students' ways of operating with fractions may influence their algebraic reasoning (Lamon, 2007).

The purpose of this paper is to examine the distributive reasoning with fractions and unknowns of 18 middle and high school students who participated in a clinical interview study. The study was designed to investigate relationships between students' quantitative reasoning with fractions and their algebraic reasoning in the area of equation solving. In this paper we assess whether students with different multiplicative concepts demonstrated the use of a distributive operation in their quantitative reasoning with fractions and their algebraic reasoning.

The research questions addressed in this paper are:

1) Do middle and high school students show evidence of a distributive operation in solving problems that involve sharing multiple units fairly? If so, how do they reason distributively?
2) Do middle and high school students show evidence of a distributive operation in solving problems involving unknowns and fractions? If so, how do they reason distributively?
3) What differences are there in the distributive reasoning of students with different whole number multiplicative concepts?

A Quantitative and Operational Approach

Quantitative Reasoning

Following Thompson and colleagues (1993; Smith & Thompson, 2008), we conceive of quantity and quantitative reasoning as a basis for helping students build fractional knowledge and algebraic reasoning. A *quantity* is a property of one's concept of an object, and to conceive of a quantity requires a person to conceive of a measurement unit, of the property as subdivided into some number of these measurement units, and of a way to enumerate the number of these units to find a value of the quantity (cf. Thompson, 1993).

Approaching fractions as quantities means that we pose problems to students in which fractions are measureable extents, or lengths; these lengths may represent other quantities as well (e.g., weight). Approaching algebraic reasoning from a quantitative perspective means that unknowns are quantities for which a value is not known, but for which a value could be determined. So unknowns are potential values of quantities. In working with students we routinely ask them to make drawings of quantitative relationships, and we aim for students' fraction and algebraic notation to reflect the quantitative reasoning in which students engage.

Operations and Schemes

Our work is also based on conceiving of mathematical thinking in terms of people's mental actions, or *operations* (von Glasersfeld, 1995). Operations that are critical for fractional knowledge include *partitioning*, or marking a part into some number of equal pieces, as well as *iterating*, or repeatedly instantiating a part to make a larger fraction. The operation under investigation in this paper is a *distributive* operation. At the start of the study we characterized this operation as applying a process to a composite unit (a unit of units) by applying that process to some segmentation of the units that make up the composite unit. For example, to take 2/5 of 3 pounds a student could take 2/5 of each of the unit pounds that make up the 3 pounds. Our findings address whether this conceptualization of a distributive operation is sufficient.

Operations are the components of *schemes*, goal-directed ways of operating that consist of an assimilated situation, activity, and a result (von Glasersfeld, 1995). For example, if a student has constructed a partitive fraction scheme, then a situation of the scheme is a request to make a new length that is 2/5 of a foot. The activity of the scheme involves partitioning the foot into five equal parts, disembedding one of those parts, and iterating the part twice. The student then assesses this result in relation to what she expected. We take students' *reasoning* to be the functioning of their schemes and operations in on-going interaction in their experiential worlds.

Multiplicative Concepts

For us, a *concept* is the result of a scheme that people have *interiorized*—i.e., reprocessed so that the result is available prior to operating. We use students' whole number multiplicative concepts as a tool for understanding differences in how students build fractional knowledge (e.g., Hackenberg, 2010). These multiplicative concepts are based the number of levels of units a student has interiorized; to progress from one concept to the next requires a major reorganization of operations. Although this study included students with three different multiplicative concepts, this paper reports only on students with the second and third multiplicative concepts, as derived from prior research (Hackenberg & Tillema, 2009; Steffe, 1994), because students with the first multiplicative concept did not show evidence of a distributive operation.

Students with the second multiplicative concept (MC2 students) have interiorized two levels of units. These students have the potential to treat a length as a unit containing some number of equal units, and they can do so prior to operating in a situation. For example, these students can treat a length that represents one foot as a unit containing five units—a unit of units structure—without having to actually make the partitions. These students can also make three levels of units in activity. For example, they can insert three parts into each of the five parts in the 5/5-foot segment and determine that they have made 15 parts in all. However, once they do so, the 15/15-foot segment is not likely to retain its structure as unit of five units each containing three units (Hackenberg, 2010; Hackenberg & Tillema, 2009).

Students with the third multiplicative concept (MC3 students) have interiorized three levels of units. Prior to operating, these students can treat lengths as a unit containing some number of units, each of which contains some number of units—a unit of units of units structure. So, in the example above, MC3 students can do what MC2 students do, but they can also retain views of the 15/15-foot segment as a unit of 5 units each containing 3 units, and they can switch to viewing the segment as a unit of 3 units each containing 5 units. Being able to flexibly switch between such unit structures can be useful for constructing a variety of fraction schemes (e.g., Hackenberg, 2010; Steffe & Olive, 2010).

Methods

A clinical interview study was chosen as appropriate methodology to explore the research questions because a strength of clinical interviewing is "the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques" (Clement, 2000, p. 547). Interview studies are a tool for generating scientific knowledge because dynamic, on-going discussion is a basis for formulating explanatory models (Clement, 2000; Steffe & Thompson, 2000) for experienced phenomenon.

Seven seventh grade students, 10 eighth grade students, and one tenth grade student participated in the study. Participant selection occurred via classroom observations, consultation with students' teachers, and one-on-one, task-based selection interviews to assess students' multiplicative concepts. Six students with each multiplicative concept were invited to participate. Four of the MC2 students were enrolled in an 8th grade pre-algebra class; one was taking an advanced 7th grade mathematics class; and one was taking a 7th grade mathematics class for struggling students. Four of the MC3 students were enrolled in an algebra class, and two were taking an advanced 7th grade mathematics class.

Students participated in two 45-minute semi-structured interviews, a fractions interview and an algebra interview. All students completed the fractions interview prior to the algebra interview, but the time between interviews varied from about 3 weeks to 4 months. The interview protocols were refined in a prior pilot study (Hackenberg, 2009), and quantitative situations were used as a basis for all problems. The protocols were designed so that the reasoning involved in the fractions interview was a foundation for solving problems in the algebra interview. For example, in the fractions interview students were posed this problem:

> F2. *Fair Sharing of Multiple Identical Bars Problem:* Here are five candy bars (congruent rectangles). Can you show how to share them fairly among seven people, and determine the fair share for one person? Draw the amount one person gets.

In the algebra interview, students were posed a similar problem but with each bar assigned a weight of *h* ounces:

> A1. *Weight of Multiple Identical Bars Problem:* Here are five identical candy bars (congruent rectangles). Each candy bar weighs some number of ounces. Let's say that *h* = the weight of one bar. How much does 1/7 of all the candy weigh? Write an expression for this result.

In working on A1, students were also encouraged to draw a picture to demonstrate how they saw their expression. In addition to the two interviews, students completed a written fractions assessment (Norton & Wilkins, 2009) to triangulate claims about their fractional knowledge.

Each interview was video-recorded with two cameras, one focused on the interaction between the researcher and student, and one focused on the student's written work. The videos were mixed into one file for analysis, which occurred in two overlapping phases. First, the researchers viewed video files and took detailed notes (Cobb & Gravemeijer, 2008). The aim of this phase was to formulate a model of each student's fraction schemes and equation solving activity, to the extent possible over two interactions. To develop these models the researchers built on constructs from prior models (e.g., Hackenberg, 2010; Steffe & Olive, 2010; Tzur, 2004). These models provided the basis for responding to the first two research questions.

In the second phase of analysis, the researchers looked across the students to articulate differences in how students with different multiplicative concepts solved the problems in each interview. This phase included assessing differences in students' distributive reasoning with knowns and unknowns and writing syntheses of differences for students with each multiplicative concept. This phase allowed the researchers to address the third research question in this paper.

Analysis and Findings

Four of the MC2 students demonstrated some evidence of a distributive operation in the fractions interview, while only one of the MC3 students did so. In this section we present data and analysis to support this finding, which was surprising, since prior to the study we conjectured that a distributive operation required the third multiplicative concept. Then we examine the students' work on the algebra interview to help illuminate the finding.

Distributive Reasoning in the Fractions Interview

MC2 students. Of the four MC2 students who demonstrated evidence of a distributive operation in the fractions interview, two demonstrated it immediately upon sharing three identical candy bars fairly among five people, which we'll call problem F1, as well as in working on F2. The other two students showed some evidence of a distributive operation over time in working on F1 and F2. We present an example of each of these two types of evidence.

In work on F1, Lisa immediately divided each of the three bars into five equal parts and said that each person would get one-fifth from each bar. She seemed certain, and she called the fair share that she drew three-fifths. When asked whether the share was 3/5 of all the candy or 3/5 of one bar, she said it was 3/5 of all the candy. Upon further questioning, she said 3/5 referred to "kind of both" (all the candy and one bar), because the share was 3/5 from one bar and 1/5 from each of three bars. Twenty seconds later, when asked if each person got 3/5 of all the candy, she said no. She noted that there were five people so one share was one-fifth of all the candy. She solved F2, directly following F1, in the same say. She said, "Each person will get one-seventh of all the candy, because there are seven people. And they will get one-seventh of each [bar]." She said each person would get five-sevenths.

This data excerpt demonstrates that Lisa has constructed a distributive operation of some kind, in that to take find 1/5 or 1/7 of multiple identical units she took 1/5 or 1/7 of each of those units. However, for some time in working on F1 she named the fair share 3/5 of *all* the candy. Our interpretation is that to determine a share, Lisa made a unit of units of units in activity: All of the candy was a unit of three units, and she partitioned each of those units into five units. Once she determined a share, she continued to view all the candy as the referent for the result, even though she also she viewed each bar as a unit of five units. Lisa said that one person's share was 1/5 of *all* the candy only after she thought again about five people sharing all of the bars; then she viewed the share as 1/5, consistent with viewing all of the bars as a length partitioned into five equal parts (a unit of units). So, she appeared to view the results of her activity as different two-levels-of-units views, but not as a three-levels-of-units view. A second MC2 student operated similarly to Lisa.

In contrast with Lisa, Sheila was one of the two MC2 students who seemed to develop a distributive operation in the process of the interview. Her fractions interview was one of the first in the study, and the interviewer inadvertently posed F1 with four people instead of five. Sheila solved this version of F1 by partitioning each of the first two bars in half, and then she partitioned the third bar into fourths. She said that each person would get "one-half plus one-fourth." In working on F2 Sheila first partitioned each of the five bars into fourths, and she numbered the fourths from 1 to 7 sequentially, as if trying to determine whether the there would be an equal number of parts for each of seven people. She then tried again but partitioned each bar into fifths, again numbering sequentially. On her third try, she partitioned each bar into sevenths, numbered them sequentially, and said that each person would get "five-sevenths." That is, each person would get all the pieces with the same number on them (person 1 gets all the 1's).

So, Sheila's work on F1 and F2 is not evidence of a distributive operation. However, in the course of the interview, she introduced the idea of numbering the pieces to try to pull out an equal amount for each of the seven people. This idea seems to be a possible root for a distributive operation, in that through it Sheila determined that she could make fair shares by taking one part out of seven from each of the five bars.

MC3 students. In contrast, the only MC3 student to demonstrate a distributive operation in working on F1 and F2 was Liam. In working on F1, Liam immediately said that you could divide each bar into five parts, but "that would be kind of cumbersome." So, he drew his three bars stacked vertically and then partitioned them vertically, all at once, into fifths. He then drew out one column (consisting of three parts) as the share for one person. He called one share 3/15 and then 1/5 of all the candy. When asked for the size of the share in relation to one bar, he paused and appeared to do some calculation. He justified his answer of 3/5 of one bar by saying that one column was the same as three parts horizontally, which was 3/5 of a bar. He solved F2 similarly, although he determined the size of one share in relation to one bar, 5/7, first. To determine the fraction of all the candy, he calculated 5/7 x 1/5 and found 5/35, or 1/7. He said he had multiplied because one share was 5/7 of a candy bar, and a candy bar was 1/5 of the total.

Although his process is somewhat different from Lisa's, it still implies the construction of a distributive operation because one column consists of one part from each bar. However, Liam may not have been aware of this pattern in his thinking. For example, when asked in F2 whether his result of his calculation to determine the fraction of all the candy, 1/7, made sense, he pointed out that the seven shares in the five bars could be seen in five horizontal rows (5/7 of each bar) and then two more columns, each containing five parts. He did *not* point to each column.

The other MC3 students did not reason distributively in the fractions interview, although some of them made sophisticated solutions. For example, Suzanne tried out a fractional amount of each bar as the share for one person and then looked to see what was leftover. In F2 she first tried a share of 2/3 of a bar, and knew she could give that to 5 people, because there were 5 bars. She then saw that she had five-thirds leftover, which left an extra one-third after two more people each got 2/3 of a bar. So, then she tried a share of ¾, followed by 4/5, followed by 5/7.

Distributive Reasoning in the Algebra Interview

In working on A1, five of the MC2 students did not demonstrate distributive reasoning with unknowns. Only Sheila's work on this problem indicates that she was continuing to develop the distributive reasoning that had emerged in the fractions interview. First we present Lisa's response to A1, followed by an analysis of Sheila's response. Then we discuss the ways of operating of the MC3 students on A1, again highlighting the work of Liam and Suzanne.

MC2 students. After being posed A1, Lisa again demonstrated a possible distributive operation by partitioning each of the five bars into seven equal parts and saying that she would take "one from each." However, when asked to draw out that amount, she believed that it would be a whole bar. After she had insisted on this result for over two minutes, the interviewer counted the pieces when taking one from each bar, and Lisa said that the result would only make 5/7 of a bar. When asked how she knew that result was one-seventh of all of the candy, she repeated her process (taking 1/7 of each, and that will be 5/7). The interviewer pressed for another fraction name for the result in relation to all of the candy, and she said 5/7. Then the interviewer asked whether the result would be more or less than the weight of one bar, *h*. She said less, because it was less than one bar. When asked for an expression, she said she did not know how to write it, and she wrote down "H <".

This piece of data sheds more light on the nature of Lisa's distributive operation. Her initial response to A1 was similar to her initial response to F2, allowing us to corroborate that Lisa has constructed a mental action—accompanied by material manifestations—for taking 1/7 of all of the bars by taking 1/7 from each bar. However, in working on A1 she did not appear to realize for some time that that her action would of necessity produce *five* pieces; she thought it would be a whole bar. So she appeared to lack awareness of the result of her operation. In addition, once she had made the result, she did not appear to have any way outside of her initial action to justify that the share she produced was one-seventh of all the candy. In fact, in A1 it's not clear that Lisa viewed the result as one-seventh of all the candy. It was almost as if the initial problem of finding one-seventh of all of the bars became lost to her once she arrived at her result, five-sevenths. We account for this as we did with her work on F2, by appealing to her multiplicative concept. Lisa made three levels of units in activity by partitioning each of the five bars into seven. However, once she did that, she did not appear to retain the structure of the bars as a unit of five units each containing seven, and we have no evidence that she switched to a view of the entire collection as a unit of seven units each containing five.

In working on A1, Sheila operated similarly to Lisa in that she immediately divided each bar into seven equal parts and colored the first part of each bar. When asked for the weight of that, she said "There are five, so five-sevenths of one candy bar." The interviewer asked her to write an expression using *h* to show the weight of this amount. Sheila wrote 5/7*h*. In explanation, she said that she had multiplied five-sevenths and *h*.

Because Sheila had worked so swiftly on A1, the interviewer posed A2:

> A2. *Weight of Three Different Bars Problem:* There are three candy bars on the table, each of different weight (three non-congruent rectangles). The first bar weighs *a* ounces; the second bar weighs *b* ounces; and the third bar weighs *c* ounces. How much does three-fifths of all the candy weigh? Write an expression to show your result.

Sheila partitioned each of the three bars into five equal parts, colored the first piece of each bar, and wrote "3/5*b*". Then she tried to fit the bars together in some way. The interviewer clarified that the bars had no particular relationship to each other and decided to pursue finding just one-fifth of the weight. Sheila then wrote that the weight was "3/15*b*." When asked what 1/5 of the first bar weighed, Sheila answered "1/5*a*." The interviewer asked similar questions for the other two bars, and Sheila answered similarly. When asked about the weight of all three parts together, Sheila pointed to her expression "3/5*b*." The interviewer then asked her to write an expression that included *a*, *b*, and *c*, using her ideas about the weights of one-fifth of each bar. Sheila wrote "3/5·c·b·a" and said, "multiplying 3/5 by the third bar, by the second bar, and by the first bar."

Sheila's work on A2 helps illuminate the nature of her distributive operation that appeared so fluidly in her work on A1. Like Lisa, Sheila had constructed a mental action that we refer to as distributive. Unfortunately, on A1 the interviewer did not press Sheila for the fraction name of the result in relation to all the candy, so we cannot draw conclusions about whether she could justify that 5/7 of the weight of one bar was 1/7 of the weight of all of the bars. However, in working on A2, like Lisa, Sheila did not appear to be aware of the result of her activity. Furthermore, in A2 Sheila's work does not indicate that she viewed 1/5 of the weight of all the candy either as the sum of 1/5 of the weight of each bar, or as 1/5 of the sum of the three weights. We account for this by appealing to Sheila's multiplicative concept: Once she had partitioned each bar in A2, the problem likely became a unit of 15 units, and so a response like "3/15*b*" was a reasonable response for the weight of 1/5 of those units.

MC3 students. In contrast with the MC2 students, four MC3 students demonstrated distributive reasoning with unknowns in the algebra interview. Liam's work on A1 showed a distributive operation more clearly than his work on F2. He wrote the result as $(5h)/7$ and $5h \cdot 1/7$. When asked to show that in the picture, he partitioned each bar into sevenths and said to take one from each, gesturing to the first part from each bar. He drew out this amount, and when asked how much that was of *h*, he said 5/7. However, in his work on A2, Liam only arrived at $(a + b + c) \cdot 3/5$ to express 3/5 of the weight of three bars of different weight. Despite several opportunities, he did not ever articulate 3/5 of the weight of the bars as $3/5a + 3/5b + 3/5c$.

In her work on A1 and A2, Suzanne seemed to develop her distributive reasoning even more than Liam. On A1 she first wrote $5h \cdot 1/7$ as her expression. To make her drawing, she partitioned each bar into seven equal parts and marked the last part of each bar. She said you could put all of those parts together to make 1/7 of all the candy. She drew out that amount and wrote $5/7h$ for its weight. When asked whether the two expressions were the same, she noted that multiplying 5 times 1/7 produced 5/7. On A2, Suzanne wrote two expressions for the weight of the three different bars and explained that the first one, $3/5(a + b + c)$, meant that you added all the bars together and took 3/5 of it. The second one $(3/5a + 3/5b + 3/5c)$ meant that you took 1/5 of each weight and then added. Based on this work, we attribute a distributive operation to Suzanne in which she had developed awareness of the result of the mental action.

Discussion and Conclusions

In this section we summarize our responses to our three research questions. First, MC2 students can develop a distributive operation in which they take a fractional amount of each unit to make a fractional amount of multiple units, and they may believe that this action accomplishes this goal. However, these students do not necessarily distinguish the fractional meaning of the result in relation to the unit versus the multiple units. In addition, they do not necessarily have a way to justify that their method works. We attribute this to their interiorization of two, but not three, levels of units, as we have discussed above. The main issue is that after partitioning multiple units, the multiple bars becomes a unit of units for these students, rather than a three-levels-of-units structure. This finding causes us to refine our definition of a distributive operation to include being able to distinguish different referent units in the results of operating.

Second, our study supports the conclusion that MC3 students have constructed the means for constructing a distributive operation. Our analysis of what is required to develop awareness of the result of the action, and a way to justify that the action produces a correct result, relies on being able to switch three-levels-of-units structures—a hallmark of the third multiplicative concept. However, third, our study also shows that even though MC3 students have this kind of power at their disposal, they will not necessarily use it to construct a distributive operation unless given opportunities to do so, i.e., unless instruction and tasks are organized favorably for this.

So, one implication of this study for teaching is to work on sharing multiple units in such a way that students construct a distributive operation that they can further develop in algebraic contexts. This can make it possible for discussions of whether 5*h* * 1/7 is the same as (5/7)*h* to have quantitative, not just numerical, meaning—an issue that appears to be within the cognitive powers of MC3 students. Although such work may be challenging for MC2 students, an open question is whether it may be influential in helping them advance their multiplicative concepts.

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STUDENTS' COMBINATORIAL REASONING: THE MULTIPLICATION OF BINOMIALS[1]

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Three clinical interviews were conducted with each of 15 sixth grade students to test conjectures about the relationship between their level of multiplicative reasoning and their solution of combinatorics problems that could involve single and multi-digit multiplication. The problems that involved multi-digit multiplication were designed with the intent of investigating whether students engaged in binomial multiplication. The conjectures about the relationship between students' multiplicative reasoning and their solution of combinatorics problems that could involve multi-digit multiplication were refined as a result of students' problem solving activity: Students who had not constructed the most advanced multiplicative concept were able to engage in a form of binomial multiplication.

Curriculum writers have responded to recommendations to incorporate combinatorics problems into K-12 curricula (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips 2002). These recommendations have been based on arguments that such problems have the potential to support both process and content standards outlined by the National Council of Teachers of Mathematics (Srirman & English, 2004). In response to these changes in curricula, researchers have produced a small body of research that has investigated students' combinatorial reasoning (Jones, Langrall, & Mooney, 2007). However, this body of research remains small, and so relatively little is known about when particular problems are appropriate to introduce to students, how teachers can support students' understanding of these problems, and how such problems are compatible with extant goals of the curricula. Therefore, research that aims to address these issues is critical for successfully and coherently incorporating such problems into extant curricula.

The study reported on in this paper addresses these issues by investigating how 15 6th grade students at three different levels of multiplicative reasoning solved two-dimensional combinatorics problems and how they used such problems to reason about multi-digit multiplication. The study involved three clinical interviews—one unrecorded selection interview and two hour-long video recorded interviews. Each interview was conducted one-on-one with study participants. In the first video recorded interview, students solved two-dimensional combinatorics problems that could involve single digit multiplication like the Outfits Problem.

> *The Outfits Problem:* You have three pairs of pants and four shirts. An outfit is one shirt and one pair of pants. How many different outfits could you make?

In the second video recorded interview, students solved combinatorics problems that could involve multi-digit multiplication like the Card Problem.

> *Card Problem:* You have the ace through king of hearts (13 cards). Your friend has the ace through king of clubs (13 cards). Use an array to show all of the possible 2-card hands you could make that consist of one heart and one club. On your array show the number of 2-card hands that have two face cards (Jack, Queen, King), that have exactly one face card, and that have no face cards. Use the sections of your array to determine the total number of 2-card hands you can make.

In both interviews, students were encouraged to use *arrays* to symbolize the solution of these problems. The following research questions guided the study:

1) How do students at different levels of multiplicative reasoning reason about two-dimensional combinatorics problems that can involve single digit multiplication?
2) How do students at different levels of multiplicative reasoning reason about multi-digit multiplication in the context of solving combinatorics problems?

Literature Review

English (1991, 1993) has studied kindergarten through 8th grade students' understanding of two-dimensional combinatorics problems like the Outfits Problem. She concluded that many 6th grade students were ready to begin analyzing the structure of such problems, whereas younger students were less ready to do so, even though they could solve the problems with concrete materials. However, English has exclusively studied how students could use two-dimensional combinatorics problems as a basis for reasoning about single digit multiplication.

Researchers who have studied students' understanding of multi-digit multiplication have primarily used repeated groups problems not combinatorics problems (e.g., Ambrose, Baek, & Carpenter, 2003). Moreover, they frequently have *not* used array representations for these problems (Verschaffel, Greer, & De Corte, 2007), even though array representations are often included in researchers' classification of situations involving multiplication (e.g., Greer, 1992). Izsak (2004) is one of the few researchers to investigate elementary grade students' use of arrays to represent multi-digit multiplication problems (although he did not use combinatorics problems). Izsak found that students in the 4th grade classroom he studied were more successful in solving multi-digit multiplication problems than U.S. students in earlier large-scale studies (e.g., Mullis, Martin, Beaton, Gonzalez, Kelly, Smith, 1997). This finding suggests that array representations may be fruitful for helping students develop an understanding of multi-digit multiplication. However, because Izsak's study is one of the few that has investigated how students use arrays to understand multi-digit multiplication, further research is needed (Vershcaffel, Greer & De Corte).

A second finding of Izsak's (2004) study, which is supported by Ambrose, Baek, & Carpenter's (2003) findings, is that students tended not to partition *both* the multiplier and multiplicand when computing multi-digit multiplication problems. For example, to solve 13 x 13, some students partitioned either the multiplier or multiplicand into 10 and 3, but they did not partition both 13s into (10 + 3). When students partition both numbers into two parts, they are multiplying two binomials together, as opposed to multiplying a monomial times a binomial. Problems like the Card Problem were designed for this study to open the possibility for students' to solve the problems using the multiplication of two binomials.

Methodological and Analytic Framework

Methodology

The study used clinical interview methodology (Clement, 2000). Clinical interviews allow "the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques" (Clement, p. 547). When using this methodology, a researcher formulates conjectures about these mental processes (Confrey & LaChance, 2000; Steffe & Thompson, 2000). These conjectures are formulated based on conceptual analysis of the mathematical domain and prior research with students in this domain.

Problem sequences are then carefully designed so that they allow for testing these conjectures (Cobb & Gravemeijer, 2008). During data collection and analysis, conjectures are refuted or not refuted based on how students operate to solve problems. When a conjecture is refuted, the conjecture becomes open to being refined so that it can be tested in future studies.

Analytic Tools

The two primary analytic constructs used to characterize students' mental processes were *schemes* and the *mental operations* that constitute these schemes. A scheme is a goal-directed way of operating that has three parts—an assimilatory mechanism, an activity, and a result (von Glasersfeld, 2001). When a student is presented with a problem situation, the problem situation may trigger records of prior operating, and in doing so the student may come to recognize the situation (assimilate it) as one that involves a particular type of activity. The activity of a scheme involves mental operations like partitioning, disembedding, and uniting (three mental operations, which will be discussed below). Finally, the student's activity produces a result (i.e. a solution to a problem).

Framework for Participant Selection

Steffe (1994) has identified three distinctly different levels of multiplicative reasoning for elementary grade students. These different levels of multiplicative reasoning are a result of learning and development, and they open possibilities for, as well as constrain how, students operate mathematically. The different levels of multiplicative reasoning have been used as a framework to study students' reasoning as it pertains to the solution of problems involving both whole numbers and fractions. However, these levels have not been used as a basis for studying students' combinatorial reasoning. In this section, I give a brief overview of how students operating at the different levels typically solve repeated groups multiplication problems.

Students operating at the first level of multiplicative reasoning are able to coordinate two levels of units in activity. That is, to determine the number of doughnuts a person has if the person has 4 packages, with 8 doughnuts in each package, these students coordinate two counts—one count tracks the number of doughnuts and the other the number of packages. This coordination of two levels of units often involves double counting like the following: one, two, three, four, five, six, seven, eight, that is one package; nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, that is two packages, etc. The two units that these students are able to coordinate are the number of doughnuts and the number of packages, but they have to make this coordination as part of the activity they use to solve a problem.

Students operating at the second level of multiplicative reasoning are able to take the coordination of two levels of units as given, which means that they do not have to make a coordination between two counts as part of their activity. Instead, a number word like 8 automatically means both 1 package and 8 doughnuts. Students who operate with the second multiplicative concept are likely to solve the doughnut problem by reasoning that 8 and 8 is 16 because 8 and 2 is 10 and 6 more is 16. Here the number word 16 would mean 16 doughnuts and 2 packages. In reasoning in this way, these students are able to operate on 8 by partitioning it into two parts (i.e., breaking 8 into 2 and 6), and disembedding both of the parts in order to strategically unite them with another group of 8 (e.g., 8 and 2 makes one group of 10, and 6 more makes one group of 10 with 6 ones). To finish solving the problem they would likely continue this sequential process of adding 8 more onto the previous amount, using strategic ways of partitioning the numbers to help them calculate the total amount.

Students operating at the third level of multiplicative reasoning are no longer constrained to sequentially combining amounts and using addition to solve repeated groups multiplication problems. Instead these students might determine the number of doughnuts a person has if the person has 12 packages of 8 doughnuts by reasoning that 10 packages of doughnuts would be 80 doughnuts and that 2 more packages of doughnuts would be 16 doughnuts, which would yield a total of 96 doughnuts. In solving the problem in this way, these students are able to reason that 12 groups of 8 is composed of 10 groups of 8 and 2 groups of 8, and they can use that to determine the total amount. In doing so, they treat the 12 groups of 8 as a unit of 12 units each containing 8 units. That is, they treat the 12 groups of 8 as itself a unit that can be operated on, which means that they are reasoning with a third level of unit. Reasoning with the third level of unit enables them to partition 12 groups of 8 into two parts, a unit of 10 units each containing 8 units and a unit of 2 units each containing 8 units. Then they disembed each part, evaluate each part using multiplication (i.e., ten 8s is 80 and two 8s is 16), and subsequently unite the two parts together (i.e., 80 and 16 is 96).

Students' Schemes for Solving Two-Dimensional Combinatorics Problems

In a prior study with three eighth grade students, all of whom were operating at the third level of multiplicative reasoning, Tillema (under review a) identified a scheme that students used to solve basic two-dimensional combinatorics problems. The scheme entailed students *assimilating* such situations using *two composite units* (*two input quantities*). In the case of the Outfits Problem, the two composite units were 4 shirts and 3 pants. The *activity* of their scheme involved two key mental operations—ordering and pairing. Students used an *ordering* operation when they supplied a qualitative property that they used to differentiate the units of a composite unit. For example, in the Outfits Problem when a student used colors, a qualitative property, to differentiate among the three shirts, the student ordered the shirts. Students then used a *pairing* operation when they created a correspondence between one unit of each composite unit and applied their unitizing operation to this correspondence. That is, to be engaged in pairing a student created a correspondence between a shirt and pants and then applied her unitizing operation to this correspondence to create an outfit, an output unit.

Students produced the output units by following a lexicographic ordering (cf. English's, 1991, 1993 odometer strategy). A lexicographic ordering is similar to a dictionary ordering—the word "aa" appears before the word "ab" and all words that begin with "a" appear before all words that begin with "b". So, for example, in the Outfits Problem students followed a lexicographic ordering when they created the outfit that contained the first shirt and first pants prior to creating the outfit that contained the first shirt and the second pants, and they created all outfits that contained the first shirt prior to creating any outfits that contained the second shirt. For these reasons, students' schemes for solving these problems were called a *lexicographic units pairing scheme* (LUPS).

To differentiate the extent to which students carried out the pairing operations as part of the activity of their scheme, two ways that students treated the units of an input quantity were defined. A student treated the units of an input quantity as *particular units* when she repeated the pairing operations that she used with one of the units with all of the other units of that quantity. For example, in the Outfits Problem, a student treated the pants as particular units when she paired the first shirt with the first pants, the first shirt with the second pants, and the first shirt with the third pants because she repeated the pairing operations between the first shirt and each particular pants. In contrast, when a student did not need to repeat the pairing operations with one or both input quantities, the student operated with a *representative unit*. For example, a student might operate as described above, and then say, "you could do that with all of the other shirts." In this case, the student treated the first shirt as representative of how all of the other shirts would function, which enabled her *not* to carry out all of the pairing operations with the other shirts. Based on this distinction between particular and representative units, four different levels of the LUPS were identified.

Study Design and Guiding Conjectures

Selection Interviews

The selection interviews were used to identify at least three 6th grade students at each of the three different levels of multiplicative reasoning, and involved a total of fifteen students. The students all attended a magnet school in an urban school district in the Midwest. They were selected from a pool of 65 possible students all of whom had the same 6th grade teacher.

During the selection interviews, I presented problems to students with the intent of determining whether a student were able to coordinate two levels of units in activity (the first level of multiplicative reasoning), take the coordination of two levels of units as a given (the second level of multiplicative reasoning), or take the coordination of three levels of units as given (the third level of multiplicative reasoning). The interview protocols involved both whole number and fraction problems that were intended to elicit this information, but none of the problems involved any type of combinatorics problem.

Design of the First Interview

During the first interview, students were presented with two-dimensional combinatorics problems like the Outfits Problem. They introduced themselves or the researcher introduced to them three ways—lists, tree diagrams, and arrays—to represent these initial problems. During students' solution of these initial problems, the researcher emphasized the use of arrays as a way to symbolize such problems. The intent of posing these problems was to test three conjectures about the relationship between the different levels of multiplicative reasoning and the different levels of the LUPS.

Conjecture 1: Students operating at the first level of multiplicative reasoning will be constrained to the second level of the lexicographic units pairing scheme (LUPS).

Conjecture 2: Students operating at the second level of multiplicative reasoning will be constrained to the third level of the LUPS.

Conjecture 3: Students operating at the third level of multiplicative reasoning will operate at the fourth level of the LUPS.

Design of the Second Interview

During the second interview, students were initially presented with problems like the Restaurant Problem.

The Restaurant Problem: A meal at a local restaurant consists of one salad and one entrée. The restaurant serves 6 different kinds of salad and 14 different kinds of entrées. 10 of the entrees have meat. Illustrate with an array the total number of meals that are vegetarian and the total number of meals that are non-vegetarian.

These problems all involved one two-digit number (14 in the case of the Restaurant Problem) and one one-digit number (6 in the case of the Restaurant Problem), and again the researcher emphasized using arrays as a way to symbolize these problems. These problems were used to test the following conjecture.

Conjecture 4: Students operating at the third level of multiplicative reasoning will be the only students to solve these problems using distributive reasoning.

If students were able to solve problems like the Restaurant Problem, then they were presented with problems that had the potential to involve the multiplication of two binomials like the Card Problem, which is symbolized below using an array (Figure 2).

Figure 2: An array for the Card Problem.

In the Card Problem, 13^2 can refer to the total number of 2-card hands a person could make, while $(10 + 3)^2$ refers to this same total once a person has thought about breaking the hearts and clubs into non-face and face cards. Similarly, each of the following, 10^2, (10 x 3), (3 x 10), 3^2, refers to parts of the array: 0 face cards, 1 face card, or 2 face cards. This reasoning can lead to the development of the following equivalences: $(13)^2 = (10 + 3)^2 = 10^2 + (10 \text{ x } 3) + (3 \text{ x } 10) + 3^2 = 10^2 + 2 \text{ x } (10 \text{ x } 3) + 3^2 = 100 + 30 + 30 + 9 = 169$. Problems like the Card Problem were used to test the following conjecture.

Conjecture 5: Students operating at the third level of multiplicative reasoning are the only students who will experience these types of problems as involving the multiplication of two binomials.

Initial Findings and Discussion

Based on preliminary data analysis from this study there was no evidence that refuted conjectures 1, 2, 3. That is, all students in the study operated as the conjectures predicted they would operate. However, conjectures 4 and 5 needed to be refined as a result of this study. Providing data exemplars of students' reasoning that provides evidence for these claims will comprise a major component of the presentation of this paper. Here, I present exemplars of data to discuss how conjecture 5 has been refined as a result of the study.

Conjecture 5 stated that students operating at the third level of multiplicative reasoning are the only students who will experience the Card Problem as involving the multiplication of two binomials. The findings from the study indicate that students at both the second and third level of multiplicative reasoning were able to solve problems like the Card Problem. However, there were qualitative differences between the students' solutions of the Card Problem depending on whether they were operating at the second or third level of multiplicative reasoning.

Students who were operating at the second level of multiplicative reasoning solved the problem by first pairing a particular card (e.g., the two of clubs) with a representative card from the other suit (e.g., the two of hearts), and could take that as indication of producing the first thirteen pairs (i.e., all pairs that could be made with the two of clubs). They repeated the pairing operation with each of the thirteen clubs (i.e., treated the clubs as particular units), and could then state that the total number of two-card hands would be 13 x 13 and symbolize this multiplication problem using an array. Students could then find in their array the four sections shown in Figure 2, but to quantify each of these sections they had to re-engage in pairing operations for each section of the array. So, for example to determine the number of two-card hands that contained only face cards the students had to pair a particular face card (e.g., the king of clubs) with a representative face card from the other suit (e.g., the king of hearts), which they took as indication of producing the first three two-card hands that contained only face cards. They engaged in this pairing operation again for the remaining two clubs that were face cards to establish that there would be a total of 3 x 3 or 9 two-card hands that contained only face cards. They had to repeat these operations to establish the multiplication problems for the other three parts of the array. In doing so, they established that the array could be quantified using one multiplication problem (13 x 13) or four multiplication problems (3 x 3, 10 x 10, 3 x 10, and 10 x 3), but these two ways of quantifying the array remained two separate ways to view the array and were not integrated into a single structure (scheme).

Students who were operating at the third level of multiplicative reasoning solved the Card Problem by pairing a representative unit (e.g., the two of clubs) with a representative unit (e.g., the two of hearts). Because they treated both one club as a representative unit and the one heart as a representative unit, they were able to take a single pairing operation as implying all of the two-card hands that they could make. That is, they reasoned, for example, that the two of hearts was representative of any of the hearts that could be paired with the two of clubs, and so 13 two-card hands could be made with the two of clubs. Subsequently, they treated the two of clubs as representative of any of the clubs in the deck and so could take it as indicating the number of times that they would produce 13 two-card hands without actually having to make these two-card hands by pairing cards together. They could symbolize this pairing operation using an array. Once they produced the array they were able to determine the four sections of the array without having to engage in any further pairing operations. Rather they simply partitioned the two composite units they used in assimilation, 13 and 13, into two parts, and this implied to them partitioning the array into four sections. Moreover, they could identify a multiplication problem for each section of the array without having to use pairing operations to re-establish the multiplication problem for each part of the array. This enabled them to generate the equivalence that $13 \times 13 = (10 + 3) \times (10 + 3) = 10 \times 10 + 3 \times 10 + 10 \times 3 + 3 \times 3$, and see these two ways of quantifying the array as part of a single structure (scheme).

In the presentation, there will be a discussion of each of the 5 conjectures and video data will be used to support the statement of the conjecture or to discuss how the conjecture was refined as a result of the study.

Endnotes

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WHERE'S THE PROOF? PROOF IN U.S. HIGH SCHOOL GEOMETRY CONTENT STANDARDS

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This study examined High School Geometry standards from recent individual states' and the Common Core Standards (CCSSM) featuring reasoning and proof. We examined specificity of content and use of proof or reasoning as a process in the language of standards. Results indicated standards with proof as its primary process are more likely to have higher specificity of content, and vice versa. Further, the CCSSM were found to have higher content specificity and more consistent use of proof as a primary process than the vast majority of states' standards.

Background and Objectives

Since the publication of *A Nation at Risk* almost 30 years ago (NCEE, 1983), an ever increasing emphasis has been given to assessment and content standards. The publication of the *Curriculum Standards for School Mathematics* by the National Council of Teachers of Mathematics (1989) helped initiate development of content standards written by individual states (Porter, 1994). The NCTM Standards (1989) did not have a specific standard strand for proof but included it within the Reasoning Standard. Some noted this absence, noting that proof "is the backbone of mathematics" (Wu, 1996 p. 1534), and after many "mathematicians objected to what they considered…an under emphasis on proof" (Kilpatrick, 2003, p. 1), the following *Principles and Standards for School Mathematics* (NCTM, 2000) included a more prominent role to mathematical proof across the grades through the more explicit Standard on "Reasoning and Proof." In spite of this greater emphasis on proof, some reports on individual states' mathematics content standards have shown either a lack of emphasis on proof and reasoning or vague descriptions of its use by students (Klein et al., 2005; Raimi & Braden, 1998; Reys et al., 2006). In 2010 yet a new set of standards is being put forth in the form of the *Common Core State Standards for Mathematics* (CCSSM) (CCSI, 2010b). These Standards uphold all of the NCTM Process Standards including Reasoning and Proof; they also underscore constructing and critiquing arguments as a mathematical practice for students to learn. Given past reports concerning the role of proof and reasoning in state content standards, and considering the implementation of this new set of standards across the nation, we argue that it is useful to examine both the emphasis and specification of proof and reasoning in the various states' content standards, as well as those of the CCSSM. We have chosen to focus on standards concerning High School Geometry both for the historical importance proof has taken in American Geometry curriculum (Herbst, 2002) and for the sake of clarity in comparisons of both proof and reasoning in standards across the U.S.

Theoretical Perspective

Content standards have been developed in the U.S. for at least a century (Porter, 1994), and since the publication of *A Nation at Risk* (NCEE, 1983), the pressure for standards has increased dramatically. Porter (1994) notes that the publication of the NCTMs' *Curriculum Standards for School Mathematics* (NCTM, 1989) spurred states into writing their own content standards In addition to NCTM, various organizations have focused on developing and examining standards, including the National Center on Education and the Economy (NCEE), Council for Basic Education (CBE), the Mid-continent Regional Laboratory (McRel), the Council for Chief State School Officers (CCSSO), the Thomas B. Fordham Foundation, and the American Federation of Teachers (AFT). These and other organizations provided specific standards, guidelines and/or evaluations of standards as various states continued to publish and edit their documents (Porter, 1994). Of particular interest for the current investigation are the evaluations that have been conducted of different state mathematics standards, as these will provide a frame of reference to both the historical development and evolution of such evaluations as well as the development of reasoning and proof in mathematics content standards, with specific focus on secondary Geometry.

The first evaluation of state standards was conducted by the AFT (Gandal, 1996): The report examined whether states were developing standards, how such standards were assessed and implemented, and the detail provided within the different standards. That report concluded that 30 states had sufficiently clear and specific mathematics standards. The organization's most recent report in 2001 concluded that 44 states had clear and specific mathematics standards (AFT, 2001). The CBE produced a similar finding in their 1998 report *Great Expectations?* (Joftus & Berman, 1998), suggesting that only three states (Alaska, Montana, and Nebraska) had a low level of rigor in their mathematics content standards. All of these reports have provided general descriptions of the mathematics standards and not details regarding specific topics, in particular, not specific accounts of the role of proof or mathematical reasoning in state standards.

In contrast, the Fordham Foundation sponsored a report on state mathematics standards that did provide detail on specific mathematics content. The first of these reports (Raimi & Braden, 1998) examined the standards of 46 states and the District of Columbia. Contrary to reports by AFT and CBE (Gandal, 1996; Joftus & Berman, 1998), Raimi and Braden (1998) found that many states were not providing clear or specific standards. Specifically when focusing on mathematical reasoning, the report cited only five states as having a sufficient degree of clarity and focus. Further, the report notes that only 17 states explicitly indicate that proof should be taught in mathematics. A more recent report by Klein et al. (2005), suggests that this trend has not changed much in that "the majority of states fail to incorporate mathematical reasoning directly into their content standards…many state documents do not ask students to know proofs of anything in particular. Few states expect students to see a proof of the Pythagorean Theorem or any other theorem or any collection of theorems" (p. 21).

While not focusing on High School Geometry, Reys et al. (2006) provide a similar and concurrent finding to Klein et al. (2005), stating that "reasoning is not well articulated or integrated across K-8 standards documents…most state standards fail to address reasoning aspects in a thorough and comprehensive manner across grade levels and content strands" (Reys et al., 2006, p. 9). Given that Reys et al. examined specific elements of mathematical content in a comparable manner to Klein et al., it appears that many state standards do not provide clear or specific standards on mathematical reasoning.

The reports thus far referenced provide two types of reports, those that paid specific attention to the way mathematics was described in standards (e.g., Klein et al., 2005; Reys et al., 2006) and those that did not (e.g., Gandal, 1996; Joftus & Berman, 1998). This breakdown generally describes the reports that appear most prominent in the literature. Further, there tends to be a general depiction of standards as 'mostly good' in earlier years, and 'mostly in need of improvement' in recent years. These criticisms added to the lack of commonalities have led to the development of the Common Core Standards.

The Common Core Standards

In June 2009, the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) announced a new initiative to develop a common set of standards for mathematics and language arts. This initiative was called the Common Core Standards Initiative (CCSI), and included 46 states, the District of Columbia, and two U.S. territories (CCSI, 2010a). As of June 2, 2010, this total included two additional states and a working draft of the standards had been developed. The standards were developed to be "fewer, clearer, and higher" (CCSI, 2010b, p. 1) as compared to previous standards in many states. Among the criteria used in designing the standards are the NCTM process standards and the strands of mathematical proficiency from the National Research Council's report *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001; see also CCSI, 2010b). Yet, numerous citations to various reports, assessments, and standards are cited as works consulted (see CCSI, 2010b for a full description). A review of the documents and descriptions of the Common Core Standards by CCSI suggest a specific focus on a number of topics that have surfaced in mathematics education, including mathematical modeling and quantitative literacy. Continued reference to reform-oriented documents and sources also convey the notion that the Common Core Standards are reform-oriented. Additionally, some organizations, such as NCTM "diligently monitored the development of CCSSM and advised NGA and CCSSO throughout the process" (p. ix, NCTM, 2010). Yet, CCSI (2010c) specifically states that the Common Core Standards are a set of expected learning goals for students and do not constitute a curriculum nor do they advocate specific materials or teaching practices to be used. This appears to be a different goal for designing content standards than some states have used, where content standards were written as documents intended to reform teaching more so than specify content (see Thompson, 2001 for a relevant description).

As is customary with newly published standards documents, a large amount of skepticism and criticism has been levied against CCSI. For example, Zhao (2009) stated that "the Common Core Standards Initiative or any such movement to create national standards risks America's future by destroying its traditional strengths in cultivating a diverse and creative citizenry" (p. 52). Tienken (2009) expressed concerns that the Common Core Standards would only serve to increase an 'achievement test drive mentality' in the U.S. Providing a more even-handed critique, Porter, McMaken, Hwang, and Yang (2010) compared the Common Core Standards to the state standards in place and found that "the common core does represent considerable change…" but are not "…more focused as some might have hoped" (p. 8). However, Porter et al.'s description of their analysis does not provide content-specific focus in regards to reasoning or proof. Therefore, it appears there is a need for a transparent and clear examination of the way that reasoning and proof are represented in the Common Core Standards as compared to the existing state standards.

A main objective of the CCSSM (2010b) is to provide mathematics standards that are clear, specific, and rigorous. According to some reports (Klein et al., 2005; Raimi & Braden, 1998) many of the states' standards the CCSSM will be replacing, however, have not provided clear or specific standards regarding proof or reasoning in High School Geometry. Yet, since Klein et al.'s (2005) depiction of proof and reasoning in various states' standards, many states have published revisions of their standards, and therefore may have provided more sufficiently clear and specific expectations of students regarding reasoning and proof. Additionally, there is currently no available examination of reasoning and proof in the High School Geometry standards in CCSSM. In this paper we provide a current and comparative account of the depiction of reasoning and proof in the CCSSM and various states for High School Geometry. In order to accomplish this goal, the following research questions will be examined:

1. To what extent do different state content standards describe reasoning or proof in association with specific mathematical content?
2. To what extent do different state content standards characterize reasoning or proof as processes engaged by students?
3. Is there a relationship between the extent to which reasoning or proof are characterized as processes that students engage in and the extent to which such reasoning and proof standards refer to specific mathematical content?

Methods

Standards documents from all 50 states were collected and coded for analysis. These documents were collected from the websites of each state's department of education. However, various states have different formats for organizing their standards in High School mathematics, such as by course, by grade band or benchmark, or by specific grade level. Given our focus on High School Geometry, it was decided that standards in the format of a course would have the Geometry course standards examined. For those states with benchmarks or grade level formats, the standards for the Geometry strand (or equivalent) was used for the full range of High School standards provided by each state. In some cases, advanced level standards were provided. To ensure a fair comparison, we examined only those standards expected of every student. In addition to taking into account varying format and structure, different states may represent the same content with a different number of standards. In other words, one state may say in one standard what another state says in three. This phenomenon was taken into account by recording the number of High School Geometry standards each state had, and using this number to calculate mean emphasis on reasoning and on proof.

As our focus was on the presence and emphasis of standards outlining mathematical reasoning and proof in Geometry, we examined the standards documents for the presence of standards that referred to these mathematical processes. For proof, standards were identified through the use of words such as proof(s), proving, or prove present in a standard's statement. Explicit use of these words was necessary for the standard to be counted as a "proof" standard, as this ensured that such a standard would be interpreted by those using the document in practice as referring to proof. A similar process was used for coding reasoning, by using words referring to logic as indicators of such standards. Words such as inductive, deductive, conjecture, logic, reason, and their cognates were used as indicators. Once standards were identified as being a proof and/or reasoning standard, they were coded for their specificity to which they referred to the mathematics concepts involved.

As noted by certain reports (e.g., Klein et al., 2005; Reys et al., 2006) many standards lacked a degree of specificity in descriptions of mathematical reasoning and proof. Rather, a state may provide a number of standards requiring that proof be taught, but may not specify what mathematical content should be associated with proof. Therefore, we developed three categorizations to distinguish the degree of content specificity in a standard. The first, low specification, is a standard that makes either no reference to a particular mathematical concept, or where such specification is vague, e.g., New York standard "*Write a proof arguing from a given hypothesis to a given conclusion.*" A standard with medium specification is a standard that makes reference to a set of mathematical concepts, but still lacks a degree of specificity, e.g., CCSSM standard "*Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.*" While this example indicates the action of proving relationships in the area of triangle congruence and similarity, the standard does not specify any theorems or specific properties. Standards coded as having high specification were standards that make reference to a set of mathematical concepts with specific identification of such concepts, e.g., Michigan standard "*Know a proof of the Pythagorean Theorem, and use the Pythagorean Theorem and its converse to solve multi-step problems.*" This example makes reference to a specific concept, the Pythagorean Theorem, rather than a category or set of concepts.

In addition to coding for concept specification, each standard was examined for transitivity. In Systemic Functional Linguistics (Halliday & Matthiessen, 2004) transitivity is a system of linguistic resources with which language supports the ideational metafunction (i.e., the way language represents the world). Transitivity permits to represent the world as sequences of processes involving participants and circumstances. Canonically, processes are realized by verbs and participants by nominal groups. To illustrate our analysis of transitivity note, for example, Delaware standard "*<u>Reason</u> deductively to justify a conclusion or to create a counter-example;*" this contains three verbs (reason, justify, create), but *reason* represents the transitive process associated with this standard. Similarly, in Delaware standard "*<u>Use</u> appropriate technologies to model geometric figures and to develop conjectures about them*" the verb *use* represents the transitive process for the standard. What interests us here is the difference between these two examples in the way that reasoning is expressed as an expectation of students. Whereas *to reason* is the actual process students are expected to engage in the first standard, the second standard expects students *to use* technologies in specific ways, one of which is reasoning. Reasoning in the second standard acts as what Halliday and Matthiessen (2004) would refer to as a participant in the clause. As can be noted in the two examples given, other verbs can represent transitive processes for clauses nested within the main clause (clause complex). To account for such use of proof and reasoning terms as transitive processes and/or participants, we ranked such usages of identified words and terms in regards to the degree to which they were a primary process.

The Pennsylvania standard "*<u>Write</u> formal **proofs** to <u>validate</u> **conjectures** or arguments*" contains the primary process of *to write*, which we coded as having a distance of zero from the primary process. *Proofs* is a participant in this clause, as is *to validate conjectures*. However, *proofs*, while not a primary process, is relatively closer to the primary process than *conjectures*. Therefore, we coded *proofs* as being a distance of one away from the primary process, and *conjectures* as a distance of two away from the primary process. In this way, we accounted for the semantic emphasis placed on proof and reasoning in various standards.

Results

The number of standards for High School Geometry varied dramatically from state to state, with some states having as few as 5 and others having as many as 74. The Common Core Standards included 42 individual standards for High School Geometry while the mean for states was 21.98 (SD = 15.47). This means that the Common Core Standards do indeed have more standards for Geometry than most states, although some states do have more.

Approximately 24% of the CCSSM Geometry standards (n = 10) featured proof while the average for states was 11.7% (SD = .118), indicating that CCSSM provided proportionally more standards featuring proof than the typical state document. The CCSSM had an average specificity score of 2.90 for proof standards while the mean specificity score for states' proof standards was 1.88 (SD = .59). This indicates that the CCSSM standards featuring proof tended to be more specific in regards to mathematics content than the typical state document. The CCSSM proof standards had an average transitivity distance of 0.10, while the average transitivity distance for the various states' standards featuring proof was 1.24 (SD = .60). This indicates that the CCSSM proof standards tended to treat proof as the primary transitive process while various states did not do so as consistently. Taken altogether, the CCSSM provide a greater emphasis on proof in their Geometry standards than the most current state standards, and also provides more specific and clear standards with a strong emphasis on the action of proof. While some states showed stronger tendencies on at least one of the three criteria (overall emphasis, specificity of content, transitivity distance), no state showed higher scores on more than one indicator. Some states did appear somewhat comparable to the CCSSM proof standards: California, Florida, Indiana, and Michigan. With the exception of these states, the CCSSM in proof for High School Geometry show clear and definitive improvement in specification and increase in focus.

While the CCSSM show an increased focus on and specification of proof in High School Geometry, this trend was not as evident when examining other forms of mathematical reasoning. The CCSSM emphasized logical reasoning in mathematics in 5% of their standards. While the mean for states was a 14% emphasis, there was a large degree of variance (SD = .14). However, the CCSSM Geometry standards that featured reasoning showed a transitivity distance of 1.00, while the average for varying states was 1.26 (SD = .42). Further, only two states had lower averages than the CCSSM regarding transitivity distance, indicating that the CCSSM was similar to many states in regards to specifying reasoning as the primary process in its standards. Additionally, the CCSSM appear to provide more specificity (M = 3.00) than did the typical state (M = 1.68, SD = .81) regarding mathematics content in standards featuring reasoning for High School Geometry. Overall, the descriptive statistics for High School Geometry standards featuring reasoning indicates that CCSSM provides fewer relative standards, but with more specificity and a slightly stronger emphasis on the act of reasoning.

Correlation analysis between standards' content specificity and transitivity distance indicated a strong and negative relationship for standards featuring proof ($\rho = -.42$, $p < .05$). This indicates that the closer the distance between proof and the transitive process, the more specific a standard tended to be regarding mathematics content. However, when examining the same relationship for standards featuring reasoning, virtually no relationship was found ($\rho = .00$, $p = .98$). This indicates that an excessive amount of variance is present in many of the states' content standards regarding the specification of mathematics content and the use of reasoning as a transitive process. Practically speaking, the correlation found for proof standards indicates that standards which describe proof as the primary process students are expected to engage in tend to also provide more specific connections to mathematics content, and vice versa. However, the near zero correlation found for the same relationship in reasoning standards indicates little consistency in the way reasoning is described in various High School Geometry standards across the nation.

Discussion

While many concerns have been expressed regarding the CCSSM (e.g., Porter et al., 2010; Tienken, 2009; Zao, 2009), the results presented here may give cause for some mindful consideration of potential benefits of the CCSSM in regards to reasoning and proof, at least as it concerns High School Geometry. The results presented here indicate that the CCSSM provide a greater emphasis on proof with more specificity to the content and a clearer representation of the process than the vast majority of states' standards. Further, while the CCSSM does not have, proportionally, as many High School Geometry standards featuring forms of reasoning other than proof, the standards tend to be more specific to the content and have a clearer representation of the process than the majority of states. Therefore, in regards to proof and reasoning in High School Geometry standards, the CCSSM appear to show an improvement over state standards in general.

By articulating specific aspects of mathematics content, the CCSSM provide a clearer message to what teachers are expected to have High School students prove and reason about. For example, the CCSSM provides a list of theorems for students to prove regarding triangles (e.g., base angels of isosceles triangles are congruent; the medians of a triangle meet at a point). The CCSSM do not indicate how teachers are expected to engage students in such proofs or in what format students should write those proofs. Teachers are provided not with a blanket expectation of having students “construct proofs about triangles” that might be met with token proof exercises whose conclusions are not memorable (Herbst, 2002); rather they are provided clear guidelines that tie proof practices to important content that students need to learn. As such, we consider this a general improvement over many of the state content standards currently in use.

Another aspect of the CCSSM High School Geometry standards featuring proof that we consider beneficial to both teachers and students is the predominate use of *prove* as the primary transitive process. With the exception of one out of ten standards featuring proof, the CCSSM are consistent in emphasizing proof as the primary process. The benefit of writing “*prove theorems about lines and angles*” rather than “*write proofs about lines and angles*” is the activity conveyed by one wording over the other. Many states use words such as *write*, *construct*, *know*, or *use* as the primary process for standards featuring proof, and this leads to potentially different interpretations of what students are expected to do. Do students memorize proofs when the primary process is *know*? Does their engagement in proof come in the form of writing when the primary process is *write*? While these questions are rhetorical, they demonstrate the bias that can come when using words other than *prove* as the primary process for standards concerning proof. As such, the CCSSM does not convey bias in the way students are expected to engage in proof, but only in the content they are expected to prove.

The results of the present study provide evidence in support of the adoption of the CCSSM in regards to High School Geometry standards featuring reasoning and proof. However, the present analysis is narrow in focus, and several other mathematical strands and grade levels might need to be examined in a similar manner to properly understand both the potential benefits and drawbacks of implementing CCSSM. For example, while the present results concerning proof standards were generally positive, the results concerning reasoning standards indicate a need to use reasoning as the primary transitive process in CCSSM.

The results of the current study do support the implementation of CCSSM in regards to High School Geometry standards featuring proof and reasoning, but our discussion and conclusions should not be taken as a wholesale endorsement of the CCSSM document. Rather, as specified, further examination of CCSSM is needed. Further, by examining specific aspects of the CCSSM document, rather than an overarching depiction of the document as a whole, we are better able to either validate or make recommendations for improvement out of a solid base of knowledge.

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MATHEMATICAL KNOWLEDGE FOR TEACHING PROOF: EVIDENCE FROM AND IMPLICATIONS FOR PROFESSIONAL DEVELOPMENT

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Research suggests that the enhanced role of proof in mathematics classrooms presented in current standards and reform policy poses great challenges for teachers and will require substantial teacher learning. However, to date there is little research detailing what mathematical knowledge might be useful for teaching proof or how professional development might afford such learning. This paper presents a framework for Mathematical Knowledge for Teaching Proof that couples research on justification and proof in mathematics and mathematics education with Ball and colleagues' (2008) conceptualization of Mathematical Knowledge for Teaching (MKT). An empirical study of teachers' proof-related activity in professional development demonstrates the utility of this framework and provides further insights into mathematical knowledge for teaching proof.

Few can dispute the increased attention that reasoning and proof have been afforded in mathematics education research and practice (Hanna & deVilliers, 2008; National Council of Teachers of Mathematics, 2009). Both NCTM (2009) and the Common Core State Standards (2010) recommend that all students learn to construct and evaluate mathematical arguments and demonstrate a variety of reasoning and proof techniques. More notably, these documents call for an enhanced notion of proof that elevates proof beyond a topic of study in advanced mathematics courses, to a tool for learning mathematics at all levels.

However, the research summarizing students' difficulties constructing proofs and their reliance on authority or empirical arguments to justify a general claim suggests that these recommendations are not easily accomplished (Harel & Sowder 2007; Healy & Hoyles, 2000). Similar research has shown that teachers often rely on ritualistic aspects of proof and fail to distinguish between non-proof arguments and valid deductive reasoning when both constructing and evaluating proofs (Martin & Harel, 1989; Knuth, 2002; Simon & Blume, 1996). Moreover, recent classroom studies show that teachers struggle to support students even when using a curriculum designed to promote reasoning and proof (Bieda, 2010). For example, teachers' methods of grouping and sequencing proof strategies or inability to follow up on students' ideas often inhibit students' ability to construct a valid proof (Martin & McCrone, 2003; Bieda, 2010). Other studies have linked teachers' methods of questioning and providing examples to students' development of authoritarian or empirical proof schemes (Harel & Rabin, 2010).

Clearly, supporting all students to learn about and engage in mathematical proof poses great challenges for teachers and will require substantial teacher learning. However, to date there is little research detailing what mathematical knowledge might be useful for teaching proof or how professional development might afford such learning.

Purpose

The purpose of the larger study was to detail Mathematical Knowledge for Teaching Proof (MKT for proof) and to investigate MKT for proof in professional development (PD). To advance the construct of mathematical knowledge for teaching proof, research on proof and mathematical knowledge for teaching was coordinated with an empirical study of teachers' engagement with proof in professional development. Specifically, the study aimed to address the following questions:

1. What does research suggest teachers need to know about proof and proving that would be useful for their work with students?
2. What mathematical knowledge for teaching proof is evidenced in professional development focused on justification and proof?

This paper describes how an initial framework for MKT for proof was developed based on an extensive literature review. A sample of findings from one segment of the larger investigation of teachers' proof activity in PD is then provided to further highlight elements within this knowledge framework.

Theoretical Considerations

Two ideas underlie the conceptualization of mathematical knowledge for teaching proof developed through this study: (i) an enhanced notion of proof as a means to support mathematical understanding at all levels and (ii) the important role of teachers' mathematical knowledge-in-use (knowledge accessed when engaging in acts of teaching) for considering a knowledge base for teaching.

Recognizing the critical importance of proof in learning mathematics, recent reform efforts call for proof to be a regular and ongoing part of students' K-12 mathematics experiences (NCTM, 2000). To support proof as a sense-making activity, there is a need for a definition that promotes a consistent meaning of proof throughout the grades. For this reason, Stylianides' (2007) definition of proof was adopted for this study. Put simply, proof refers to a mathematical argument that is based on accepted statements, valid modes of argumentation, and representations that are known by or are within the conceptual reach of the classroom community. Further, to identify and describe the mathematical knowledge teachers need to engage students in practices consistent with this reform-oriented view of proof, the entire range of activity associated with proving must be considered. These proving activities include identifying patterns, making conjectures, testing examples, and providing non-proof arguments, as well as constructing proofs.

Secondly, a situative perspective on learning informs both the conceptualization of mathematical knowledge for teaching proof and the ways in which it might be investigated in PD settings. This perspective takes seriously the ways in which mathematical knowledge is attuned to the specific demands of teaching (Adler & Davis, 2006). Defining mathematical knowledge for teaching proof thus involves consideration of what, when, and how knowledge is required and used to meet the demands of the classroom. Central to a situative perspective is the recognition that cognition is situated in particular contexts, social in nature, and distributed across the individual, other persons, and tools (Putnam & Borko, 2000). Accordingly, mathematical knowledge evidenced in interaction as teachers participated in proof-related activities was the focus of the qualitative analysis.

Methods

Developing MKT for Proof Framework

To develop a framework for MKT for proof, research on mathematical proof, students' difficulties with proof, and classroom studies of teaching practices related to proof were reviewed, coordinated and synthesized. Proof ideas drawn from the literature were categorized using four domains of MKT introduced by Ball and colleagues (2008). Tables 1 and 2 below provide a snapshot of elements included in this MKT for proof framework. Common Content

Knowledge (CCK), the subject matter knowledge held in common with others who use mathematics, includes the ability to construct a valid proof as well as common understandings of the nature of proof. Elements within CCK reflect essential proof knowledge and skills desired of students but also directly address findings that teachers often attend to the form of proof rather than substance and fail to see the generality in an argument (Dreyfus & Hadas, 1987; Martin & Harel, 1989).

For their work with students, teachers need more than CCK. Fundamental understandings of proof must be developed, "unpacked," and explicitly connected to the work of teaching. This Specialized Content Knowledge (SCK) is pure mathematical knowledge uniquely needed for teaching proof. While mathematicians may know how proofs depend on previously known definitions, to deal with issues that arise in the classroom, teachers also need to know a range of possible definitions and understand how an argument might look different depending on the definitions that are accepted. To guide their instruction, teachers also need to understand the relationship between the proving task and valid or efficient methods of proof such tasks evoke (Stylianides & Ball, 2008). Thus, knowing that a claim about a finite number of cases can be verified through a systematic list, but that a generic example or deductive argument would be needed to prove a more general claim would be considered SCK.

Table 1. Subject Matter Knowledge for Teaching Proof

Common Content Knowledge	**Specialized Content Knowledge**
Ability to construct valid proof Use definitions, theorems to build a logical progression of statements Analyze situations by cases & use counterexamples **Nature of Proof** *A theorem has no exceptions* *A proof must be general* The validity of a proof depends on its logic	**Explicit Understanding of Proof Components** •Accepted Statements: range of definitions or key theorems, role of language & defined terms •*Modes of Argumentation: recognize range of valid methods, distinguish between empirical and deductive arguments* •*Modes of Representation: variety of visual, symbolic, verbal methods to express general argument* **Relationship between proving tasks & activity**

Elements within pedagogical content knowledge arose from studies documenting students' typical responses when asked to produce or evaluate mathematical proofs as well as classroom studies that illustrated where proof instruction fell short. Although these studies did not specifically explore teachers' knowledge, inferences can be made regarding knowledge or resources teachers might have drawn upon to support students' understanding of proof. As indicated in table 2, Knowledge of Content and Students (KCS) includes detailed knowledge of students' thinking as well as attention to the proof-related resources available to students. Knowledge of Content and Teaching (KCT) intertwines knowledge of proof from the other domains with methods of representing or drawing out key proof ideas in classroom instruction. For example, knowing that students typically rely on authority or empirical justification (Harel & Sowder, 2007) guides instructional decisions and questions or examples teachers may use to encourage students' progression toward deductive arguments.

Clearly, no one idea within MKT for proof is unique, nor is this meant to be a comprehensive list of everything a teacher needs to know. However, this framework provides a rare synthesis of research and begins to detail the mathematical knowledge of proof that would be useful for teachers' work with students. As described next, delineating knowledge of proof in this way provided a common analytic tool to make sense of teachers' proof work across different professional development activities.

Table 2. Pedagogical Content Knowledge for Teaching Proof

Knowledge of Content and Students	Knowledge of Content and Teaching
Detailed knowledge of student thinking • Explicit knowledge of proof schemes • Recognizing characteristics of external, empirical, and deductive proof schemes • Students' tendency to rely on authority or empirical examples • *Progression from inductive to deductive proof* **Developmental aspects of proof** Ability to size up arguments in terms of: • Definitions & statements available to students • *Representations within students conceptual reach* • *Forms of argumentation appropriate for students' level*	**Relationship between instruction and proof schemes** **Questioning strategies** • To elicit justification beyond procedures • *To encourage thinking about general case* **Pivotal examples or counterexamples** • To extend, bridge or scaffold thinking • To focus on key proof ideas **Explicit knowledge of proof connections** • *Linking visual, symbolic or verbal proofs of same concept or theorem* • Comparing proofs in terms of accepted definitions and argument structure • *Lifting general argument from numerical example or specific diagram*

Investigating MKT for Proof in Professional Development

The empirical study used to test and refine this framework, drew upon an existing research project designed to support teacher leaders' understanding and facilitation of mathematically rich PD. Although teachers' knowledge of proof was not the central project purpose, the focus on generalizing patterns and number concepts and explicit attention to norms for justification and explanation made the project an ideal site to investigate MKT for proof. Within this project, K-12 teacher-leaders participated in either three or four two-day seminars during the school year. During these seminars, teachers worked on a number of mathematics tasks, analyzed videocases of PD sessions centered on the same mathematical tasks, and engaged in additional activities designed to connect these experiences to their own practice.

Given the goal of detailing MKT for proof, all math tasks, discussion prompts, video data logs, and seminar agendas were reviewed to select tasks in which teachers were asked to make conjectures, prove claims, and engage in explicit conversations about proof in the context of teaching. Two mathematical tasks, Consecutive Sums and Halving and Doubling, were selected. This paper reports on teachers' seminar work associated with the Halving and Doubling task. In particular, video data of two small groups and one whole group discussion for each of eight activities directly related to the Halving and Doubling task were analyzed. These activities included teachers 1) doing the math task; 2) viewing PD video to compare two teacher justifications; 3) viewing video to consider an additional conjecture 4) discussing MKT in relation to video; 5) revisiting video to consider facilitation of the task; 6) planning PD session using task; 7) discussing justification article (Lannin, 2005); 8) reflecting on their own PD enactment of task with colleagues.

Analysis

To address the second research question, what MKT for proof is evidenced in PD, all

Halving and Doubling video was reviewed to develop a coding scheme that would capture the nature of teachers' conversations in relation to proof. Initial coding indicated when teachers referred to visual or algebraic models, tested specific examples, explicitly used the terms "conjecture," "generalization," or "proof," as well as when discussions moved to student or teaching concerns. Studiocode (2010), a video analysis software, was used to facilitate the coding and analysis of each of the small group and whole group discussions. The data matrix features of Studiocode then allowed for both in-depth analysis of a proof topic as well as an indication of how and if this proof idea was evidenced across the range of seminar activities. For example, all instances of teachers using the word "proof" or attempting to understand a generic example could be gathered and viewed in succession to facilitate more focused coding. As patterns emerged from this analysis, integrative memos (Emerson, Fretz & Shaw, 1995) were written to summarize elements of proof consistent across teachers' discussion. It was only at this stage that proof ideas emerging from seminar activity were mapped back to the MKT framework.

Results

To illustrate this mapping, findings from analysis of teachers' seminar activity provided here are organized around the four domains of teacher knowledge: CCK, SCK, KCS, and KCT, within the MKT for proof framework. Elements in tables 1 and 2 corresponding to these reported findings have been italicized to further highlight the relationship between the teachers' proof activity within Halving and Doubling and categories of MKT for proof.

CCK: Nature of Proof

Teachers Recognized the Need for Proof to be General

Across all small and whole group conversations teachers used phrases that clearly related to the general nature of proof. Teachers recognized flaws in arguments that were based on a few specific cases or did not explicitly demonstrate that the conjecture was valid for *any* numbers. When evaluating ways of verifying the conjecture, comments such as the following were typical:

> "I like the algebraic model that you came up with and then also the area model because you can show that for *any* string of numbers if you half it and then add to the end - that is essentially what you are doing" (sg1 – doing math)
>
> "In the rectangle the use of a and b *is an attempt to generalize* and regardless of what it looks like visually, the a and the b represent variables and therefore *any case*." (wg - comparing videocase teacher work)

Teachers both pressed for generalization in their own mathematical work and recognized attempts the teachers in the PD videocase made toward presenting general arguments.

SCK: Explicit Understanding of Proof Components

Teachers Connected a Variety of Visual, Symbolic, and Verbal Methods Used to Express Argument

Teachers went beyond noting that a proof must be general to explore a variety of visual, symbolic, and verbal methods. In their own mathematical work small groups explored the problem through the use of numerical tables, array models, and algebraic expressions. More importantly, as illustrated in the two previous quotes, teachers considered how those representations could be used to express a general argument. As teachers continued to evaluate and connect various methods of verifying the conjecture, they grappled with ways in which a visual diagram might move beyond a specific case to provide a generic example or connect to an algebraic proof. Discussions such as the ones prompted by the teacher comments below firmly situated teachers in the realm of SCK for proof.

> "I don't know that your method lends itself to an algebraic proof or justification because it is only showing concrete, discrete numbers its not showing me how I would connect a variable standing for any number…
> So what would we need to add to my model, my proof to meet that criteria?" (sg2 – role playing after video viewing)

KCS: Developmental Aspects of Proof

Acceptable Proof Depends on Forms of Argumentation and Representations Appropriate to Grade Level

In both teachers' discussions and actions it was evident that when considering what might be an acceptable proof of the Halving and Doubling conjecture, teachers considered what knowledge and skills students at a given grade level typically possess. Teachers made explicit statements that what constitutes a proof might depend on the grade level you were teaching and "where your students are at." In terms of representations, teachers asked questions about whether students had a grasp of variable, or were comfortable with area or array models to represent multiplication. And in considering forms of argumentation that might be accessible to students, teachers acknowledged students' ability to see generality in the array diagrams but wondered if elementary students would be capable of constructing a deductive argument.

KCT: Explicit Knowledge of Proof Connections

Teachers Discuss Scaffolding from Examples or Visual Proof to Symbolic Representations by Capitalizing on a Key Idea or Generality

When discussing representations and forms of argument students may or may not understand, teachers often described specific teaching strategies they may employ. For example, in this statement below, a teacher is imagining how he might lead a class discussion of the problem by exploring patterns across numerical examples and then moving students toward the use of variables to make a more general argument:

> "I might bring up multiple examples, then say 'what I'm hearing you say is no matter what numbers we use, the operation stays the same so let's put a letter in'" (sg1 - doing the math)

Teachers further explored ways in which visual array or area models might be introduced to make connections between a specific example and a more general argument using variables. Importantly, teachers were able to bring out key mathematical ideas underlying the generalization (reciprocal properties, conservation of area, commutativity, etc.) when suggesting strategies to move students from example-based proofs toward more general arguments.

Discussion

Two findings from the Halving and Doubling teacher discussions in particular shed light on previous research regarding teachers' understanding of proof and what MKT for proof might entail. First, it is worth repeating that although teachers recognized essential features of a valid proof, this was not always evidenced in their acceptance of a proof. For example, as highlighted earlier, teachers made clear statements that proof must be general. It must work for *any* number and testing only specific numbers was not enough to prove the general claim. And yet teachers also discussed how in 3rd grade, it might be okay to use specific numbers in an array diagram, or how they might begin instruction by having students look for patterns in a table. Or, teachers might follow up on a valid general argument by asking how it would work for fractions, odd numbers, or negative numbers. In other words, while empirical justification clearly was not enough proof, teachers saw pedagogical value in posing questions about different classes of numbers or in exploring visual representations. As teachers, they saw the value of testing examples or using specific visual representations for helping students make sense of the mathematics involved and providing a foundation for algebraic proof. This finding makes us step back and reconsider the research on what teachers “understand” about proof. While previous research has suggested that teachers fail to recognize the generality of proof, here we see a sophisticated understanding of generality that is clearly tied to the context of teachers' work.

A second, and closely related finding, is that teachers moved fluently across the four domains of knowledge. Teacher talk within all eight seminar activities took up multiple dimensions of the MKT for proof framework. Interjections about what representations students have access to or what numbers to use in examples were intermingled with explanations of teachers' own proof constructions. Teachers' genuine questions about generic examples or the difference between inductive and deductive reasoning led directly to issues of students and of teaching. This finding further highlights the situated nature of teachers' knowledge and calls to question PD models focusing on either mathematics or pedagogy. As discussed next, this has implications for how the mathematics education field might begin to detail, research, and develop MKT.

Implications

First and foremost, this study contributes to the growing research on teacher knowledge by detailing critical aspects of mathematical knowledge of proof that support teachers' work with students. Recent research addressing teacher learning has identified *mathematical knowledge for teaching* as a key construct in teacher education (Ball, Lubienski & Mewborn, 2001). However, most attempts to specify mathematical knowledge for teaching have focused on elementary content. This investigation extends the work to mathematical knowledge useful for promoting proof, a key mathematical practice, K-12.

Further, this research provides a common framework for analyzing MKT for proof in both classroom and PD settings. The framework supports research on the teaching of proof by serving as an analytic tool to identify mathematical resources teachers access as they engage students in proof activity. The framework can also be used to make sense of teachers' proof-related activity in PD by connecting specific teacher activity such as constructing proofs versus comparing proofs presented in a videocase to the domains of teacher knowledge evidenced in discussions. This work supports further research on what “effective” PD to develop teachers MKT for proof might entail.

Finally, the findings related to teachers' fluent movement across knowledge domains help advance theory about the complex relationship between subject matter and pedagogical content knowledge. Investigation of conversations in which teachers move fluidly from talking about the mathematical ideas embedded in a proof to thinking about students and teaching can help the field better understand those connections between the four domains of knowledge - CCK, SCK, KCS, and KCT. This close relationship has implications for both researching and designing activities that might support teachers' development of MKT.

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REASONING-AND-PROVING IN GEOMETRY TEXTBOOKS: WHAT IS BEING PROVED?[1]

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As calls are made for reasoning-and-proving to permeate school mathematics, several textbook analyses have been conducted to identify reasoning-and-proving opportunities outside of high-school geometry. This study looked within geometry, examining six geometry textbooks and characterizing not only the justifications given and the reasoning-and-proving activities expected of students but also the nature of the mathematical statements around which reasoning-and-proving takes place. The majority of reasoning-and-proving exercises focused on particular mathematical statements, whereas the majority of expository mathematical statements were general in nature. Although reasoning-and-proving opportunities were numerous, it remained rare for reasoning-and-proving to be made an explicit object of reflection.

Background

Mathematicians and mathematics educators are calling for reasoning-and-proving[2] to become a central component of the mathematical experiences of students (Hanna, 2000; Stylianou, Blanton, & Knuth, 2009). One argument behind this call is that reasoning-and-proving is integral to the discipline of mathematics and thus an essential piece of an "intellectually honest" (Bruner, 1960) mathematics education. Such a perspective is reflected in the mathematical practices of the *Common Core State Standards for Mathematics* (2010), which include abstract reasoning, the construction of viable arguments, and the critique of others' reasoning. Another argument for the inclusion of reasoning-and-proving throughout the school mathematics curriculum is that, by reasoning through and proving mathematical results, students can develop deeper conceptual understanding of mathematical ideas as well as greater procedural fluency (de Villiers, 1995; Dreyfus, 1999; National Council of Teachers of Mathematics, 2009).

In the United States, however, reasoning-and-proving has not been ubiquitous in school mathematics but has traditionally been confined to a single geometry course in high school (Herbst, 2002). To document this current landscape and to prepare the way for a more comprehensive treatment of reasoning-and-proving, researchers have recently been studying the opportunities that exist for reasoning-and-proving in curriculum materials other than geometry. Kristen Bieda (personal communication, January 18, 2011) is in the process of coding a variety of elementary-level textbooks and Stylianides (2009) has examined an NSF-funded middle school series. At the high-school level, Davis (2010) explored an integrated textbook series and Senk, Thompson, and Johnson (2008) analyzed non-geometry courses, such as algebra, advanced algebra, and precalculus, from six different textbook series. Davis found that 12% of the problems in a particular integrated textbook related to reasoning-and-proving. Senk and her colleagues found this number to be only 6% in the textbooks they analyzed, even though their analysis focused on the chapters where reasoning-and-proving seemed most likely to occur. Senk, Thompson, and Johnson noted that the exposition sections of non-geometry textbooks gave more attention to reasoning-and-proving than the exercises; nevertheless, 40–50% of the stated mathematical properties were left unjustified.

In this study we contribute to the efforts described above by characterizing the nature of reasoning-and-proving opportunities in geometry textbooks themselves. Although we agree with the premise that it is important to understand the current state of reasoning-and-proving outside of geometry as efforts are undertaken to integrate reasoning-and-proving into those domains, we would add that it is equally important to understand reasoning-and-proving opportunities in geometry, where they are most plentiful. In other words, we should strive to understand and reflect upon the way we are handling reasoning-and-proving in geometry so that we may inform the process of expanding it to other courses and grade levels.

Analytic Framework

All of the curriculum analyses cited above have focused on the types of reasoning-and-proving activities that students are expected to perform, presenting data on how often students are asked to notice patterns, make conjectures, test conjectures, or develop arguments. Although many of the documented difficulties that students have with reasoning-and-proving (see Harel & Sowder, 2007, for a review) may be attributed to insufficient opportunity to engage in such practices, it also seems to be the case that students have fundamental misunderstandings of reasoning-and-proving, even after significant exposure. For example, Chazan (1993) found that some geometry students do not understand what has been proven by a deductive argument. Soucy McCrone and Martin (2009) reported on students, also from geometry, who viewed the purpose of proof to be the mere application of recently learned theorems, similar to the way in which recently learned formulas are applied in subsequent student exercises.

We employ the necessity principle as an interpretive frame to make sense of these phenomena and to guide our analysis. The *necessity principle* (Harel & Tall, 1989) is a standard for pedagogy that involves presenting subject matter in a way that encourages learners to see its intellectual necessity, "[f]or if students do not see the rationale for an idea, the idea would seem to them as being evoked arbitrarily; it does not become a concept *of the students*" (p. 41). The fact that many students do not understand the role of reasoning-and-proving in mathematics and view it as being required of them arbitrarily (Tinto, 1988) is evidence that the necessity principle is being violated with respect to reasoning-and-proving. Furthermore, the well-documented overreliance on empirical forms of argumentation (Harel & Sowder, 2007) suggests at least two possibilities: (a) students do not recognize the limitations of empirical reasoning or the intellectual necessity of deductive reasoning, or (b) students recognize the need for deduction but lack the resources or capabilities to successfully develop such arguments and so give an empirical argument rather than leave an item blank. If the latter is the case, then it is important to continue examining the opportunities that exist for students to engage in various reasoning-and-proving activities. But if it is the former, we must push further and consider whether or not those reasoning-and-proving activities are necessitating deductive reasoning.

With these considerations in mind, we developed our analytic framework by building upon past frameworks (particularly Senk et al., 2008) with the addition of a dimension for the mathematical statement around which the reasoning-and-proving activities are taking place. In particular, we distinguish between general and particular mathematical statements (see Figure 1). Our rationale is that general statements intellectually necessitate deductive forms of reasoning because empirical means cannot establish truth for an infinite class of objects. Thus, having students engage in reasoning-and-proving around general mathematical claims has the potential to better satisfy the necessity principle than particular mathematical claims. (This also happens to better align with the disciplinary practices of mathematicians.) We do not mean to imply that all reasoning-and-proving opportunities should be around general statements or that there is no benefit to exercises of a particular nature; we are simply arguing for the value of including this dimension when examining reasoning-and-proving opportunities in textbooks.

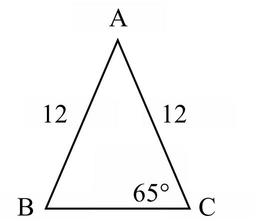
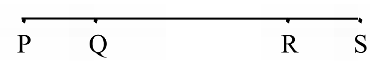
Type of Statement	Definition	Examples
General	A statement made about an infinite class of mathematical objects or an infinite number of mathematical situations.	(1) All isosceles triangles have congruent base angles. (2) Any lines l and m that are perpendicular to the same line n are parallel to one another. (3) Let a, b, and c be natural numbers. Then $a^{b+c} = a^b \cdot a^c$.
Particular	A statement made about a single or finite number of mathematical objects or situations. Note: A particular statement may be easily translated into a general statement, as in (2) to the right which may be reinterpreted as a statement about any such overlapping segments. This awareness of generality is non-trivial, however, and thus statements such as (2) remain categorized as particular.	(1) In the given diagram, angle *ABC* has a measure of 65 degrees.  (2) If $PR = QS$, then $PQ = RS$.  (3) If $2x+y=10$ and $y=4$, then $x=3$.

Figure 1. General and particular mathematical statements.

Method

This study focused solely on stand-alone high-school geometry textbooks and so did not include analysis of geometry units within integrated textbooks. The six textbooks included in our analysis were *CME Geometry* (CME Project, 2009), Glencoe McGraw-Hill *Geometry* (Carter, Cuevas, Day, Malloy, & Cummins, 2010), Holt McDougal *Geometry* (Burger et al., 2011), Key Curriculum *Discovering Geometry* (Serra, 2008), Prentice Hall *Geometry* (Bass, Charles, Hall, Johnson, & Kennedy, 2009), and UCSMP *Geometry* (Benson et al., 2009). These were chosen to overlap series as much as possible with previous analyses (i.e., Senk, Thompson, & Johnson, 2008) so that comparisons would be possible. The six included series together span nearly 90% of the U.S. high school population (Dossey, Halvorsen, & Soucy McCrone, 2008).

Within each chapter of the six student edition textbooks, we randomly selected for analysis a minimum of 30% of the canonical sections (i.e., not special exploration or technology investigation sections). Additionally, chapter review exercises were coded for each chapter as a representation of the textbook authors' own identification of key ideas. This process resulted in an actual sampling of 44% of sections across the textbooks, totaling 285 sections and 12,468 exercises. Within the sampled sections, both exposition and student exercises were coded by the authors using the framework in Figure 2. Double-coding was performed on a 20% sample of the included sections yielding 95% agreement on statement-type and 91% agreement on justification-type within exposition sections, and 92% agreement on statement-type and 93% agreement on activity-type within student exercises.

	Exposition	Student Exercises	
	Properties, Theorems, or Claims	Related to Mathematical Claims	Related to Mathematical Arguments
Mathematical Statement or Situation	• General • Particular	• General • Particular • General with particular instantiation provided	• General • Particular • General with particular instantiation provided
Expected Student Activity		• Make a conjecture or refine a statement • Fill in the blanks of a conjecture or statement • Investigate a conjecture	• Develop a mathematical proof • Develop a rationale or non-proof argument • Outline a proof or develop a proof given an outline • Fill in the blanks of an argument or proof • Evaluate or correct an argument or proof • Find a counterexample
Type of Justification (or environment for exploration)	• Deductive • Empirical • Outline • Past or future • Left to student • None	• Deductive (explicit) • Empirical (explicit) • Implicit	• Deductive (explicit) • Empirical (explicit) • Implicit
	• Statements *about* reasoning-and-proving	• Exercises *about* reasoning-and-proving	

Figure 2. An analytic framework for reasoning-and-proving in geometry textbooks.

Results

As shown in Table 1, student exercises involving reasoning-and-proving were much more prevalent in geometry textbooks than in even the most reasoning-and-proving focused units of non-geometry or integrated high-school textbooks. CME contained the most reasoning-and-proving exercises with nearly 38% falling into at least one of the reasoning-and-proving activity categories from Figure 2. The other geometry textbooks ranged from approximately 20% to 27% of exercises related to reasoning-and-proving.

Table 1. Percent of student exercises involving reasoning-and-proving.

Textbook Series	Geometry		Non-Geometry	
	No. of Exercises Analyzed	Reasoning-and-Proving (%)	No. of Exercises Analyzed	Reasoning-and-Proving (%)
CME	1058	37.8	--	--
Glencoe	2730	24.3	2117[a]	3.7[a]
Holt	2531	23.6	2042[a]	3.7[a]
Key Curriculum	1489	26.7	916[a]	8.0[a]
Prentice Hall	2479	19.5	2446[a]	5.6[a]
UCSMP	2181	27.6	1739[a]	6.2[a]
Core Plus	--	--	1114[b]	12.3[b]

[a] Senk, Thompson, & Johnson (2008).
[b] Davis (2010) differed from the other studies in this table by including patterns as reasoning-and-proving.

The types of reasoning-and-proving activities expected of students are presented in Table 2. In four of the books, 13–15% of the reasoning-and-proving exercises (or 3–5% of the total exercises) involved students *developing a mathematical proof*. In two books, CME and Glencoe, such items comprised 25% and 28%, respectively (or approximately 7% of the total). The most common reasoning-and-proving activities were to *investigate a statement* (i.e., determine the truth-value of a mathematical claim) and to *develop a rationale* (i.e., to explain or justify an answer or result in a manner that is not necessarily a proof).

Table 2. Nature of reasoning-and-proving activities expected of students.

Textbook Series	Type of Activity by No. of Reasoning-and-Proving Items (%)					
	Develop a Proof	Develop a Rationale	Find a Counter-example	Investigate a Statement	Make a Conjecture	Other
CME	25	36	7	45	16	3
Glencoe	28	48	4	30	17	5
Holt	13	42	4	39	8	16
Key Curriculum	13	52	26	42	20	6
Prentice Hall	15	54	1	46	19	11
UCSMP	14	44	3	36	14	5

Note: Rows sum to more than 100% because exercises often involved more than one activity. "Other" includes fill-in-the-blanks of a proof, provide or argue from a proof outline, and evaluate a given proof.

Table 2 does not answer the question, what sorts of mathematical claims do students have opportunities to reason about or prove? Figure 3 depicts the percentages of reasoning-and-proving exercises that involved a *general mathematical statement*. This graph excludes general statements for which a particular instantiation was provided to the student, because in such cases the student may reason about the given object as they would any particular object without realizing the general implications (as was found by Chazan, 1993). Figure 3 compares the percentages of general statements in exercises with the percentages in textbook exposition.

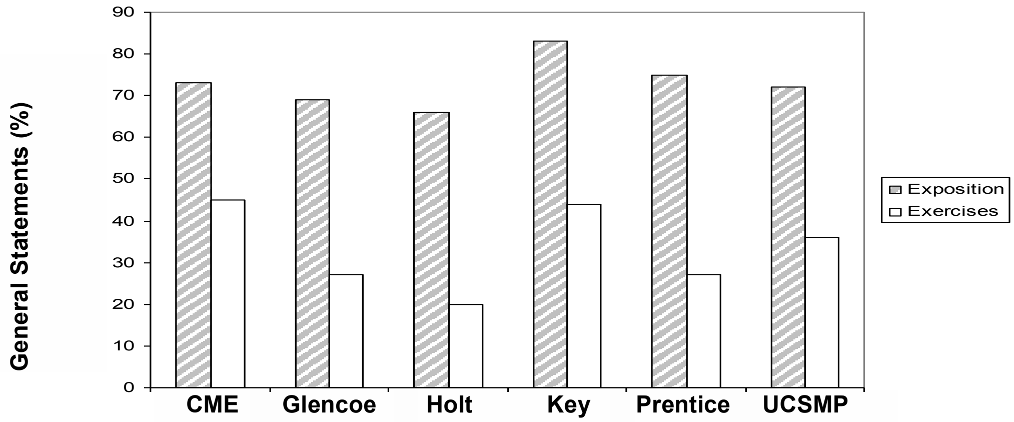


Figure 3. Nature of mathematical statements in exposition versus student exercises.

Most reasoning-and-proving exercises involved particular mathematical statements. In Glencoe, Holt, and Prentice Hall, approximately two-thirds of the reasoning-and-proving exercises were of a particular nature. In UCSMP, the number is 58%. CME and Key Curriculum had lower percentages of particular-type exercises—52% and 48%, respectively—but even in these textbooks, general statements were used in less than half of the reasoning-and-proving exercises. In textbook exposition, on the other hand, at least 66% of the statements containing a mathematical claim or result were of a general nature, with most textbooks falling above 70% (see Figure 3). Expository statements of a particular nature were especially infrequent in Key Curriculum and UCSMP, but were noticeably present in Glencoe (26%), Holt (24%), and Prentice Hall (20%). These particular statements that did appear in the exposition were almost always in the form of "worked examples." From this perspective, particular statements essentially appeared in exposition only when the textbook authors were modeling the behavior of students, for whom reasoning-and-proving exercises around particular statements are common.

Finally, we note results with respect to statements and exercises *about* reasoning-and-proving (see Table 3). For example, an exposition section may note that a deductive argument builds upon definitions or previously proved theorems, or an exercise may ask a student to write about the process of proof by contradiction. Within the 285 coded sections (out of 653 total sections), there were only 98 statements that made reasoning-and-proving an explicit object of reflection, and nearly half of these were found in a single book, UCSMP. Of the 12,468 coded exercises, only 67 asked students *about* the reasoning-and-proving process (as opposed to asking them to engage in that process). Therefore, although we saw in Table 1 that reasoning-and-proving is relatively common in geometry textbooks, opportunities are rare even in geometry to step out of the process and reflect on the core mathematical practice of reasoning-and-proving.

Table 3. Reasoning-and-proving not as an activity but as an object of discussion or reflection.

Textbook Series	No. of Statements *about* Reasoning-and-Proving	No. of Exercises *about* Reasoning-and-Proving	Percent of Total Exercises Analyzed
CME	13	4	0.38
Glencoe	8	17	0.62
Holt	15	17	0.67
Key Curriculum	14	13	0.87
Prentice Hall	7	2	0.08
UCSMP	41	14	0.64

Discussion

In this paper, we have presented top-level results of an investigation of the reasoning-and-proving opportunities in six different high-school geometry textbooks. Even in geometry, the traditional home of reasoning-and-proving, students were asked to develop a mathematical proof in less than 7% of the textbook exercises, and statements or questions about reasoning-and-proving as a mathematical practice were rare. The most common reasoning-and-proving activities were to provide a rationale (not necessarily a proof) and to determine the truth-value of a mathematical claim. Interestingly, students were expected to make judgments about truth much more frequently than they were expected to provide deductive arguments—the disciplinary process by which truth is established. With respect to the prominence of rationale exercises, one might contend that an "explain" prompt provides students with an opportunity to develop a proof because a key function of mathematical proofs is explanation (de Villiers, 1995). However, the question remains: Do students realize that a proof would be an effective response to an "explain" exercise? Answering this question would take us beyond the realm of textbook analysis.

The necessity principle (Harel & Tall, 1989) implies that it would be beneficial to help students recognize the intellectual need for deductive forms of reasoning by, for example, providing them with opportunities to reason around general mathematical claims for which empirical arguments falter. Our analysis has revealed that the majority of reasoning-and-proving exercises in geometry textbooks are around particular, not general, mathematical statements. In exposition sections, on the other hand, the majority of mathematical statements are general in nature. This discrepancy may shed light on such phenomena as geometry students believing that proof is merely an application of recently learned theorems (Soucy McCrone & Martin, 2009), because indeed students are applying the theorems presented in exposition sections to prove things about contrived, particular situations, or geometry students believing that mathematical knowledge is created by others and not themselves (Schoenfeld, 1988), because indeed the most significant mathematical results are general in nature and likely found in textbook exposition.

Pursuing these potential connections requires further research, and one might be skeptical of the merits of this course of study. Perhaps it is necessary for key results to be explicated in exposition sections so that they may be officially established in the classroom canon. Moreover, one could argue that it is necessary to provide students with numerous particular statements to prove because practice is essential and there are not enough relevant general statements to allow for this practice. In response to these points, we would again cite the research literature which shows that the status quo of reasoning-and-proving in geometry is not producing the student outcomes the mathematics education community hopes to see. Yes, it is important to establish important results into a collective space, but it is also important to reflect upon how the process of establishing those results may influence students' notions of who is capable of generating mathematical knowledge. Yes, it is important to allow students to practice reasoning-and-proving, but it is also important to consider whether the nature of the practice we afford them aligns with actual mathematical practice. In the end, our goal is for students to have success with reasoning-and-proving but also to see its intellectual necessity and value.

Endnotes

1. This work was supported by a grant from the College of Natural Science at Michigan State University. We thank Kristen Bieda and Sharon Senk for their insightful feedback.
2. We join Stylianides (2009) in hyphenating reasoning-and-proving to emphasize the inseparability of the reasoning process that leads to a proof and the resulting proof product.

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USING CONCEPTUAL BLENDING TO ANALYZE STUDENT PROVING

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We explore ways that university students handle proving statements that have the overall structure of a conditional implies a conditional, i.e., $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. We structure our analysis using the theory of conceptual blending. We find conceptual blending useful for describing the creation of powerful new ideas necessary for proof construction as well as for describing the creation of blends that slow or hinder student efforts at proof construction.

Introduction

The purpose of this paper is to illustrate the power of the theory of conceptual blending to clarify issues that students have in proving statements having the overall structure of a conditional implies a conditional, i.e., $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. This logical structure occurs often in statements to be proven at the university level. For example, since the definition of A is a subset of B ($A \subseteq B$) is a conditional statement ($x \in A$ implies $x \in B$), then a simple set theory statement such as "If $A \subseteq B$, then $A \cup B \subseteq B$" has this logical form.

The research literature indicates that students' misunderstanding of logical rules and misinterpretation of logical statements result in their difficulty with structuring their proofs (Brown, 2003; Duran-Guerrier, 2003; Harel, 2001; Roh, 2010; Selden & Selden, 1995). Students tend to structure their proofs in terms of the chronological order of their thought processes instead of rearranging it with careful consideration of proper implications (Dreyfus, 1999). The literature also shows that students are often unable to bring useful syntactic knowledge to mind. Such knowledge includes formal definitions (Knapp, 2006) as well as theorems and properties (Weber & Alcock, 2004) of the mathematical concepts. Likewise, research calls attention to various forms of personal knowledge of mathematical concepts. Such knowledge is internally meaningful to an individual student (Pinto & Tall, 2002; Vinner, 1991), and helps a student recall conceptual ideas to apply when attempting to construct a proof (Knapp & Roh, 2008). Because of its private and informal nature, students' personal knowledge is often insufficient for them to know how to get started on a proof (Moore, 1994). Raman (2003) suggested the *key idea* as a means of connecting personal intuitive ideas and procedural knowledge when constructing a proof. When students possess a key idea for a proof, it gives them conviction and the basis for the formal mathematical proof.

Theoretical Background: Conceptual Blending

Fauconnier and Turner (2002) posit conceptual blending as a powerful unifying theory to describe how people think across multiple domains. They argue that blending "makes possible … diverse human accomplishments … [in] language, art, religion, [and] science [as well as being] indispensable for basic everyday thought" (p. vi). This theory has begun to be used to describe student understanding of mathematical concepts (Gerson & Walter, 2008; Megowan & Zandieh, 2005; Núñez, 2005). In this section we give a brief example describing three of the main mechanisms of the theory of conceptual blending.

Conceptual blending is a subconscious process that entails the blending of two or more mental spaces (inputs) to form a new stable conceptual model for use in reasoning (See Figure 1). A mental space consists of an array of elements and their relationships to one another, being activated as a single unit. Two (or more) mental spaces are activated and crucial elements of each are integrated and *mapped* to a third space to form a blended space. As part of *completing* the blend, a conceptual frame may be recruited to help organize the information in the blend. Once the blend is complete it can be manipulated to make inferences or answer questions. This manipulation is referred to as *running* the blend. The blended concept is treated as a simulation that can be run imaginatively according to principles and properties that the input spaces bring to the blend.

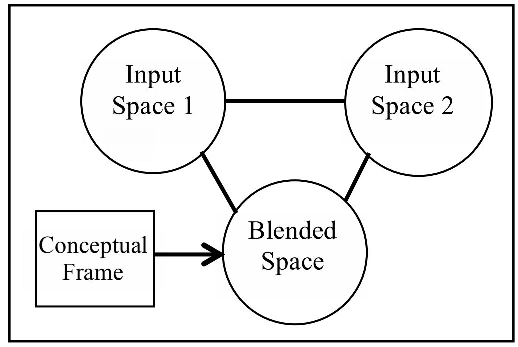

Figure 1. Generic blending diagram

For example Coulson and Oakley (2001) consider the nursery rhyme "the cow jumps over the moon." Children easily comprehend this statement by blending an input space of animals which includes cows, a second input space for the moon and sky, and a conceptual frame of jumping. In the blended space the cow is mapped to the thing that jumps and the moon is mapped to an object which is jumped over. Whereas children easily construct this blend, adults might have to inhibit their notions of reality and instead bring to bear a "nursery rhyme" frame which allows them to think of real things, the cow and moon, in impossible situations. Running the blend might include imagining the cow taking off from the ground, being over the moon, and landing on the ground on the other side of the moon.

Methods and Setting

The data for this study was originally collected as part of a semester long teaching experiment (Cobb, 2000) in an upper division geometry course at a university in the USA. Data consisted of videotape recordings of each 75 minute class session as well as copies of student written work. For the purpose of this paper we chose to analyze one day of class where we recognized something powerful was happening with student reasoning. Maher and Martino (1996) refer to such occasions as "critical events." The class period consisted of a brief introduction of the problem by the teacher (the first author), followed by small group work on the problem and whole class discussion. For the purpose of this paper we focus on the small group consisting of students we call Andrea, Nate, Paul and Stacey. The curriculum consisted of a series of activities in which students would need to define, conjecture, and prove results in geometry on the plane and the sphere (Henderson, 2001). This study focuses on one day late in the semester in which students were asked to prove either Euclid's Fifth Postulate (EFP) implies Playfair's Parallel Postulate (PPP) or PPP implies EFP.

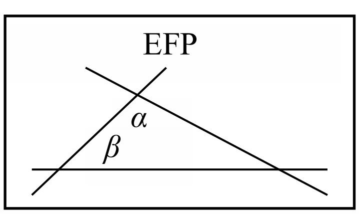

Figure 2. Euclid's Fifth Postulate

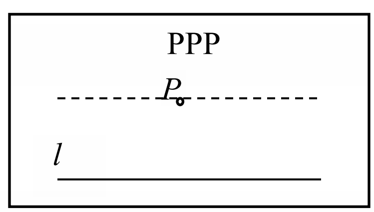

Figure 3. Playfair's Parallel Postulate

Henderson (2001) states EFP as, "If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then the two lines (if extended indefinitely) will meet on that side on which are the angles less than two right angles," (p. 123) and PPP as, "For every line *l* and every point *P* not on *l,* there is a unique line *l'* which passes through *P* and does not intersect (is parallel to) *l*" (p. 124). The instructor told the students that the two postulates are equivalent and gave them the option to "*use*" EFP in order to prove PPP or vice versa. In her introduction of the task the instructor drew the two figures shown (see Figure 2 and Figure 3) while explaining each of the postulates. The instructor's initial drawing showed only the part of the statement that was given. For example, for the PPP picture (see Figure 3), she initially just drew the bottom line, *l*, and a point, *P*, not on that line. However, when she explained the conclusion of each statement she completed the picture and these completed pictures were left on the board for students to reference. A second visual reference was available to students. The two pictures in the book for these two statements were also completed pictures similar to what the teacher had drawn.

Results and Analysis

In our first pass through the data, we noticed that elements of conceptual blending occurred both in students structuring of their proof and in their combining of the pictures and statements of EFP and PPP. We then read through the data looking to specify what blends the students were creating. We noticed the students were creating three types of blends: structural, geometric, and a combination of the two. We also noticed that the same blends occurred whether students were attempting to prove EFP implies PPP or PPP implies EFP. To better illuminate when each blend occurred in the data, each of the authors color coded a portion of the data and then all three authors came to a consensus on the coding for each of the following aspects:

- How students were blending the pictures associated with EFP and PPP: the key geometric blend (KGB) used by most students or Stacey's geometric blend (SGB).
- The logical construct that the students were using to frame their proof: a simple proving frame (SPF) or a conditional implies conditional proving frame (CICF).
- The direction of the proof: EFP implies PPP (EtoP) or PPP implies EFP (PtoE).

As we coded the data it became clear that there were four combined blends each of which could be described as an episode. In Figure 4 we summarize the evolution of student thinking through the four episodes, highlighting the three aspects of the combined blend: a structural blend (SPF or CICF), a geometric blend (KGB or SGB) and the direction of the implication (EtoP or PtoE). In addition to the main blend, we note a secondary blend if there were contrasting remarks or questions from other students during the episode that seemed to refer to a different combined blend.

Episode		*Main Blend*				*Secondary Blend*			
	Time	*Structure*	*Direction*	*Geometry*	*Presenter*	*Structure*	*Direction*	*Geometry*	*Presenter*
E1	9:00-	SPF	EtoP	KGB	Paul	CICF	EtoP	KGB	Nate
E2	15:13-	SPF	EtoP	SGB	Stacey				
E3	17:48-	SPF	PtoE	KGB	Andrea	CICF	EtoP	KGB	Nate
E4	24:38-	CICF	EtoP	KGB	Nate	SPF	EtoP	KGB	Paul

Figure 4: Summary of the progression of student proving ideas.

As illustrated in Figure 4, we found conceptual blending useful for describing the evolution of student thinking while proving. Using the three aspects of the combined blends, we were able to track the progression of the main thrust of the small group discussion as well as the contrasting voices in those discussions. In a longer paper (Zandieh, Roh, & Knapp, 2010) we describe the blends involved in each episode. Here we focus on three salient examples that illustrate the power of blending to describe student reasoning that moves the proof construction forward as well as student reasoning that slows proof construction.

The Key Geometric Blend (KGB)

From the beginning of Episode 1, students looked for a way to blend together the picture and statement of the two postulates geometrically and conceptually. They did this by creating the key geometric blend (KGB) which turned out to be the key idea of the proof for the students.

Stacey: Because no matter what we can put any P out there [reaching out her arms to touch her finger tips together] at our point of intersection [pointing to Andrea's notebook]. [Nate: That's a good point.] And then we know that, that it is unique [tracing a line with her pen], that it is not going to come back and intersect somehow. [...]

(Silence 1 minute -- Paul flips pages, Stacey flips pages (x2))

Paul: Well, if you assume the first one [EFP], would there be three cases that $\alpha + \beta < \pi$, $\alpha + \beta = \pi$, or $\alpha + \beta > \pi$? And then the uniqueness part of it would be proved by the $\alpha + \beta = \pi$ and in that case they wouldn't meet.

As students flipped pages between the picture and statement for EFP and the picture and statement for PPP they began *mapping to a blend.* The two figures in the text functioned as input spaces for a blended space (see Figure 5). Paul's three cases take the EFP picture and lay it on the PPP picture such that the bottom line from each of the input spaces (*m* from EFP and *l* from PPP) is mapped into the bottom line of the blended space. The transversal (*n*) from the EFP input space is included in the blended space, and the top line from each space is mapped into a line in the blended space. Finally, the point *P* from the PPP space is mapped onto the intersection of the transversal and the top line from EFP in the blended space (see Figure 5). To *complete the blend* the students brought to bear their previous geometric knowledge of lines and angles. Notice that Paul imagined three different possible locations of the top line in the blended space and coordinated the different geometric positions with different sums for $\alpha + \beta$. We would say Paul's idea of three cases comes from *running the blend*. After Paul's comment, Andrea and Nate also contributed to running the blend by imagining that if $\alpha + \beta > \pi$, then the lines would intersect on the other side.

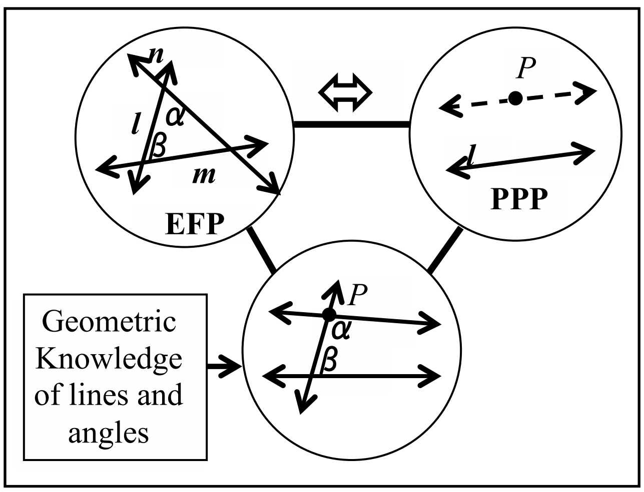

Figure 5. Key Geometric Blend (KGB)

The KGB was also involved similarly when the students attempted to prove the other direction, PPP implies EFP, in Episode 3. The main difference for the students in Episode 3 was that the blending of the two pictures was created by a slightly different mechanism. Andrea started with the PPP picture and constructed a transversal to create line *l* of the EFP picture. This construction does not change the basic content of the KGB, but it is significant in that it allowed students to see additional relationships in the geometric blend focusing on the case when the two lines are parallel transports of each other.

Andrea: If we assume this [points to PPP picture] and draw a transversal through here, through P and through this line. So we know that since these are parallel that they add to 180, right? Because they are supplementary, whatever.

Having agreed upon the sum of the angles being 180 degrees, Andrea then made her full argument more clearly as follows:

Andrea: So say that we rotate this line so that it makes it less than 180, the sum less than 180. [Nate: Yeah]… Oh, so, right! So that since we know that this is a unique line because we've assumed that there is only one line. So we rotate it so that it is a different line, then we know that it is going to intersect because there is only the one line that won't intersect.

The students continued to discuss the issues involved in Andrea's proof idea for a few more minutes, but ultimately returned to trying to prove EFP implies PPP. Although Andrea's proof was not completed, the KGB was fundamental to this discussion as it was in Episode 1 when Paul introduced the idea. In addition, both discussions of the KGB (in Episodes 1 and in Episode 3) played into Nate's idea for the proof that was discussed in Episode 4. Examples of this occur in the next two sections. The KGB was fundamental to discussions of the proof in Episodes 1, 3, and 4 and in the final write up of the proof. The KGB was the central idea of the proof, the key idea of the proof in the sense of Raman (2003). So the result of the blend was powerful for proving. In addition, the students ran the KGB over and over imaginatively as they worked out issues in the proof.

Simple Proving Frame (SPF) vs Conditional Implies Conditional Frame (CICF)

We introduce and compare two proving structures that the students used in their proofs of statements of the form $(p \rightarrow q) \Rightarrow (r \rightarrow s)$: the Simple Proving Frame (SPF) and the Conditional Implies Conditional Frame (CICF). By SPF, we refer to a proving frame where there is a given statement (premise), then a series of implications, then a conclusion (see Figure 6). There is nothing inherently wrong with the SPF or trying to apply it to a conditional implies a conditional statement. However, unless a student has particular theorems to work with that allow a direct proof from $(p \rightarrow q)$ to $(r \rightarrow s)$, then a simple proving frame (SPF) may be inadequate.

Generic SPF	Case 1: $(p \rightarrow q)$	Case 2: $(p \rightarrow q) \Rightarrow (r \rightarrow s)$
Given $\boxed{}$ … Series of Implications … Then $\boxed{}$	Given p … Series of Implications … Then q	Given $(p \rightarrow q)$ … Series of Implications … Then $(r \rightarrow s)$

Figure 6: Simple Proving Frame (SPF)

The students began to work on the task by looking at the pictures and statements of the two postulates described on two sequential pages in their book. As they began to think about which direction might be easier to prove and how to prove it, three of the four students each flipped back and forth between the two pages multiple times. We describe their deliberations in terms of a simple blend in which the two input spaces are the two postulates (see Figure 7). The students then were bringing the SPF to bear on the problem by putting EFP in the place of what is given and PPP in the place of the conclusion. To the extent that there is resolution in this early discussion the students seem to have created a blended space that is structured by the SPF with EFP as the given and PPP in the conclusion.

On the other hand, the CICF is substantially different from the SPF that the students used in Episodes 1-3. By CICF we refer to proofs of statements of the form $(p \rightarrow q) \Rightarrow (r \rightarrow s)$, where one starts with *r* and uses a series of implications including $p \rightarrow q$, to reach the conclusion, *s* (see Figure 8). In Episode 4 Nate began to lay out his case for the CICF more directly.

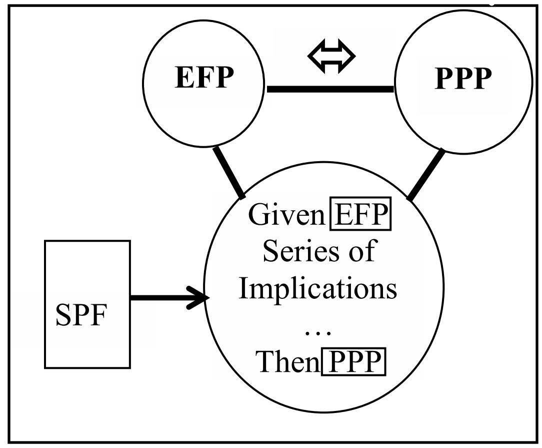

Figure 7. A structural blend of SPF from EFP to PPP

Nate: […] If we assume this [points to a drawing of EFP] is true? [Andrea: Okay, so this way?] This [EFP] is true for a moment. [Andrea: Okay.] Now we have our little point over there and we draw this line. We know that if this line is such that the angles on the one side are less than 180, that it is not parallel based on this assumption [EFP]. We know that if they are greater than 180, we can apply this assumption again and show that they do [intersect] on the other [side]. Can we use our parallel transport proof to show that the boundary condition when they are equal to 180, that this angle is congruent to this angle and therefore they are parallel and therefore they don't intersect?

Nate first established that he was assuming EFP is true and therefore was proving PPP. Next, he hinted at the premise of PPP, "we have our little point over there and we draw this line," suggesting that he was starting with *r* of the $(p \rightarrow q) \Rightarrow (r \rightarrow s)$. He then stated how he could use EFP in the middle of the proof, "we can apply this assumption." This is the use of $(p \rightarrow q)$ in the series of implications. He also hinted at what else may be needed to get to the conclusion of PPP. As we will see in the next section, Paul, Stacey, and Andrea initially struggled with Nate's proof structure. Towards the end of the discussion they began to think Nate's idea might work. This was aided by the teaching assistant visiting the group and being supportive of Nate's idea. Following her departure the group prepared a presentation for the class based on Nate's idea. As the group began work on their presentation the three dissenters made contributions indicating that they understood Nate's blend.

Generic CICF	For the case of $(p \rightarrow q) \rightarrow (r \rightarrow s)$
Given … Use … Thus	Given *r* … Then *p* Since *p* and $(p \rightarrow q)$ Then *q* … Thus *s*

Figure 8. The CICF

Blending the Premise and Conclusion

As explained above, students initially, from Episode 1, wanted to use a simply proving frame (SPF) that would start with EFP and end with PPP. Since the SPF is a legitimate proof technique in many situations and one that students were very familiar with, this is understandable, and might have even led to a proof if the students had appropriate theorems for this. However, in this case, the problem with using SPF was compounded by students blending the premise and conclusion of EFP in a way that lost the implication structure. We illustrate this with a transcript from Episode 4. As explained in the previous section, Nate had suggested his proof idea which used CICF. Other students in this group saw a similarity between their and Nate's idea in the sense of using three cases (KGB). However, they struggled with Nate's proof structure.

Paul: So, we don't even need to necessarily have the three cases, do we? Just we need to prove that one case – uniqueness on that one. Because we are assuming they meet on this side so I don't think it really matters if. [Stacey: Exactly. Agreed.] […]

Nate: I disagree because you are applying EFP to your specific cases. You're not assuming that α and β are less than. In order to apply it, you have to show that α and β are less than. […]

Paul: I think you are assuming that.

Stacey: Yeah, it assumes. Because we are assuming the whole EFP.

Andrea: We're assuming EFP.

Stacey: We're saying if we got two lines that are going to intersect on this one side, the interior ones right there are going to be, they have to be less than π.

Nate: Right, right. But now we are drawing a picture where we're going to say that α and β are less than and therefore we can apply EFP to show that they intersect. […]

Paul: If we're assuming that, then how can we say that they are going to be equal to π? Do you know what I am saying?

Nate: Well, that's my point. You have to draw three cases. You have to draw when they sum less than, when they sum equal, and when they sum greater than. And you have to apply EFP to two of those cases.

For Paul, Stacey and Andrea assuming EFP meant that they were assuming both the premise and the conclusion of EFP. In part, we see this as a faulty use of the SPF proof structure, the notion that we are starting with "all" of EFP and we will end the proof with the statement of PPP. In addition to the problem of using the SPF structure there is more specifically the idea that assuming "all" of EFP causes EFP to lose its implication structure. We describe this as a blending of the premise and the conclusion of EFP. Using blending theory we would say that there is a blend with two input spaces, (1) the premise of EFP ($\alpha + \beta < \pi$) and (2) the conclusion of EFP (the two lines intersect on the same side as α and β). The relationship between these two input spaces is that of an implication. However, when the two spaces are mapped to the blend, they are mapped to the same diagram with the implication having been compressed to an "and" or a simple coexistence without any implication structure. When the students were running this blend in the context of proving EFP to PPP using the SPF, they concluded that both the premise and conclusion of EFP were given, so it was not necessary to consider the cases when $\alpha + \beta > \pi$ or the lines didn't intersect, since students were assuming as part of EFP that the lines intersected and $\alpha + \beta < \pi$.

Summary

We find conceptual blending useful for describing the creation of powerful new ideas necessary for proof construction as well as for describing the creation of blends that slow or hinder student efforts at proof construction. We noted how students blended the two pictures of EFP and PPP and ran that blend in ways that allowed them to create a key idea (KGB) for the proof. This blending continued to serve as the foundation for the proof even through its final configuration. In addition, Nate's use of CICF was eventually blended with the KGB to construct the final, correct proof that this group presented to the class. On the other hand, there were two cases of student blending that served to hinder or slow their proving. In the first case we saw that students' initial use of an SPF structural blend hampered their efforts to structure their proof. The students did not have the necessary theorems to complete a proof of this conditional implies conditional statement using an SPF. In a second case we saw students treating "all" of EFP as given. EFP was treated as a collection of parts without maintaining the appropriate implication structure between these parts. As a result, the students' conceptual blends led them to blend the premise and conclusion in ways that obscured the implication relation between them. Consequently, their heavy reliance on this proof frame in the initial discussions slowed their efforts. It is our suggestion that instruction should be more explicit in contrasting the use of SPF and CICF.

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LEARNING TRAJECTORIES: FOUNDATIONS FOR EFFECTIVE, RESEARCH-BASED EDUCATION

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Approaches to standards, curriculum development, and pedagogy are remarkably diverse; however, recent years have seen a growing movement to base each of these on learning trajectories. In this paper, I discuss and compare the various terms and conceptions of this construct, present our definition, differentiate between our conception and that of others', and briefly review some of our recent evidence in the area of early childhood mathematics paper.

Throughout history, approaches to standards, curriculum development, and pedagogy have been remarkably diverse. Recent years, however, have seen a growing movement to base each of these on *learning trajectories*. Examples include the National Council of Teachers of Mathematics' *Curriculum Focal Points* (2006) to the National Research Council's report (2009), and most notably the Common Core State Standards (CCSSO/NGA, 2010, for which the "progressions" of a learning trajectory were developed first—the standards followed). Here I compare and contrast different notions of this important concept and summarize results of recent empirical work illustrating its potential.

The term "curriculum" stems from the Latin word for racecourse, referring to the course of experiences through which children grow to become mature adults. Thus, the notion of a path, or trajectory, has always been central to curriculum development and study. In his seminal work, Simon stated that a "hypothetical learning trajectory" included "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (1995, p. 133).

Building on Simon's definition, but emphasizing a cognitive science perspective and a base of empirical research, "we conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (Clements & Sarama, 2004, p. 83). The term "learning trajectory" reflects its roots in Simon's constructivist perspective (in emphasizing students' learning). However, although the name appears to focus on learning more than teaching, both Simon's and our definitions clearly involve teaching and instructional tasks. Some interpretations and appropriations of the learning trajectory construct emphasize only the "developmental progressions" of learning (what Simon calls hypothetical learning processes) during the creation of a particular curricular or pedagogical context. That is, they only describe levels of thinking through which students develop, which we believe is but one part of the learning trajectory construct. Some terms, such as "learning progressions" are used ambiguously, sometimes indicating only developmental progressions, and at other times, also suggesting a sequence of instructional activities. Although studying either psychological developmental progressions or instructional sequences separately can be valid research goals, and studies of each can and should inform mathematics education, we believe the power and uniqueness of the learning trajectories construct stems from the inextricable interconnection between these two aspects. Both these aspects (developmental progressions of thinking and instructional sequences) serve the most important, but often least discussed, aspect of learning trajectories—the goal. Our learning trajectories base goals on both the expertise of mathematicians and research on students' thinking about and learning of mathematics (Clements, Sarama, & DiBiase, 2004; Fuson, 2004; Sarama & Clements, 2009a). This results in goals that are organized into the "big" or "focal" ideas of mathematics: overarching clusters and concepts and skills that are mathematically central and coherent, consistent with students' (often intuitive) thinking, and generative of future learning (Clements, Sarama, et al., 2004). Once the mathematical goals are established, research is reviewed to determine if there is a natural developmental progression (at least for a given age range of students in a particular culture) that can be identified within theoretically- and empirically-grounded models of children's thinking, learning, and development (Carpenter & Moser, 1984). That is, researchers build a cognitive model of students' learning that is sufficiently explicit to describe the processes involved in students' progressive construction of the mathematics described by the goal across several qualitatively distinct structural levels of increasing sophistication, complexity, abstraction, power, and generality.

What, if Anything, is "New" in the Learning Trajectories Construct?

When we discuss learning trajectories, some (commendably) skeptical colleagues ask what is really different. If curricula have always been "courses" or paths (and frequently "horse races" through them), and if psychological and educational theories always postulated series of goals, then is this not simply renaming old (and palpable) ideas? At certain simple levels, the answer is positive. Most of these notions describe or dictate a series of educational goals. All have some theoretical perspective on why one goal might follow another.

In contrast, these theories often differ markedly on the details, and the learning trajectories construct as we define it builds upon theories and research of years past, as any theory should, but is distinct from previous formulations and constitutes a substantive contribution to theory, empirical research, and praxis. For example, early educational psychology considered educational series or sequences on the accumulation of connections. "We now understand that learning is essentially the formation of connections or bonds between situations and responses, that the satisfyingness of the result is the chief force that forms them, and that habit rules in the realm of thought as truly and as fully as in the realm of action" (Thorndike, 1922, p. v). Thus, curricular sequences could be logically arranged to establish connections between simple situations (addends) and responses (sum) and then later connect these and other bonds to complete more difficult tasks (e.g., multidigit addition) and even to develop mathematical reasoning. However, conceptual, meaningful learning was not the focus, but rather simple paired or associated learning. Also, potential differences and nuances of learning in different subject matter domains were not considered.

Bloom's taxonomy of educational objectives and Robert Gagné's "conditions of learning" and "principles of instructional design" (Gagné, 1965; Gagné & Briggs, 1979) postulated that Thorndike's theory was too simple and that there were "types of learning" and that certain types, such as stimulus-response learning (e.g., Thorndike's "bonds") were prerequisite to other types (e.g., discrimination learning, concept learning, rule learning, and last, problem solving). For a specific topic, or a specific domain within a topic such as mathematics, these could be assembled in "learning hierarchies"—sequences of pairs consisting of a subordinate skill whose acquisition is hypothesized to facilitate the learning of a higher-level skill. These, then, specified a "learning route"—certainly one early form of a learning trajectory. Such routes would be determined by logical analysis (logically identifying what subordinate competence is required by a superordinate competence) and empirical task analysis (Gagné, 1965/1970).

In a similar manner, others continued to promote task analysis as a way to develop complex hierarchies of skills. Some researchers similarly based these hierarchies on logical and task analyses, but gave more weight to extant findings in educational—and especially psychological—research to perform “cognitive” or “rational” analyses—with follow-up empirical validation studies whenever possible (Resnick & Ford, 1981). Work from this perspective increasingly used the computer metaphor (i.e., information-processing theories), and often actual computer models, in their analyses (Hoz, 1979; Klahr & Wallace, 1976).

These approaches determined hierarchies of educational goals and were the basis of many “scope and sequences” in the educational literature (see Baroody, Cibulskis, Lai, & Li, 2004, for an extended discussion and somewhat different perspective). The view of learning of the earlier approaches was generally that of knowledge acquisition, with the environment providing input that was “received”—that is, imitated and mentally recorded by the student.

Other researchers attended more to students’ thinking and cognitive development. Some devised developmental learning theories in attempts to integrate structural views such as those of Piaget with views based on task analysis and information-processing models. Later theoretical efforts in cognitive science extended these efforts to focus on the importance of domain-specific learning and development (Davis, 1984; Karmiloff-Smith, 1992).

In historical parallel, several theories, from Piagetian (Piaget & Szeminska, 1952) to field theories (Brownell, 1928; Brownell & Moser, 1949) and later developmental and cognitive science theories (Case & Okamoto, 1996) emphasized students as makers of meaning. Similarly, cognitively- and constructivist-oriented research programs explicated the concepts and skills children build as they move from one level to the next within a mathematical domain (Baroody, 1987; Carpenter & Moser, 1984; Steffe & Cobb, 1988). Unfortunately, those applying these studies practically often oversimplified and misconstrued their results and implications, emphasizing laissez–fair or outdated “discovery” approaches (Clements, 1997).

Learning trajectories as we have defined them (and our overaching theory of Hierarchic Interactionalism, see Clements & Sarama, 2007a; Sarama & Clements, 2009a) owe much to these previous efforts, which have progressed to increasingly sophisticated and complex views of cognition and learning. However, the earliest applications of cognitive theory to educational sequences tended to feature simple linear sequences based on accretion of numerous facts and skills. This was reflected in their hierarchies of educational goals and the resultant scope and sequences. Learning trajectories include such hierarchies, but are not as limited as these early constructs to sequences of skills or “logically” determined prerequisite pieces of knowledge. Learning trajectories are not lists of everything children need to learn, as are some scope and sequence documents; that is, they do not cover every single “fact” or skill. Most important, they describe children's *levels of thinking*, not just their ability to correctly respond to a mathematics question. They can not be summarized by stating the mathematical definition, concept, or rule (cf. Gagné, 1965/1970). So, for example, a single mathematical problem may be solved differently by students at different (separable) levels of thinking in a learning trajectory. Levels of thinking describe *how* students think about a topic and *why*—including the cognitive actions-on-objects that constitute that thinking.

Further, the ramifications for instruction from earlier theories were often based on transmission views, which hold that these facts and skills are presented and then passively absorbed. In comparison, learning trajectories have an interactionalist view of pedagogy.

To further elaborate these differences, consider the three components of learning trajectories.

Goal

The explication of the *goal* is important and distinguished from previous theories of learning that tended to either (a) apply the same theories and procedures to all domains, ignoring subject matter, or (b) accept the goal as arbitrary or "given" by existing standards or curriculum. In contrast, as stated, our learning trajectories base goals on both the expertise of mathematicians and research on students' thinking about and learning of mathematics. Thus, in contrast to earlier approaches, both domain-specific expertise and research on students' thinking and learning in that domain play a fundamental role in determining the mathematical goal—the first component of learning trajectories.

Developmental Progression

The *developmental progressions* of learning trajectories are much more than linear sequences based on accretion of numerous facts and skills. They are based on a progression of levels of thinking that (as does the goal) reflects the cognitive science view of knowledge as interconnected webs of concepts and skills. It is important to describe the nature of these levels and differentiate them from 'stages' (such as Piaget's).

A *level* is a period of time of qualitatively distinct cognition, as are stages; however, there are at least four important distinctions between levels and stages. First and most important, they do not apply across domains but only within a *specific* domain. Second, the period of time is generally far shorter, and can be months or days (especially given efficacious instruction), rather than a period of years for stages.

Third, although—like Piaget— Hierarchic Interactionalism postulates that subsequent levels are built upon earlier levels, there are two important differences. (a) The order of magnitude of difference in durations indicates a distinctly different cognitive "distance" between successive states. Informally, the "jump" between contiguous levels is far smaller than the jump between Piagetian stages (admittedly, measuring such distances, for this distinction and related theoretical notions such as Vygotsky's Zone of Proximal Development, remains an open problem). (b) The Hierarchic Interactionalism theory of levels makes no commitment (as does the Piagetian theory of stages) that the actions-on-objects of level $n + 2$ must be built from those of level $n + 1$. In Piagetian theory (Piaget & Szeminska, 1952), for example, stages are long periods of development characterized by cognition across a variety of domains qualitatively different from that of both the preceding and succeeding stages. Further, in Piagetian theory, stage $n + 2$ necessitated passing through stage $n + 1$ *because* stage n + 1 constructed the elements from which stage $n + 2$ would be built.

Levels in Hierarchic Interactionalism are not "stages." Rather, in many cases the cognitive material may be present at level n, requiring only a greater degree of construction or generalization to construct the pattern of thinking and reasoning defining level $n + 2$. We return to this issue when we discuss students "skipping" a level or "jumping ahead."

Fourth, although levels of thinking can be theoretically viewed as nonrecurrent (Karmiloff-Smith, 1984), students not only can, but frequently do, "return" to earlier levels of thinking in certain contexts. Therefore, Hierarchic Interactionalism postulates the construct of nongenetic levels (Clements, Battista, & Sarama, 2001), which has two special characteristics. (a) Progress through nongenetic levels is determined more by social influences, and specifically instruction, than by age-linked development. (At this point, this only implies that progression does not occur by necessity with time, but demands, in addition, instructional intervention, although certain levels may develop under maturational constraints.) (b) Although each higher nongenetic level builds on the knowledge that constitutes lower levels, its nongenetic nature does not preclude the instantiation and application of earlier levels in certain contexts (often, but not necessarily limited to, especially demanding or stressful contexts or tasks). There exists a probability of evoking each level depending on circumstances. Again, Figure 1 illustrates that earlier levels do not “disappear”; people do not “jump” from one type of thinking to a separate type, but rather build new ways of thinking upon the previous patterns of thinking. This process is codetermined by the probabilities of instantiation and conscious metacognitive control, which increases as one moves up through the levels, allowing more intentional application of various cognitive strategies. Therefore, students have increasing choice to override the default probabilities. The use of different levels is environmentally adaptive; thus, the adjective “higher” should be understood as a higher level of abstraction and generality, without the implication of either inherent superiority or the abandonment of lower levels as a consequence of the development of higher levels of thinking. Nevertheless, the levels would constitute veridical qualitative changes in thinking and behavior.

Each level in Hierarchic Interactionalism’s developmental progressions is characterized by specific mental objects (e.g., concepts) and actions (processes) (e.g., Clements, Wilson, & Sarama, 2004; Steffe & Cobb, 1988). Specification of these actions-on-objects allows a degree of precision not achieved by previous theoretical and empirical works. Further, the research methods that generate and test these mental models are distinct from methods used in earlier research. Strategies such as clinical interviews are used to examine students' knowledge of the content domain, including conceptions, strategies, intuitive ideas, and informal strategies used to solve problems. The researchers set up a situation or task to elicit pertinent concepts and processes. Once an initial model has been developed, it is tested and extended with teaching experiments, which present limited tasks and adult interaction to individual children with the goal of building models of children’s thinking *and learning*—that is, transitions between levels are the *crux* of these studies—which is another way learning trajectories differ from many earlier research programs. Once several iterations of such work indicate substantive stability, it is accepted as a working model. Thus, the developmental progressions’ levels of thinking and explication of transitions between levels describe in detail the following: (a) what students are *able* to do, (b) what they are *not yet able* to do *but should be able to learn,* and (c) *why*—that is, *how* they think at each level and how they learned these levels of thinking. This distinguishes learning trajectories’ developmental progressions from earlier efforts to develop educational sequences that, for example, often used reductionist techniques to decompose a targeted competence level only into subskills, based on an adult’s perspective.

Instructional Tasks

The *instructional tasks* of learning trajectories are much more than didactic presentations or external “models” of the mathematics to be learned. They often include these elements, but they are fine-tuned to develop the level of thinking that a particular student needs. Learning trajectories differ from instructional designs based on task (or “rational”) analysis because they are not a reduction of the skills of experts but are models of students’ learning that include the unique constructions of students and require continuous, detailed, and simultaneous analyses of goals, pedagogical tasks, teaching, and children’s thinking and learning. Such explication allows the researcher to test the theory by testing the curriculum (Clements & Battista, 2000).

This early interpretive work evaluates components using a mix of model (or hypothesis) testing and model generation strategies, including design experiments, as well as grounded theory, microgenetic, microethnographic, and phenomenological approaches. The goal is to understand the meaning that students give to the instructional objects and tasks. The focus is on the *consonance* between the actions of the students and the learning trajectory; that is, does the instruction task engender, in a student at level *n*, the cognitive actions-on-objects that are described as accounting for the type of thinking and problem-solving at level *n* + 1. If not, other tasks can be tried, based on a detailed account of the students' responses. (Discrepancies may also reveal a need to alter the developmental progression.) Questions such as the following direct the inquiry. Do students use the tools provided (e.g., manipulatives, tables or graphs, software tools or features) to perform the actions, either spontaneously or only with prompting? If prompting is necessary, which type is successful, and does this differ for different students? Are students' *observable* actions-on-objects enactments of the desired cognitive operations in the way the model posits, or merely trial-and-error manipulation? Are there indications of an internalization of these; that is, indications that students are building mental actions-on-objects and thus developing *n* + 1 level of thinking? In this way, the developer/researcher creates more refined models of the thinking of particular groups of students (the developmental progression) and describes what elements of the instructional tasks, including specific scaffolding strategies, are observed as having contributed to student learning. The objective is to connect the developmental progression with the instructional tasks.

The tightly interwoven and interacting connections among the three components of a learning trajectory—goal, developmental progression, and instructional tasks—encompassing levels from the microscopic and individual student's cognition to the cultural surround, are a major distinguishing features of the learning trajectory construct. There are not two different paths (see footnote 1)—a learning path and a teaching path—but one *learning trajectory* with three components borne of the same theoretical and empirical parents.

Scientific experiments that examine, evaluate, and extend these connections and components include conceptual analyses and theories. They are tested *and iteratively revised* in progressively expanding social situations, which results in greater contributions to both educational theory and practice (Clements, 2007).

Empirical Support

We initially reviewed research in early mathematics because we believed that learning trajectories should be the backbone of our *Building Blocks* research-and-development curriculum project (Clements & Sarama, 1998), which was developed based on a Curriculum Research Framework (Clements, 2007) that itself puts learning trajectories at the core. Our work in that and several subsequent projects convinced us of the usefulness of the construct, with effect sizes from .72 to 2.12 (Clements & Sarama, 2007b; Sarama & Clements, 2009b; Sarama, Clements, Starkey, Klein, & Wakeley, 2008). The effect size of the *Building Blocks Pre-K* curriculum was .72. Longitudinal analyses with follow-up interventions focused *only* on learning trajectories (i.e., the teachers in kindergarten and first grade used their regular curriculum, but studied the research-based learning trajectories) continues these gains (Sarama, Clements, Wolfe, & Spitler, 2011). We believe these results indicate that the use of learning trajectories in curriculum development and professional develop have consistent, substantial, benefits.

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RESEARCH IN MATHEMATICS EDUCATIONAL TECHNOLOGY: CURRENT TRENDS AND FUTRUE DEMANDS

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This systematic review of mathematics educational technology literature identified 1356 manuscripts addressing the integration of educational technology into mathematics instruction. The manuscripts were analyzed using three frameworks (Research Design, Teacher Knowledge, and TPACK) and three supplementary lenses (Data Sources, Outcomes, and NCTM Principles) to produce a database to support future research syntheses and meta-analyses. Preliminary analyses of student and teacher outcomes (e.g., knowledge, cognition, affect, and performance) suggest that the effects of incorporating graphing calculator and dynamic geometry technologies have been abundantly studied; however, the usefulness of the results was often limited by missing information regarding measures of validity, reliability, and/or trustworthiness.

Educational technology (i.e., digital technology, as opposed to other forms of educational tools such as overhead projectors or physical manipulatives) is promoted to mathematics teachers as a research-based strategy for improving student outcomes. Although research on mathematics educational technology appears at first glance to be ubiquitous, the usefulness of this research to practitioners and researchers is limited by lack of attention to research design and validity, reliability, and threats to validity (Rakes et al., 2011). Additionally, much of the research appears to be unorganized, with topics such as graphing calculators studied often, while other topics such as virtual manipulatives understudied (Ronau et al., 2010). The purposes of this systematic review were to (1) examine the evidence of technology impact on the teaching and learning mathematics in K-13, graduate, teacher development, and adult education using three frameworks (Comprehensive Framework of Teacher Knowledge [CFTK], Research Design, and Technology, Pedagogy, and Content Knowledge [TPACK]) and three supplementary lenses (Data Sources, Outcomes, and NCTM Principles) and (2) assess the utility of each framework for guiding the synthesis of mathematics educational technology research.

Theoretical Framework and Background

Three frameworks were applied to the analysis in the set of mathematics educational technology studies discovered by the systematic review: Research Design, CFTK, and TPACK. The research design framework was used to guide the investigation of the types of research approaches used in mathematics educational technology research. The complex nature of questions pertaining to educational technology effectiveness requires a variety of research designs such as (1) experimental or quasi-experimental studies, (2) large-scale studies, (3) studies with sufficient statistical information to be included in meta analysis and mixed-methodology studies, (4) studies with rich analysis of student content knowledge, and (5) studies that address the complexities of learners, classrooms, and schools (Bell, Schrum, & Thompson, 2008; Means, Wagner, Haertel, & Javitz, 2003). However, without explicit attention to the alignment of the research design to the questions of interest, the validity and reliability of the measures used, and the threats to validity within the chosen design, the reported outcomes will be less likely to have been founded on scientific principles or to be replicable (Shadish, Cook, & Campbell, 2002). The usefulness of such studies to practitioners and researchers will be, therefore, limited without robust attention to research design issues. The Research Design framework (Ronau et al., 2010) was compiled from several sources to address pertinent issues across a wide range of research types (e.g., Creswell, 2009; Shadish et al., 2002; Shavelson & Towne, 2002; Teddlie & Tashakorri, 2009). For a detailed description of the Research Design framework, see Rakes, Wagener, and Ronau (2010).

Within the past few years, two new teacher knowledge frameworks have been proposed that have the potential to support the research community in responding to questions on the impact of technology on learning. The Comprehensive Framework of Teacher Knowledge (CFTK; Figure 1) provides a three-dimensional model for addressing multiple aspects of teacher knowledge and their interactions (Rakes, Ronau, & Niess, 2010; Ronau, Rakes, Wagener, & Dougherty, 2009; Ronau, Wagener, & Rakes, 2009). This model transforms current understanding of teacher knowledge from a linear structure to a three dimensional model, as shown in Figure 1, by pairing six inter-related aspects into three orthogonal axes: 1) Field, comprised of Subject Matter and Pedagogy; 2) Mode, comprised of Orientation and Discernment; and 3) Context, comprised of Individual and Environment. For a detailed description of the CFTK aspects and dimensions, see Ronau and Rakes (in press).

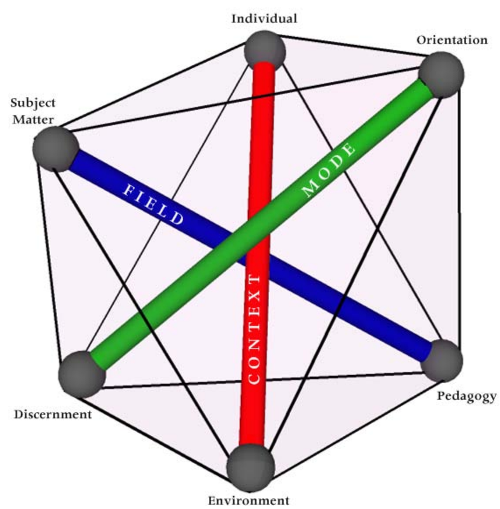

Figure 2. CFTK framework of teacher knowledge as a three-dimensional structure.

The Technology, Pedagogy, and Content Knowledge (TPACK) framework defines the knowledge needed by teachers to integrate technology into the pedagogy of particular subject matter (e.g., Mishra & Koehler, 2006; Niess, 2005). In its entirety, TPACK consists of a set of descriptive knowledge components embedded in an educational Context, Content Knowledge (CK), Pedagogical Knowledge (PK), and Technological Knowledge (TK), and a series of interactions, Pedagogical Content Knowledge (PCK), Technological Pedagogical Knowledge (TPK), Technological Content Knowledge (TCK), and TPACK. The initial TPACK framework has been extended to provide benchmarks of the development of this knowledge as shown in

Figure 2, including recognizing, accepting, adapting, exploring, and advancing (Niess, Lee, & Sadri, 2007).

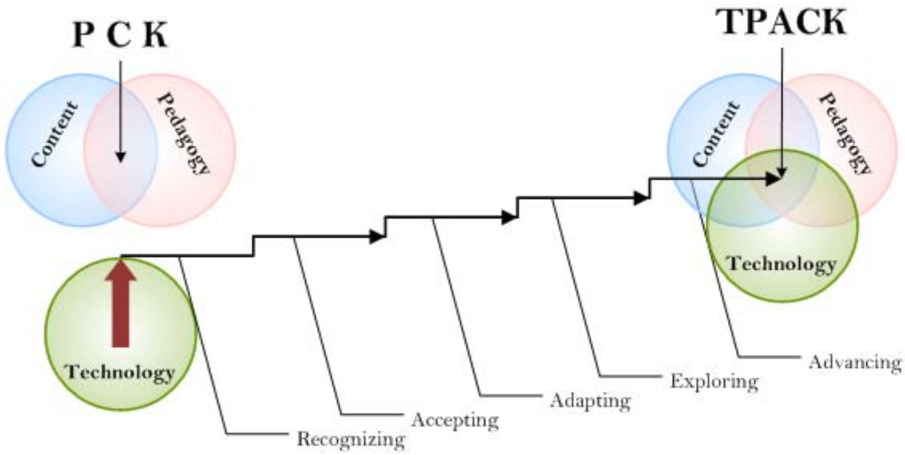

Figure 3. Model of teacher thinking and understanding as that knowledge develops toward the intersection identified as important by TPACK.

Based on feedback through peer debriefing (Rakes et al., 2010), three additional lenses (Data Sources, Outcomes, and NCTM Principles) were added to provide a more comprehensive snapshot of the research landscape, guiding practitioners to choose best practices and for guiding future research directions.

Using these frameworks and lenses, we began our investigation with two overall questions: (1) To what degree do the three frameworks, Research Design, CFTK, and TPACK, capture the scope of mathematics educational technology research? (2) What Data Sources, Outcomes, and NCTM Principles are addressed in mathematics educational technology research? To what degree, and how, implicit/explicit?

Method

A research synthesis (Cooper & Hedges, 2009) was conducted to address the two overall questions. To identify the most representative sample that was relevant to the questions of interest (i.e., construct validity), a wide array of databases were searched using terms to restrict the sample based on three inclusion criteria: (1) The study needed to examine a technology-based intervention; (2) The intervention needed to target the learning of a mathematics concept or procedure; (3) The manuscript needed to be available in the English language. The database platforms and individual databases included EBSCOWeb (ERIC, Academic Search Premier, PsychInfo, Primary Search Plus, Middle Search Plus, Educational Administration Abstracts), JSTOR (limited to the following disciplines: Education, Mathematics, Psychology, and Statistics), OVID, ProQuest (Research Library, Dissertations & Theses, Career & Technical Education), and H. W. Wilson Web (Education Full Text). From these databases, 1356 manuscripts (journal articles, book chapters, technical reports, conference proceedings, master's theses, and doctoral dissertations) were identified as being potentially relevant to the questions of interest.

The initial coding database was pilot tested with three articles and two coders to help refine the coding database. Refinements based on the results of this pilot test were examined with all six researchers coding the same, original three articles. This process was repeated through three more iterations of refinement and coding of 27 more articles (i.e., 30 articles were group coded).

After the analysis of 473 manuscripts, a number of coding issues emerged that required attention. Extensive team discussions led to a number of coding clarifications to improve team alignment, including how to: code studies with meta-analyses and/or systematic reviews, mixed methodology designs and single subject designs, action research, and survey research; code purposive and convenience sampling, subject dialog data, and modified and validated instruments; and record evaluative comments when deemed necessary. Finally, a number of coding form issues was addressed. The existing database of 473 studies was aligned with this new set of procedures and understandings, and the team developed a new process of coding the remaining studies that paired each of the six coders with all the other coders to provide a mixed set of double coding for the remainder of the studies. The new coding design created a completely counter-balanced design with all six coders that provided greater inter-rater reliability and content validity of the coding. With this plan, every manuscript was coded by two members of the coding team, and each member coded 59 studies with every other member. Any discrepancies between coder and re-coder were recorded and discussed by the pair and by the full team as needed.

Preliminary Results

At the time of the writing of this paper, 473 manuscripts have been coded and cross-validated (i.e., double coded and checked for accuracy). Twenty four of these manuscripts were screened out because they were not relevant to the questions of interest (i.e., did not address the learning of mathematics concepts and procedures, did not involve technology, or was not available in English), leaving 449 manuscripts in the sample. Initial results were examined by grouping the manuscripts into four categories (not mutually exclusive) of outcomes: Student Achievement and Learning; Student Orientation, Discernment, and Learning Behavior; Teacher Knowledge; and TPACK. The manuscripts and their 449 characteristics were analyzed through descriptive statistics as an initial method for interpreting the landscape of mathematics educational technology research.

The Role of Educational Technology in Student Achievement and Learning

Of the relevant manuscripts, 218 addressed educational technology in mathematics with a view of improving student achievement and learning. As shown in Table 1, over half (N=113) were dissertations and many of the remainder (N=69) were journal articles. Over half (N=118) were purely quantitative studies. The remaining studies were qualitative (N=36), mixed methods (N=43), non-research (N=9), meta analysis/systematic review (N=6), literature driven (N=5), or single subject (N=1). Performance assessments (e. g., tests, performance tasks, grades, GPA, etc.) were the most common sources of data used in these manuscripts, while journals (all types), focus groups, and non self-report surveys were the least used.

The manuscripts in this subsample most commonly addressed the Algebra NCTM Content Standard (N=159) and the Problem Solving Process Standard (N=89). Graphing calculators (N=43), tutorial software (N=39), and dynamic geometry software (N=25) were the more regularly studied technologies. In regards to information on measures of reliability and validity for the quantitative and mixed methods studies, approximately 41% of the manuscripts addressed reliability, 30% addressed validity, and 57% addressed threats to validity. For the qualitative and mixed methods studies, 60% attended to trustworthiness where approximately 62% of these studies attended to only one form of trustworthiness.

Type of Manuscript by Research Design	Non-research	Quali-tative	Quanti-tative	Mixed Methods	Single Subject	Meta-Analysis/ Sytematic Review	Liter-ature	Total
Book Chapter	0	1	1	0	0	0	1	3
Conference Paper	1	2	1	2	0	0	1	7
Disserta-tion	0	15	59	34	1	4	0	113
Journal	8	16	35	6	0	1	3	69
Master's Thesis	0	2	18	1	0	0	0	21
Report	0	0	4	0	0	1	0	5
Grand Total	9	36	118	43	1	6	5	218

Table 1. Student achievement and learning manuscripts by research design.

The Impact of Educational Technology on Student Orientation

Of the relevant manuscripts, 126 had examined the impact of educational technology on student orientation (i.e., the affective domain), discernment (i.e., the cognitive domain), and learning behaviors (e.g., student dialog and collaboration). The manuscripts consisted of 71 dissertations and 39 journal articles. Of these, 62 used purely quantitative analyses, 28 used purely qualitative methodologies, and 27 used mixed methodology. Self-report orientation survey data and performance assessment data were the two most common types of data sources; the top seven data sources are listed in Table 2. The Algebra NCTM Content Standard (N=32) and the Problem Solving Process Standard (N=23) were the most commonly addressed NCTM standards. Graphing calculators (N=24) were the most common type of calculator-based technology, followed by non-scientific calculators (N=16). The three most frequently used nonweb-based software were tutorial software (N=15), dynamic geometry (N=13), and algebra (N=12). Distance learning stood out among the web-based technologies (N=10) as these other technologies were often not addressed or addressed in only one or two manuscripts. Only 27% of the quantitative and mixed methods studies addressed issues surrounding validity, 61% addressed threats to validity, and 42% addressed reliability. Of the qualitative and mixed methods studies, 73% attended to trustworthiness, with 55% that attended to only one form of trustworthiness.

Data Sources	Number of Studies
Self-Report Orientation Survey Data	83
Performance Assessment Data	76
Observation Data	42
Interview Data	37
Content Analysis Data	28
Self-Report Polls and Census Survey Data	13
Researcher Journal Data	8

Table 2. Top seven data sources for student orientation.

The Interaction of Teacher Knowledge Aspects in Educational Technology Research

Teacher Knowledge was examined as an outcome in 72 manuscripts, of which 39 were journal articles and 26 were dissertations. The most common research design was qualitative (N=35), followed by non-research (N=15), mixed methods (N=11), and quantitative (N=8). The four most frequently used data sources were observation data (N=42), interview data (N=30), content analysis data (N=29), and self-report orientation survey data (N=24). Graphing calculators, dynamic geometry software, and spreadsheets were the most commonly studied technologies. The Algebra Content Standard (N=20) and the Problem Solving Process Standard (N=11) were the most commonly addressed NCTM standards. The distribution of CFTK aspects and interactions from this sample was compared to the distribution reported in Ronau and Rakes (in press) for teacher knowledge studies across multiple subject matter domains. The average number of aspects examined in mathematics educational technology ($\bar{x}$ = 1.53, SE = 0.207) appeared to be smaller than the average number of aspects examined across multiple subject matter domains ($\bar{x}$ = 2.16, *SE* = 0.119). This difference appeared to be statistically significant ($t_{df=71}$=3.74). This result may indicate that the research field in mathematics education technology may be considering less complex perspectives of teacher knowledge than other subject matter domains. Reliability was attended to in 32% of the quantitative and mixed methods studies, with validity and threats to validity attended to in 11% and 53%, respectively. Trustworthiness was addressed in 65% of the qualitative and mixed methods studies, where 60% addressed only one form of trustworthiness.

	Graphing Calculators	Dynamic Geometry	Spread-sheets	Graphing Software	Algebraic Software	Statistics Software	Tutorials
Number of Studies	22	19	14	7	5	5	5

Table 3. Top seven technologies for teacher knowledge.

The Use of TPACK to Guide Educational Technology Research

TPACK, as a guiding framework of the knowledge teachers need for integrating educational technology in mathematics, was employed either explicitly or implicitly, in 219 manuscripts, of which 102 were journal articles and 85 dissertations. Of these manuscripts, 72 used purely quantitative methodologies, 48 used purely qualitative methodologies, 27 used mixed methodologies, and 61 did not employ any type of research design (i.e., anecdotal support of hypotheses or descriptions of techniques). Performance assessment was the most commonly used data source (N=86). The NCTM Content Standards considered most often were Algebra (N=67), Geometry (N=45), and Number & Operations (N=37), while Problem Solving (N=35) was the most commonly considered Process Standard. Graphing calculators (N= 54) and dynamic geometry software (N=49) were the two most common types of technology used. Only 24% of the quantitative and mixed methods studies addressed validity issues, 81% addressed threats to validity, and 39% addressed reliability. Of the qualitative and mixed methods studies, 65% attended to trustworthiness, with 63% that attended to only one form of trustworthiness.

Summary

Several patterns were common among all four outcome groups. Dissertations and journal articles were the two most common types of manuscripts, with quantitative studies being the most prevalent research design among three of the four areas (all except teacher knowledge outcomes). Also among these three areas, performance assessment and self-report orientation survey were the two most often used data sources. The most common NCTM Content and Process Standards addressed in all four areas were Algebra and Problem Solving, respectively. Graphing calculators were consistently the most frequently used calculator-based technology, while dynamic geometry was either the most commonly used or second most commonly used non-web-based technology for all four areas. Web-based technologies were the least frequently used type of technology among all four areas.

Missing information regarding measures of validity, reliability, and/or trustworthiness was prevalent. Overall, approximately 40% of the quantitative and mixed methods studies addressed reliability, 27% addressed validity, and 64% addressed threats to validity. Trustworthiness was attended to in approximately 65% of all qualitative and mixed methods studies.

Discussion

The completion of the study will provide a searchable database of educational technology studies from 1968 to 2010, containing key information organized by three frameworks and four lenses. With this data, we will be able to better describe the landscape of educational technology research in mathematics, providing significant detail about the type, quality, content, and alignment of the studies. Doctoral students and advisors will benefit from an analysis of the 600+ dissertations in the sample, providing a guide to over- and under-studied dissertation topics (e.g., impact of graphing calculators in algebra). The research team will be able to identify gaps in the research base with respect to a number of study characteristics organized by the frameworks and lenses described above, as well as the depth and quality of areas well-studied. Detailed coding of research design features will allow for rigorous examination of the evidence currently available to the field in the form of meta-analysis and qualitative research syntheses. The systematic nature of this review will provide a foundation for future investigations and replication. Additionally, once the initial coding is completed, the task of updating the database with newly released manuscripts can easily be accomplished. Finally, this study will also include an evaluation of the utility of each of the three frameworks and four lenses used in the analysis to capture the perspective and the critical details of educational technology research.

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OPPORTUNITIES TO CONCEPTUALIZE LINEAR RELATIONSHIPS IN UNITED STATES MATHEMATICS TEXTBOOKS: BEYOND Y=MX+B

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This study examined opportunities provided for students to conceptualize linear relationships, as reflected in five United States mathematics textbooks. Texts represented a broad spectrum of types: commercial, so-called "back to basics", and NSF funded. Analysis of allocation, topic choice, presentation, context, and cognitive level was completed. Analysis results revealed that students are being asked to grapple with linear relationships at increasingly younger ages, limits in the models they are asked to use, limits on discussion of concepts and connections, lack a real world context for most problems, and lower levels of cognitive expectation. The results indicate a significant gap between learning goals from intended curricula, and the potentially implemented curricula contained in many current U.S. textbooks. Conclusions suggest ways in which the present curriculum may be transformed.

Linear relationships are important because they measure a basic way in which one quantity changes in relation to another. Frequently expressed as equations, graphs of lines and tables, almost all are functions. Linear functions are one of the foundational types of functions for students to understand in mathematics. As early as the 1920s, it was recognized that "without functional thinking there can be no real understanding or appreciation of mathematics" (Breslich, 1928, p.42). In the present day, calls for reform recognize that "the concept of function is an important unifying idea in mathematics (NCTM, 1989, p. 154).

Along the same line, recently published Common Core State Standards (2010) makes it clear that 8th graders should be able to: "Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values" (Common Core Standards, 2010, mathematics, grade 8, function, para. 4).

This standard is noticeably written in terms of concepts: "rate of change", "initial value", and "situation it models" and puts the emphasis on connections between descriptions, tables, and graphs. Reasoning and sense making, highlighted here, are cornerstones of mathematics (NCTM, 2009), and so must be integral to learning about linear relationships. Then how should linear relationships be presented in classrooms?

Despite a robust link between instructional attention to concepts and students' level of understanding, Hiebert and Grouws (2007) reported that "typical classrooms in the United States focus on low-level skills and rarely attend explicitly to the important mathematical relationships"(p. 2). Students tend to regurgitate "y = mx + b" and manipulate equations mechanically. Is it possible that textbooks have contributed to this problem by limiting expectations and opportunities for students? Have textbooks provided opportunities for students to learn about linear relationships in conceptual, connected terms?

Textbooks, as potentially implemented curricula, give messages of what students might know and be able to do. Although various factors influence student learning and teachers deviate their textbooks, Donavan, Bransford & Pelligrino (1999) reported that students' learning, in fact, is highly correlated with curricular treatment of topics. Many researchers generally agree that the curriculum, especially the curriculum embodied in textbooks, has a large influence on learning and teaching (Son & Senk, 2010; Valverde, Bianchi, Wolfe, Schmidt & Houang, 2002). Therefore, recognizing the influence of textbooks on student learning, we sought to examine the following question: "What opportunities for students to learn about linear relationships are presented in five commonly utilized United States textbooks?" Various types of mathematics textbooks, such as commercial, so-called "back to basics", and NSF funded were examined. Answering this question will help us find a better way to enhancing student learning about linear relationships through textbooks.

Theoretical Framework

A growing body of research has analyzed textbooks in order to understand their potential effect on students' mathematical learning. While some studies focus exclusively on content analysis (e.g., Fuson, Stigler, & Bartsch, 1988), other researchers examined problems presented in textbooks (e.g., Li, 2002). This study examined both the content and problems presented in textbooks. In analyzing the content, we looked at allocation, and topics. In analyzing problems, we looked at context, response type and cognitive level, which were identified by researchers (e.g., Li, 2002; Son & Senk, 2010). In particular, to analyze cognitive level, we built on Webb (2000)'s framework as shown in Table 1, which will be discussed in Methods.

Level	**Characteristics**
1: Recall	Recall of a fact, information, or procedure
2: Skill/Concept	Use information or conceptual knowledge, two or more steps etc.
3: Strategic thinking	Requires reasoning, developing plan or a sequences of steps, more than one possible answer
4. Extended thinking	Requires investigations, time to think and process multiple conditions of problems

Table 1. Depth of Knowledge Levels

Methods

In this study, we analyzed five textbooks, as addressed in Table 2. Similar to Slavin and Lake's framework (2007), three categories of textbooks were selected: (a) "Back to Basics," which originated in the 1970s and early 1980s, (b) "National Science Foundation (NSF) funded" texts, which originated in the 1990s in keeping with mathematics reform standards and (c) "Commercial Texts", which are typically reissued every five to seven years. In addition we selected, (d) a historic text, which was which originated in the 1960s. We felt these categories allowed us to examine representative types of texts currently available to students and the historic text allowed comparison. Each textbook was part of a series and represented the point in the series where linear graphs were first introduced.

Code Name	Origins	Date of Edition	Grade Level	Categorical Description (quotes: Slavin & Lake, 2007)
Historic	1960s	1965	9th	Commonly used Algebra text: 1960s & '70s
Back to Basics	Early 1980s	2001	8th	"A back-to-the-basics curriculum that emphasizes building students' confidence and skill in computations & word problems."
NSF Funded	1990s	2003	9th	"Textbooks developed under funding from the National Science Foundation (NSF), that emphasize constructivist philosophy… problem solving, manipulatives, and concept development …with relative de-emphasis on algorithms."
Commercial Text One	various	2005	8th	"Commercial textbook programs which include computational fluency but also are written to help students develop concepts."
Commercial Text Two	various	2005	8th	"Commercial textbook programs which include computational fluency but also are written to help students develop concepts."

Table 2. Textbooks Examined

- Historic = Dolciani, M. P. (1965). *Modern Algebra Book 1.* Boston, MA: Houghton Mifflen.
- Back To Basics = Saxon, J. H. (2001). *Pre-Algebra.* Norman, OK: Saxon Publishers, Inc.
- NSF Funded = Fendel, Resek, Alper, & Fraser. (2003). *Interactive Mathematics Program: Integrated High School Mathematics: Year 1.* Emeryville, CA: Key Curriculum Press.
- Commercial Text One = Larson, Boswell, Kanold, & Stiff. (2005). *Pre-Algebra.* Evanston, IL: Houghton Mifflen.
- Commercial Text Two = Malloy, Boswell, Willard, & Sloan. (2005). *Pre-Algebra.* New York, NY: Glencoe/McGraw Hill.

Analysis involved two broad foci: content analysis and problem analysis, as presented in Table 3. Content analysis included examination of (a) allocation and (b) topic presentation. Problem analysis included examination of (a) contextual features, (b) mathematical features, and (c) cognitive performance requirements.

Analysis Category	Description
Content	
● Allocation	(a) What are the number of problems allocated to the topic? (b) What percent of the textbook does the topic occupy? (c) How much time do students spend on the topic?
● Topics	(a) Sub-topics presented including slope and y-intercept (b) Breadth of presentation of further topics
Problem	
● Context	(a) Real world context (b) Exclusively numeric exercises
● Response Type	(a) Single response number, point, or vocabulary word (b) Equation only (c) Graph only (d) Table only (e) Extended response, such as “explain” or “describe” (f) Mixed response, i.e., some combination of the above
● Cognitive Level	(a) Recall (b) Skill/Concept (c) Strategic Thinking (d) Extended Thinking

Table 3. Analytical Foci of the Study

Summary of Results

Content Analysis Results

Results from Analysis of Allocation

We examined allocation, which involved the number of problems, as well as the proportion of time and text devoted to the topic. Table 4 shows both trends and some interesting constants from historic to current textbooks.

Text	Historic	Back to Basics	NSF Funded	Commercial Text One	Commercial Text Two
Grade Level	9^{th}	8^{th}	9^{th}	8^{th}	8^{th}
Total Number of problems	315	44	44	304	305
Percent of text devoted to linear relationships	7.3%	1.1%	8.7%	8.0%	9.6%
Approximate class time	Under 3 weeks	N/A	About 3 weeks.	About 3 weeks.	Just under 3 weeks.

Table 4. Results from Analysis of Allocation

The historic textbook and two commercial texts have much in common with respect to the total number of problems given, percent of text devoted to the topic and approximate class time. Each used a very similar number of problems; about 310, and similar timing: about 3 weeks. In fact, all but the “back to basics” text introduced linear relationships by spending about 3 weeks on them, devoting an entire unit (chapter) to the topic. The percent of material showed a slight general increase: from 7.3% in 1965 to 9.6% in 2005. This may indicate growing emphasis on linear relationships in modern and more technological times.

The introduction to graphing linear equations has been "moved down" to the 8^{th} grade level from historic to current textbooks. This seems to be part of a general trend. Indeed, Common Core Standards now expect that students will see the material in 8^{th} grade.

Compared to other textbooks, the back to basics text did not use chapters, but reviewed many topics each day. It was not possible to determine the exact amount of time focused on linear relationships, hence the "N/A" in the table above.

Results from Analysis of Topics

Table 5 summarizes the various separate objectives pertaining to linear relationships in the five textbooks. We further analyzed topics presented in textbooks, the breadth of presentation by answering the following questions: (1) Were a large number of sub-topics presented, stretching beyond such things as equation, slope, intercept, graphs and tables? (2) If so, this might affect the depth to which those more foundational conceptual topics could be covered.

Textbook	**Objectives beyond points, equation, slope, intercept, graphs & tables**
Historic Text	Equation of a line through two points, a line with a given slope through a given point, and a line parallel and/or perpendicular to a given line through a given point. Points where lines intersect. Graphing parabolas.
Back to Basics	Students do not go beyond plotting three points and sketching a line.
NSF Funded	Students do not go beyond the topics above, though they explore these in conceptual depth.
Commercial Text One	Equation of a line through two given points, a line though one point, though a given parallel and/or perpendicular line. The equation of a line of best fit. Graphing inequalities.
Commercial Text Two	Direct variation and inverse variation. Lines of best fit. Equation of a line through two points and equation of a line from a table. Graphing systems of equations, and inequalities.

Table 5. Breadth of Presentation: Topics

As shown in Table 5, large numbers of sub-topics (objectives) were included in a majority of the texts. The back to basics text, however asked only that students produce a table with three points, graph the points, and draw a line through the graph in their initial foray into linear relationships. The commercial and historic textbooks included numerous sub-topics.

The United States curriculum has often been criticized as "a mile wide and an inch deep" (Common Core State Standards, 2010). We examined how this tendency toward broadness remains from historic to current textbooks. Surprisingly, a very large number of objectives are presented in the commercial texts, just as in the historic text of about forty to fifty years ago. A plethora of topics was not attempted by the back to basics and NSF funded text, however.

Problem Analysis Results

Results from Analysis of Context

Table 6 shows the frequency of two categories of contextual features. Contextual features involve whether a setting is given for the problem, or whether it is devoid of real world context and merely in the form of an "exercise".

Text	With Context	Without Context
Historic	1%	99%
Back to Basics	0%	100%
NSF Funded	100%	0%
Commercial #1	15%	85%
Commercial #2	12%	88%

Table 6. Problem Context

There was a great deal of variation between texts when it comes to providing real world context. The back to basics and historic texts essentially did not include it. In more recently published commercial texts, a middle ground approach was utilized, with an average of 12.5% of problems including some context. In contrast, all problems in NSF funded reform texts were presented in regard to a real-life situation. In this case, the situation involved one over-arching unit theme problem. Thus, students could relate all problems in this text to real life applications. This indicates that students with the NSF funded text will learn linear relationship differently from students with other textbooks.

Results from Analysis of Response Type

Response types were categorized as single number/word, equation, graph, table, written word, or mixed. Single point coordinates were coded with single numbers. Written words (plural) in response might have required a phrase or a sentence to explain or justify, but sometimes called for a longer response in the form of sentences or paragraphs. A mixed response was some combination of responses, for example, “Make a table, then plot a graph”.

Text	Single Number	Equation	Graphic	Tabular Response	Written Words	Mixed Response
Historic	50%	17%	19%	0%	6%	7%
Back to Basics	0%	0%	39%	0%	0%	61%
NSF Funded	34%	21%	9%	2%	30%	5%
Commercial #1	37%	20%	14%	1%	24%	4%
Commercial #2	48%	11%	18%	0%	11%	11%

Table 7. Response Types

Regarding limits on these models, it was striking that single numbers were the most common type of response in most of the texts. Equations and graphs tended to be the second most common. Tables were rare as an exclusive response type, but were often included in a mixed response. The NSF funded text allowed for the greatest number of written word responses, which were almost entirely absent in historic and back to basics texts, and comparatively rare in at least one of the commercial texts. Written word responses and mixed responses, the response types most oriented toward multiple representations and connections between models, were generally limited, except as noted. Kilpatrick, Swafford and Findell (2001) stress that multiple representations and connections are important for fostering understanding. Analysis of response type indicates that students with the NSF funded textbook will be provided more opportunities to learn linear relationships with multiple representations and connections.

Results from Analysis of Cognitive Expectation

Cognitive Expectation refers to the type of problem solving required which consists of four levels: recall facts (level 1), concepts (level 2), strategic thinking and evidence (level 3), and extended thinking (level 4). Table 8 shows the percentage of each cognitive expectation in each textbook.

Text	Level 1: Recall	Level 2: Skill/Concept	Level 3: Strategic thinking	Level 4: Extended thinking
Historic	83%	11%	7%	0%
Back to Basics	100%	0%	0%	0%
NSF Funded	0%	0%	0%	100%
Commercial #1	61%	20%	16%	3%
Commercial #2	71%	11%	17%	1%

Table 8. Results on Cognitive Expectation

The large numbers of problems devoted to recall were linked to a procedural presentation of how to do these problems in textbook examples. Thus, the NSF funded textbook was able to attempt greater depth of presentation in comparison with other texts. It should be noted that all 44 problems in the NSF funded text were grouped into a single theme, and so they were coded as one, single, extended problem with 44 parts. However, none of problems in the historic and back to basics texts were at Level 4, and thematic connections between problems, such as those in the NSF funded text, were rare.

Regarding Level 3 "Strategic Thinking," there was an increase in problems of this level from the 1960s to the present. This shows that we may presently be moving toward problems of somewhat higher cognitive demand. Even though an average of only about 34% of problems were conceptual and strategic in the commercial texts, these problems take more time than mere recall problems. Thus, while it is impossible to say for sure, it is possible that students in commercial texts have the opportunity to spend well over half their time or more on conceptually oriented problems.

It is striking, however, that a large percentage of problems in the four non-NSF funded texts were categorized into Level 1, demanding no more than 'recall'. Also, it is comparatively rare for students to work on extended thinking problems if teachers use the historic, back to basics, or commercial textbooks.

Discussion and Implications

This study presents different student opportunities to understand linear relationships in a majority of texts examined from historic to current textbooks. There were numerous similarities between the historic textbook and the present day commercial texts. These included the number of problems, and the time devoted to the topic. The method of presentation was either largely or moderately procedural in these texts. The NSF funded text, on the other hand, showed the greatest difference from the historic text. All of its problems involved real world context and were geared toward extended thinking, in the form of a project, something the historic text did not include.

Regarding content, students using commercial texts who are just being introduced to linear equations are expected to master a broad array of objectives, such as "find the equation of a line between two points" and "find the equation of a line through a given point with a given slope". A deeper curriculum, providing more time for actual understanding of concepts such as slope and y-intercept, may be in order. This finding suggests that students may be limited with regard to models, timing, topics and expectations. In contrast, however, students may be receiving an overly broad a treatment of the topic, a treatment which lacks sufficient depth.

In particular, this lack of depth can be seen in problem presentation. The back to basics and historic texts showed a high degree of orientation toward mere procedural presentation. The NSF funded text, in contrast, was highly oriented toward conceptual problems, with commercial texts somewhere in between. Similarly, regarding real world problem context, historic and back to basics texts tended to provide little, commercial texts a moderate amount, and the NSF funded text provided real world context for all problems. Finally, continuing this theme, cognitive expectations tended to be largely procedural in the historic and back to basics text, to include some mix of procedural and conceptual orientation in commercial texts, and to be almost entirely conceptually oriented in the NSF funded text.

This study also revealed that there was a difference between a high degree of cognitive expectation and a high degree of symbol manipulation. A problem on finding the equation of a line between two points, for example, requires lengthy manipulation of equations, but can be done with little or no conceptual basis. Are we failing to give students enough problems with high cognitive expectation, all the while "feeling good" because we ask more of them in terms of what they do with symbols? It is important for future researchers to address such questions. None of the texts examined appeared to sufficiently address all five strands of mathematical proficiency articulated by Kilpatrick, Swafford and Findell (2001). The NSF sponsored text came the closest, addressing adaptive reasoning, strategic competence, conceptual understanding and productive disposition. However, it failed to address procedural fluency.

This study has implications for curriculum developers, school administrators and teachers. All three groups can help students gain better opportunities to understand linear relationships. This study stresses the important role of teachers. Teachers need to recognize the gap between the intended and potentially intended curriculum and modify their textbook. They can use supplementary materials for the NSF text, for example, providing practice with procedures. Supplementary materials for commercial texts might provide practice with extended thinking problems and additional real world context.

Curriculum developers also should recognize the gap between the intended deeper curriculum in such documents as the Common Core State Standards and the potentially implemented curricula in current textbooks. They should try to minimize this gap by providing context, attending to depth of conceptual topic presentation, calling for a balance of response types, including sufficient procedural practice, and calling for appropriately deep levels of cognitive expectation.

School administrators need to consider the types of textbooks they select. If they want to increase student understanding, reasoning and sense making, they need to choose textbooks which emphasize all five strands of mathematical proficiency, where possible, like those suggested above.

Future research may reveal optimal blends for the factors examined in this study. For example, while the study reports on the percent of problems with a real world context, it is up to future research to reveal whether or not it is helpful for all problems to be set in a context.

To sum up, the data revealed some limited learning opportunities. They indicate that though students are being asked to grapple with linear relationships at increasingly younger ages, that they are limited in the models they are asked to use for linear relationships, that discussion of concepts and connections is often limited, that the majority of problems lack a real world context, and that a majority of problems address lower levels of cognitive expectation. Often, linear relationships are presented in textbooks in a way that, while very broad, in keeping with detailed and lengthy individual state standards, does not aim for depth of understanding. The results therefore revealed a significant gap between learning goals as set forth in intended curricula, and the potentially implemented curricula contained in many current U. S. textbooks.

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INSTRUCTOR AND STUDENT PERCEPTIONS OF MATHEMATICS FOR TEACHERS COURSES

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This cross case analysis synthesizes results from two qualitative studies on mathematics content courses for prospective elementary teachers: one from the U.S. interviewed instructors, one from Canada interviewed students. Results were examined for common themes. Salient commonalities were found. Two will be discussed here: the role of affect in student learning and the role of connections to the elementary classroom.

Pervasive concerns about the adequacy of the mathematical preparation of elementary teachers (Ball, Hill, & Bass, 2005; Rowland, Huckstep, & Thwaites, 2005) have prompted many institutions of higher education to require specialized mathematics content courses for prospective teachers. These courses, referred to here as *Math for Teachers* (MFT) courses, aim to provide deep understandings of elementary mathematics concepts in order to develop prospective teachers' confidence and flexibility in teaching mathematics (Kilpatrick, Swafford, & Findell, 2001;Williams, 2008). MFT courses are most often taught in mathematics departments by mathematics faculty.

The research described in this paper presents results from a cross-case analysis of two existing studies on MFT courses. The two studies are briefly described here.

The *instructor-focused* study examined ten instructors' perspectives on a MFT course at several institutions in southwestern Canada (Oesterle & Liljedahl, 2009). The purpose of the study was to provide insights into the instructors' approaches in the course and how their beliefs impacted pedagogical decisions. Data from the semi-structured, individual interviews (1 hour in duration) were analyzed for emergent themes using constant comparative analysis. These themes include: instructor identity, tensions, resources, student knowledge, student affect, orientation to mathematics, orientation to teaching, and classroom environment.

The *student-focused* study explored the perspectives of 12 elementary education majors (i.e., prospective elementary teachers) who had completed MFT courses at a university in the southeastern United States (Hart & Swars, 2009).The study was inspired by concerns over the poor success rates of elementary education majors enrolled in these courses. Data collection included semi-structured, individual interviews (approximately 1 hour in duration), and constant comparative analysis was applied to the data, revealing three major themes: (1) domains of mismatch (2) affective reactions, and (3) classroom practices. The domains of mismatch theme had three sub-themes: mismatch with elementary classroom, mismatch in programmatic emphasis, and mismatch in mathematics content.

Overview of Relevant Research

These two extant studies were framed by prior research on mathematics knowledge-for-teaching (e.g. Ball & Bass, 2003) and instructor beliefs about mathematics (Ernest, 1989). Of relevance to both studies, and of particular interest in this paper, is the literature pertaining to: the role of affect, beliefs and efficacy in prospective elementary teacher learning (e.g., Di Martino

& Sabena, 2007; Thompson, 1992; Swars, Hart, Smith & Smith, 2007); post-secondary mathematics content courses for elementary teacher (e.g., Lubinski & Otto, 2004; Philips, 2007); and perspectives on and characteristics of effective university mathematics instruction (e.g., Schulze & Tomal, 2006; Weinstein, 2004). While there is an abundance of literature on the significant role of beliefs and affect in the pedagogical practices of teachers, far fewer studies are available on mathematics content courses for elementary teachers and those found were primarily limited to outcomes of reform-based approaches in these courses. Scant studies have looked at characteristics of effective post-secondary mathematics instruction in traditional (i.e., non-MFT) mathematics courses.

Theoretical Perspective and Research Question

This cross-case analysis, as well as the two original studies, is grounded in phenomenological interpretation (Burch, 1990). Students and instructors individually reported their perspectives on the MFT courses. Through the interviews, participants reflected on and retrospectively identified the significant or memorable events from their MFT experiences. Through this process, they recovered and verbally reenacted the meaningful components of their lived experiences.

For this cross-case study, we were interested in determining what intersections might exist between the perspectives of instructors of a MFT course and the perspectives of students in the MFT courses. More specifically, we asked this research question: *What common themes exist in instructor perspectives and student perspectives on mathematics content courses for prospective elementary teachers?*

Methods

Participants and Setting

The student-focused study involved twelve students (11 females and 1 male) from one urban university in the southeastern U.S. The students had completed three or four MFT courses. They were randomly selected from 4 cohorts of students in the elementary teacher preparation program with a combined size of 99 students, thus representing approximately 12% of the total population. Collectively they had taken 42 sections of MFT courses. At the time of the original study, all of the students were in the last semester of the program and completing student teaching.

Although the instructor-focused study gathered data from ten instructors, for the purpose of this paper examples will be drawn from the transcripts of only two: Harriet and Bob (pseudonyms). The divergent perspectives they offer on teaching the MFT course provide a sufficient basis for illustrating the common themes identified in this analysis. Harriet and Bob are both experienced instructors, having taught in mathematics departments for 22 and 13 years respectively. Harriet is relatively new to teaching the MFT course but had taught the course six times over three years, while Bob taught it nine times over nine years. Both have Master's degrees in mathematics but neither took mathematics education courses nor had formal teacher training. Harriet was initiated into teaching the MFT course by a colleague with a Master's degree in mathematics education who taught MFT courses for many years. Bob's first forays into teaching the course were guided by his institution's curriculum, the textbook, and informal discussions with colleagues.

Data Collection and Analysis

Data collection for the two extant studies was described in the background section of this paper. To conduct the cross-case analysis for this present study, we created a matrix with the results from the two studies, specifically examining the data for convergence. This analysis revealed two commonalities across the themes, including *connections to the elementary classroom* and *student affect*, as shown in Figure 1.

Student-focused Study	Instructor-focused Study
• *affective reactions*	• *student affect*
• classroom practices	• instructor identity
• domains of mismatch	• resources
a. elementary classroom	• *orientation to teaching*
b. programmatic emphasis	• tensions
c. mathematics content	• orientation to mathematics
	• student knowledge
	• classroom environment

Figure 1. Themes and commonalities of the two extant studies.

Cross-case Results

The Students' Perspectives on Connections to the Elementary Classroom

After experiencing other courses in their teacher preparation program, the students were acutely aware of disconnections between their experiences in the mathematics content courses and other experiences in the program. The students frequently described an inability to position the mathematics content coursework within their growth as educators, which led to perceptions of lack of usefulness or relevance of the courses as evidenced by this statement: '*I mean a lot of us were always questioning, you know, why we have to take these math courses…It's not even necessary.*' Similarly, another student stated, '*The reason why we were taking those courses was never brought to our attention. We had no clue why we were taking those classes, no clue…It seemed very unnecessary.*' Another student said:

> *It [mathematics courses] had no connection to elementary schools. Anybody could take those courses. I don't think we ever talked about kids. I seriously don't think we ever talked about teaching or students or anything like that. I just don't remember ever that connection.*

Another student explained the lack of connection with elementary classrooms as:

> *In our [elementary] classrooms right now, you know, they're not graphing how many bagels and coffee people are going to eat and drink tomorrow. It was just not very logical for, you know, a kindergartner.'*

The following statements further support this sentiment:

> *They're [mathematics instructors] blind to what we are actually doing with our lives'; '[Elementary] Students were never even brought up… I mean, students or when you get your own classroom were never brought up'*; and '*We're thinking we're learning something about how to be teachers. But, in reality we're learning how to get through their math courses.*

The participants did indicate positive experiences in some of the courses. In particular these related to modeling pedagogical methods that might be used in an elementary classroom or using materials that elementary students use. For example, one student commented:

> [The Number & Operation course instructor] *was big on you know . . . we would do that in class, it wasn't like, take this home and do this but we actually* [did activities in class] *and we did a lot of group work and that was really good too.*

One student provided a specific example of making strong connections:

> *We had to do a project in geometry where we had to go to like a specific county and get a book and find a geometry lesson, but that was just a project and we presented it to the class . . . so just different things like that, just to make it real to us.*

One student noted that the positive experiences and factors that made it positive occurred in her methods course, not in the content course.

> *No* [not in the content], *this happened in my methods course, mainly because she* [the methods instructor] *took out the manipulatives, showed us what they were going to be doing, she basically took curriculum from those grade levels and put it in front of our faces.*

Ideas for changing the courses were proffered, including*: 'Get some teachers who were actually qualified in elementary [teaching]… They actually know what children are going through and that would help;' 'They [mathematics instructors] could talk to elementary teachers;'* and '*Maybe have us students say, hey, this is what is going on in my [elementary] classroom.'*

The Instructors' Perspectives on Connections to the Elementary Classroom

Harriet's descriptions of her goals and strategies for teaching the MFT course are permeated with comments related to mathematics-for-teaching knowledge (Ball & Bass, 2003) and how her students' learning relates to their future as teachers. When asked if there is anything that she teaches MFT students about fractions that she would not teach other students, she states:

> *The fact that there are different models, there are different ways of picturing what's going on, and that they are appropriate for [...] what may work well for some situation, or for some [elementary school] student, may not work for some other one*.

She also emphasizes connections between mathematical ideas both within and across grade levels. She explains:

> *At all times I connect it [the course content], as far as I can, to what goes on at different levels. What you might do with a grade 1 class, how that connects to what they're going to see in, you know grade 4 or 5 or something like that, how that connects to what they might do in high school and how that connects to what I'm doing in Calculus. Because they've got to see how it's connected, and how we build bigger and bigger [...] understandings of sets of numbers, or calculations.*

Harriet does not just pay lip-service to these ideas. She describes assignments that allow her students to build their mathematics-for-teaching knowledge, such as analysis of pupil errors and discussion of alternative solutions.

In contrast, Bob makes very little reference to mathematics-for-teaching knowledge. His emphasis is instead on developing a strong understanding of fundamental mathematics and communication skills. Varieties of algorithms and models form part of his course content, but he does not specifically address how they can be applied differently at various grade levels.

Bob needed to be pressed by the interviewer to consider what aspects of the course content might be particularly relevant to prospective teachers as opposed to general learners of mathematics. Initially his comments revolve around his teaching methods, such as the use of group work and manipulatives, but he makes no reference to any special mathematics knowledge for teaching. Eventually he describes challenging his students to think about the kinds of questions that they will encounter as teachers:

> *. . . what kinds of questions will you encounter? And why is it important that you to be able to communicate your ideas effectively, […], why should you understand this material to the most, [...], fundamental and basic level, and understand all of the structure?*

He adds:

> *when you get some of these obtuse questions, that are seemingly [...] obtuse, you have to be able to appreciate it and be able to differentiate whether that's something that can lead you into a teachable moment*

His response appears to be a justification for his goals of developing strong mathematics content knowledge and communication skills. For Bob, mastery of the subject content along with general pedagogical skills, seem to be sufficient for the teaching of mathematics—a traditional and prevalent point of view (Hill et al., 2007).

The Students' Perspectives on Student Affect

A second theme across the interviews was students' affective reactions to the coursework experiences. Many statements described negative emotions, for example, they used words such as '*emotional wreck', 'so stressed', 'very belittling', 'discouraged', 'terrified', 'struggling',* and *'frustrating'.* A student asserted, '*I felt like I was just hanging on. Just trying to dig myself out of a hole, and I kept falling down.'*

The students also portrayed the courses as having deleterious influences on their mathematics teaching efficacy beliefs and self-efficacy beliefs, which were often linked with the classroom practices of the instructors. Most often, descriptions of ineffective pedagogy were related to traditional approaches to instruction. The students mentioned a preponderance of 'lecture,' 'note-taking', and 'power point presentations,' and asserted the 'classes were not hands-on.' In describing how the courses impacted teaching efficacy beliefs, a student stated, '*I felt less confident [about teaching mathematics] when I walked out of those classes because it's just so much and it just seemed so unnecessary… It was just very discouraging.'* In response to a question on how the courses prepared her to teach elementary mathematics, a student stated the courses made her, '*Feel less prepared. Feeling more scared, definitely.'* One student attributed this negative impact on her teaching efficacy to the attitude of the instructor of the course, '*The attitude was if you don't get this [math content], you won't be able to teach it, basically.'*

Students also commented on how the experiences in the courses influenced their self-efficacy beliefs in mathematics, as represented by this student's statement:

> *[I felt] terrified, struggling, especially in geometry. It was just, it was very frustrating because I didn't get it. I didn't understand why we're doing what we were doing, how we were coming out with the answer, and especially if I didn't get the answer right.*

Further, another student stated:

> *Like geometry… I came out of there in tears. I felt very disappointed. I felt stupid. I felt alone. And, I know that I am an intelligent person, or I have the potential to learn something. If I don't know it, I'm willing to give up my time and my efforts. But, I felt like my efforts didn't matter.*

Similarly, another student said:

> *It (mathematics courses) made me feel so low in math. Even though I knew those math courses, I would never be teaching that stuff… It totally lowered my self-esteem in mathematics*.

The Instructors' Perspectives on Student Affect

Both Bob and Harriet describe their students as suffering from mathematics anxiety and lacking confidence in their ability to do mathematics. However, there are considerable differences in their perspectives and pedagogical approaches to these negative affective states.

Harriet observes that her students: '*are very anxious around problem solving. They are just* *terrified, most of them, of a problem they haven't seen before*.' Her efforts to address this seem to be centered on changing their ideas of what the enterprise of mathematics is all about. She tries to convince them that '*we're supposed to have fun with this*' and tells her students that '*you may never have seen it; you might not get all the way through it. But what I'm looking for is how far did you get, and how well can you explain what it is that you got*', shifting the focus away from getting the right answer toward less threatening goals.

By the end of the course she hopes her students have grown in confidence and also '*they have more of a sense of play [...] I think they're more flexible. They think they're more flexible. They're not as scared if [...] that someone will ask them a question that they can't answer.*'

Bob describes his students as believing that mathematics is arbitrary and incomprehensible: '*So many things seem magical to them'*. He affirms that '*it's not your standard sort of math group, it's one that has encountered some challenges along the way, and it hasn't always left them with a positive impression of mathematics.'* In his view, their confusion and anxiety is closely linked to their skills:

> *In many cases, some of the very elementary arithmetic operations are in fact, confused in their minds and so when they hit upon things, in particular when you hit rational numbers, as an example, that's one place where students have a great deal of anxiety and they would demonstrate poor understanding of ideas.*

More than once he describes the MFT course as a second start for these students. He attempts to reshape their beliefs and attitudes by providing them with opportunities to see the logical structure of mathematics and deepen their understanding. For Bob, the course '*focuses on a very sound fundamental ability to appreciate it [mathematics], in a theoretical way, why things work, as opposed to technical aspects of how do you do mathematics.*' However, although he believes that improved skills will lead to increased appreciation and confidence, he confesses that the realities of the course conspire against this occurring. Early in the interview he expresses a wish that his MFT students develop a love of math, but when asked about whether this goal is accomplished, he admits: '*in terms of the other goal, for love of math? Unfortunately, the course is so packed, that in some ways, I think they do get a little bit beaten by the end, and they're just tired*.' This statement illustrates Bob's realization that the volume of content covered in a limited time is at odds with his affective goals.

Discussion

Although the two groups of participants in these studies were in different settings, they provide two distinct viewpoints on a similar experience: MFT courses. When juxtaposed the data reveals salient commonalities. The findings provide important insights into issues and concerns around creating experiences in MFT courses that best support elementary prospective teachers' learning of mathematics; they also enrich our understandings of the realities of MFT classrooms, revealing both the affordances and the constraints.

The student voices emphatically call for the need for connecting the mathematics to the elementary classroom. Without this connection, the students were not able to find relevance in their learning. This need is recognised in the literature (Philip, 2007). Ball and Bass (2003) also strongly advocate for this link:

> *Practice in solving the mathematical problems they will face in their work would help teachers learn to use mathematics in the ways they will do so in practice, and is likely also to strengthen and deepen their understanding of the ideas.* (p. 13)

The instructor-focused study reveals how differently instructors may perceive the need for incorporating these connections. Harriet is very aware that these links help to motivate her students, helping them to see why a deeper understanding of mathematics is required of them in this course as compared to their previous mathematics courses. For Bob, making these connections is not an explicit part of his course. One reason for this may be that as a mathematician his lack of experience in elementary classrooms limits his ability to do so. However, Harriet also lacks such experience. Another possibility is that Bob takes such connections for granted. His inability to identify content in his course that would be particularly relevant to future teachers of mathematics as opposed to general mathematics students reflects a lack of awareness of specialized mathematics knowledge-for-teaching. For Bob, subject content knowledge and pedagogical knowledge are distinct. He sees his role as supporting the development of the former.

With regard to the theme of affect, the student-focused study reports an alarming number of negative comments, indicating increases in students' anxiety and decreases in self-efficacy. Both instructors were acutely aware of the impact of affect and described their students as coming into the course with high mathematics anxiety and lack of confidence. However, their perceptions about the cause of the anxiety and strategies for addressing it were quite different. For Bob, the source is students' lack of fundamental skills. As a result, his solution is to help them see the logical structures of mathematics and develop these skills, though he acknowledges that the sheer volume of the material he must cover, in fact, adds to his students' stress. For Harriet the source is negative past experiences and a perception of mathematics as rigid. Her efforts focus on moving students away from the 'one right answer' view of mathematics, helping them develop more flexibility in approaching mathematical problems and to just have *fun*.

From the students we hear that traditional instructional methods of lecture, power point presentations, and drill and practice tended to elevate anxiety and decrease efficacy, while reform approaches such as small group work, hands-on learning, and opportunities to share and discuss were less stressful and increased efficacy. They also shared that the instructor having a caring manner, an approachable demeanour, and a perceived willingness to help supported their learning. This echoes Schulte & Tomal (2006) cited above. Regardless of their perceptions of the source of their students' anxieties, knowledge of this research could help inform instructor choices with respect to how to address concerns around student affect.

The voices of the students in the student-focused study lend support to concerns that mathematicians in mathematics departments may be unprepared to take on the task of preparing elementary teachers. The lack of connections of content with the elementary classroom and traditional teaching approaches seem to contribute to frustration and anxiety as well as decreased self-efficacy. However, the instructor-focused study shows that though lack of explicit connections to elementary learning may occur, this need not be so. The differences between Harriet and Bob in this regard may have been the result of the mentorship Harriet received, suggesting a potential means for supporting the mathematicians who teach these courses.

Another side of this issue is that mathematicians, at their best, have much to offer future teachers, even at the elementary school level (Hodgson, 2001; Williams, 2008). Jonker (in review) describes mathematicians in mathematics departments as 'stewards of their discipline,' 'passionate about mathematics', and 'eager to share their excitement with students and concerned about the place of mathematics in the world.' The challenge is to create opportunities for conversations between mathematics educators and mathematicians so that students in MFT courses are better prepared to teach mathematics to elementary children.

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REFRAMING FAILURE: AN ANALYSIS OF HIGH SCHOOL MATHEMATICS TEACHERS' LEARNING

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This paper examines how teachers engaged in equity-oriented reforms learn through interactions in teacher groups. Analysis of teachers' framing of a freshman mathematics student failure problem showed that over time teachers' frames shifted from invariable framings based on student characteristics and systemic issues to actionable framings based on classroom systems contributing to student failure, thereby promoting teachers' concentration on courses of action linked to instruction. By joining the frame analysis and community of practice literatures, this study contributes an empirical example of development within teacher community alongside analytic tools for documenting teachers' learning within these groups.

Research suggests that teachers' participation in a strong teacher community has the greatest potential for yielding the kinds of teacher learning that produces equitable student outcomes, though what that learning is or how it might be taking place is largely unaccounted for in the literature (Gutiérrez, 1996; Horn, 2005; Little, 2003; McLaughlin & Talbert, 2001). This "black box" of teacher learning (Little, 2003) has resulted in schools and districts moving forward with well intentioned yet underconceptualized reforms involving *professional learning communities,* based on an inferred causal connection between teachers' participation in these communities and improved student achievement. Many teachers have nevertheless been required to spend professional development time participating in what Grossman and colleagues (2001) call *pseudocommunities*, characterized by members "playing community" and behaving "*as if* we all agree" (p. 955), which surely sidesteps the important work that needs to be done inside of the professional learning community to achieve equitable student outcomes. This paper addresses this phenomenon by examination of the professional community as a learning resource for teachers. The research question driving this investigation asked: *How do teachers engaged in equity-oriented reforms learn through interactions in teacher groups?*

Theoretical Framework

Teachers' Learning in a Community of Practice

Teachers' participation in their professional communities is a social endeavor. This activity catalyzes a dual process of participation and reification, which is the fundamental process through which learning happens (Wenger, 1998). This learning-as-a-social-phenomenon stance supports a more general conception of teacher community, meaning that these groups do not necessarily have a certain level of functioning, improvement-oriented stance, or meet some other criterion; rather, they are the places where learning unfolds (Coburn & Stein, 2006). By adopting a *community of practice* perspective – which Wenger (1998) characterizes as communities where members are mutually engaged in an activity, held together by a joint enterprise, and have a shared repertoire of customs for praxis – *learning* is defined as a change in participation within that community. This definition of learning recognizes the co-construction and distribution of knowledge across teachers and takes the wider social context into consideration. This framing is critical for this study because it allowed for an equity-oriented description of the teacher group in my study with the understanding that such a description is neither unitary nor consistent, which helped me see teacher groups for what they are: key sites for negotiation of meaning related to their joint enterprise (Coburn & Stein, 2006).

Frame Analysis as a Means for Capturing Learning

This study aimed to understand teachers' learning through interactions in a community of practice context, and so conceptual tools that capture learning as changes in participation within the group are needed. Theoretical and empirical work on frame analysis proved useful for making sense of these interactive learning processes as they unfold (Benford & Snow, 2000; Goffman, 1974; Snow & Benford, 1988). Snow and Benford (1998) use the verb *framing* to conceptualize the signifying work of spreading ideas, meaning making, and mobilizing others into action. I borrowed conceptual tools from this work and coupled this literature with the communities of practice literature because of their common interest in understanding the processes surrounding participants' interaction, with focus on "how people use interpretive frames strategically to shape others' meaning-making processes in an effort to mobilize them to take action" (Coburn, 2006, p. 347). By analyzing teachers' framing processes, I gained analytic purchase on making sense of teachers' engagement in these negotiation of meaning processes, such as how participation in teacher groups and reification of equity-oriented reforms shaped their ideas and guided the community's action. It stands to reason that teachers' collective engagement in framing processes is likely to generate evidence of and describe changes in teachers' participation in a community of practice (Wenger, 1998), which I interpret as evidence of learning. Thus, examination of the ways in which teachers engage in framing processes through interactions in teacher groups stands to result in more manageable units of interactions for the analysis of their learning.

Methods

Context, Settings, and Participants

This research takes place in the context of *Adaptive Professional Development,*[1] a larger design-experiment project situated in part at Clark High School (all names are pseudonyms), a diverse, large, urban comprehensive high school in a large northwestern school district in the US. Our research team worked with the Clark mathematics teachers using a mutual appropriation approach – that is, we collaborated with the teachers to create activities that fit theoretical principles about equitable mathematics teaching while serving the teachers' goals (Cole, 2006). Our precepts included pedagogical principles about equitable mathematics teaching, such as the use of pedagogical strategies to engage learners in important mathematical ideas (Boaler, 2002; Horn, 2006; Moses, 2001). In addition, we used learning principles for teachers, such as prioritizing providing teachers with collaborative time in the school day to make sense of new practices in their classrooms (Horn, 2005, 2007; Horn & Little, 2010).

During the 2004-2005 school year, I followed the interactions of the mathematics department at Clark in my role as a researcher. I observed classrooms, attended department meetings, and provided classroom support to teachers. Susan, a veteran teacher, confided that she struggled with issues related to students, teaching, and mathematics. She asked for my help and so I provided her with additional classroom-based support several days per week, such as co-planning instruction, revisiting content, modeling teaching, making sense of student work, and interpreting student interactions. However, even with this support Susan still faced a crisis: over 75% of her freshmen students were failing her first-year mathematics course. This crisis caused the other teachers of first-year mathematics to examine their pass rates, and the results were stunning: more than 50% of students taking the first year (9th grade) mathematics course at Clark were failing. Teachers were in a panic over these data and asked our research team to help them make changes to their existing curriculum and pedagogy with the aim of improving all the success of all students.

Realizing the ambitious nature of the Clark teachers' plans for implementing starkly different pedagogical and curricular equity-oriented reforms, our team designed an intervention for the 2005-2006 school year to support their reforms. We created the "Freshman Team" intervention by providing the four teachers of first year mathematics with an extra planning period (in addition to their personal planning period) so that they would have dedicated time each day during the school day to collaborate around issues of teaching and curriculum. We also helped Clark find a new teacher trained in equity-geared teaching practices who could take on the "missing" four first year classes, in addition to being a part of the collaborative team and having her own personal planning period. The Team was composed of five teachers: Susan (10+ years experience), Zack (3+ years experience), Rose (30+ years experience), Julie (5+ years experience), and Linda (new teacher). The Team met during every sixth period meeting, and in a typical week they had three 50-minute meeting and one 110-minute meeting.

Research Strategies

I crafted a case study around the Freshman Team at Clark because this method focused the investigation and analysis on the complexities and particulars of teachers' learning around about struggling students in context of equity-oriented reforms (Merriam, 1998). Sustained attention to one group and context fostered in-depth exploration and analysis of the "richly brewed particulars" (Dyson, 2005, p. 2) of teachers' learning. These choices ultimately allowed me to use the case of Clark to theorize about teacher learning inside teacher community more generally and respond to a need for case studies of this nature (NAE, 2008).

My study targeted high school mathematics teachers because they teach a high status, high stakes content area that consistently plays a gatekeeper role for students (Moses, 2001; NRC, 1989; Schoenfeld, 2002). Making matters worse is the fact that a disproportionate number of poor and minority students compose this group, meaning that working-class students and students of color are marginalized in their mathematics classes more than their peers (Moses, 2001). These harsh realities have renewed interest and urgency in creating *equitable mathematics classrooms*, which I characterized as spaces where we cannot distinguish high performers from low performers based on race and social class (Schoenfeld, 2007). Following Martin's (2006) lead, "race is viewed here as socially, politically, and relationally constructed so that issues of marginalization, power, dominance, and devalued social status assume prominence" (p. 198). Moreover, these classrooms are spaces where "mathematical identities, excellence, and literacies of marginalized students" (Gutierrez, 2008, p. 357) are supported.

I focused my study on teachers engaged in equity-oriented reforms for two reasons. First, in keeping with prior reasoning, a specific portrait of teacher learning about equity-oriented reforms directly speaks to single-system attempts to change disparities in student achievement by educators. To achieve this goal, I selected a group of teachers who not only chose to engage in equity-oriented reforms but who also had some success with their efforts to improve equitable outcomes. This particular group is made more exceptional as case of teacher community because it was designed for optimizing teachers' learning (e.g., attending to issues of equity through conversations about curriculum and pedagogy became a part of teachers' daily work) and had considerable external support by our research team.

Second, there is presumably a greater impetus for teachers engaged in equity-oriented reforms to question their assumptions and practices, thereby rendering their learning more visible. I made this assumption because a major goal of equity-oriented reforms involves providing *all* students with opportunities for making sense of essential mathematics ideas. It follows that the conditions surrounding teachers' enactments of reform, such as instruction and classroom culture, must also align with this goal in order to yield equitable outcomes. As such, my focus on a group of well-resourced, highly motivated teachers collectively engaged in equity-oriented reforms was a strategic choice for increasing observable instances of teachers' collective sensemaking about these reforms.

I collected a variety of qualitative data about the teachers' work, including audio records and fieldnotes of Team meetings, artifacts from Team meetings and activities, and teacher interviews. Primary data were transcriptions of audio records from Team meetings. The data corpus was designed to capture teachers' framing of the struggling student problem over time in context of their equity-oriented reforms. The data set included 35 records of Freshman Team meetings, 31 of which were from weekly long meetings. Of the 35 meeting records, 26 had fieldnote records and 32 meetings had audio records.

Data Analysis Procedures

I began data analysis by strategically reducing my data set using my unit of analysis, *episodes of pedagogical reasoning* (EPRs), which Horn (2005) defines as "units of teacher-to-teacher talk where teachers exhibit their reasoning about an issue in their practice" that are "accompanied by some elaboration of reasons, explanations, or justifications" (p. 215). My decision-rule for locating EPRs was based on topical shifts related to struggling students. After, four episodes were selected for closer analysis because they contained extended talk about the struggling student problem. I selected episodes that (a) represented development across time, (b) had three or more Team members present, (c) had quality records available, and (d) had substantive discourse dominated by teachers.

Date	Oct. 2005	Jan. 2006	March 2006	May 2006
Diagnostic Frames for Struggling Students	Fixed Student Attributes	Personal and Systemic Issues	Classification Schemes for why Students Struggle	Status, Race, and Tracking
Proposed Solutions (Prognostic Frames)	Inform parents of problem and changes students need to make	Conceptual solutions (e.g., see students as capable, students should take math in evening)	Student and peer observations; Email students' other teachers for strategies and extra support	Expand implementation of and training in Complex Instruction
Location of Problem ⇩ Location of Solution	Student *Invariable*	Personal Circumstances, The System *Invariable*	Abilities Students are not Good at Yet ⇩ *Actionable* ⇩ Classroom-based Interventions	Classroom Enactments of Society and The System ⇩ *Actionable* ⇩ Classroom-based Instructional Responses

Figure 1: Overview of Teachers' Framing Practices

The transcripts and corresponding meeting summaries for selected EPRs were coded using three core framing tasks identified by Snow and Benford (1988) to help identify how teachers framed the struggling student problem in context of their equity-oriented reforms. Specifically, I looked for (a) *diagnostic frames* to understand how teachers conceptualized the struggling student problem; (b) *prognostic frames* to understand how teachers conceptualized interventions related to the struggling student problem; and (c) *motivational frames* to understand how teachers made a case for their framing. After coding the transcripts of the EPRs for these framing moves I looked for themes within and across the data (see Figure 1 for an overview).

Results

Finding 1: Evidence of Within-Group Development in a Teacher Community

Teachers' Shifting Frames Show Development within a Teacher Community over Time.

The Freshman Team teachers concentrated on diagnosing struggling students in the episode from October (EPR 1), including invariable student characteristics related to work ethic and classroom behavior. Though teachers agreed to send home good news cards, call parents, and mail home letters with improvement strategies, by locating their diagnoses within the context of fixed student characteristics teachers had little access to actionable responses proximal to everyday instruction. In January (EPR 2) teachers' initial diagnoses of struggling students were challenged by information learned from reviewing academic histories of identified struggling students: ELL students and special education students were not supported when they were mainstreamed into a regular classroom, some students only recently performed poorly, and some struggling students had slipped by unnoticed. Participants put forward conceptual strategies such as encouraging students to take an evening class, moving students to work with another teacher, and seeing students as capable. Teachers were once again left with little actionable responses proximal to everyday instruction, though their diagnoses generally shifted away from simple assignment of blame, such as struggling students are students who "choose to fail," and trended towards external factors that influenced students' performance, such as lack of ELL support and special education transitions.

In March (EPR 3) teachers used classification schemes (Horn 2005; 2007) to diagnose the "core group of struggling students that seem to drive the whole school crazy," which included locally meaningful categories like the Taylors (smart students who are lazy) and the Paiges (students who are very far behind their peers academically). Teachers went beyond these categories and offered reasons why these students perpetually struggle, such as with students like Autumn, who had never experienced what it felt like to be successful in math class. Teachers presented myriad strategies for action, including the systemic response of generating a school-wide list of struggling students so that a comprehensive effort could be made to help them. More proximal classroom responses included strategies such as using a new teacher to help run triage with difficult students, peer and student observations, and emailing struggling students' other teachers for ideas and to garner additional support for students. This episode highlighted teachers' increasing specificity with their diagnoses of why students' struggle, resulting in a larger repertoire of actionable responses and focus on classroom-based intervention strategies.

In May (EPR 4) teachers considered the effects of status and race on the struggling student problem. Rose made the diagnosis that "white kids automatically have more status" because of Clark "not having a middle" and "not having a good pool of white kids" in Math 1. Though Rose's prognosis was unclear, Julie extended the status problem idea Rose originated by diagnosing the growing racial divide of students caused by the tracking practices that were in place at Clark. Julie made meaning of Rose's "not having a middle" and "not having a good pool of white kids" comments by linking these problems to tracking (as opposed to suggesting abstractly that they need more "good" white kids in Math 1 at Clark). Teachers talked through the pros and cons for a variety of solutions to the tracking problem, such as offering a within-class Honors option, using harsher grading schemes, calling all classes Honors, giving placement tests, and detracking altogether. Rose ended the episode with the diagnosis that the most important strategy for these issues relates to groupwork, thereby linking the problems of status, race, and tracking with the Team's pedagogical reforms. Though teachers' diagnostic frames reflected classroom enactments of society and the system, their solution frames reflected classroom-based instructional responses.

What is important to note is that teachers' diagnoses in later episodes contrasted earlier diagnoses, especially the in the first and second episodes where teachers were focused on understanding the attributes of struggling students. While teachers continued to use frames related to fixed attributes and personal circumstances, their primary diagnoses shifted towards the nuances behind why particular students struggle and status-related issues. While the former frames places the onus for achievement primarily on the students, the latter provide teachers a means for action. Teachers' frames became more nuanced in their representation of the struggling student problem, disentangling issues of ability from school-savvy and systemic problems like racism and tracking. Thus, there is a preponderance of evidence that shows how teachers developed and drew upon a larger repertoire of frames of the struggling student problem over time, with focus and priority given to solution frames that were related to everyday instruction. More generally, by tracking teachers' shifting frames around problems of practice, the case of the Clark Freshman Team offers a counternarrative to static characterizations of teacher community and shows within-group development in a teacher community over time.

Finding 2: Evidence of Teachers' Collective Learning

Teachers' Collective Learning is Manifested through their Framing Practices.

Examination of teachers' framing practices across episodes yielded teachers' evolving narratives of the struggling student problem, which I claim made teachers' learning in the Freshman Team community of practice more transparent. Teachers' participation in the Team gave them the opportunity to engage in extended talk about the struggling student problem. Participation in these conversations fostered teachers' reification of the struggling student problem, evidenced by the different frames they used to diagnose problems, propose solutions, and give rationale for their ideas. The participation and reification processes worked in tandem to coordinate and localize the meanings of these frames. The prior result showed how teachers' reification of the struggling student problem changed through interactions in their teacher group. These shifts served to mark and describe changes in teachers' participation in a community of practice, which is a process Wenger (1998) characterizes as learning. This analysis thus showed that the Clark teachers' collective learning was manifested through their framing practices.

As an illustration, in October (EPR 1) each teacher participated in the Team debriefing and reified the struggling student problem through diagnoses that centered on fixed student attributes. In January (EPR 2) teachers developed a community-owned framing for why students struggle (struggling students must not understand anything and must be used to failing, which is a status problem), which depended on teachers modifying another teacher's framing. In March (EPR 3) teachers reified potential classroom-based interventions using specific students as representative cases for a larger group of struggling students. Teachers intertwined their category systems and frames and then negotiated meaning around these frames through the processes of participation and reification, out of which their classroom-based interventions emerged. This resulted in conversation turns that contained different kinds of linked frames that had the overall effect of building resonance for their ideas with the Team. In May (EPR 4) teachers again intertwined and linked their framing practices to negotiate the meaning of status, race, and tracking in context of the struggling student problem. What is significant is how the Team negotiated the meaning of all these frames; the Team haggled over every offered frame. This important process helped the Team iteratively make progress on their collective understanding of these issues.

Teachers increasingly used a variety of linked frames to help build resonance for their ideology, courses of action, or rationale for action, which likely contributed to which frames were picked up or ignored by the Team. What is more, teachers moved towards using the group to negotiate the meaning of these frames, which corresponded to episodes with more actionable solution responses. By examining shifts in the Team's framing of the struggling student problem in the context of their engagement in fundamentally temporal, fundamentally social learning actions, this dissertation documented the Team's movement towards more classroom-based actionable responses and away from less obvious actions based on invariable student characteristics. These shifts accounted for a change in participation in the Freshman Team, which contributes an empirical example for teachers' learning in a community of practice.

Discussion

This paper aimed to describe high school mathematics teachers' learning as they took on issues of equity in their workplace group. As with many stories of learning, my data tell a developmental story, though not a linear one. My analysis thus required tools that could preserve the messiness of learning and at the same time tell a nuanced learning story in a productive way, which I accomplished by joining the communities of practice and frame analysis literatures. Analysis showed that teachers' frames shifted from invariable framings based on student characteristics, personal circumstances, and systemic issues to actionable framings based on classroom systems that contributed to student failure, which promoted teachers' concentration on courses of action linked to their instructional practices. As a result of this analysis, this study yielded significant findings about teachers' shifting framing practices and learning, and advanced the literature with analytic tools for making sense of teachers' development, shifting frames, and learning in context of teacher groups.

As is often the case with the study of rich data, this analysis raised issues that merit further study. For example, the process of offering frames appears to catalyze a response by the group, creating potential for group interactions around this framing. Through more research is needed to confirm my speculation that offering and then discussing rams around problems of practice yields productive learning opportunities inside of teacher community, my hypothesis is that this avenue of research would generate findings that help explain how opportunities to learn connect with teachers' learning inside teacher community. A limitation of my analysis is that I do not make claims about teacher's individual learning, though my speculation is that an individual's consistent orientation to invariable frames closes off learning opportunities readily available and important to the group. This raises an emerging analytic issue concerning learning opportunities alongside analysis of individual learning that merits further study. Moreover, I predict that my use of key conceptual tools from frame analysis was but the tip of the iceberg when it comes to the application and utility of the larger social movement literature to the teacher community and teacher learning literatures; in any case, more research needs to be conducted to understand the interplay of these literatures and open up an entirely new way for conceptualizing this work.

Endnotes

1. Ilana Horn, Principal Investigator.

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PERSPECTIVES ON INSTRUCTOR MODELING IN MATHEMATICS TEACHER EDUCATION

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We unpack the notion of mathematics teacher educators modeling teaching in mathematics teacher education courses. Specifically, we investigated what practicing teachers gained from mathematics teacher educators' modeling by examining: (1) What do mathematics teacher educators believe they model about effective instructional practice? (2) What do practicing teachers notice about the mathematics teacher educator's pedagogy and identify as effective mathematics teaching? (3) In what ways do these perspectives align in mathematics courses for practicing teachers? The results provide insight into what and how teachers learn from engaging in inquiry-oriented teaching. We discuss the implications for the education of mathematics teachers.

Introduction

While research has pointed to the benefit of student-centered instruction and collaborative, inquiry-oriented learning environments in both K-12 and college mathematics courses (Boaler, 1998; Bowers & Nickerson, 2001; Goos, 2004; Rasmussen, Kwon, Allen, Marrongelle & Burtch, 2006), many mathematics courses are still taught rather traditionally. Borg (2004) points to what Lortie (1975) calls the 'apprenticeship of observation' to explain the prevalence of traditional mathematics instruction. With respect to teaching practices, the notion of apprenticeship of observation suggests that teachers learn about teaching long before they enter the classroom. Their education as teachers starts when they themselves are students and their experiences influence the ways in which they think about the teaching and learning of mathematics (Ball, 1988). Many teachers' experiences as learners of mathematics, from elementary through college, primarily involved their teacher in the role of the provider of information, in what has been called a factory model of education (Callahan, 1962). Since teachers bring their own experiences as learners of mathematics to their practice, it is not surprising that the way teachers teach mathematics is not often substantially different from how they learned mathematics (Hiebert & Stigler, 2000; Sowder, 2007; Thompson, 1992). The fact that teachers were educated in a traditional system has been described as "…perhaps the greatest obstacle to these reforms" (Simon, 2008, p. 17).

It has been suggested that if teachers learn from their experiences as mathematics students then mathematics teacher educators should model desired instructional practices in teacher education and professional development programs (Borasi, Fonzi, Smith, & Rose, 1999; Loughran & Berry, 2005; NCTM, 1991; Simon & Schifter, 1991; Sowder, 2007). The implication is that an important factor in helping teachers enact these shifts is to engage them as learners in inquiry-oriented mathematics communities. How might this learning occur and what might be learned?

We report on instructor modeling in three mathematics courses for practicing teachers. We investigated what mathematics teacher educators believed they model, what practicing teachers noticed, and the ways in which these perspectives were aligned. The mathematics courses for practicing teachers discussed in this report are a part of a university-based professional development group. As with many mathematics professional development programs, the goal is to move teachers forward in their thinking about content and student learning so teachers can work to help increase student achievement in mathematics (Nickerson, 2010, Sowder, 2007). These professional development programs are designed to provide extra preparation for teaching mathematics by providing opportunities for teachers to deepen their content knowledge and by collaboratively reflecting on their teaching and student learning.

Theoretical Perspective

We view learning as inherently social and seek to account for individual perspectives within evolving social practices. We are concerned with the "negotiation of meaning." We focus on individuals' perceptions of the mathematics teacher educator's pedagogy to give an account of what mathematics teacher educators believe their instruction conveys, what individual teachers notice and describe the social conditions in which the teaching acts of note were situated. Neither the social processes nor the individual's interpretations can be considered without the other.

The notion of *perceptual lived experience* (Loughran & Barry, 2005) suggests that some learning occurs by "living through" experiences. Loughran and Barry give an example in which student teachers develop perceptual rather than conceptual knowledge of a situation. In their example, the student teachers as students in a classroom setting experience the emotions, images, needs, values, volitions, and frustrations of individuals in the situation, which develops their perceptual knowledge of learning environments instead of their conceptual knowledge. While this kind of learning is rather passive, the learner gleans ideas about the "how to" of the activity by simply being a part of the situation.

Intent participation (Rogoff, Paradise, Arauz, Correa-Chávez & Angelillo, 2003), a related, more deliberate form of learning from experiences in a situation, describes learning from keen observation and listening to ongoing activities in which the learner participates or expects to participate in the future. Both perceptual lived experience and intent participation suggest that learning occurs through the (implicit) modeling of an expert, where the expert provides an example of the required performance. From these perspectives, teachers have the opportunity to learn about the practice of teaching by engaging in an inquiry mathematics classroom.

Another way teachers can learn about the practice of teaching through modeling in teacher education is that the teacher can learn as an *apprentice* to an "expert" teacher. Here the learner is not unaware of the intentional modeling of practice. Apprentices learn through methods of observation, scaffolding, and increasing independent practice (Collins, Brown & Newman, 1989; Lave & Wenger, 1991). Student teachers and observers can be viewed as apprenticing from master teachers, teacher coaches, and teacher educators. The learner can be seen as a cognitive apprentice (Collins et al, 1989; Schoenfeld, 1992) where instructional practices are learned through observation, guided experience and participation.

In the mathematics teacher education courses in this study practicing teachers participated in mathematics classes, and thus learning about instructional practice looked quite different. In mathematics teacher education like the courses in this study, the mathematics teacher educators are not teaching along side teachers in teachers' classrooms. The mathematics teacher educators must make connections to K-12 classrooms through anticipation exercises, discussions of possible trajectories in classroom situations, and reporting back experiences. In these courses, the connections often took the form of commentary by the mathematics teacher educator that was related to a mathematical activity the teachers themselves were engaged in. The mathematics teacher educators prompted discussion among the teachers, asking them to anticipate how it might apply in a classroom situation. Also, the teachers discussed their own teaching as they reported on a predetermined task that all of the teachers in the class tried with their own students, called "try-ons" in this context. This anticipation and reporting back on teaching experiences with a more experienced mathematics teacher educator can provide a means of scaffolding the learning of reform teaching.

The constructs of perceptual lived experience, intent participation, and apprenticeship share a situated perspective on learning from the milieu. They all suggest that knowledge is developed and deployed in activity and is not separable from or ancillary to learning and cognition (Brown, Collins, & Duguid, 1989). Thus, learning the teaching profession stems, at least in part, from the teaching teachers see and experience as learners and the activity they engage in as professionals. Moreover, how teachers learn about practice affects how they view the practice of teaching.

Method

In this study we observed the mathematics professional development of a large urban university in the southwestern United States. Data was collected in three cohorts: a primary elementary cohort (grades k-3), an upper elementary cohort (grades 4-6), and a middle school cohort. Our data encompassed teachers' and mathematics teacher educators' reflections on a year of professional development and with respect to two classes. We drew on a three-step methodology (Busse & Ferri, 2003) and complementary accounts research methodology to develop a rich account from the data (Clarke, 1997). The facets of the methodologies include: observation, interview, stimulated recall and analysis of complementary perspectives.

The data set consisted of:

1) Audio recorded pre-session interviews with four mathematics teacher educators.
2) Video of the six individual classroom sessions (two consecutive classes for each of the three cohorts).
3) Stimulated recall interviews (SRI) with the mathematics teacher educators (MTEs), one for each class session.
4) Semi-structured interviews with a subset of the practicing teachers, part of which included stimulated recall interview.
5) Researcher fieldnotes of the six classroom sessions and interviews.
6) Reflection surveys, given to all 49 practicing teachers in the three cohorts at the end of the first session.
7) Quantitative surveys, given to all 49 practicing teachers in the three cohorts at the end of the second session.

The mathematics teacher educators were interviewed pre and post observation. In pre-session interviews, MTEs were asked to reflect generally on what instructional practices they believe they model as they teach. Then classroom data was collected in two consecutive classes for each of the three cohorts. The class sessions were videotaped and a researcher took fieldnotes. Both the MTEs and the classroom teachers were interviewed within a day or two using stimulated recall interviews (SRI). In the SRIs, it was the interviewees that pointed to noteworthy aspects. The surveys asked all the teachers to reflect on what was modeled throughout the year.

The videos of the classroom sessions were reviewed to create a descriptive timeline of classroom events to aid in analysis. Several participants were interviewed to enable the coding and subsequent creation of an integrated data set of complementary perspectives. The initial interviews with the MTEs suggested the coding categories aligned with the NCTM Professional Standards for Teaching. The transcript of the classroom sessions were analyzed in a cyclical process of coding and search for confirming and disconfirming evidence (Strauss & Corbin, 1990) to delineate the categories of modeled instructional acts. Once we developed what we thought to be an exhaustive group of codes, we coded a few episodes separately and compared codes for inter-rater reliability. The coders were in agreement 78% of the time and discussion resolved discrepancies. The primary causes of discrepancies were related to sub-codes of the categories.

The data allowed for analysis on two levels, (1) globally on modeling throughout the year of professional development and (2) locally on modeling in two consecutive class sessions. The pre-session interviews with the mathematics teacher educators, the reflection surveys and the individual interviews with eight practicing teachers provided data on perspectives on what the mathematics teach educators modeled about practice in general. The stimulated recall interviews with the mathematics teacher educators and the practicing teachers provided data on specific events. Both levels were important to examine. Globally, with respect to the year, because though a specific instructional act may not have occurred in a particular session or one may not have noted certain aspects of instruction in a given event does not imply the action is not common or that it has not been noted elsewhere. Locally, with respect to two consecutive sessions, is important to examine because interpretations of a specific event may vary. An analysis of both levels has the capacity to highlight how implicit and explicit modeling conveys teaching practices.

Analytical Framework

An initial analysis of the classroom videos and the interviews began with the NCTM *Professional Standards for Teaching Mathematics* with respect to the four categories of tasks, discourse, environment and analysis. Using a cyclical process for confirming and disconfirming evidence (Strauss & Corbin, 1990) the pre-session interview data and the classroom data (fieldnotes and classroom video) informed the elaboration of the four categories derived from the NCTM *Professional Standards for Teaching Mathematics*. This process allowed for the emergence of parallel categories for describing the mathematics teacher educators' pedagogy. The parallel categories characterize the mathematics teacher educators' pedagogy as the participants are engaged as mathematics learners and as mathematics teachers. The emergent codes were used to categorize and coordinate the mathematics teacher educators and the practicing teachers' perceptions of classroom events.

> Teachers' learning about instructional practice in mathematics teacher education is often not restricted to learning about classroom mathematics instruction. There are also opportunities for teachers to learn about other facets of the teaching profession. The current reform movement in mathematics education has a strong underlying theme of the professionalism of teaching. This view recognizes the teacher as a part of a learning

community that continually fosters growth in knowledge, stature, and responsibility (Dufour, 2005; NCTM, 1991, Author). Reform recommendations suggest that teachers ought to collaboratively plan instruction, reflect on practice, create and reflect on new practices, and support one another's professional growth (NCTM, 1991). Such interactions allow teachers in a school and its administrators to continuously seek and share learning and then act on what they learn. This type of communication and collaboration with a focus on inquiry about student learning are important aspects of what researchers call a "professional learning community" (Dufour & Eaker, 1998; Astuto, Clark, Read, McGree & Fernandez, 1993; Hord, 1997).

Results

An initial analysis of the pre-session interview data revealed that the mathematics teacher educators modeled more than the teaching of mathematics, they also modeled participation in the collective inquiry process of a community of mathematics teachers. The mathematics teacher educators were not explicit in their discussion about modeling the collective inquiry part of the teaching profession; however, their discussion related to developing the teachers' pedagogical content knowledge pointed to the teachers' engagement as professionals in the mathematics teacher education sessions. The emergent framework, summarized in table 1 (see next page), characterizes the mathematics teacher educators' pedagogical moves with respect to the participants' engagement as learners and teachers.

This first result, the emergent analytical framework, made it possible to characterize the mathematics teacher educators' pedagogy as described by the mathematics teacher educators and participating teachers and observed by the researchers. These characterizations describe the mathematics teacher educators' pedagogy as they engage teachers as learners and as professionals.

Teachers can learn about the practice of teaching from MTE modeling. In the Reflection Survey 1 and teacher interviews, the participants noted teaching actions related to the categories worthwhile tasks, discourse, tools, and learning environment in the analytical framework. Reflection Survey 1 asked the teachers to describe what they have done in their classes to support student learning drawing from what they have learned in prior mathematics content sessions. The teachers reported that they had incorporated specific activities and tools that they found useful in their own learning in the sessions. The teachers reported that they had adopted practices like engaging students in mathematical conversation (discourse), specifically, through questioning and prompting students to share out and provide justification. The teachers also reported incorporating wait-time and letting students struggle with difficulty, facets of the learning environment from the analytical framework.

While both the mathematics teacher educators' and the participants' utterances point to the mathematics teacher educators' pedagogy as the participants were engaged as both learners of mathematics and as mathematics teachers, the emphasis was on participants' engagement as learners, as opposed to their engagement as teachers. This was more so the case for the teachers than for the mathematics teacher educators. The teachers pointed to the pedagogy as they were engaged as a learner in about 86% in their reflections on the year and 88% in their reflections on the two sessions.

The teachers in the primary and upper elementary cohorts noted pedagogy related to their engagement as learners almost exclusively, while the middle school teachers reported it in only about 71% of their coded utterances. On average in the pre-session interviews where the mathematics teacher educators reflected on the year, they emphasized mathematics pedagogy they believe they model as the teachers were engaged as learners in about 71% of the coded utterances. In the stimulated recall where the mathematics teacher educators reflected on the two sessions noted such engagement in about 78% of the coded utterances. Like the teachers, the mathematics teacher educators from the elementary cohorts more frequently discussed instructional acts where the participants were engaged as learners. The middle school cohort facilitator, MTE-Karla, emphasized pedagogy related to the participants' engagement as teachers and as learners almost equally, reflecting the dual purpose of the mathematics teacher education. This points to the likelihood that the difference is due to the intended purposes of the mathematics professional development sessions and the mathematics courses.

The pedagogical moves most noticed by both the mathematics teacher educators and the teachers were characterized under the categories of tasks and discourse. The mathematics teacher educators and the teachers highlighted the way the mathematics teacher educator modeled the use of tasks, specifically noticing tasks that problematized the mathematics and prompted the teachers to make connections between mathematical ideas and among representations. These tasks were seen as promoting discussions of the mathematics. Both the mathematics teacher educators and the teachers pointed to the scope of tasks, particularly the large amount of time spent on a single task devoted to developing mathematical concepts in learning situations.

Table 1: Summary of Analytical Framework.

	Classroom Community of Mathematics Learners	Community of Mathematics Teachers
Tasks	Worthwhile **Tasks** are the projects, questions, problems, constructions, applications, and exercises in which teachers engage intended to develop teachers' mathematical content knowledge.	**Tasks** are the projects, questions, problems, constructions, applications, and exercises in which teachers engage intended to develop the teachers' understanding of the students' mathematics.
Discourse	Classroom **Discourse** refers to the ways that mathematics teacher educators mediate discourse about mathematical ideas to focus discussion on concepts and solution paths instead of answers.	Classroom **Discourse** refers to the ways that mathematics teacher educators mediate discourse about practice to focus discussion on teaching, student learning and/or thinking.
Tools	**Tools** are objects, tangible or intangible that the MTE and/or teacher's use in learning situations to reason, make connections, solve problems, communicate and enhance discourse.	**Tools** are objects, tangible or intangible that the MTE and/or teachers use to reason or enhance learning about or discussion of lesson planning, student learning, thinking and understanding, and so on.
Environment	**Environment** represents the setting for which the development of each teachers mathematical power is fostered.	**Environment** represents the setting for which the development of each teacher's knowledge for teaching is fostered.
Analysis	Ongoing **Analysis** is the systematic reflection in which mathematics teacher examine relationships between what they and their students are doing and what students are learning.	Ongoing **Analysis** is the systematic reflection in which mathematics teacher educators engage in analysis to foster teachers' participation in the Collective Inquiry Process

While the teachers pointed to tasks and discourse, some pointed to how the coordination of the pedagogical moves related to structuring learning environment, specifically structuring the sessions. The teachers' discussion about discourse often pointed to discourse in task situations, collaborative work on tasks and sharing ideas. The teachers specifically mentioned how the mathematics teacher educator pushed for justification and asked questions as opposed to giving answers. As a part of this, the teachers did point to a facet of the learning environment category, reporting that the mathematics teacher educators allowed them to struggle with difficulty. The teachers characterized the mathematics teacher educators' instruction as Socratic, that is, the mathematics teacher educators asked questions that helped them shape their thinking and move them forward in their thinking. The teachers reported that they try to incorporate these facets of the mathematics teacher educators' pedagogy in their own teaching.

There are aspects of ongoing analysis that the mathematics teacher educators believe they model that are not noticed by the teachers. There is a belief among the mathematics teacher educators and the teachers that a deeper understanding of the mathematics content better prepares teachers to present challenging tasks and ask questions that help students move forward in their thinking. However, the mathematics teacher educators' report that one of the purposes of questioning as the teachers are engaged in tasks is ongoing analysis of the teachers' thinking. The mathematics teacher educators note that their questions are driven by the need to develop conceptual models of the teachers' thinking. The teachers did not note developing conceptual models as a reason for being able to ask "good" questions; the teachers pointed to experience, mathematical knowledge, and being well prepared for the lesson as key to the mathematics teacher educators' ability to ask questions that move them forward in their thinking. This perhaps points to one of the reasons for differences in the emphasis the mathematics teacher educators and the teachers put on the category of ongoing analysis.

Conclusions

In general the mathematics teacher educators and practicing teachers reported that the mathematics teacher educator modeled student-centered instruction as conveyed in the NCTM *Professional Standards for Teaching Mathematics* (1991), but also facets of the collaborative work of teachers outside of the classroom. This work outside of the classroom might include teachers collaboratively planning instruction, reflecting on practice, creating and reflecting on new practices, and supporting one another's professional growth.

This research informs the body of knowledge about teaching the practice of teaching mathematics. The work described in this report makes recommendations regarding modeling by mathematics teacher educators based on an analysis of empirical data. It provides insight into *what* and *how* teachers learn from modeling reform teaching. Specifically, the study draws on an analysis of data from three classrooms to illustrate how explicit discussion about mathematics teaching and implicit modeling of instruction supported practicing teachers' noticing of the instruction they experienced, and how engaging in facets of the teaching profession that take place outside of their interactions with students has the potential to foster the enculturation of teacher into a professional learning community of mathematics teachers.

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AN ANALYTIC FRAME FOR EXAMINING TEACHERS' COLLABORATIVE MATHEMATICS WORK TO DEVELOP SPECIALIZED CONTENT KNOWLEDGE

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A common practice in mathematics professional development to enhance teachers' content knowledge is to engage them in solving and discussing mathematics tasks. Yet, what is entailed in these experiences and how teacher discussions afford or constrain opportunities to develop teachers' knowledge of mathematics is not fully understood. In this paper, the authors provide a research-based framework to analyze teachers' mathematical productions (verbal reasoning and written inscriptions) used in collaborative mathematical work that illustrates the complexity of considering teachers' opportunities to learn and how discussion of mathematics may or may not be resources for learning.

Introduction

To better understand the mathematical knowledge teachers need to prepare them for their work, Ball and colleagues (Ball, Thames, & Phelps, 2008) have analyzed detailed records of practice to identify the content and pedagogical knowledge demands of mathematics teaching developing a practice-based theory of mathematical knowledge for teaching (MKT). From this work, some of the *mathematical* demands that are unique to the field of teaching have been outlined and include: recognizing what is involved in using a particular representation, linking representations to underlying ideas and to other representations, and giving or evaluating mathematical explanations (Ball et al., 2008). These demands become even more prevalent when teachers facilitate the development of students' mathematical understanding by selecting high-level mathematics tasks, orchestrating student discourse around the task, advancing students' mathematical justifications, and helping students seek connections between previous and new knowledge.

Mathematics professional development (PD) is considered one setting in which teachers may deepen their content knowledge used for teaching (Ball & Cohen, 1999). While mathematics content in PD is often overlooked or superseded by other issues of teaching (Hill, 2004), one common way to keep mathematics central is to engage teachers with activities and discussions around mathematics tasks (Jaworski, 2007). These activities often situate mathematical work in examinations of student work or representations of practice such as narrative or video cases (Kazemi & Franke, 2004; Silver et al., 2007). While *practice-based* (Ball & Cohen, 1999) activities are invaluable to mathematical and pedagogical learning, there is still much to learn about how teachers collectively do mathematics in ways that support their mathematical learning (Suzuka et al., 2010). This paper highlights research on teachers' collective opportunities to learn mathematics and shares a framework for analysis of teachers' mathematical work.

The most common rationale for engaging teachers in mathematical work is to advance teachers' mathematical content knowledge. However, this goal takes on varying levels of focus and specificity across studies. Recent research and development efforts have advanced that the purpose of doing mathematics in professional education (teacher preparation and PD) is to develop teachers' specialized content knowledge (SCK) needed to meet the demands of teaching (Ball et al., 2008; Elliott et al., 2009a; Lo, Grant, & Flowers, 2008; Suzuka et al., 2010). Programs like those discussed by Lo and her colleagues (2008) and Suzuka, Sleep, Ball, and their colleagues (2010) engage teachers in solving mathematics tasks and analyzing solutions, alternative strategies, and multiple representations in order to enhance teachers' content knowledge and connect it to the demands of teaching.

To develop teachers' knowledge, researchers have also asserted the need for teachers to justify their own mathematical reasoning. Simon and Blume (1996) considered how pressing on preservice teachers' mathematical justifications supported the emergence of learners' understandings of mathematical concepts. Other researchers have found that teachers' and teacher-leaders' justifications invoked or developed MKT through the unpacking of key mathematical ideas (Elliott, Lesseig, & Kazemi, 2009). Such mathematical work provides teachers with both a deeper understanding of the mathematics they use in the classroom as well as more facility to use this knowledge in a classroom in which students' reasoning is central (Stylianides & Ball, 2008).

Fostering meaningful mathematics learning for teaching in PD is no easy task. One prevailing issue that impacts this work is that doing mathematics with teachers is different from doing mathematics in the classroom (Elliott et al., 2009a). This is due in part to the fact that teachers tend to hold mathematical knowledge differently than students. While teachers may have previous experience with mathematical content, they often do not have a deep understanding of or facility with developing justifications that explain mathematical ideas (Lo et al., 2008). To meet the mathematical demands of teaching, teachers need not only *a* method for solving a math task, but a repertoire of solutions and justifications that unpack why methods work. Yet, a number of studies have found that teachers were less likely to positively evaluate justifications that use non-standard methods, those potentially more likely to be accessible to students who are not formally trained in proof, or to distinguish arguments that provided a mathematically valid reason rather than a thorough explanation of *how* a solution was developed (Bieda, 2010; Knuth, 2002).

Despite the struggles researchers have uncovered on engaging teachers in mathematical work that unpacks key mathematical ideas the importance of providing teachers with opportunities to do this work in PD is not diminished. Indeed, these struggles point to a need for more research to be done to understand the potential learning opportunities made available to teachers during collaborative mathematics work. If we expect teachers to have opportunities to learn mathematics for teaching in PD, leaders need understandings of what makes a discussion productive and how discussion might stay focused on key mathematical ideas. Similarly, the research community needs tools for analyzing discussions that coordinate argumentation with attention to teachers' specialized content knowledge. Detailed images of mathematics discussions among teachers could serve professional educators learning how to facilitate PD oriented to SCK and researchers who are trying to build a knowledge base on teacher learning.

We advance that researchers need opportunities to share the analytic work entailed in examining collaborative mathematical activity in order to construct more rigorous means of evaluating how conversations advance teachers' learning of mathematics for teaching. To this end, we share our coding system and examine the utility of it. Using data from a teacher-leader research project, we will illustrate how the framework examines what is entailed in teachers' mathematical discussions and how key mathematical ideas are potentially drawn upon. The framework and the data illustrations are a basis for a discussion on the import of frameworks that examine the depth of conversation, key mathematical ideas, and teachers' opportunities to learn MKT, in particular SCK. As such, we intend that this discussion will open up a dialogue for other researchers to consider how we might investigate teachers' collective mathematical work.

Context

The seminars from which data for this paper are drawn are part of a five-year research and development project, *Researching Mathematics Leader Learning* (RMLL) investigating how teacher-leaders develop mathematically rich learning environments for teachers. The project's framework and design evolved to focus on developing teachers' SCK as a key purpose for engaging in mathematics during PD. RMLL seminars provided opportunities for teacher-leaders, who were also teachers of mathematics, to do mathematics, discuss the mathematics, and consider the mathematical entailments of the tasks they solved (other activities were part of RMLL seminars, but not central to this paper). For this paper we will refer to RMLL participants as teachers since we investigate their collective mathematical work and not activities central to their leading. Participants in this study (n = 70) were volunteers involved in one of two phases of RMLL seminars and were becoming, or already were, leaders in their schools, districts, and regions charged with facilitating mathematics PD. Most participants were current part- or full-time classroom K-12 teachers. A few participants were full time leaders of mathematics PD.

Using data on participants' mathematical productions in small and whole group discourse, RMLL researchers have focused on justification as a means to understand the mathematical work at play and the normative aspects of these interactions (Elliott et al., 2009b). This previous work inventoried what participants reported as acceptable justifications, what participants' solution methods entailed, and made claims about how justifications allowed for an understanding of the underlying mathematics (Elliott, Lesseig, & Campbell, 2010). The intent of our previous work was to provide images of mathematical justification productions that invoke and develop SCK.

We have come to realize that a focus on teachers' solution methods and justifications without considering the collective talk elaborating teachers' solutions severely limits how we might understand doing mathematics in PD and teachers' opportunities to learn. As a result, we consider teachers' justification episodes through a new lens drawing on research on mathematical discussions of both students and teachers.

Theoretical Considerations

We are interested in the collective work of teachers solving mathematics tasks with a facilitator. Guided by a situated learning perspective (Putnam & Borko, 2000; Wenger, 1998), this study examines the entailments of learners' participation in collective activity. Specific to our study, we investigate the collective work of a group of teachers solving mathematics tasks and the potential resources made available through their work to learn mathematics for teaching.

Previous studies have considered the nature of students' mathematical discussions as well as the mathematical learning opportunities they afford or constrain. In an account of students' collective mathematical learning, Cobb (2002) considers the mathematical ideas that students take up, entailments of classroom discussion, and how students use tools and inscriptions. We saw these foci as central in how to examine the collective participation of teachers in mathematical activity. Considering community discourse in terms of the three foci allows for the documentation of emergent practices, which account for mathematical learning of the community (Cobb, 2002). In his study Cobb uses Toulmin's (1958) scheme for argumentation to examine classroom discussion. Similarly, Forman and colleagues (1998) in their analysis of a mathematics classroom advance that Toulmin's argument analysis is a valuable tool for examining the mathematical reasoning used to solve tasks. In general, both research projects use a classification of utterances as *claims*, *warrants*, and *backings* to examine the structure of conversation to infer what we consider to be depth of a conversation. *Claims* are assertions in need of argumentative support because the audience has called them into question or the rules for social interaction require such support. *Warrants* explicate how conclusions have been drawn from accepted givens. *Backings* strengthen the acceptability of warrants, essentially serving as the "why" that refers to the structure of the key mathematical ideas at play.

Considering the benefits of the approach to analyzing classroom mathematical practices used by Cobb (2002) and Forman and colleagues (1998), we took into consideration recent studies that have taken a similar theoretical and analytical stance in examining teacher discourse around mathematics tasks. Steele's (2005) analysis of mathematical and pedagogical discussions in a mathematics methods course also used Toulmin's (1958) argument model as a basis for analysis. In addition to the focus on claims, warrants, and backings (referred to as "bridge statements" and "evidence"), Steele also considered the potential responses to an argument adding "challenges" and "connections" as elements of the community's discourse structure and important for considering the depth of a discussion. Implicit in Steele's analysis is how deep conversations afford teachers opportunities to learn mathematics, although he does not take this up directly. Steele's work instead considers how argumentation may be driven by the underlying disciplines informing the conversation to explain how mathematical discussion differ from pedagogical discussions. We have found Steele's analysis helpful to consider the depth of conversation, and we use a similar analytic tool to explicitly examine learning opportunities for teachers.

Researchers who directly consider the entailments of teachers' discussions have reported conflicting results when examining the nature of mathematical and pedagogical conversations to advance teachers' learning. Crespo (2006) dubbed teachers' talk during mathematical work to be *exploratory*, in that it tended to be interactive, consisting of unprompted interruptions, disagreements, and tentativeness, and pedagogical talk as *expository*, more definitive in nature. Exploratory discussion allowed teachers to make public numerous mathematical ideas, but here the analyses did not explore the nature of the mathematical ideas, nor how ideas contributed to teacher learning. Crockett (2002), in another study, noted that teachers working on a mathematics task were so invested in reaching a correct answer that they missed out on examining the mathematical value and assumptions in the problem. Steele, Crespo, and Crockett provide valuable insights on teacher discourse of mathematics. However, each shares slightly different levels of detail and conflicting results for supporting teacher learning through teachers doing and discussing mathematics.

To fill the knowledge gap in understanding how teachers doing mathematics in PD may contribute to their development of SCK, we came to understand that our analyses must take up discursive interactions as mathematical justifications unfold in relationship to drawing on key mathematical ideas. Research on students' and teachers' mathematical discourse allowed us to conceptualize teachers' math argumentation accounting for both how and what math ideas are justified.

Methods

Our work looking at teachers' discussions around mathematics tasks and our consideration of how to analyze such work has been an iterative process. We developed our coding scheme based on the theoretical considerations uncovered in the literature and by reviewing data first individually, then together. An initial selection of video-data that included a variety of teachers across our two-phase research design, two different mathematics tasks that center on algebraic reasoning, and instances from early and late in RMLL seminars were examined to develop the window. After a code window was developed, an individual researcher would code the data and a second researcher would verify codes. Any differences in coding were resolved through discussion to arrive at 100% agreement.

Figure 1 illustrates the coding window used to analyze teachers' mathematical discussions in both small and whole group settings. Data were analyzed using the video analysis software, *Studiocode*. Participants' written mathematical inscriptions were also examined to look closely at what is referenced in video and the mathematical nature of the inscription.

Each video episode was coded as follows: Each discussion was coded into *idea units* based on the main speaker and the claim being argued or challenged. Each idea unit was then further described using elements of the argument structure: claim, warrant (referred to here as *evidence-how*) and backing (referred to here as *evidence-why*). Evidence-why codes were distinguished by levels of justification (Simon & Blume, 1996) to describe the backing for a solution. In addition to those elements of an initial argument we have also included challenges to a claim (e.g., press for justification, stumbling) and connections (e.g., to other solutions, across multiple representations) from the main speaker or the rest of the group. Finally, we coded when teachers explicitly identified key math ideas or implicitly discussed key math ideas embedded in their arguments. In using this code, we are not suggesting that an individual teacher knows a particular key idea. Instead, we are highlighting the ideas that are made public that have potential to serve in the development of SCK. Because our teachers, at times, would narrate their experience by talking about themselves as learners or how the work connected to other experiences we coded their narrations as *meta-talk* and *accounts of experience* to capture teachers' disclosures made during math activity.

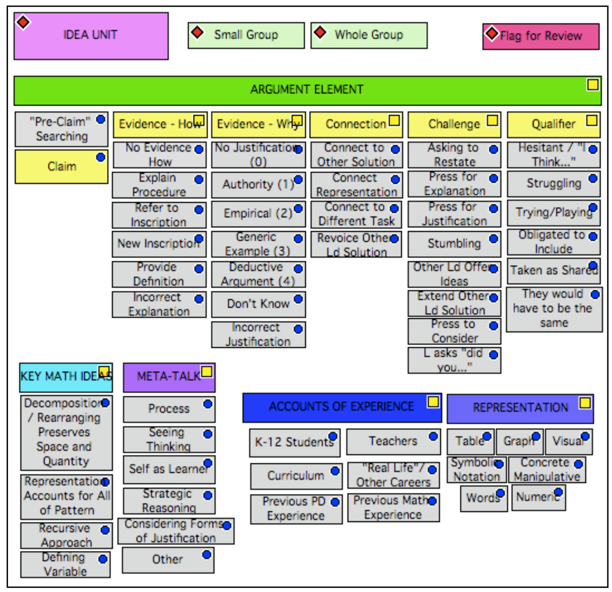
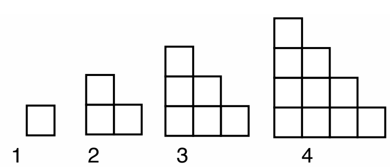


Figure 1. *Studiocode* Coding Window

Discussion of an Illustration of Teachers' Collaborative Mathematical Work

To illustrate how a consideration of both teachers' justification productions and the argumentation structure of a teacher group support the interpretation of complex discourse of teachers in PD, analyses from our larger project will be discussed here. Data for the discussion in this paper came mid-way through the RMLL seminars where participants were engaged in collaborative discussions of solutions to a math task (the *Staircase* task; see figure 2). The *Staircase* task is a commonly used task in PD with the potential to highlight important mathematics ideas such as understanding how models can be decomposed and rearranged while preserving space and quantity and mapping relationships among representations, in particular how the composition of a figure is accounted



1. Build the next two staircases.
2. Describe the 10th staircase.
3. What is the relationship between the staircase number and the number of cubes it takes to build the staircase? See if you can find a rule to represent that relationship.

Figure 2. The *Staircase* Task

for in a symbolic expression.

The episode shown here is a discussion among three Phase I participants: Wendy, Hal, and Erica (all pseudonyms). The transcript of the segment, our coding of this instance, as well as the relevant inscriptions can be found in Figure 3. In the episode, Wendy is presenting her solution to the *Staircase* task, which she mentions relates to a solution that Hal just presented. She begins by verbally recounting the way she recomposed the visual model in order to create a "duplicate" staircase. In the process of drawing a connection between her visual model and her symbolic solution, Wendy stumbles. Her next move we code as a new solution, because it uses her inscription differently. Wendy considers the re-conceptualizing of the inscription and its correspondence to a symbolic expression. Her restated idea exhibited a more transparent link between her representations and the idea of a duplicate staircase (and, thus, needing to divide by two). This move makes available the idea that decomposition preserves quantity. Hal's revoicing of Wendy's idea provides evidence that this connection has been taken up publicly.

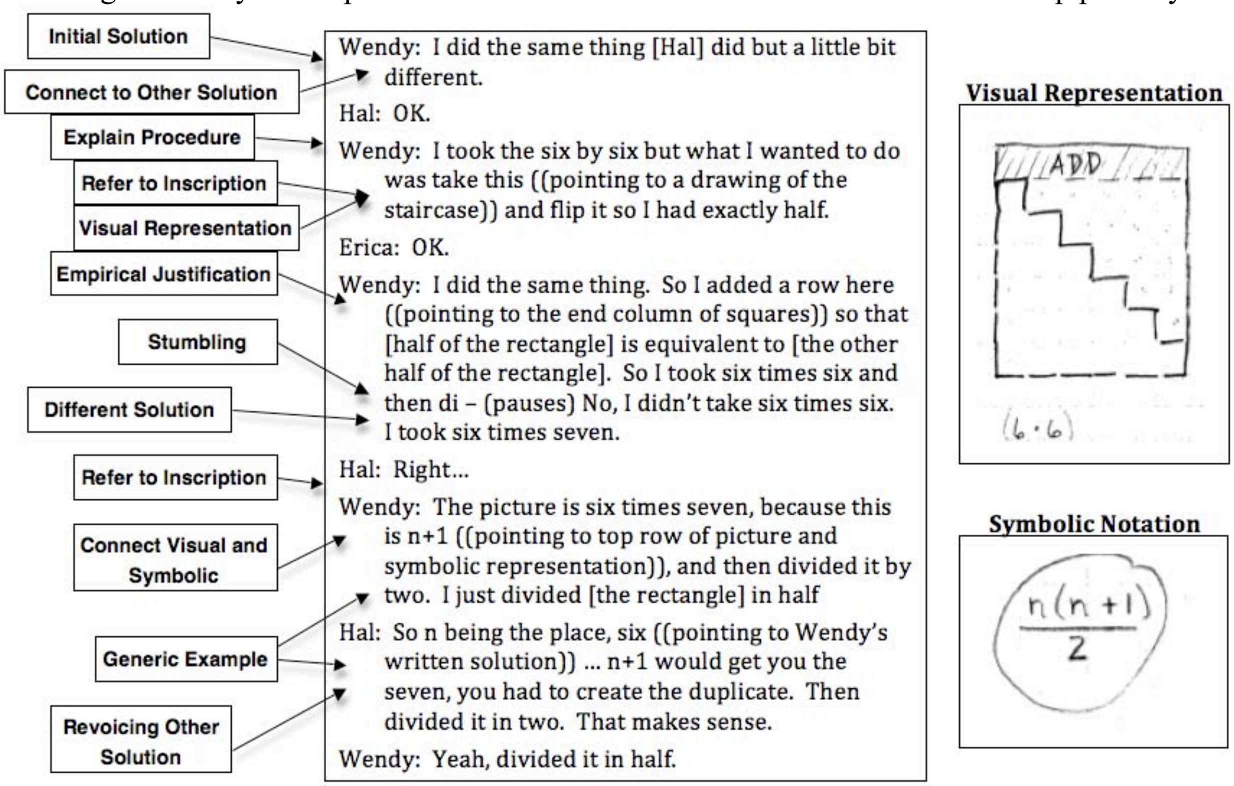


Figure 3. Teachers' Discussion and Inscription Around *Staircase* Task

What does this say about doing math in PD? In Wendy's first solutions we do not see her connect to a symbolic expression in a transparent way but she provides an explanation with a mention of another solution. We see that stumbling acts as a challenge that has opened up her reasoning as she connects her model to a symbolic expression such that the decomposition of the figure corresponds and is accounted for symbolically. Hal's later revoicing allows us to see his reasoning connected to her reasoning. As we examine the claims that these teachers are making – how they get supported, challenged, and used in the collective space – we are able to see the mathematical content that is emerging in the collective through the argument, interactions, and use of inscriptions. Further analyses of subsequent data will allow us to trace mathematical ideas across teacher discussion and consider what mathematical ideas are put on the table by this group, which seem to be taken up and unpacked through considerable discussion of claims, warrants (explaining how), backings (explaining why), challenges, connections and qualifiers. This analysis allows us to move our examination beyond an inventory of solutions and representational use. We are teasing out what is entailed in doing mathematics, the role of inscriptions and discourse. Subsequent data analysis will also examine the role of facilitation in teachers' collective mathematical work as the small group moves to a whole discussion.

Conclusion

In this paper we have provided the theoretical and analytic considerations for examining teachers' collective mathematical work accounting for argumentation, levels of justification, and key mathematical ideas. Previous research has accounted for teachers' argumentation without incorporating attention to justification or key mathematical ideas emerging in teachers' collective mathematical work. Important to note in these analysis is the role of our video coding tool, *Studiocode*, that makes it possible to examine over 20 episodes of teachers doing mathematics and provides methods for examining patterns across the 20 episodes to consider how teachers' mathematical learning emerges in the community.

The framework we have presented here affords ways of capturing the complexity of teachers' collaborative mathematics talk. With this framework, we are able to look at a wide range of data to better understand the aspects of collaborative talk that advance mathematical learning. The excerpt of data we chose for this paper also illustrates the framework's ability to highlight the opportunities for key mathematical ideas to emerge in discussion, such as "stumbles", outside challenges, and building connections across representations and solutions. As we continue to move forward with our examinations of teachers' collaborative mathematics talk in order to better understand these opportunities we plan to consider more deeply what stumbles or challenges provide teachers in their mathematical learning and uncover more aspects of collaborative work that facilitate the development of SCK.

Acknowledgements

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GROWTH IN SECONDARY TEACHERS' CONTENT KNOWLEDGE AND PRACTICE IN DISCRETE MATHEMATICS

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Thirty-five teachers teaching advanced topics in high school mathematics were engaged in a year-long intensive professional development project intended to increase their Mathematical Knowledge for Teaching in the area of discrete mathematics. Project activities embodied a models and modeling perspective on pedagogy and learning. Results showed that participating teachers' content knowledge improved significantly in discrete mathematics, their classroom practice improved significantly. Additionally, participating teachers' students outperformed their counterparts in a matched-control group in achievement. Results provide evidence of the effectiveness of mathematical modeling for developing teachers' mathematical knowledge for teaching, and on improving secondary mathematics teachers' practice.

With the recent push to increase the general mathematical competence of US high school graduates, the State of Arizona has instituted new standards incorporating significant aspects of discrete mathematics, including graph theory, combinations and permutations applied to the solution of conflict problems, probability and statistics, and shortest path algorithms (Standards Committee, 2008). This subject matter is becoming increasingly important in scientific and economic modeling, and is now considered essential background for college- and work-readiness.

Two problematic issues arise when attempting to infuse discrete mathematics content into the current high school curriculum. The first concerns the ability of teachers to effectively deliver high quality instruction given that discrete mathematics has historically been given only cursory (if any) treatment in their college-level mathematics courses (Rosenstein, Franzblau, & Roberts (1991, Eds). The second issue, and one that is at least as important as teacher knowledge is the fact that there is no coherent set of curricula (i.e., texts, materials) for teachers to use to structure their implementation of the new standards.

To address these concerns, a partnership was formed between a large Southwestern University, and a large urban school district to substantially increase the pedagogical content knowledge of high school mathematics teachers in the areas of discrete mathematics and mathematical modeling, and to create a coherent course of study around discrete mathematics appropriate for a 4th year of high school mathematics. The project worked toward four measurable goals and used formative evaluation as a guide for on-going program development and refinement to meet the goals.

Goals

1. Increase mathematical knowledge for teachers (MKT) for Junior-Senior level teachers (e.g., Hill, Ball, & Schilling, 2008);
2. Create a coherent, rigorous curriculum for fourth year courses that develop student knowledge and skills in Discrete Mathematics and modeling (e.g., Zawojeski, Lesh, & English, 2003)
3. Increase student achievement specifically focusing on fourth year courses that develop student knowledge in Discrete Mathematics and modeling
4. Develop student's content knowledge through the use of workplace technology

Goals were designed to ensure that this learning and growing expertise was sustained and shared with district instructors and colleagues.

Project Activities

The partnership identified 10 campus instructional leaders and an additional 25 mathematics teachers proportionally distributed by student population across high schools. The project developed an integrated system of PD, curriculum, and technology implementation in the area of Mathematical Modeling. Mathematical Modeling was chosen because it integrates all aspects of the College and Work-Readiness Standards in Arizona, because it draws easily upon students' prior knowledge of Algebra and Geometry, and because it provides important contexts for learning and applying discrete mathematics in realistic scientific and workplace scenarios Figure 1 (below) shows project activities arranged across the 14 months of project funding.

The meat of the program has been a monthly series of 8-hour professional development sessions designed to increase teachers' pedagogical content knowledge in modeling discrete mathematics topics. These sessions are bookended by week-long (40 hours each) Summer

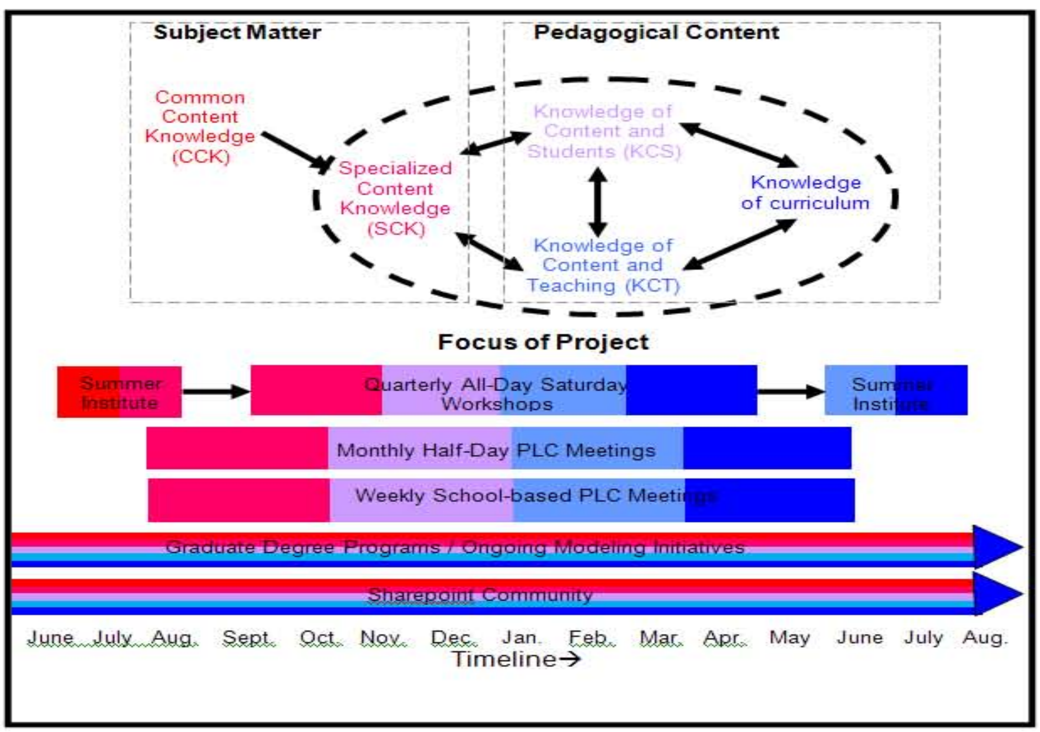

Figure 1. Professional development foci and timeline

Workshop. Teachers are organized across schools into curriculum development teams, tasked with designing the 4th-year curriculum structure and tasks for subsequent implementation in the 2010/2011 academic year (see Figure 1).

Initially, forty two teachers were provided 135 contact hours of professional development utilizing two 40-hour intensive summer workshops that introduced participating teachers to the content of discrete mathematics, including graph theory, combinatorics, probability and statistics. Institute faculty emphasized a variety of researched best practices, focusing on modeling high cognitive demand tasks in problem-based-learning scenarios. The use of readily-available software and mathematical technology was emphasized throughout these sessions. Technology included simulation tools, probeware and data analysis tools, and graphical/visualization tools. Throughout the school year, additional content instruction was provided to teachers in each of 9 additional 8-hour Saturday sessions.

Also throughout the school year teachers implemented these best practices by developing and piloting curriculum for a new 4th-year course designed to embody the College Ready and Work-Ready Standard for Arizona Mathematics. This course is targeted towards non-calculus-intending students, and is pedagogically based on mathematical modeling. Teachers engaged in iterative design cycles (Lamberg & Middleton, 2009) to develop and test lessons and whole units in their classrooms, videotaping their practice, and collecting student work samples for use in subsequent professional development. These data records were used to improve the designed courses and develop teachers' guides to assist other teachers in implementing the College and Work Ready Standards in their classrooms, focusing on discrete mathematics, probability and statistics. All work products were uploaded to the project Sharepoint® website to enable ready access for teacher teams to updated materials.

Evaluation

Participants

The project was evaluated using a pre-post, quasi-experimental design with a matched comparison group. Each school was asked to provide a comparison teacher for any participating teacher. Matched comparison was based on number of years teaching, grade level taught, and math courses taught. Twenty-seven project teachers completed all of the 135 hours of professional development, and had complete data. Forty teachers, who received no professional development in discrete mathematics or mathematical modeling comprised the comparison sample.

Measures

To measure teachers' practical change, the Reformed Teaching Observation Protocol (RTOP) was employed. The RTOP was developed as an observation instrument to provide a standardized means for detecting the degree to which K-20 classroom instruction in mathematics or science is "reformed". Pre-RTOP data have been used to assess the quality of instruction against an external metric of excellence, and fidelity of implementation. In addition, for teacher content knowledge, a cognitive growth instrument, The Discrete Math Content Measure (DMCM), was developed (the pre-test was administered in 2009 and the post-test administered in 2010). The overall reliability (Chronbach's Alpha) of the exam as a whole was 0.70. Student achievement was measured with AIMS scores and CRT (course) scores yearly.

Observational Protocol

There were five observers that completed the RTOP evaluations for the participant group. The observers completed RTOP training through center for Research and Innovation in Mathematics and Science education (RIMSE), with Arizona State University. Each participant and contrast group teacher was observed in a typical class where they taught discrete mathematics topics: Once in the Fall semester, 2009, once in the Spring Semester, 2010. There were no major differences in the timing or method of the administration of the RTOP evaluations in the pre or post RTOP.

Results

Teacher Knowledge

Results from the DMCM analysis shows a significant difference between the 2 groups (*p*-value = .00) as post-test scores for the Participant (*M* = 18.29) were higher than the comparison group (*M* = 11.95) See Table 1 Results from the DMCM analysis for the 27 program Participants show that there were gains from pre (*M* = 10.47, *SD* = 3.41) to posttest (*M* = 18.29, *SD* = 3.29), and the gains were significant (*p*-value = .00). Results from the analysis for the comparison group revealed that there was little change between pre-test mean scores (*M* = 11.42, *SD* = 2.72) and post test mean scores (*M* = 11.95, *SD* = 3.17) and the difference was not significant (*p* = .24).

Discrete Mathematics Content Measure: Participant Group			
Group	**Pre-score Mean (SD)**	**Post-score Mean (SD)**	**Significance p < .05**
Participant	**10.47** (3.41)	**18.29** (3.30)	**.00**
Control	11.42 (2.72)	11.95 (3.17)	.24

Table 1. DMCM comparison of participant and comparison groups

Teacher Change in Practice

Differences in pre- and post-classroom observation (RTOP) average scores indicate a positive gain in those items relating to reformed and inquiry-based classroom from mean of 59.78 at the beginning of the MSP to a mean of 68.56 following the MSP professional development intervention. The gains were significant (*p* = .028). The Post RTOP score (*M* = 68.56, *SD* = 17.29) revealed a nine-point gain from the Pre RTOP score (*M* = 59.78, *SD* = 17.59), and the gain was significant (*p* = .028). PUHSD had a 15% change in pre-post RTOP scores.

With regards to the relationship between teachers' RTOP scores and their student achievement, we found that, for our data, the Spearman rho between the RTOP & post-DMCM was to be .50, a moderate-to-high, statistically significant correlation (p = .007). This finding indicates that project teachers who learned more mathematics, also taught mathematics in a more reformed manner on average, and that this reformed practices was related to higher student achievement.

Student Achievement

The project measured student growth using the State Arizona Instrument to Measure Standards, a Criterion-Referenced test bench-marked on the State Mathematics Standard. Unfortunately, in the middle of the project, the State elected to change the test, and so pre- and post administrations are not measured on the same scale. Overall in the State, scores on the 2010 AIMS are lower than on the 2009 administration due to this change. For the 2010 administration of the Arizona Instrument to Measure Standards (AIMS), there were 610 students in participating classrooms, primarily 10th graders, taking the test for the first time. Of those students, 338 students (55%) scored proficient or above and 272 students (45%) scored basic or below. The comparison group teachers tested 899 students that were 10th graders. Of those students, 405 students (45%) scored proficient or above and 494 students (55%) scored basic or below. This provides some evidence to suggest that the teaching practices in the MSP program contributed to student learning.

Course	Group	Mean	Std. Deviation	N
Algebra 1	Control	36.59	11.041	191
	Particip	42.26	11.502	111
	Total	38.68	11.525	302
Geometry	Control	51.80	13.510	574
	Particip	52.96	15.038	353
	Total	52.24	14.115	927
Algebra II	Control	66.83	10.567	81
	Particip	73.40	7.066	70
	Total	69.87	9.661	151
Pre-Calc/ Calculus	Control	73.67	5.508	3
	Total	73.67	5.508	3
Academic Decathlon	Particip	70.80	9.834	5
	Total	70.80	9.834	5
AIMS prep Math	Control	44.34	12.992	50
	Particip	50.37	10.622	49
	Total	47.32	12.200	99
Enrichment	Control	47.64	11.582	22
	Total	47.64	11.582	22
Total	Control	49.57	15.201	899
	Particip	53.10	15.731	610
	Total	51.00	15.510	1509

Table 2. Descriptive Statistics on the AIMS-Mathematics test for Students of Participating and Control Group Teachers

A univariate post-test, experimental/control group model was applied to the analysis of student achievement data. The dependent measures included students' AIMS mathematics raw score, the ACT composite mathematics score, ACT Algebra/Coordinate Geometry score, and ACT Plane Geometry/Trigonometry score. It was hypothesized that students of teachers participating in the Modeling MSP would show greater student achievement on these outcome measures than teachers who had not participated in the program. In addition, because typical high school courses (e.g., Algebra 1, Geometry, Algebra II, etc) would show different levels to which course content fit the content in the MSP, we analyzed differences in student achievement by course, and we accounted for the interaction effect between project participation and course by performing a factorial analysis. All family-wise alpha coefficients were set to .05 (1-tailed). Table 2 shows results for the analysis of the AIMS-mathematics test by course and by participation.

Results indicate that students of participating teachers outperformed their peers in classes taught by control group teachers. When analyzing the main effect for course, as expected, students in higher level courses showed greater AIMS achievement. Most importantly, the main effect for overall mathematics achievement as measured by the AIMS-mathematics test was found to be superior for students of participating teachers when compared to their matched control group (F(1,1497) = 8.58, p = .048). In analyzing the interaction of the two conditions, the data show that for every class where comparison was possible, participating teachers' students outscored control group students. Overall, these differences are statistically significant (F (3,1497) = 3.87, p = .009). Details of the analysis are presented in Table 3 below.

Source		Type III Sum of Squares	df	Mean Square	F	Sig.	Observed Power
Intercept	Hypothesis	153458.779	1	153458.779	329.403	.002	1.000
	Error	1000.119	2.147	465.870[b]			
Course	Hypothesis	99905.196	7	14272.171	36.117	.001	1.000
	Error	1966.882	4.977	395.161[c]			
Particip	Hypothesis	4396.818	1	4396.818	8.585	.048	.575
	Error	1867.314	3.646	512.157[d]			
Course * Particip	Hypothesis	1945.671	3	648.557	3.866	.009	.825
	Error	251110.353	1497	167.742[e]			

Table 3. Univariate ANOVA--Effects of participation and course on student achievement on the AIMS-mathematics test.

These findings are in alignment with the goals of the project. Since all project activities were designed explicitly to embody the Arizona Mathematics Standard, it was expected that on a test of this Standard, students of teachers who had more experience learning the mathematics related to the Standard would perform better. Additionally, it provides evidence that modeling as an environment for enhancing teachers' understanding and pedagogical skill in applying the Standard content is a viable strategy.

In addition to the AIMS data, the ACT was utilized as a nationally-validated outcome measure to augment the AIMS data. We analyzed student achievement using three of the ACT scale-scores: ACT Mathematics Composite, ACT Algebra/Coordinate Geometry subscale, and ACT Plane Geometry/Trigonometry subscale. Results are mixed, showing no overall effect of project participation on ACT Mathematics Composite scores, or on ACT Plane Geometry/Trigonometry scores. A main effect for Course was detected for each of these subscales, with students enrolled in higher courses performing better on both (See Table 4 below).

On the ACT Algebra/Coordinate Geometry subscale, a significant main effect was found for Course, but not for Participation. The interaction effect, however was found to be statistically significant. A Scheffe post hoc test revealed that in Algebra II, the largest represented course in the study, Participating teachers' students outperformed those of Control teachers ($p < .05$).

Source		Type III Sum of Squares	df	Mean Square	F	Sig.	Observed Power
Intercept	Hypothesis	3107.319	1	3107.319	798.648	.000	1.000
	Error	3988.149	1025.040	3.891[b]			
Course	Hypothesis	1401.903	7	200.272	15.917	.001	.999
	Error	84.387	6.707	12.582[c]			
Particip	Hypothesis	.023	1	.023	.003	.957	.050
	Error	284.312	35.854	7.930[d]			
Course * Particip	Hypothesis	75.155	5	15.031	2.425	.034	.771
	Error	8028.120	1295	6.199[e]			

Table 4. ANOVA--Effects of participation and course on student achievement on the ACT algebra/coordinate geometry subscale.

Discussion

Results show that teachers, given an intense, sustained set of professional development experiences organized around mathematical modeling, can learn significant new content, change their practice to reflect modeling instruction, and increase the mathematical achievement of their students in the process. Several problems, however, have been identified related to attracting and retaining teachers over the long haul. In particular, retention of participants was an ongoing challenge. The number of original participants was identified as 60; however 43 instructors participated. Participants dropped out for various reasons: one teacher passed away, one was taking care of a dying parent, one started their Master's degree and had a time conflict, three teachers were involved in remediation and wanted to discontinue, two were getting paid to do Saturday School so there was a conflict with their schedules, and others stopped coming without explanations.

Moreover, implementing a two-week summer workshop may not be the most effective way to provide the bulk of the learning. Teachers experience a high level of mental fatigue and express feelings of being overwhelmed using this format.

RTOP observations and teacher comments continue to demonstrate the need to model and explicitly identify reformed teaching practices and an understanding of how students learn mathematics content. During the final weeks of the project, teachers were expected to implement specific instructional strategies in their classroom. During the RTOP Evaluation Dissemination session, were able to share experience and ask follow-up questions regarding each RTOP category and item. We conducted the RTOP Evaluation Dissemination after the first RTOP. The team felt that the process of identifying each item, addressing strengths and weaknesses in the overall group and modeling exemplar teacher performance (with MSP teacher participants) would help the majority of teachers be more successful in terms of developing pedagogical content knowledge and providing more effective mathematics instruction for their students.

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LEARNING TRAJECTORIES AND KEY INSTRUCTIONAL PRACTICES

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As learning trajectories gain traction in mathematics education, we seek to understand the ways in which teachers may use them in interactions with students. This paper reports on one group of elementary teachers' use of their emerging knowledge of a learning trajectory to examine key pedagogical practices. Findings suggest that a learning trajectory helps teacher place student thinking at the center of these practices.

In this paper we examine the role a learning trajectory (LT) that characterizes students' progression from less to more sophisticated levels of understanding plays in teachers' analysis of classroom discourse orchestration. In the introduction of the *Common Core State Standards* (CCSSO, 2010), the authors of the document highlighted the role LTs played in the initial development of the new standards and reported that the development of the standards began with research-based trajectories.[1] LTs have been demonstrated effective in assessment design (Battista, 2004), curriculum development (Clements, Wilson, & Sarama, 2004), and as a tool for teachers to identify students' ideas in instruction (Furtak, 2009; Mojica, 2010; Wilson, 2009). By providing a framework for aligning evidence of student cognition with research findings on likely tendencies for the development of children's mathematical conceptions, LTs support teachers in locating students' ideas within a range of conceptual development and offer a theory of how mathematical ideas develop overtime (Wilson, 2009).

The recent release and adoption by many states of the Common Core State Standards recalls the question about the relationship between standards, professional development, and instructional practice. Cohen and Hill (2001) showed that in the absence of focused professional development on the content and the ways in which students learn that content, it is unlikely for new standards to result in changes in teachers' practices or increases in student achievement. Further, these authors indicated that professional development related to changes in standards should provide teachers with opportunities to explore and understand new standards through examining student thinking, work samples, and curricular connections if meaningful change is to occur. Thus, we propose that professional development focused on LTs can provide opportunities for teachers to examine both changes in content and the ways in which students think about that content in light of the new standards. Further, we seek to understand how participation in professional development that attends to LTs supports participants' examination of classroom mathematics instruction during whole class discussions.

Perspectives

Learning trajectories are descriptions of the qualitatively different levels of reasoning sophistication through which students' concepts pass as they evolve from informal ideas to complex understandings. With roots in Simon's (1995) hypothetical learning trajectory, LTs bring clarity to the intermediate understandings a student may have of a mathematical concept. LTs outline how those intermediate understandings relate to previous ideas as well as to those that will evolve from them. In our work we use Confrey et al.'s (2009) definition of a learning trajectory: "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (p. 347). In this definition, the role of instruction is highlighted, making teachers and the practices they employ to create learning environments for students central in the development of students' understanding of mathematical concepts.

Researchers have recently begun to investigate the ways in which teachers may use LTs in their practice. For instance, Wilson and Mojica (2010) reported that knowledge of an LT supported both practicing and prospective teachers in constructing models of students' thinking. Furtak (2009) concluded that knowledge of an LT helped teachers in viewing students' conceptions along a continuum of proficiency rather than simply "right" or "wrong." Wilson (2009) described how LTs assisted teachers in interpreting evidence of cognition when analyzing student work and in interacting with students during instruction. He suggested that LTs sensitize teachers to a variety of strategies used by students when gaining proficiency with a mathematical idea. He also conjectured that LTs provide a framework for teachers to organize students' work in class discussions. The research reported in this paper investigated this last conjecture further.

This study is part of a larger project that aims to conceptualize the notion of *Learning Trajectories Based Instruction*, broadly defined at the outset of the project as the ways in which teachers use their own knowledge of a LT to organize their instructional practices and participate in their professional communities. As the context for our investigation, we are working with one LT: the learning trajectory for equipartitioning (EPLT), which is the set of cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing 'fair shares' for each of a set of individuals (Confrey et al., 2009). The EPLT describes how children's informal experiences with creating fair shares evolve over time to a robust understanding of partitive division. The proficiency levels of the EPLT document the strategies, practices, emerging relationships, generalizations and the misconceptions children experience through instruction as they mature to an understanding of $a \div b$ as creating b equal-sized parts of a.

As students engage in equipartitioning tasks, the EPLT describes how they work to coordinate three equipartitioning criteria: (1) students must create the *correct number* of groups or parts; (2) students must create *equal-sized groups or parts*; and (3) students must *exhaust the whole or collection*. Also, the EPLT outlines how different parameters affect the level of difficulty of the task, such as sharing a whole among eight people is less difficult than sharing a whole among five people. The trajectory goes on to characterize the practices of naming and justifying answers to equipartitioning problems. For example, children proceed from nonmathematical names for each share such as parts or pieces, to counts and then to numbers in relation to a unit. Three important emerging relationships are also considered in the trajectory: compensation, the idea that producing more equal-sized parts decreases their size; composition of splits, the idea that one split across another split affects the size of every part in a multiplicative way; and transitivity, the idea that non-congruent parts resulting from the same split on the same whole are of equal size.

To frame our examination of connections between teachers' experiences with EPLT and pedagogical practices that support the orchestration of productive mathematics discourse in the classroom, we draw on the work of Stein, Engle, Smith, & Hughes (2008). These authors proposed a set of five practices to examine teachers' work in supporting discourse: anticipating, monitoring, selecting, sequencing, and connecting. In this paper, we focus on the last three of these practices. *Selecting* is the practice of choosing particular students to share their work with the rest of the class to get certain ideas on the table while remaining in control of which students present and what the mathematical content of the discussion will likely be. *Sequencing* is the practice of making decisions about how to order the students' presentations to maximize the chances that the mathematical learning goals for task are achieved. *Connecting* is the practice of helping students develop relations among presentations and judge the consequences of different approaches for various problems, making sure students' presentations build on each other to develop powerful mathematical ideas.

Methods

The specific research question addressed in this study is: In what ways do teachers use their understandings of EPLT to select, sequence and connect students' mathematical ideas for whole class discussion? We conducted a design experiment within a school-based professional development setting. Design experiments are used to provide "systematic and warranted knowledge about learning and to produce theories to guide instructional decision making" (Confrey, 2006, p. 136). They "entail both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003, p. 9). The data set for the study is comprised of videos of professional development meetings and audiotapes (with transcripts) of small group discussion during the professional development. As it is usual for design experiments, data analysis consisted of constant comparison methods (Glaser, 1992; Strauss & Corbin, 1998), which allows for the creation of emerging categories in the data analysis and the refinement of these categories as they are contrasted with new information. The various sources of data were also used for triangulating information (Miles & Huberman, 1994) in search of both confirming and disconfirming evidences. As we examined the ways in which teachers engaged with one particular professional learning task, we coded the data for teachers' use of their knowledge of particular aspects of the EPLT to justify their arguments about how to select, sequence, and connect students' responses in whole group classroom discussion.

Participants

We partnered with one elementary school in a mid-size urban area in the southeast. Twenty-four K-5 teachers from this school volunteered to participate in the professional development. Only three of these teachers had less that 3 years of teaching experience and about half of the teachers had more than 10 years of experience. Many of the teachers have been at this school for a few years. Overall, this school has a yearly turnover teacher rate of 4% and a student population of approximately 600 students. The student population in the 2010-2011 school year was 35% Caucasian, 29% Hispanic, 25% African American, 7% Asian, and 4% other, with 54% of the children on free or reduced lunch.

Professional Development Design and Professional Learning Task

The professional development was designed to offer a total of 96 hours of work on the concept of EPLT over a period of 12 months. The initial 30 hours of professional development were offered as a Summer Institute, followed by 54 hours of face-to-face and web-based activity during the school year, ending with 12 hours the following summer. This report is based on the work conducted during the Summer Institute. The 30 hours of work for the institute were spread over 6 days, immediately before the 2010-2011 school year started. We worked with the teachers for 6 hours during the first 3 days, and then for 4 hours during the following 3 days. All meetings took place at the school, giving teachers the opportunity to also work on organizing their classrooms after the professional development meetings.

The research reported in this paper concerns a professional learning task that was posed to the teachers on the fourth day of the Summer Institute. On Days 1–3 teachers had investigated the concept of equipartitioning, the proficiency levels of the EPLT, and the parameters affecting the difficulty of tasks at a particular level. Through analysis of student work and video clips of children solving various equipartition tasks for collections and single wholes, teachers had discussed the equipartitioning criteria, naming practices, and emerging relations such as compensation, composition of splits, and transitivity.

We began Day 4 of the Summer Institute with a task that aimed to provide teachers with an opportunity to make explicit connections between their learning in the professional development and their classrooms instruction. In this task, we purposefully asked participating teachers to select, sequence and connect student work. Teachers were given a scenario where students in a second grade classroom had worked on the task of fairly sharing a rectangular cake among four people and the second grade instructor saw four different solutions to the task (represented in figure 1, labeled A, B, C, and D). In cross-grade-level groups, teachers were asked to discuss the following question: “How would you select and sequence the presentation of these solutions for a whole class discussion? Why?”

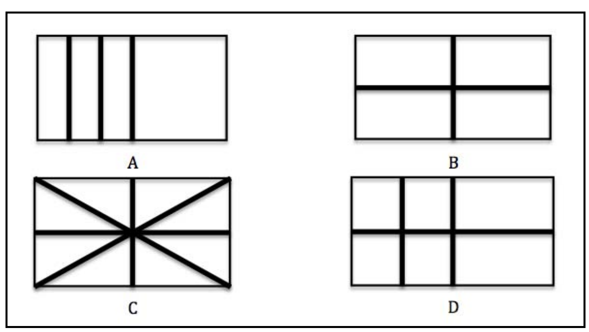

Figure 1. Representations of the approaches to sharing a whole among four.

After the small group discussions, the teachers viewed an 18-minute video recording of the actual second grade classroom. In the video the classroom instructor starts by having students share solution C. The students indicated that each person would get two pieces and that each piece was one-eighth. Next, a student shared approach A, and the instructor guided the students to see that each of the four pieces needed to be the same size and that if they were of equal size, then each could be called one-fourth. Approach D was shared next and students told the class that they realized that although they had not used all of the cake. The instructor concluded the sequence by having a pair of students share approach B, and made connections to the two-eights created in approach C and the parts created in approach D.

In the whole group discussion of the video, teachers examined the instructor's choices in selecting, sequencing, and connecting her students' responses. After each solution was shared with the second grade class in the video, the recording was paused and the teachers were asked to discuss the mathematical issues that arose as students shared a particular solution. They conjectured what they would do at that particular moment in the lesson if they were teaching it and examined the connections the actual instructor made.

Results

Selecting and Sequencing

Selecting and sequencing student work was not a trivial task for participating teachers. It elicited a rich discussion in the small groups and only one of the cross-grade-level small groups actually agreed on a sequence for presenting students' answer: D, A, B, and then C. One group reached consensus that they should begin with approach A, and another group agreed they would begin their sequence with approach D. The two other groups failed to reach a consensus, with one not being able to solve whether they would begin with A or C and the other debating whether to begin with A or D. In the process of these discussions, three aspects of the EPLT emerged in teachers' justifications for how they would select and sequence these particular students' solutions: the three equipartitioning criteria, students' misconceptions, known strategies for creating equal parts.

All of the small groups made reference to the three equipartitioning criteria as they worked to make sense of the four approaches. The teachers used the criteria both in understanding solutions A through D and in examining how they might sequence these solutions for discussion. In selecting a response to begin a classroom conversation, one teacher told her group, "I would start this one [*D*] just because I think this is the problem that…that they're not using the whole. They are thinking it's a cake and they've made four pieces but they've not used the whole." In another group, a teacher stated, "I would probably share the top left corner [A] also and then talk about, you know, first that you did use a whole; that you do have the right amount of pieces, and then really look into 'are the pieces equal?'" In trying to decide which of the criteria she should address first, one teacher questioned her group, "Did we ever establish that there is necessarily a hierarchy of those three criteria?" In this, the teacher was not only recalling the 3 criteria from the EPLT but was also adopting a perspective of student thinking as a continuum of least to most sophisticated. These examples are representative of the teachers' comments across the small groups. Being a pre-requisite for obtaining fair shares, all teachers considered that these criteria needed to be explicitly addressed in the classroom as they made sense of various students' responses.

Concerning the three criteria, approach C generated interesting discussion as one teacher questioned whether the student had created the correct number of parts. Other teachers wanted to use approach C to think about a ratio relationship. "The reason I would do it [C] is because it seems like you can turn that into a higher order, you know, solution. You can say, Okay, I see this one has eight wedges, whatever. Eight triangles. There are eight pieces and we have four [people]." For these teachers, although approach C showed eight pieces, it represented an opportunity to explore equivalent fractions. In another group, a teacher noted the same idea: "So that would be a really good teaching point if you want it. I mean, that… showing the eight could be a really powerful thing."

Teachers also made use of known misconceptions considered in the EPLT as they discussed how they would select and sequence students' solutions in their classrooms. Four of the small groups referred to the misconception that when creating equal-sized parts of a whole some children believe the number of cuts they make is the same as the number of parts they wish to create, usually in the context of making parallel splits. Though the work samples used in the activity did not depict this exact misconception, teachers recalled it in their group conversations. In particular, teachers were concerned that approach C could foster such misconception, as one teacher argued: "One of the dangers is them equating making just four cuts and then to always…it'll always work." Later she repeated: "there is a risk in this [C] if they're going to think 'oh, make four cuts, no matter what'." Similar exchanges happened in three other small groups. As teachers discussed their sequences of approaches for class discussion, erroneous conceptions documented by the EPLT influenced their analyses and considerations of how to chose and order the students' approaches.

In relation to students' equipartitioning strategies, two of the small groups explicitly discussed but all groups made at least some reference to students' use of "halving" and "repeated halving" as a strategy, as well as the relative ease of equipartitioning wholes into two equal parts. In the previous days of the Summer Institute, teachers had learned a variety of different strategies students use to equipartition a whole described in the EPLT, but in examining the given sample of student work, they repeatedly related to halving and repeated halving. As one teacher observed, "I guess given what we talked about last week, halving and halving is easiest. But I mean, like, you know, what would my reasoning be for sharing the easiest?"

Connecting

When watching the actual classroom video, the teachers drew on several ideas from the EPLT to describe the connections the instructor made and to conjecture others. As in the selecting and sequencing small-group exercise, teachers referred to the three equipartitioning criteria to understand the instructor's sequencing and to describe the relationships among the approaches. For instance after sharing the first approach [C], one teacher suggested that the teacher should focus on the number of parts created in subsequent discussion. In response to a question from the researcher about what the instructor may be thinking and where she could take the discussion, the teacher said, "Counting, like 'how many pieces did you make? Let's count them' and like see that there's more than four." After viewing the instructor's discussion of approach D in relation to C, and how two of the pieces created in C matched into one of the pieces created in D, one teacher commented, "And it was really nice how she made the connection back to the previous work to show the change." The three equipartitioning criteria helped teachers notice connections made by the instructor as well as suggest other connections that could have been made.

Another way that the EPLT informed connections teachers noticed was by calling attention to the issues of naming equal-sized parts in relation to the original whole. As the teachers observed the connections the classroom instructor was making in the video, they noted the different naming practices and commented on the instructor's support of its refinement. For example in approach A, a student first referred to a share as "a piece." One teacher commented, "Well she saw that you got a, uh, qualitative answer. We saw that before. A piece. And then she was trying to go for the quantitative one." Another teacher followed up on this comment about naming by pointing out the clarity of the instructor's language. She stated, "And she made a kind of subtle distinction too, because the child who attempted to answer what the second one was, said, 'A second fourth?' And so, she used that but she said, 'a second one-fourth.' Just to kind of clarify that." Thus, the teachers' exploration of the subtleties and challenges of naming parts of wholes supported their noticing of the connections the instructor made.

Perhaps the most profound way that the EPLT affected the connections the teachers suggested was their use of emerging relationships to examine links across students' approaches. While the instructor's connections in the video centered on the three equipartitioning criteria and on naming, participating teachers also focused on emerging relationships. For instance, regarding transitivity, after viewing how two of the small pieces in approach D produced a fourth, one teacher suggested posing a question relating these results to the eights generated from approach C. She stated, "It would have been interesting, I mean, if we had said, okay, well are these…are these two pieces the same [D]? Is it the same fair share as these two pieces [C]?" This led to a discussion among the teachers and PD facilitators about how to show that non-congruent parts are actually the same. Also, the teachers connected the issue of two-eighths and one-fourth to the ideas of compensation and composition of splits. In referring to approach B, one teacher summarized, "Composition of splits - so, if you make a new cut then it affects all the other pieces - but it also is compensation because now there are more people sharing the smaller one." This teacher was noticing that by splitting along the diagonals in approach B, the single split acts on multiple parts, the essence of composition. Further, it creates more equal-sized parts but that each of the parts is smaller. By providing teachers with a description of these relationships that emerge from engaging in equipartitioning tasks, the EPLT assisted the teachers in connections that may support students in gaining proficiency with these emerging relationships.

Discussion

In this paper we examined the ways in which teachers who participated in a 30-hour Summer Institute on the EPLT used their understandings of this particular LT to engage with the three key pedagogical practices of selecting, sequencing and connecting students' mathematical ideas. We noted that teachers used the three criteria for equipartitioning to support their work in all of these three key pedagogical practices and, in fact, these criteria guided most of teachers' decisions when engaged in these practices. Teachers highlighted the importance of supporting students as they develop these criteria by proposing to select and sequence tasks that highlight the criteria and also to use the criteria as a venue for connecting the selected strategies. Teachers also suggested ways to assist students in coordinating the criteria. We also noted that teachers used known misconceptions and splitting strategies when selecting and sequencing students' solutions, whereas they used their knowledge of naming and emerging mathematical relations to examine connections across solutions. Teachers suggested ways to confront students' misconceptions as well as to support the development of emerging relations, which represent more sophisticated levels of thinking in the EPLT.

We started our design experiment using Wilson's (2009) revised conjecture that LTs provide a framework for teachers to organize students' work in class discussions. By examining how teachers used the EPLT in three specific pedagogical practices, we can now further refine the initial conjecture. We contend that LTs allow teachers to put students' mathematics at the center of their instructional decision, bringing specificity to the ways in which teachers' engage with the three key pedagogical practices of selecting, sequencing, and connecting (Stein et al. 2008) student work to generate productive mathematics discourse. The trajectory allows teachers to focus on students' solutions and thought processes to orchestrate discourse, instead of attending to non-mathematical ideas such as what the majority of the class did or what the teachers wanted students to do. Additionally, LTs support teachers in drawing connections among students' ideas and in identifying emergent relationships that build toward more sophisticated understanding.

Endnotes

1. Although the CCSS used the term learning progression, we take learning trajectories and progressions to be synonymous and we have opted to consistently use the latter in our work.

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PRESERVICE ELEMENTARY TEACHERS' USES OF MATHEMATICS CURRICULUM MATERIALS: THE ROLE OF DISEQUILIBRIUM AND HUMAN RESOURCES

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Over the course of two years, data was collected focused on preservice elementary teachers' engagement with mathematics textbooks and curriculum materials in two contexts: (1) a university mathematics education course, and (2) student teaching. To frame the results and suggest opportunities for learning, Feiman-Nemser and Buchmann's (1985) critical analysis of experience in teacher education is utilized. These "pitfalls" of teacher education, or particular types of "inappropriate learning" are used as frames to help highlight important ideas from this research. Results indicate high levels of disequilibrium surrounding preservices teachers opportunities for learning, and also point to the importance of human resources during university coursework and fieldwork. Implications for teacher education are shared.

In 1996, Ball and Cohen asked a critical question: what is, or might be, the role of curriculum materials in teacher learning? To address this question, growing numbers of teacher education courses engage preservice teachers in textbook analysis and adaptation, as well as in the use of *Standards*-based curriculum materials (Frykholm, 2005; Tarr & Papick, 2004). Moreover, many student teachers are placed in school settings where the use of *Standards*-based curriculum materials has been mandated (e.g., Van Zoest & Bohl, 2002). As Ben-Peretz (1984) points out, however, "The ability to grasp the full meaning of curriculum materials is a prerequisite for their professional use in classrooms. This ability has to be developed in pre- and inservice teacher education programs" (p. 11). If teacher education is to play a pivotal role in helping teachers learn from the use of mathematics curriculum materials, it is important to examine carefully the typical experiences and learning opportunities embedded in preservice programs.

Research Context and Results

Over the course of two years, data was collected focused on preservice elementary teachers' engagement with mathematics textbooks and curriculum materials in two contexts: (1) a university mathematics education course, and (2) student teaching.

Preservice Teachers Curriculum Use in a Mathematics Education Course

Research Context

Participants in the first part of the study were 23 preservice elementary teachers in an undergraduate mathematics course titled *Geometry and Computing for Teachers.* As part of a larger research study, students were engaged in using reform-based middle grades curriculum materials as the basis for all mathematical learning. During the fifth week of a 15-week semester, the preservice teachers were given copies of selected student pages from two different sets of instructional materials, one reform-based curriculum, and one traditional curriculum. The preservice teachers were asked to first give an open-ended analysis of each set of instructional materials, and then to respond to 10 questions focused on the comparison of the two sets of instructional materials.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Methods

Data for the first part of this research consists of the 23 preservice teachers' written responses to the assignment described above. To synthesize and interpret the teachers' written analyses of the two sets of instructional materials, the entire collection of papers was reviewed several times by two researchers. In particular, the reviews aimed to identify the criterion the teachers seemed to be using for reading and evaluating the instructional materials, as well as those factors that seemed to be primary in their ultimate decisions about which set of instructional materials they preferred. After these two large reviews, each of the 23 papers were more carefully examined to confirm the tentative themes developed during initial reviews. In all, each individual paper was reviewed at least four times to develop major themes.

Results

To synthesize and interpret the teachers' written analyses of the two sets of instructional materials, common themes were developed across all written components of the project. First, I found that the preservice teachers were in search of familiar (traditional) components in both sets of instructional materials. The majority of the teachers specifically cited their own past experiences with traditional mathematics textbooks and lessons as major influences on their interpretations of the two sets of instructional materials. Second, I found that the teachers tended to use traditional expectations to judge both sets of instructional materials. The influence of preservice teachers' familiarity with more traditional instructional material components was, at times, so strong that it led them to inaccurately describe the two sets of instructional materials or to arrive at questionable conclusions about the materials. Finally, teachers attempted to justify the differences between the two sets of materials. Although they were not asked to do so, the teachers were compelled to discuss why such different sets of instructional materials might exist. In one way or another, they each communicated a belief that each set of materials had been created for a different type or level of learner.

Preservice Teacher Curriculum Use During Student Teaching

Research Context

The participants in the second part of the study were two elementary student teachers, Heather and Bridget. Heather completed her student-teaching internship in Jameson County, where the *Standards*-based *Everyday Mathematics* [*EM*] curriculum program (University of Chicago School Mathematics Project [UCSMP], 2001) was in use, while Bridget completed her internship at a Coopersburg Schools, where the teachers utilized materials from the commercially-developed *Silver Burdett Ginn* [SBG] (Fennell et al., 1999) textbook series.

Methods

The majority of the data was collected through classroom observations (11 observations total) and both informal and semi-structured interviews (7 interviews total). Most observations took place in 2-3 day consecutive blocks and fieldnotes were recorded throughout the entire block of time devoted to mathematics. Artifacts and documents included Heather's lesson plans and student-teaching journals. Analysis of data began at the start of data collection for the study. All fieldnotes were typed within 48 hours of each classroom observation, coupled with analytic notes and memos written at the end of each file. More extensive analysis took place following the completion of data collection. Separate files were created to group data according to developing themes (e.g., lesson pacing, teacher direction, lesson objectives, curriculum script). As major themes developed, lesson segments and interview quotes that appeared to highlight the major aspects of Heather's curriculum use were selected for inclusion in this report.

Results

Below, I briefly describe Heather and Bridget's use of their mathematics curriculum materials for the design and enactment of instruction during their student-teaching internships.

Heather's use of standards-based curriculum materials. Each weekend Heather prepared for the upcoming week's mathematics activities using a copy of the *EM* teacher's guide to develop general plans for her lessons: "On the weekend I'll do an outline for the week and write down roughly what I'm going to do." Heather explained that she looked at the teacher's guide again each morning before teaching: "During specials or snack time, I'll just review the lesson for that day." Heather felt that detailed lesson plans were unnecessary because when she taught, she had "the teacher's manual up there." Although Heather typically planned on her own as she read through the lessons in the teacher's guide, she also consulted her cooperating teacher: "I would ask my cooperating teacher about any questions that came up when I was planning, like about different games or just questions that come up."

Typically, Heather attempted to conduct her mathematics lessons in the specific ways recommended by the 4 to 5 page lesson plans found in the first grade *EM* teacher's guide. She used the guide during instruction to refer to specific tasks and questions to ask students as well as the overall organization of lessons. Heather explained that she tended to rely on the book during instruction because of the detailed, scripted nature of the information contained in the teacher's guide: "I feel like the teacher's guide is a script, so I always have it with me. A lot of times, I feel like if I miss a paragraph in the book then maybe that will throw the lesson off." When Heather adapted the recommendations, her changes usually related to the amount of time to spend on each lesson component. She often experienced difficulty carrying out her lessons in the timeframe she had allotted and, as a result, she sometimes changed the nature of the activities to "make up time."

Bridget's use of a commercially-developed textbook. During her internship, Bridget used the workbook component of the SBG curriculum program and supplemented the workbook with additional tasks and activities. Each week, Bridget met with three other kindergarten teachers to plan for upcoming lessons. The focus of these planning meetings was on the selection of SBG workbook pages and worksheets: "I've been told several times that I needed to make sure that [the students] are getting plenty of paperwork." The teachers used a year-long curriculum plan to identify which pages of the SBG workbook could be used to address the state curriculum standards. Bridget explained that "the principal likes to know what [state standards] we're covering which day."

Although Bridget found the planning meetings to be helpful, she consistently made her own plans after the meetings. As Bridget explained, "The truth is, I am trying to use what they're giving me and add to it where I think it's lacking." For each lesson, Bridget evaluated the SBG workbook offerings according to her informal assessment of students' knowledge, the objectives presented in the state curriculum framework, and her own visions of mathematics instruction. To develop new mathematics activities for use in conjunction with the SBG worksheets, Bridget first consulted the state curriculum framework to identify specific mathematical content, and then tapped other resources, including pages from *Everyday Mathematics* that she copied from teachers in Jameson County, for instructional ideas that would address the needs of her students.

Discussion

The descriptions of preservice teachers' experiences with mathematics curriculum materials not only add detail to what we know about teachers' interactions with and uses of curriculum materials, but also have the potential to offer insight into the role of teacher education in guiding and supporting teachers' ongoing learning with these materials. To frame these results and suggest opportunities for learning, I use Feiman-Nemser and Buchmann's (1985) critical analysis of experience in teacher education.

Pitfalls of Experience in Teacher Education

Although a trust in firsthand experience in learning to teach is common, Feiman-Nemser and Buchmann (1985) examine early experiences in teacher education with a critical eye. The authors ask, "Is experience as good a teacher of teachers as most people are inclined to think?" (p. 53). To explore this question, Feiman-Nemser and Buchmann discuss teacher learning in the moment as well as the "potential learnings – insights, messages, inferences, reinforced beliefs – about being a teacher, about pupils, classrooms, and the activities of teaching" (p. 54). These "pitfalls" of teacher education, or particular types of "inappropriate learning" as described by the authors, are outlined in Table 1. I use these frames to help highlight important ideas from this research.

Table 1: Pitfalls of Experience (from Feiman-Nemser and Buchmann, 1985)

Pitfall	**Description of experience**
Familiarity pitfall	The familiarity pitfall stems from the tendency to trust what is most memorable in personal experience…. Ideas and images of classrooms and teachers laid down through many years as a pupil provide a framework for viewing and standards for judging what [is seen] now (p. 56).
Two-worlds pitfall	The two-worlds pitfall arises from the fact that teacher education goes on in two distinct settings and from the fallacious assumption that making connections between these two worlds is straightforward and can be left to the novice (p. 63).
Cross-purposes pitfall	The cross-purposes pitfall arises from the fact that classrooms are not set up for teaching teachers (p. 63). The legitimate purposes of teachers center on their classrooms, which generally are not designed as laboratories for learning to teach (p. 62).

Learning to Challenge what is Familiar about Curriculum during University Coursework

Research indicates that teachers teach in the ways in which they were taught (Ball & Feiman-Nemser, 1988; Lortie, 1975). Feiman-Nemser and Buchman's (1985) "familiarity pitfall" highlights this idea. The authors suggest that *unquestioned* familiarity is a pitfall in that it "arrests thought and may mislead it" (p. 56). The authors further emphasize, "People do not recognize that their experience is limited and biased, and future teachers are no exception. The 'familiarity pitfall' stems from the tendency to trust what is most memorable in personal experience" (p. 56).

The preservice teachers in my studies experienced the familiarity pitfall. Many brought ideas and images from their own schooling experiences to their teacher education coursework. When we asked our preservice teachers to evaluate and compare mathematics lessons – two fairly self-contained sets of instructional materials that dealt with the same mathematical topic but in different ways – many of their views about what should be in a lesson related very closely to what they had experienced as students themselves. Their past experience not only limited their view of what could possibly be incorporated into a mathematics lesson, but the strength of their conceptions also tended to cloud their interpretations of some qualities of the less familiar lesson activities.

The familiarity pitfall suggests the need for activities such as the mathematics lesson comparison. The selection of the instructional materials that we asked teachers to analyze for this lesson comparison was very deliberate – material sets with distinctly different conceptions of teaching and learning, but also sets of materials that were familiar and comfortable as opposed to materials that were unfamiliar and more closely aligned with the current reform movement in mathematics education. Contrasting familiar materials with newer, more innovative materials not only provided insight into the *strength* of the apprenticeship of observation (Lortie, 1975) as preservice teachers found traditional elements even when they were not there, but also created an entry point for discussion related to the power of past experience.

Teacher Learning about Curriculum Materials across Two Distinct Settings

Feiman-Nemser and Buchman (1985) also describe the "two-worlds pitfall" in teacher preparation. As suggested by the authors, preservice teachers will need guidance in recognizing how what they have learned as university students can help shape their perspectives and practices as teachers. Making these connections are not necessarily easy or automatic.

I examined the experiences of two elementary student teachers who taught in different classroom contexts and utilized different instructional resources to teach mathematics. I found that, in contrast to the inservice teachers in Remillard and Bryans' (2004) study who drew upon their own instructional repertoires as they interpreted and used their curriculum materials, the student teachers in my study turned to their cooperating teachers, peers, teacher education experiences, and other textbooks and materials. This finding suggests that resources such as these may be critical supports for student teachers when they use curriculum materials for mathematics instruction for the first time. Bridget, for example, relied heavily on her teacher education experiences – she pulled in activities from her mathematics methods courses and from the instructional resources she had come to believe were more innovative and closely aligned with her new views of mathematics teaching and learning. In addition to relying on ideas from her teacher education coursework, Bridget also needed to use the mathematics instructional resources mandated by her placement school – materials she felt were inappropriate for her students learning. She was caught in the "two-worlds pitfall" as she taught with mathematics instructional materials for the first time. Bridget worked hard to fulfill the requirements of her internship site by using the required workbook, but also needed to find ways to incorporate new instructional ideas she had learned throughout university coursework.

Classrooms as Sites for Teacher Learning about Curriculum Materials

As preservice teachers first enter classrooms, they are confronted with the responsibility of teaching while still learning how to teach. Feiman-Nemser and Buchman (1985) describe this experience as the "cross-purposes pitfall." This pitfall suggests the frequent disconnect between the responsibility of teaching and the need for critical reflection on teaching. It also highlights the idea that classrooms are not set up for teaching teachers.

During student teaching, Heather was caught in the "cross-purposes pitfall" as she found herself placed in a classroom with a cooperating teacher who had set routines and guidelines for students, and who used a detailed, *Standards*-based mathematics curriculum for instruction. When Heather entered her student teaching experience in the middle of the year, she easily assimilated into the order already established in her cooperating teacher's classroom. Heather was able to observe her cooperating teacher teach with the detailed mathematics curriculum, and was then able to step in to the already established instructional routine. Heather was afforded an opportunity to consider and learn about the complicated nature of *Standards*-based curriculum program enactment as she worked to understand how to use a particular set of curriculum materials well. For student teachers, relying heavily upon a curricular guide or on predetermined classroom norms might limit opportunities to move moment to moment and constrain certain aspects of learning to teach. Heather's experience with a *Standards*-based curriculum provided her an opportunity to understand the complicated nature of curricular resources, but also limited her chances to reflect critically on other aspects of curriculum enactment.

Teacher Learning with Mathematics Curriculum Materials

The preservice teachers in the two studies presented here found themselves immersed in professional development with mathematics curriculum materials, textbooks, and state curriculum guides during coursework and fieldwork experiences. To respond to Ball and Cohen's (1996) question, curriculum materials indeed played a substantial role in preservice teacher learning in my studies. Preservice teachers had the opportunity to:

- develop an understanding of the variety of mathematics instructional resources available for teaching *that were different from what was familiar and comfortable*;
- *negotiate balance* between university experiences and personal expectations for instructional resources and the expectations of schools in regard to mathematics curriculum;
- consider and learn about the *unexpectedly complicated nature* of *Standards*-based curriculum program enactment; and
- make decisions regarding lesson *adaptation* from a variety of mathematics instructional materials for particular students and for particular classroom contexts.

The results not only illustrate teacher learning with and about curriculum materials, but also point out opportunities within teacher education for preservice teachers to question well-established beliefs and practices regarding mathematics teaching and mathematics instructional resources. In other words, the opportunities for learning afforded to these preservice teachers as they interacted with mathematics curriculum materials display a common theme of *disequilibrium*. These studies also highlight possible missed opportunities for learning and the importance of *human resources* within teacher education as it relates to preservice teachers' encounters with disequilibrium. These ideas are explored further below.

The Role of Disequilibrium

As Wheatley (2002) describes, disequilibrium among preservice and inservice teachers is caused by a challenge to teachers' beliefs about their existing practices. Wheatley further suggests, "The psychological need to resolve such disequilibrium often pulls teachers into learning and change" (p. 9). It was when our preservice teachers encountered materials and practices different from what they were expecting or accustomed to that opportunities for learning seemed to arise. For example, for many of the preservice teachers who participated in the lesson analysis assignment, exposure to new curriculum materials allowed them to consider the methods in which they were taught mathematics. Many of the preservice teachers examining the curriculum materials encountered disequilibrium as they questioned prior certainties about effective lesson structures for the teaching of mathematics. This lesson analysis assignment positions teachers to encounter disequilibrium and confront tacitly-held beliefs about teaching built on what is inherently familiar and comfortable.

Heather too encountered disequilibrium as she utilized Standards-based mathematics curriculum materials during student-teaching. Quite the opposite of what she expected, Heather discovered how difficult it was to plan and teach with what she described as a detailed, scripted mathematics curriculum. What Heather thought she knew as she entered student-teaching – that mathematics would be planned – was challenged as she discovered the work and reflection necessary to use the curriculum materials effectively. Further, when Heather was encouraged to plan from alternative instructional resources for just one lesson, she felt more attuned to her students and the overall learning objectives. Heather's feeling of disequilibrium when using the EM curriculum, coupled with the opportunity to plan a lesson using different curricular resources, helped Heather to evaluate more critically her understanding of a Standards-based curriculum program prior to beginning full-time teaching.

The Role of Human Resources

Bumping up against disequilibrium when using mathematics curriculum materials for teaching and learning is not surprising. There is potential for learning when preservice teachers are asked to consider materials different from what they are used to, teach with methods and materials different from their philosophies about teaching and learning, and teach with complicated and detailed curriculum materials. However, as preservice teachers encounter disequilibrium amongst the pitfalls of experience in teacher education, it is important to articulate the role of human resources in this learning. As our understanding of the role of curriculum materials in teacher learning matures, it is important to reconsider the role of teacher educators, cooperating teachers, and field supervisors.

The familiarity pitfall stems from teachers' tendency to trust what is most memorable from past schooling experiences. Left unaddressed, preservice teachers may have a hard time viewing other alternatives as valid possibilities for their future teaching. This common pitfall within teacher education emphasizes further the importance of many instructional activity comparison assignments designed to enhance and challenge curricular knowledge, and the critical role of teacher educators in both the design of and facilitation of reflection surrounding such activities. Teacher educators need to help preservice teachers make sense, in a deep and conceptual way, of the variety of curriculum resources available to them both before and during student teaching experiences. They need to help preservice teachers realize that what they have experienced with mathematics curriculum materials and instructional resources as students is only one option amongst many possibilities in their future use of mathematics curricular resources.

To address the two-worlds pitfall, teacher educators might position themselves as critical supports, or safety nets, for preservice teachers as they make the transition from university coursework to classroom-based fieldwork. Helping teachers make connections between philosophical beliefs and actual classroom practice with mathematics instructional materials and pushing them to "act with understanding" (Feiman-Nemser & Buchmann, 1985, p. 64) is critical. For mathematics curriculum materials and instructional resources to play a role in teacher learning throughout student-teaching, teacher educators and university supervisors might need to expand their support and redefine their roles to stretch far beyond the walls of university classrooms. Cooperating teachers might also reconsider their role in the education of preservice teachers. If cooperating teachers viewed themselves as teacher educators rather than model teachers, they might be better positioned to help preservice avoid the "two-worlds" pitfall as it relates to developing curricular knowledge.

In light of the "cross-purposes" pitfall, cooperating teachers might also search out ways to support novice teachers to move beyond mere imitation towards purposeful and reflective decision-making with curriculum materials. Critical examination and use of many types of instructional resources during student teaching might help us work towards a compromise between the necessary responsibility of a preservice teacher to teach and his or her ultimate goal of learning about teaching. Although these modified roles might create new challenges within current teacher education practices, we must not underestimate the importance of human resources as we consider opportunities for teachers to learn with mathematics curriculum materials.

Final Thoughts

In order to position curriculum materials as tools for teacher learning, we need to move beyond mere exposure to specific materials and on curriculum use strategies, towards a focus on the critical analysis of curriculum materials and their use. Helping preservice teachers to (a) understand the philosophies that underlie curriculum materials, (b) make sense of their use of materials both before and during student teaching as they transition from the university to school settings, and (c) navigate the pitfalls of experience as they encounter learning opportunities in real classrooms is critical. As we design opportunities for preservice teachers to engage with mathematics curriculum materials, we must position all players in the preparation of teachers as critical supports amongst the pitfalls of experience within teacher education. With these human supports in place, engaging preservice teachers in activities and learning opportunities with the potential to create disequilibrium and reflection may position mathematics curriculum materials as clear tools for teacher learning and as vehicles for renewal and innovation in the teaching of mathematics.

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GEOMETRY AND MEASUREMENT CONTENT KNOWLEDGE OF PRESERVICE K-8 MATHEMATICS TEACHERS: A SYNTHESIS OF RESEARCH

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In this paper, we summarize the findings of a research synthesis specifically related to the geometry and measurement content knowledge of elementary/middle school preservice teachers. Findings from this synthesis show that preservice teachers have weak conceptions in geometry and measurement content knowledge. However, instructional strategies that incorporate technology (e.g., dynamic geometry software or virtual manipulatives) or analyzing children's work, have been shown to strengthen preservice teachers' understanding.

Introduction

For the past several years, a working group has met at the Psychology of Mathematics Education-North American Chapter (PME-NA) annual conferences. With members sharing their individual research, common themes and interests emerged and discussions turned towards needs for future work. In order for the group to build on and extend the knowledge base in the area of mathematical content knowledge for preservice teachers, a common need that emerged was knowledge of the status of current related research. Thus, a synthesis of the current research was essential for the following reasons:

1. To design a program of study to develop elementary/middle school preservice teachers' mathematical content knowledge for teaching, we need to understand *what* content knowledge preservice teachers bring to their courses. "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (CBMS, 2001, p. 17).
2. In addition to knowing *what* knowledge the preservice teachers bring to their coursework, we need to understand *how* to build on that knowledge in meaningful ways to further develop the preservice teachers' mathematical content knowledge for teaching. We need more empirical evidence of what learning opportunities contribute most to more knowledgeable and confident teachers in order to make more informed changes to our preparation programs (Mewborn, 2000).
3. To identify future research directions, we need to know *what* has been established in terms of preservice teachers' content knowledge and *how* that knowledge has been developed and *what* we still do not know or understand.

Theoretical Framework

There has been a recent emphasis in the mathematics education community in describing the needed and desired mathematical content knowledge for teaching, with various descriptions emerging from research (e.g. Hill, Rowan, & Ball, 2005; Ma, 1999; National Research Council, 2001; Shulman, 1986). Building on the work of Shulman (1986), Hill, Ball, & Shilling (2008) describe a framework for distinguishing different types of knowledge included in a construct of mathematical knowledge for teaching. This framework distinguishes between subject matter knowledge and pedagogical content knowledge. The interest of the working group centered on synthesizing research on subject matter (content) knowledge described in this framework. Understanding this content knowledge is vital for mathematics educators as we develop content (and methods) courses to meaningfully prepare preservice teachers as well as knowing *how* to help preservice teachers understand the needed content knowledge for themselves.

Our guiding questions for this synthesis were: *What research has been conducted on elementary/middle school preservice teachers' content knowledge?* and *What is known from this research about preservice teachers' content knowledge?* This paper reports on the work related to the subgroup focusing on preservice teachers'geometry and measurement content knowledge.

Methods

We began our work using the ERIC database focusing on the years 1998 to 2009. For dissertation work, we chose to review those from 2006 to 2009 and conference proceedings 2004-2009. In addition to the publication date requirement, each research study had to be published in a peer-reviewed journal and focus on elementary or middle school preservice teachers' geometry and measurement content knowledge. Since not all countries use the same grade-level classification as in the US, we decided to look at children aged 3 - 12 (possibly up to 14) when including international studies. Key words used in various combinations during this search process included geometry, preservice/pre-service, elementary, teacher, mathematics, measurement, length, volume, area, and angle. This produced a total of 30 studies.

Each of the 30 studies went through an independent review that detailed the research questions, study type and research design, lens and/or approach used, selection and description of participants, conditions of and procedures for data collection, data analysis, findings, and conclusions/implications. When there was a discrepancy on the inclusion or exclusion of a study, discussion allowed for mutual consensus on a final decision. Excluded from our synthesis were studies that had: (a) a general description of content knowledge that lacked specific attention to geometry or measurement, (b) a selection of inservice teachers or college students majoring in mathematics as opposed to mathematics education, (c) a sole focus on perceptions about mathematics not connected to content knowledge needed for teaching, and (d) a focus on describing classroom practice or activities with a lack of attention to research design methods. After examining the 30 studies, only 11 met all of the review criteria.

When the PME-NA Working Group met again to discuss the status of all content area reviews, the group decided to conduct another search through the literature. This second phase of the search focused on the 25 journals that provided relevant research studies for any of the content areas during the first phase. This second phase was initiated to search for relevant research published in 2010 and to see if new studies surfaced that possibly were missed in the initial search. Two new studies were found for geometry and measurement; one was published in 2010 and the second was published in a year not included in the ERIC database for that particular journal. Thus, in our final analysis, only 13 studies matched our criteria to be included in our analysis of research describing elementary preservice teachers' knowledge of geometry and measurement.

Results

Characterizations of the data

Eleven studies focused on geometry and only two studies reported on measurement. Of the geometry studies, seven focused on preservice teachers' content knowledge relative to shapes and their attributes (diagonals, reflective symmetry, etc.), one examined visualization skills, and four focused on other general knowledge of geometry. Studies on shape focused on quadrilaterals, rectangles/rhombi, and triangles. Studies on measurement focused on perimeter, area, and volume.

In order to synthesize findings across the 13 studies, we examined study types and characterized the research questions that dealt with preservice teachers' geometry and measurement content knowledge. Three broad categories emerged: descriptive studies that examined the *status* of mathematical knowledge and understandings of preservice teachers; studies that *explored* the impact of some "treatment" in a mathematics content or methods course; and finally, comparison studies that examined *differences* between ideas/features, groups or looked for relationships between two entities. Italicized text emphasizes the key features of the questions in the studies for these three categories.

The Status of Preservice Teachers' Content Knowledge in Geometry and Measurement

Six studies had at least one research question that focused on what preservice teachers understand about specific topics in geometry and measurement. These topics included quadrilaterals, reflective symmetry, triangle altitudes, volume, perimeter, and area.

Across all of these studies, we found preservice teachers' understanding of geometry to be weak or limited. "Preservice teachers' understanding of geometry was limited with shapes and measurement aspects of shapes" (Aslan-Tutak, 2009, p. 158); "it was found that a large portion of preservice teachers has lack of content knowledge of reflective symmetry" (Son, 2006, p. 149); "the results presented in this study are that most of these participants lack the ability to articulate complete descriptions of rectangle and rhombus" (Pickreign, 2007, p. 6); "we found that many preservice teachers had the same poor concept images as primary or secondary students" (Gutierrez & Jaime, 1999, p. 272); "the diversity of responses offered by preservice teachers in this study could be comparable to what would be expected from students in an upper primary classroom" (Zevenbergen, 2005, p. 21); and "yet, even with an apparently better foundation in mathematics, the students appeared to have poor conceptual understanding (*in perimeter and area*)" (Menon, 1998, p. 365).

Pickreign (2007) attends to how teacher preparation programs might address the concern of such limited geometrical understanding of preservice teachers, very similar to our initial quote from CBMS, working from what preservice teachers do know:

> "This, however, raises serious questions to be answered if we are to affect the content knowledge of teachers of mathematics: Can sufficient experience with these geometric ideas be provided in teacher education programs to lead to more profound understandings? Are there other mathematical ideas that require such experience? What should characterize these experiences? Can these experiences be provided without adding time or credit hours to teacher certification programs? One option in addressing these questions might be to not address them. Instead, accept teacher education programs as entry level preparation and utilize continuing professional development to address the growth of teachers' profound understanding of fundamental mathematics." (p. 6)

Time appears to be a critical component. Perhaps teacher preparation programs will not be able to provide the needed extent of time to make significant change but it can lay important groundwork.

Exploring the Impact of a Treatment

Five studies had questions of an exploratory nature, investigating "what happens if", four in the area of geometry and one looking at volume.

Geometry studies explored how dynamic geometry softwares impact preservice teachers' understanding of proof, the influence of the analysis of children's work with quadrilaterals on preservice teachers' understanding of geometry content knowledge, the impact of instruction utilizing graphic organizers and concept attainment strategies on preservice teachers' understanding of altitudes of triangles and diagonals of polygons, and the influence of digital and concrete tangrams on preservice teachers' 2D visualization on tangram designs.

Findings from Christou, Mousoulides, Pittalis, and Pitta-Pantazi (2004) show dynamic geometry softwares, along with appropriate questioning, motivated the students in the study to justify their conjectures and enabled students to pass from "exploratory" geometry to deductive geometry. Aslan-Tutak (2009) found that incorporating analysis of children's work into activities while preservice teachers make sense of quadrilaterals had a significant positive affect on the preservice teachers' learning as determined by a repeated measures ANOVA. Cunningham & Roberts (2010) explored the impact of a treatment lesson involving instructional strategies designed to assist the development of preservice teachers' concept images and concept definitions related to altitudes of triangles and diagonals of polygons. These strategies included the use of graphic organizers along with the concept attainment model in the development of definitions. The combination of these strategies resulted in some improvement for the preservice teachers' understanding of triangle altitudes and diagonals of polygons. Spencer's work (2008) found that using both digital and concrete tangrams in a methods course did help improve 2D visualization skills of preservice teachers which, in turn, had a significant positive impact on their attitude toward geometry.

Zevenbergen's (2005) study explored the impact of various learning dispositions that were emphasized within a mathematics course module on volume. These dispositions included developing mathematical meaning of volume as opposed to only using algorithmic methods, emphasizing the development of measurement and spatial sense, and developing the capacity within the preservice teachers to identify errors in children's mathematical thinking. However, despite the various methods used in the course to develop these dispositions, there was a "worrying number of students" who had not achieved them (p. 21), with some students quite resistant to alter their thinking about how to learn mathematics. Responses to interviews of students in the course highlighted the power of the teaching practicum, with preservice teachers rejecting the nature of the work done in their mathematics course due to their experiences in the schools. "Ideally, it would be useful to expose students to schools and classrooms that demonstrate the values embedded within teacher education courses if such courses are to effectively change teaching practice" (p. 21).

These few studies suggest various instructional strategies can make positive change on preservice teachers' geometrical understanding, such as those incorporating technology, making use of graphic organizers and concept attainment, and analyzing children's work. Field work that provides similar experiences as found in preservice teachers' mathematics courses and the length of time devoted to concept development may be critical factors to maintain these positive changes.

Examining Associations and Differences

Six studies included research questions that examined relationships and/or differences. Studies examined differences in geometric reasoning levels between elementary and secondary preservice teachers and male and female preservice teachers, differences in content knowledge based on instruction, relationships between visualization skills and attitude towards geometry, differences in student learning environment, and differences in students' conceptions of geometry concepts.

Halat (2008) administered a *van Hiele Geometry Test (VHGT)*, based upon work of Usiskin (1982), to 281 Turkish elementary and secondary preservice teachers. He found "no statistically significant difference in regard to the reasoning stages between the pre-service elementary school and secondary mathematics teachers, and that although there was a difference with reference to van Hiele levels between male and female pre-service secondary mathematics teachers favoring males, there was no sex-related difference found between male and female pre-service elementary school teachers" (Halat, 2008, p. 1).

With respect to examining the impact of instructional strategies, Aslan-Tutak (2009) considered the impact of having supplemental geometry activities in a two-week unit during a methods courses, using control and treatment groups. The results showed the geometry knowledge of students, as measured by the *Content Knowledge for Teaching Mathematics (CKT-M)* test (Hill & Ball, 2004), in either group was increased significantly, however the grouping did not have any statistically significant affect on participants' knowledge growth. Lundin (2007) also used the *CKT-M* test to measure content knowledge differences of preservice teachers from two groups: those who had completed mathematics courses designed for elementary education majors and those preservice teachers who had completed the mathematics requirement through the completion of other mathematics courses. She found no statistically significant differences between the two groups based upon students' mean scores. Lundin suggests the lack of difference could be attributed "to the likelihood that mathematical content gained in regular math courses taken by the participants was as beneficial as the content knowledge gained in the courses examined in this study" (p. 78) or that the subjects in her study "did not make a transition from applying general mathematical content knowledge to the mathematical content knowledge required for teaching" (p. 79).

Spencer (2008) found a positive relationship between attitude toward geometry and 2D visualization with increased levels of 2D visualization corresponding to an increased attitude towards geometry. Two-dimensional visualization skills were improved by the use of concrete and/or digital tangrams.

Gerretson (2004) examined to see whether or not there was a difference in elementary preservice teachers' performance on similarity tasks when using dynamic geometry software as compared to paper-and-pencil learning environment using traditional tools (e.g., compass, ruler). Using a pre/posttest control group experiment using randomized blocks controlling for initial performance, she found a statistically significant difference in learning environments between the two treatment groups. "Fundamentally, software users outperformed non-software users even when prior knowledge variability was taken into consideration." (p. 18). Analysis suggests elementary preservice teachers using a paper-and-pencil learning environment encountered more difficulties particularly situated around similarity properties of unfamiliar shapes, whereas preservice teachers using dynamic geometry software had "acquired a greater knowledge base to access, network, and apply" (p. 19).

Fujita and Jones (2006) explored the nature of preservice teachers' *personal figural concepts* and *formal figural concepts*, particularly in the area of quadrilaterals, building on the work of Tall & Vinner (1981) with respect to concept image and concept definition and Fischbein's figural concept (1993). The study examined data from two groups of preservice teachers in Scotland. The groups were based on their year of study in the teacher education program; either first or third. For either group, the data indicate a gap exists between the *formal figural concepts* of preservice teachers and their *personal figural concepts* "such that their images are so influential in their personal figural concepts that they dominate their attempt to define basic quadrilaterals" (Fujita & Jones, 2006, p. 131). Preservice teachers rely on their images of shapes to construct definitions rather than examining and using properties of shapes.

In several of these comparison studies, despite further evidence of preservice teachers' limited conceptual understandings in geometry, growth in content knowledge was demonstrated. Growth occured where technology was incorporated into the learning environment and when supplemental learning activities examining children's work were utilized.

Discussion and Conclusion

Our synthesis was guided by two questions, determining what research has been done related to geometry and measurement content knowledge and synthesizing what we know from this research regarding preservice teachers content knowledge. We organized the research found into three categories; briefly, these were status, explorations, and comparisons.

Based upon the research examining the status of preservice teachers' knowledge of geometry content, there is a need for these courses to take the time to develop foundational knowledge of geometry and measurement.

Preservice teachers enter their teacher preparation programs with limited geometry and measurement content knowledge, relying more on their understandings from middle school than their high school geometry course and operating on low van Hiele levels of understanding, relying on their concept images of geometrical concepts, with poorly constructed concept definitions. Although one study found no significant difference in mean scores between preservice teachers who had taken mathematics courses designed for teachers and those who had not, others had positive findings. Studies that explored alternative methods of instruction with the use of technology, such as dynamic geometry systems and virtual manipulatives, provided encouraging results related to improving deductive thinking and 2D visualization skills. Other instructional strategies in this synthesis that promoted preservice teachers' learning of shape and their properties included using a graphic organizer and concept attainment model, and analyzing children's work. The importance of a "supportive" field component was also noted to help sustain the values of the teacher education program, otherwise it is likely preservice teachers would reject constructive methods of teaching mathematics.

Limitations to our synthesis include using only one database. Using multiple sources may have produced more studies. Expanding our search to include more than the previous ten years would also have included more research and perhaps more seminal pieces in the area of geometry and measurement.

This summative look at research on preservice teachers' geometry and measurement content knowledge shows a paucity of research published in peer-reviewed arenas in the past ten years. However, this limited research consistently shows that preservice teachers do not have a solid understanding of several fundamental concepts within geometry and measurement. In order to work from what preservice teachers *do* know (CBMS, 2001), it appears that a greater focus on developing strong foundational knowledge is needed in geometry and measurement courses. Yet this raises some questions:

- What do we perceive, as a community of mathematics educators, as the essential foundational geometry and measurement knowledge that content courses should include?
- If such "core" foundational knowledge can be determined, what are sufficient

experiences that would enable preservice teachers to acquire profound understandings of this core knowledge and when do these experiences occur in a teacher preparation program?

Further work of researchers mapping out the core mathematical content knowledge for teachers will be helpful in highlighting the gaps between what content knowledge is *needed* by preservice teachers and what knowledge they bring with them to their teacher preparation programs.

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MAINTAINING COGNITIVE DEMANDS OF TASKS THROUGH SMALL GROUP DISCUSSIONS IN PRESERVICE ELEMENTARY MATHEMATICS CLASSROOMS

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Enacting tasks at high levels of cognitive demand helps preservice teachers make sense of mathematical ideas and serves as a model for instruction. Small group discussions can be useful pedagogical tools for maintaining a task's cognitive demands. In this article, we contrast two small group discussions within a preservice elementary classroom to illustrate characteristics of productive small group discussions.

Introduction

Current research on mathematical tasks examines the levels of cognitive demands at which tasks are written, set up, and implemented in elementary and middle school classrooms in the hopes of shedding light on effective classroom instruction (Stein, Smith, Henningsen, & Silver, 2000). Tasks that require high levels of cognitive demands prompt students to explain their reasoning and make connections between different representations or procedures. Since reasoning and sense making are critical components of conceptual understanding, finding ways to maintain high levels of cognitive demand during task implementation is important. Research shows, however, that teachers often inadvertently reduce the cognitive demands of a task during its implementation by making problems easier to solve, often by providing students with too many hints or by removing prompts for student-constructed justifications (Stein, Grover, & Henningsen, 1996).

One way to maintain high levels of cognitive demands may be through the use of mathematical discourse. There is evidence demonstrating the benefits of using discourse in the mathematics classroom (Walshaw & Anthony, 2009). When students are pushed to articulate their ideas and listen to the thinking of their peers, they are able to make better sense of their own thinking. Serving as a basis for current reform efforts, the National Council of Teachers of Mathematics (2000) identifies communication as an essential standard of mathematics education, stating that students must learn to communicate their mathematical thinking to others and evaluate the mathematical thinking of their peers. Nevertheless, many classrooms continue to implement traditional teacher-centered instruction (Hiebert et al., 2005).

In this research report we will present an analysis of preservice elementary teachers' small group discussions regarding a mathematical task written at a high level of cognitive demand. Through careful examination of videotaped discussions, we will identify characteristics of small group interactions that maintained the level of cognitive demand inherent in the task. We will contrast them with other small group interactions that lowered the task's cognitive demand level. Focusing on small group discussion yielded new information as small group interactions provide opportunities for learning that whole class formats do not (Yackel, Cobb, & Wood, 1991). For example, preservice teachers may feel more comfortable sharing their ideas in a small group setting. Although our original intent was on examining the role of whole-class discussion in fostering preservice teachers' conceptual understanding, we began to quickly notice that small group interactions played a key role in helping preservice teachers make sense of important mathematical ideas.

Theoretical Framework

Research on mathematical tasks and their cognitive demands has been on-going for over twenty years. Mathematical tasks play a large role in determining the mathematics that students will see in the classroom (Doyle, 1988). Tasks that emphasize computation and memorization result in students acquiring procedural skills without understanding why they work; tasks that focus on solving rich, contextual problems help students attend to the concepts underlying the problems (Stein & Lane, 1996).

Research suggests that preservice teachers should become accustomed to solving cognitively demanding problems, since the expectation is they will be posing cognitively demanding problems to their own students (Stein et al., 2009). The cognitive demands of a task refer to the kinds of thinking processes entailed in the task (e.g. memorization, use of procedures, use of complex thinking and reasoning strategies). Stein and colleagues suggest that the cognitive demands of a task can change as the task moves through three phases of development – writing, set up, and implementation (Stein et al., 1996; Stein et al., 2000; Boston & Smith, 2009). For example, a mathematical task may be written at a high level of cognitive demand, but presented and/or implemented at a lower level of cognitive demand.

Rubrics have been developed to assess the cognitive demands of tasks as written and as implemented (Boston & Smith, 2009). The *Instructional Quality Assessment – Academic Rigor* (IQA-AR) rubric includes two forms, the *Potential of the Task* and *Implementation of the Task.* Both forms scores range from 0 to 4. A score of 0 indicates that the lesson tasks were nonmathematical in nature. Score levels 1 and 2 represent low-level cognitive demands. Score levels 3 and 4 represent high-level cognitive demands in which the connections to meaning and understanding are implicitly (score level 3) or explicitly (score level 4) required by the task.

When comparing tasks' levels of cognitive demands at the written and implementation phases, Suzuka and colleagues (2009) found that teacher educators have difficulty maintaining a task's cognitive demands when implementing the task in a preservice teacher classroom. Two primary reasons could explain their difficulty: sometimes teacher educators focus too heavily on the mathematics of the task, thereby disconnecting the task from the work of teaching; or, work focuses too heavily on pedagogical concerns without engaging preservice teachers in understanding mathematical content. In order to keep tasks focused on MKT development, the authors suggest asking preservice teachers to attend to their classmates' thinking, pressing preservice teachers to explain their own thinking, and being explicit about the connections between mathematical tasks and teaching.

While reform efforts during the past twenty years call for the use of classroom discourse (Senk & Thompson, 2003), research has also recently begun examining classroom communities where discourse is used to promote mathematical learning. Hufferd-Ackles, Fuson, and Sherin (2004) created the *Levels of Math – Talk Learning Community* framework, which identified four components of a math-talk learning community: questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning. Each component is rated from level 0 to 3 to reflect changes in how teachers and students interact to make sense of mathematical ideas. Classrooms at a level 0 are teacher-centered, where the teacher is the only questioner, students provide short answers and never elaborate on their thinking, the teacher is the only source of mathematical ideas, and students are passive listeners to what the teacher is saying. Classrooms at a level 3, however, are primarily student-centered, where students and teacher question one another, students provide more elaborate justifications of their mathematical ideas, and students take responsibility for helping each other learn mathematics.

Though research on mathematical discourse and small group interactions is abundant at the K-12 level, little is known about discourse in preservice teacher classrooms. Dixon, Andreasen, and Stephan (2009) conducted a classroom teaching experiment in a preservice elementary teacher mathematics content course in order to understand how classroom norms are established. They observed that making sense of others' reasoning was one of the most difficult, yet important, norms to establish. Research on discourse in preservice teacher classrooms also showed that math-talk learning communities (as defined by Hufferd-Ackles et al., 2004) can be established by prompting preservice teachers to question, explain, and clarify each others' thinking (Rathouz & Rubenstein, 2009). These studies, however, did not focus on small group interactions. As such, we do not yet have a clear picture of the characteristics of student-to-student talk in preservice teacher classrooms.

Methods

The study took place in the spring of 2010, during a semester-long preservice elementary mathematics content course at a large private university in the northeastern United States. It is the second course in a series of two required mathematics content courses for undergraduates seeking state certification as elementary or special education teachers. The course emphasizes preservice teachers' conceptual understanding of content and seeks to develop productive mathematical talk as a classroom norm. Each section consisted of approximately sixteen participants who were randomly assigned to four small groups. Two classroom sections of this course were videotaped and audio taped during eighty minute class sessions over a two-week period.

The curriculum used in the study was a four-part unit on geometric measurement created by the researchers. Our analysis focuses on a two-day lesson about surface areas of prisms and cylinders. Day one required participants to work through mathematical tasks regarding the lateral and total surface areas of prisms. Day two required participants to work through mathematical tasks focused on lateral and total surface areas of cylinders, as well as the relationship between the surface areas of prisms and cylinders. The tasks were written to be cognitively demanding, with repeated prompts for participants to justify their reasoning to others and make generalizations across topics.

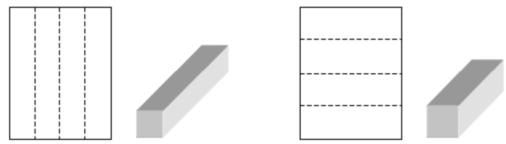
We used two sets of rubrics to collect data about each task and its implementation. We used the IQA-AR rubrics for the potential of the task and for the implementation of the task in order to assess the cognitive demands of the tasks before and during implementation, respectively (Boston & Smith, 2009). In order to assess the nature of student-to-student talk within small group discussions, we used the *Levels of Math Talk* framework (Hufferd-Ackles et al., 2004).

Results

During day one of the surface area task, small groups in both sections worked on a mathematical task consisting of four questions (shown below). Each group had to fold one piece of paper lengthwise and a second, identical, piece of paper widthwise, find each prism's dimensions, and calculate their surface areas. Participants then needed to identify similarities and differences between the surface area calculations of both prisms. Each group had approximately fifteen minutes to complete the task.

3. Take two pieces of $8\frac{1}{2}$ in. by 11 in. paper and fold them into two different rectangular prisms with square bases. Label one, Prism X, and the other, Prism Y. The large rectangle that is used to form each prism is called the "lateral surfaces rectangle." Note that the prisms are

missing their bases. On the drawing below, label the dimensions of both prisms.



Prism X Prism Y

4. a) What are the dimensions of the lateral surfaces rectangle for Prism X? How can we use the dimensions of the prism to find these measures?
   b) Calculate the area of the lateral surfaces rectangle of Prism X, the area of the bases of Prism X and the total surface area of Prism X.
5. a) What are the dimensions of the lateral surfaces rectangle for Prism Y? How can we use the dimensions of the prism to find these measures?
   b) Calculate the area of the lateral surface rectangle of Prism Y, the area of the bases of Prism Y and the total surface area of Prism Y.
6. Explain any similarities and differences between the surface area calculations in questions 2 and 3.

Cognitive Demand Analysis for Potential of Task

We rated this task at a level 4 cognitive demand using the IQA-AR rubric for potential of the task. The task engages participants in complex thinking because they are asked to compare and contrast two different orientations of seemingly identical prisms. They are asked to follow a procedure (creating two different prisms from the same sized sheet of paper) to illustrate an important mathematical concept (prisms can have the same lateral surface area but different total surface areas). In other words, the numerous computations of total and lateral surface areas required by the task serve a greater purpose – conceptual understanding of the differences between total and lateral surface areas of prisms. In the last question of the task, participants are prompted to provide explicit evidence of their understanding by comparing and contrasting the calculations that were performed on the two prisms. In doing so, they are making connections between two representations of a rectangular prism.

Instructor F's Small Group Discussion

Participants create the two paper rectangular prisms using sheets of paper with dimensions 8 ½ inches by 11 inches, following the instructions written in question 3. In the following small group discussion, participants are finishing their calculations of prism Y's surface area. After having found that the length of a side of prism Y's base is 2.75, they are helping each other with the rest of the calculations that need to be done in order to answer question 5. Once this is complete, they, move on to a discussion of question 6.

S1: We're saying that the lateral surface area would be 93.5 because it's the same piece of paper but the bases are going to change.
S4: So the base is?
S1: It would be 2.75 squared.
S4: 7.5625.
S2: Plus 2 times 7.56.

S1: Put another one [prism Y base area] because you need 2 of those. 108.625. So it's not the same.
S3: Well yeah because the squares [points to paper rectangular prism] are different sizes.
S3: The lateral surface areas are the same, but…
S2: Yeah, the lateral surface areas are the same.
S3: But the bases…
S2: The bases are different.
S3: …are different sizes, so the larger base is going to have a larger surface area?
S2: Yeah.
S3: Um, what number are we supposed to go up to?
S1: I think up to 6.
S3: Oh, okay…are you sure?
S1: Yup, that's why we had to do different squares on the…
S4: So why is the surface area different?
S1: Because this [points to paper rectangular prism]…they have the same lateral surface area [points to top of paper rectangular prism], but the boxes [points to base of paper rectangular prism] of the squares of the bases are different. Because these [points to bigger paper prism] squares are bigger so it makes it more.

In this discussion, the small group interactions result in a shared understanding of why the total surface areas of the two prisms differ. Initially, S2 and S1 notice that S4 has only added one base area to prism Y's lateral surface area, explaining two base areas are needed to accurately calculate total surface area. S3 revoices S1's remark that the total surface areas of the two prisms are not the same since their base areas differ. S3 and S2 again revoice the idea. Although S3 is satisfied with this explanation, S4 is not and asks the group to explain why the prisms' surface areas differ. S1 restates the explanation in his own words using folded paper prisms to illustrate his reasoning.

We rated this small group discussion at a level 4 in cognitive demand using the IQA-AR rubric for implementation of the task. There is clear evidence of participants' reasoning, as group members articulated an explanation for why the surface areas of the two prisms are different. When S3 responds to S1's initial comment that the surface areas are not the same, she is making sense of the mathematics. She could have simply stated that the surface areas differ because the two prisms are different, but this would have shown vague understanding. Instead, she states that the difference is a result of different sized bases. This shows that she understands that it is the base areas of the two prisms that distinguish one prism's total surface area from the others' – the lateral surface areas are the same. Also, S1 is making an explicit connection between representations when he lays one paper prism next to the other and points to their different sized bases.

We rated the small group discussion at a level 3 for questioning, explaining mathematical thinking, source of mathematical ideas, and responsibility for learning using the *Levels of Math Talk* framework. These high levels of math talk are due to the comfort with which group members express their ideas and help one another understand them. Participants initiate their own student-to-student talk without prompting from the instructor, and they do not hesitate to ask questions of each other to clarify ideas. Participants freely interject their own ideas throughout discussion, either correcting each others' mistakes as evidenced by S2's and S1's noting that S4 omitted one base to calculate the total surface area. Participants repeatedly justify their ideas and with little prompting from each other, as seen when S3 spontaneously began to articulate her explanation for why the surface areas differ. Participants in this group take responsibility for their own learning, as seen when S4 asks for clarification after the other group members were finished.

Instructor C's Small Group Discussion

Participants in this small group also worked on the same mathematical task as the group in Instructor F's class. In the following discussion, the group members are finishing their computations of the total surface areas of both prisms and need to explain any similarities and differences between the surface areas of the prisms.

S2: So we know the lateral surface area….
S1: Is 8.5, oh wait sorry what did you get for the lateral surface area?
S2: Is 93.5
S3: What's 93.5?
S2: Is the lateral surface area….one piece of paper.
S1: Yeah same as the last one.
S2: Right. And then to that we have to add the bases [points to bases of prism] which are now different.
S3: The bases are….2.75?
S2: Yeah. So 7.5…now you got to multiple that by 2…15.
S3: What was the area of one of them?
S2: One of them is 7.56. So the total is 108.425, which is bigger.
S1: Okay, so similarities and differences between the surface area calculations…the lateral surfaces were the same, but the bases were different and therefore the total surface area…
S1: Okay, are we supposed to keep going?
S3: Well he said till 8.

In this discussion, the group focuses on making correct calculations of prism Y's total surface area. S1 asks S2 for the lateral surface area of prism Y (93.5), after which S3 asks S2 what 93.5 represents. As S2 begins to articulate the next step in the procedure for calculating total surface area, S3 asks her for the area of the base of prism Y. The only discussion surrounding mathematical concepts occurs at the end, when S1 states that the difference in surface areas is due to the difference in base areas. None of the other participants follows up on this claim at all.

We rated this small group discussion at a level 2 in cognitive demand using the IQA-AR rubric for implementation of the task. The purpose of this group's talk is to make sure that all group members arrive at the same numerical results for the calculations of total surface areas of prism X and prism Y. Since the group as a whole does not engage in any discussion around S1's statement, it is not apparent that the other group members agree with this statement or understand why it is true. Throughout their discussion, participants follow a well-known procedure for calculating surface areas; they are not engaged in any complex thinking or meaning making.

We rated this small group discussion at a level 1 using the *Levels of Math Talk* framework. Participants ask each other questions, but they do not listen attentively since questions are repeated (S3 asks for the lateral surface area immediately after S1 asks the same question). S1 is the only participant who describes her non-computational thinking, but her claim is not explored any further by the other students. In fact, small group discussion focuses solely on procedures for finding total surface areas, rather than the concepts underlying the problem.

When comparing Instructor F's small group interactions to Instructor C's small group interactions, clear differences appear. In Instructor F's class, group members maintain the task's high level of cognitive demand during task implementation. They do this by engaging in thoughtful discussion about the key mathematical ideas of the task, making sure to provide reasoning when appropriate. In Instructor C's class, however, group members are focused on getting the correct answer to the problem and neglect the last part of the question that asks for an explanation of the similarities and differences between calculations.

Differences in both group's level of math talk are also evident. Instructor F's group is at the highest level of math talk because they make repeated efforts to explain their thinking. In fact, when analyzing classroom video of these interactions, one notices a sense of urgency, among the group members, to make sure everyone understands the concepts under investigation. They are not afraid to interject their own ideas into a conversation because they know their ideas will be valued. On the other hand, Instructor C's group members exhibited a low level of math talk. Since participants were focused on getting the correct answer instead of making sense of their answers, questions were short and only elicited superficial responses. Also, since S1's claim is not explored by other group members during the small group discussion, one might conclude that the group uses another source of mathematical ideas, such as the instructor or whole class discussion, instead of each other.

Discussion

In this article, we provided examples of how differences between two small groups of elementary preservice teachers working on the same task lead to maintaining or lowering the task's cognitive demands. In Instructor F's small group discussions, participants maintained the cognitive demands of the task by fully answering the questions posed and justifying their answers. In Instructor C's small group discussions, participant discussion resulted in lowered cognitive demands because this group only answered the computational portion of the questions posed and neglected to provide explanations of why they performed their computations or reached their conceptual conclusions.

We also evaluated the two small group discussions using a rubric for levels of math talk (Hufferd-Ackles et al., 2004). We found that Instructor F's small group discussions were rated at a higher level of math talk than Instructor C's small group discussions. Participants in Instructor F's class were actively involved in asking each other questions and defending their own ideas, while participants in Instructor C's class rarely asked each other to explain their thoughts and were hesitant to pose clarifying questions.

Based on our analysis, we believe that the focus of preservice teachers' attention during small group discussion can impact whether tasks' cognitive demands are maintained or not. Small groups who focus on providing explanations, justifying procedures, and making generalizations appear to maintain a task's high level of cognitive demand. We observed a shared desire among all group members in Instructor F's class to make sense of the task. Participants were consistently asking each other for explanations and were often dissatisfied with correct solutions that they did not understand clearly. In contrast, small groups who focus solely on finding correct answers, do not justify their procedures, and do not attempt to generalize their ideas beyond specific cases risk lowering the cognitive demands of a task. In Instructor C's small group, there did not appear to be a shared need to develop an understanding of the mathematics of the task. Participants seemed satisfied when they arrived at the same numerical answer, and chose not to push for meaning or clarification.

Although it is unclear why one group developed a shared need to make sense of the task while the other did not, it seems that a small group working together in a preservice Instructor Course can develop its own culture that helps to determine how the group functions. Future research should explore the source of small group cultures. Do small groups adopt the culture of their classroom? Does what the instructor of the course value – respectful discourse and focus on meaning – dictate how small groups operate? In our study, Instructor F focused whole-class discussion on sense making while Instructor C focused his discussions on answer checking.

It is also possible that maintaining high levels of cognitive demand is associated with high levels of math talk during small group discussions. A future study could investigate the interplay between a task's cognitive demands and the levels of math talk used to solve the task: does a small group's success in solving cognitively challenging problems relate to the degree to which they use productive math talk during discussions? Our study suggests that math talk can help maintain high levels of cognitive demand, and that using contrasting cases might be a useful method for investigating this further.

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PRESERVICE ELEMENTARY TEACHERS DEVELOPING UNDERSTANDINGS OF SELF AS MATHEMATICS TEACHER AND TEACHING IN CONTEXT

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As preservice elementary teachers (PSTs) examine their personal readiness, instructional practices, and their agency in enacting those practices, a ubiquitous question is "What can I do?" to support all students in learning mathematics amidst social, political and institutional dynamics present in today's classrooms. This paper explores how PSTs are developing understandings of self as mathematics teacher and of teaching through participation in a seminar designed to support critical examination of themselves as mathematics teachers, and if and how this self-examination is consequential for their understandings of teaching, particularly as within complex realities of schooling and attention to equity and access.

> I think I am experiencing an identity crisis as a teacher. …As far as me as a math teacher, I am not sure where I see myself. I want to believe that I will be using problematic tasks and encouraging my students to explore and to question. And I *hope* that I won't be skilling and drilling my students. I *hope* I won't be using mind numbing workbooks and textbooks. I *hope* my students are learning and growing in my classroom. I *hope* I continue to see them as unique individuals, not just as test takers. I *hope* I am still reflective and responsive to my students' needs. I *hope* that I am flexible and constantly strive to improve my teaching. I *hope* I am not stuck in a rut. I *hope* I haven't succumbed to the pressures of high stakes testing. And, I *hope* that I will think of the administrators and other teachers at my school as allies, not adversaries. (Sarah, March 29, 2010)

Sarah wrote this vision statement during the last semester of an intensive elementary master's certification program after completing over two semesters of coursework toward her degree and interning for seven months in a third-grade classroom. Sarah's writing highlights the complexity of teaching mathematics in the realities of her classroom context. Sarah essentially asked, "What can I do?" a complex question heard in preservice teacher education as preservice elementary teachers (PSTs), such as Sarah, question their personal readiness to engage in instructional best practices, their understanding of what those practices are, and their agency in enacting those practices in school contexts. As she discussed in other writings, for example, Sarah remained uncertain about how to actually work towards equity in mathematics by employing teaching practices such as engaging students in conceptually rich mathematics in this era of high-stakes testing. While it is generally accepted that teacher preparation needs to include an emphasis on best practices (Ball & Forzani, 2009) and classroom analysis (Hiebert, Morris, Berk, & Jansen, 2007), it also must prepare PSTs to navigate the many social, political, and institutional dynamics in classrooms so they can enact equitable teaching practices that best support *all* students in learning with understanding. Successful negotiation of these dynamics requires that teachers have an examined vision of their goals of mathematics teaching, the social and political contexts of schooling, and the realities of their school contexts, that is, understanding of themselves as mathematics teachers. The paper explores how PSTs develop understandings of themselves as mathematics teachers, and if and how this self-knowledge is consequential for their understandings of mathematics teaching, particularly as situated within the complex realities of schooling and with attention to equity and access.

Elementary teacher preparation is further complicated by elementary teachers' positioning as generalists (Brown & McNamara, 2005), such that they may not identify as mathematics teachers or attend specifically to creating rich mathematical experiences. Elementary teachers' existing beliefs and their personal, perhaps damaging, mathematical experiences also influence their interpretations of curricula and reform (e.g., Drake, 2006) and conceptualizations of mathematics (Gellert, 2000). In the complexity of their mathematics teaching environments and their complex positioning, PSTs highlight different tensions through three iterations of the question, "What can I do?": a) What can *I* do?, with emphasis on personal readiness; b) What can I *do*?, focusing on actionable steps; and c) What *can* I do?, attending to personal agency (Pollock et al., 2010)? In response, teacher preparation needs to emphasize dispositions and knowledge that relate specifically to mathematics and mathematics teaching for understanding, while also supporting elementary PSTs in understanding themselves, their students, and the social, cultural, and intellectual contexts of teaching.

Without approaches in mathematics teacher education that address the complex situations or multiple dynamics with which PSTs will contend as novice teachers, PSTs are unprepared for the realities of public schooling. In particular, PSTs need approaches that attend to issues of equity, self-understanding, and the social and political contexts of teaching within content-specific teacher education (Crockett & Buckley, 2009). The consequences of leaving teachers unprepared for the realities of schooling are damaging, not only for those teachers, but also for students, as teachers who are unprepared may be unable to provide all students opportunity to learn or to enact equitable mathematics teaching practices. Developing an understanding of self as mathematics teacher, which for Sarah means examining her own relation to the complexities in her context, may build capacity for making sense of one's relation to these dynamics and the realities of schooling and critically analyzing teaching decisions. This understanding of self as mathematics teacher is related to what the research literature frames as teacher identity. The theoretical perspective of this study extends current perspectives of identity, as simultaneously personal and social and in constant negotiation (e.g., Wenger, 1998; Holland, Lachicotte, Skinner, & Cain, 1998), but emphasizes the importance of individual's awareness of the construction of identity and active engagement in its negotiation (Butler, 1999).

I contend that engaging PSTs in understanding that *mathematics teacher* is a complex construction supports them in making sense of their relation to the multiple dynamics and forces present in their mathematics classrooms, or in understanding themselves as mathematics teachers. This understanding should be framed by prevailing discourses of teaching, mathematics, and students, and by local discourses within their teaching contexts. Further, it builds PSTs' capacity for navigating their contexts and taking up particular mathematics teaching practices that support access to mathematics and equitable learning environments, such as teaching students through methods of inquiry and problem solving (NCTM, 2000). As Butler (1999) argues, understanding how identity is situated and negotiated—simultaneously, asserted and threatened—may open possibilities for PSTs to reconstitute mathematics teacher identity with a different set of attributes and engagements.

In applying Butler's (1999) perspective of gender identity to elementary PSTs, the following questions guide this research: 1) How does the seminar support PSTs to critically examine their understandings of themselves as mathematics teachers and the discourses that shape those understandings?, and 2) How are PSTs' understandings of themselves as mathematics teachers and of related social and political discourses consequential for whether and how they take up new understandings of self as mathematics teacher and teaching in context?

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Theoretical Framework

Conceptions of teacher identity assume that identity is dependent on and formed within contexts, and is relational, shifting, and multiple (Rodgers & Scott, 2008). Becoming a teacher is not developing *an* identity, but is developing *identity* as a continuous process of constructing and deconstructing understandings within the complexities of social practice, beliefs, experiences, and social norms. This is similar to Butler's (1999) description of the process of becoming a woman: "If there is something right in Beauvoir's claim that one is not born, but rather one *becomes* a woman, it follows that *woman* itself is a term in process, a becoming.… As an ongoing discursive practice, it is open to intervention and resignification" (p. 45). Being and becoming a teacher also should be seen as open to continuous identification and reidentification. For example, Sarah engaged in this ongoing process during her internship, identifying how her mentor's actions were shaping her interactions with her students, and, thus, how she saw herself as a teacher in relation to her mentor:

> I find myself yelling at my students and I'm like, "Why am I yelling at them? They just don't get long division yet!" But … I feel my mentor, her voice, and it's like, "Is that me?" because I used to not be a yeller. Have I become *that* teacher? …Have I turned into my mentor? (April 27, 2010)

What Sarah is doing—struggling with the internal and external forces that are shaping her understandings of herself as a mathematics teacher—is identity work. Understanding of self as mathematics teacher is situated within and shaped by historical, political, and social contexts, and also influenced by other individuals and imposed constraints. Prevailing discourses about mathematics teaching and learning and local discourses within their teaching contexts influence PSTs' understandings of mathematics teaching (e.g., de Freitas, 2008) and may shape their engagement with teaching and learning to teach. These may include: institutional discourses around curriculum and testing (e.g., Brown & McNamara, 2005); social discourses around race, class, and student abilities (e.g., de Freitas & Zolkower, 2009; Sleeter, 2008); or, discourses of teacher as "savior" (e.g., Britzman & Pitt, 1996).

My theoretical premise is that deconstructing these contexts, expectations, and constraints and challenging PSTs to question them and the dominant discourses that frame mathematics teaching—the "culturally established lines of coherence" (Butler, 1999, p. 33)—is encouraging PSTs to be actively involved in the process of identity work. Developing awareness of these dynamics means "unpacking the invisible knapsack" (McIntosh, 2001, p. 180) of the implicit discourses that have shaped and are shaping their understandings of being mathematics teachers and making these discourses explicit. For Sarah, that may include thinking about *why* she may "succumb to the pressures of standardized testing" or otherwise lose the flexibility or responsiveness she aspires to demonstrate in her mathematics teaching. According to Butler (1999), deconstruction, as a process of critically examining the meaning and "substantive appearance of gender", identifying "its constitutive acts, and locat[ing] and account[ing] for those acts within the compulsory frames set by the various forces that police the social appearance of gender" (p. 45), can support resignification. As PSTs question these discourses and *deconstruct* (Butler, 1999) what it means to be mathematics teachers in context, they are examining themselves as mathematics teachers. Making sense of one's relations to the multiple dynamics in the mathematics classroom and developing understanding of self as mathematics teacher builds capacity for navigating these dynamics, attending to issues of equity, and being critical analytic about teaching decisions. Thus, this critical examination may lead PSTs to take up new understandings of being a mathematics teacher and of mathematics teaching.

Methods

Data collection took place during a seminar in the final semester of a 15-month elementary master's certification program and the required school-based fieldwork. Ten female PSTs, 23- to 34-years old, volunteered to participate in the eight 2-hour group sessions of the seminar and the study. One PST self-identified as African-American, one as an immigrant from Argentina, and eight as White.

Seminar Design

Seminar sessions engaged the participating PSTs in *deconstruction*, a process of critical examination of their understanding of themselves as mathematics teachers, teaching, and the political and social discourses that shape their understandings. The seminar design situated critical pedagogy (e.g., Ellsworth, 1989; Freire, 1970/2000; Kumashiro, 2000) in mathematics teacher education. Although PST education has attended to metaphors (e.g., Fenstermacher & Soltis, 1986) and the construction of a professional identity (e.g., Peressini et al., 2004), this seminar engaged theories of deconstruction and wove issues of self-understanding into content and context. PSTs analyzed case studies, transcripts, and videos, which provided opportunities for them to name and complexify prevailing discourses and positionings in relation to mathematics teaching and learning and to examine their own and others' practices. PSTs also completed writing assignments to relate these activities to their own positioning as mathematics teachers. Rather than reaching a predetermined understanding of certain content, the goal was for PST's self-understanding, where the PST asks how he or she is positioned and how others are positioned within discourses and "brings this knowledge to bear on his or her own sense of self" (Kumashiro, 2000, p. 45). That is, sessions served to open possibilities for PSTs to position themselves differently. Meeting as a group generated a sense of common cause and created a space for PSTs to voice their concerns as well as strategize; sessions, thus, became legitimate and valued spaces where PSTs could speak and others would listen.

Data and Data Analysis

I collected data from the seminar to examine: a) how PSTs develop understanding of themselves as mathematics teachers, b) how this is consequential for whether they take up new understandings of mathematics teacher and teaching; and, c) how the seminar supported PSTs' self-examination. Data include interviews, discussion transcripts, and PSTs' written work from the seminar.

Attention to PSTs' understandings of themselves and their take up of new understandings was analyzed through the particular discursive practices of "working difference" (Ellsworth & Miller, 1996). How PSTs are engaging in the particular discursive practices of working difference that attend to context, their positioning, and the positioning of others around an issue that emerges as salient speaks to how the seminar is supporting critical self-examination. If understandings of self are conceptualized as positioning, then repositioning and negotiating positions suggests new understandings of self and personal readiness. How PSTs are taking up *new* understanding of mathematics teacher is operationalized through how PSTs are engaged in the particular discursive practice of showing the active labor of working difference (Ellsworth & Miller, 1996), that is, how they are repositioning themselves.

To understand how PSTs are taking up new understandings of mathematics teaching, I followed how individuals are problematizing one emergent issue through the sessions, attending to content of their conversations about teaching and the processes they engaged in their conversations. Problematizing teaching includes how PSTs are attending to the work of teaching and how they are negotiating three specific levels of attention— attention to core principles of teaching, strategies for instruction, and actions specifically for tomorrow's classroom (Pollock, 2008)—and using particular discursive practices in taking up and engaging in problems of practice (Horn & Little, 2010). Pollock's (2008) three levels of talk serve to structure the analysis of the content of the conversation, while attention the discursive practices of normalizing, specifying, revising, and generalizing (Horn & Little, 2010) are the processes of the conversation that the analysis follows.

Results

The following analysis focuses on two PSTs' understandings of self as mathematics teacher and how the seminar was consequential for their new understandings of mathematics teacher or teaching. The analysis continues the case of Sarah and presents a contrasting case of Brooke. Issues of assessment emerged as salient to both Sarah and Brooke; Sarah engaged in particular discursive practices that suggest her taking up new understandings of mathematics teacher and mathematics teaching, while, in contrast, Brooke did not. That is, the seminar did not support Brooke's new understandings of mathematics teachers or teaching in the same way.

Sarah's initial vision statement presented her "hopes" for herself as a mathematics teacher and her teaching. She presented the complexity of mathematics teaching and tensions with standardized testing and how testing frames students and their achievement: "I *hope* I continue to see them as unique individuals, not just as test takers." In Session 2, Sarah also shared how she was uncomfortable in hearing her mentor's voice in her own teaching, asking herself, "Is that me?" when she yelled at her students because of their difficulties with long division. My analysis suggests how Sarah felt uncomfortable with how her mentor was positioning students and their abilities and with how she was also positioning her students when they were having difficulties with long division. Sarah's discursive practices suggest that she struggled with concurrently supporting her students' learning and assessing them, either in mathematics class or in relation to standardized testing. Sarah engaged in unpacking her positioning by attending to institutional pressures of testing and pressures in her context.

Issues of assessment also emerged as salient to Brooke's understanding of herself as a teacher. In the following excerpt, Brooke emphasized, but resisted her positioning by issues of assessment, even minimizing its influence on her teaching:

> But see, I don't feel like [curriculum pacing] influences me all that much because like I'm spending five days on this stupid volume lesson, which really doesn't have the huge practicality of counting these imperfect block shapes, but I'm doing it because it is going to be on their unit assessment. But the majority of the stuff I teach is all beyond what is being assessed on those tests. So I feel like the test doesn't necessarily, like I make sure that I hit

> those things but it is only like, you know, a couple of things that I need to do and I have a lot more time that I get to do a lot more enriching things with my kids.

Brooke attended to how she is teaching “a stupid volume lesson” because the content is on the assessment but also emphasized that she is not positioned by assessment: “It doesn’t influence me all that much.” In this way, Brooke seems to resist how she is positioned by district-mandated assessments, although she is spending a week of instructional time on a topic that she does not value. Brooke did not attend to these tensions or to the political contexts of her teaching or immediately engage in discursive practices of working difference (Ellsworth & Miller).

As the sessions continue, Sarah and Brooke engaged in different discursive practices with working difference, that is, in understandings themselves as mathematics teachers in relation to discourses about assessment. In a particular conversation in Session 4, PSTs highlighted challenges and tensions with engaging students in rich tasks and assessments while grading students in line with institutional constraints on teachers’ flexibility with assigning grades. In response to these tensions, PSTs began to complain and share categorical advice, which are discursive practices that Horn and Little (2010) suggest are not conducive to the taking up of problems of practice. Sarah, however, took up the tensions of assessment and shared a practice in her classroom that aligned with principles of seeing assessment as an ongoing practice of actively attending to student thinking in the context of her test-driven school culture:

> You have to give them opportunity. What I do a lot of times, as part of the morning routine, I don't give them morning work sometimes if I'm handing back like a big writing assignment or something. I'll leave it on the reading group table. Come and pick up your paper, read my comments, raise your hand if you have questions, and that's their morning work for the day. And you know, if you're monitoring and you're making sure they're not chit-chatting and they're, you know, staying on task, I've found that that's helpful. My mentor thinks it's a waste of time, she thinks it's stupid. I did [this] when I was in takeover, and I thought it was really helpful because I actually did have some kids come up to me and say, “What do you mean I'm not specific?” and like, “What does that mean?” I'm like, “You've got to give details.” You explain it them, like so, “Oh, this is what you wanted.” So I had like, you know, a couple of little writing conferences, and did it make big difference for everyone in the class? I don't know, but I do know it made a difference for two students in particular.

In attending to student thinking and supporting her students’ continued reflection on their work, Sarah presented assessment as a complicated process that she engaged in with students and within her school context. Sarah engaged in discursive practices of problematizing practice by specifying the problem through adding details of how she enacted this practice in her classroom and by generalizing: “You have to give them opportunity.” Additionally, Sarah wove together attention to different levels of Pollock (2008) through presenting principles of assessment, a strategy of how to enact this in her classroom, and the particular actionable steps she took. This conversation was significant for how Sarah demonstrated her developing positioning in relation to assessment, and it also created an opportunity for other PSTs to think about assessment, as evidenced by further conversation.

Other PSTs followed by asking Sarah questions. Then, the conversation shifted to discuss practices in mathematics assessment in primary grades as Candice asked about how to grade first-grade students on their understanding of place value when they are just learning how to count. Brooke interjected and shared how she graded second grade students’ homework:

> I'll be honest sometimes that it really kind of depends on my mood. Like, sometimes I'll be like, you know [shrugs shoulders], it's just that kind of day that I'm willing to be harsh on everybody. Which is bad. I know that we should be really consistent, but some days—

Brooke presented a solution grounded in her local context to the exclusion of attending to the complexity of assessing students. This is similar to her earlier comment about her volume lesson, where she did not attend to the prevailing discourses of testing, but focused on her current students. While assessment emerged as an issue that was salient to her understandings of herself and her teaching, analysis does not suggest that Brooke took up new understandings of assessment as situated within prevailing institutional discourses or problematized assessment or her current practices in this session or subsequent sessions in relation to these prevailing issues.

In her final presentation, Sarah, however, situated discourses of assessment into her mentor's current teaching context and its test-drive culture. Sarah emphasized how she did not appreciate her mentor's classroom management style or her mentor's strict and even negative interactions with students: "I struggled with [her mentor's style] throughout the year. She was too heavy handed with the kids. She relied on humiliation and fear." She ended her presentation, however, by presenting her understanding of assessment and her mentor in context:

> Even though my presentation probably started very critical of her, she's an amazing teacher. She's just operating within this Title 1 environment, where there is all this pressure on teachers to maintain pace with the curriculum guides and FAST test. … I actually calculated the amount of hours and amount of days [of testing]. I think that it was something like over a month of school days were disrupted by testing. So, overall, I think that kind of informs my critique of her. I had a really good year, and I look forward to next year. (June 21, 2010)

Sarah noted how her mentor operates within particular institutional discourses and is positioned in specific ways by the curriculum guides, testing, and her students' high test scores, which made her an "amazing teacher" by the administration's standards. In response to her initial criticism of her mentor's management style, Sarah related her mentor's positioning and the pressures she felt, as brought on by testing and the Title 1 environment, to how her mentor positioned and responded to students. In this way, Sarah presented awareness of the complexity of the assessment and the work of mathematics teaching in challenging contexts and how being a mathematics teacher is contingent on and in response to contexts.

In addition to her new awareness of the complexity of being a mathematics teacher, Sarah demonstrates awareness of the value in the process of deconstruction and critical self-examination. Her deconstruction supports her in understanding the complexity of the particular interactions in her classroom, her mentor's positioning and how this positioning influences her expectations, her students and their learning. That is, Sarah's presentation suggests that she valued the process of deconstruction, as it allowed her—and will continue to allow her—to explore and question how students and teachers are positioned and navigate the social and political dynamics in schools and classrooms. Deconstruction is now a tool that she can use to inform future decision-making, supporting her agency in reidentifying with particular discourses, repositioning herself, and redefining herself as mathematics teacher.

Discussion

As a seminar for mathematics teacher education, analysis suggests that opportunities for PSTs to analyze discourses and positioning and discuss their implications on students and teachers may support PSTs' critical examination of their understandings of themselves as mathematics teachers and their teaching *when* there are continued opportunities to connect to their own contexts and experiences, such as in seminar discussion prompts and writing assignments. Critical self-examination supported Sarah in taking up new understandings of mathematics teacher and mathematics teaching. Brooke characterized her positioning as *localized*, or in relation only to her particular field placement and current practices under her mentor, to the exclusion of how she positions herself towards or is positioned by broader social and political discourses. This suggests how some PSTs are strongly attending to their local contexts to make sense of how they are being positioned as mathematics teachers and more work with attending to the social and political discourses in her *specific* context may have supported her in attending to her complex positioning. Facilitating PSTs in connecting these discourses and positionings to their teaching contexts may empower PSTs, such as Brooke, to understand how they are positioned, a question of personal readiness, and to exercise agency—that is, what *can* I do?—in responding to prevailing discourses by taking up or rejecting certain positions. That is, deconstruction can become a practice used in mathematics teacher education for engaging PSTs in thinking critically about teaching and themselves and also a tool for supporting PSTs' own critical analysis of teaching practices.

Embedded in a conception of teacher identity as fluid and negotiated is "the implicit charge that teachers should work towards awareness of their identity" (Rodgers & Scott, 2008, p. 733). Teacher education has a responsibility to support PSTs in this identity work, that is, in navigating the social, political, and historical forces that are shaping their identity and the multiple demands in schools and mathematics classrooms, as this work is necessary for teachers to have the capacity to address issues of equity and access in teaching mathematics. By operationalizing identity and critical pedagogy in mathematics teacher education, this study contributes to research in mathematics teacher education by providing new tools for mathematics teacher education and attending to enduring questions of how to prepare teachers to enact equitable teaching practices in the complexity of today's classroom contexts.

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TRUE OR FALSE? PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' STRATEGIES FOR EVALUATING STATEMENTS

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To implement current reforms regarding proof and reasoning in secondary school mathematics successfully, pre-service secondary mathematics teachers must have adequate understandings of concepts across mathematical domains for their future teaching. This paper examined the strategies that pre-service secondary mathematics teachers had for evaluating statements in the domains of algebra, analysis, geometry, and number theory. The results suggest that most pre-service teachers rely on examples and their (partially) correct understandings of mathematical facts when determining the statement's veracity. The results also suggest that the majority of pre-service teachers accurately evaluate the validity of statements.

Introduction

Reasoning and proof have been receiving an increasing level of attention in mathematics because they are considered important components of understanding concepts mathematically. Recent reform documents from the National Council of Teachers of Mathematics (NCTM, 2000) and the Mathematical Association of America (MAA, 2004) have placed emphasis on teaching and learning about verifying statements in secondary school mathematics and undergraduate mathematics. The *Principles and Standards for School Mathematics* (NCTM, 2000) suggests that students should be able to evaluate conjectures by the end of secondary school. The *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide* (MAA, 2004) recommends that undergraduate students need to "learn a variety of ways to determine the truth or falsity of conjectures" (p. 45). Research investigating secondary students' and pre-service secondary mathematics teachers' abilities to decide the truth and falsity of statements, however, suggests that many students and teachers have difficulty doing such tasks (Hoyles & Kuchemann, 2002; Ko & Knuth, 2009; Riley, 2003). Hoyles and Kuchemann (2002) found that 40% of 1,984 eight-grade students inaccurately determined the geometric conjecture's veracity. Riley's (2003) result indicated that roughly 57% of 23 prospective secondary mathematics teachers believed that a false statement in geometry was true. Ko and Knuth (2009) reached a similar finding, reporting that 20% of 35 pre-service secondary mathematics teachers who answered believed one false statement about differentiation to be true.

The aforementioned findings should come as no surprise because students are usually asked to prove a true statement rather than to disprove a false one in the mathematics classroom (Buchbinder & Zaslavsky, 2007). Under such a learning environment, it is likely that most students are easily convinced that mathematical propositions are true (Smith, 2006). Limited research has documented evidence that undergraduate students might rely on examples, deductive reasoning, both example-based reasoning and deductive reasoning, or their familiarity with content when asked to evaluate the statement to be true or false (Gibson, 1998; Goetting, 1995; Weber, 2009). No studies to date have investigated the mathematical thinking behind the strategies pre-service secondary mathematics teachers use to determine the validity of statements in different mathematical domains. Such practices are particularly important for secondary mathematics education majors, as they need to understand the conventions of proving in a variety of domains to implement current reform recommendations about reasoning and proving in secondary school mathematics. Thus, the main purpose of this study was to examine strategies pre-service secondary mathematics teachers had for evaluating various statements in the domains of algebra, analysis, geometry, and number theory. This study was guided by two research questions: (1) In what ways did pre-service secondary mathematics teachers verify each given conjecture to be true or false? and (2) Which strategy did lead pre-service secondary mathematics teachers to make a correct determination of the statement's veracity?

Theoretical Framework

Evaluating the truth and falsity of statements accurately is a complex problem-solving process, as individuals should have adequate understandings of mathematical concepts related to problems and be able to apply such knowledge flexibly. Although individuals need to be able to correctly decide the truth or falsity of a given proposition before constructing a proof or generating a counterexample, such a decision really depends on whether or not individuals believe the statement to be true or false. When evaluating purported statements, some individuals are inclined to base their determination on example-based reasoning strategies. Moreover, they tend to use random, general, or specific examples related to statements during the processes of verifying the statement's validity (Alcock & Inglis, 2008; Gibson, 1998; Goetting, 1995; Harel & Sowder, 1998).

When asked to evaluate the validity of statements, others use their understandings of true known definitions, theorems, or axioms involved in problems (Alcock & Inglis, 2008) or start constructing a proof and then find a counterexample if they get stuck in the proof (Weber, 2009). Still others make judgments based on their past memories of similar conjectures, so they begin producing a proof, looking for a counterexample, or testing a couple of numbers and then making an attempt at a proof (Goetting, 1995). Indeed, verifying statements across mathematical domains is a vital practice for pre-service teachers to foster their reasoning and comprehend the concepts. Particularly, undergraduate mathematics is an important period for pre-service teachers to build knowledge to advance mathematical reasoning for learning more advanced mathematics in secondary-school teaching. The goal of the paper reported here is to shed light upon the processes of evaluating practices associated with conjectures.

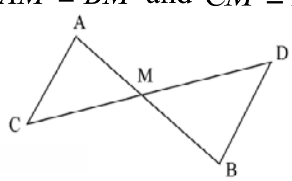
Methods

Eight secondary mathematics education majors from a large Midwestern university in the United States participated in this study. One was a third-year, five were fourth-year, and two were fifth-year undergraduate students. All of the participants had taken a number of upper-level, non-proof intensive mathematics courses that included the topics of analysis, combinatorics, differential equations, linear algebra, modern algebra, number theory, probability, or statistics. They had also taken at least two courses involving mathematical proofs about the topics of analysis, linear algebra, geometry, modern algebra, or transition to proofs. Since number theory, geometry, continuous functions, differentiation, and one-to-one functions were addressed in undergraduate mathematics courses as well as in pre-calculus and calculus courses in high school, all of the pre-service teachers participating in this study had some relevant domain knowledge.

The primary source of data was one-on-one semi-structured interviews. Interviews were audio-recorded, lasted approximately 90-120 minutes, and included a focus on the evaluation of the statement being true or false and the productions of proof and counterexample. Because the focus of this article is on pre-service teachers' strategies for verifying the validity of statements, the results presented and the subsequent discussion focus exclusively on data about their proof and counterexample productions. During the interview, the pre-service teachers were handed each statement (see Table 1), one at a time, and were asked to think aloud how to determine each statement to be true or false. The instrument, comprised of six mathematical statements that were adapted from existing literature and textbooks, was designed to assess pre-service teachers' abilities to evaluate the veracity of statements about algebra, analysis, geometry, and number theory. The instrument was finalized after receiving feedback from three mathematics professors as well as one mathematics education professor and pilot testing with engineering and mathematics graduate students.

The interview transcripts and participants' written responses were summarized initial impressions and highlighted interesting issues regarding the participants' statement verifications. Coding of the data began with a set of external codes that were derived from the theoretical framework. By examining the data and reviewing the transcripts, themes emerged in participants' statement verifications. After proposing these internal (data-grounded) codes, each transcribed interview was reexamined and recoded to incorporate these new codes.

Table 1. The instrument.

Question	Statement	Domain	True or False
1.	If $n \in N$, then GCD(n, $6n-5$) $=1$.	Number Theory	False
2.	If $\overline{AB}$ and $\overline{CD}$ intersect at the point M, $\overline{AM} \cong \overline{BM}$ and $\overline{CM} \cong \overline{DM}$, then $\overline{AC} // \overline{DB}$.  (Adapted from Yang & Lin, 2008, p.65)	Geometry	True
3.	Let $f: D \to R$ be a function and $x_0 \in D$. If $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$, then f is continuous at x_0.	Analysis	False
4.	If $n \in N$, then $n^3 + 44n$ is divisible by 3. (Adapted from Smith, Eggen, & St. Andre, 2006, p. 109)	Number Theory	True
5.	If $h: A \to C$ and $g: B \to D$ are both 1-1 functions, $A \cap B = \phi$, and $C \cap D = \phi$, then $h \cup g: A \cup B \to C \cup D$ is a 1-1 function. (Adopted from Smith et al., 2006, p. 204)	Algebra	True
6.	Let $f: D \to R$ be a function and $a \in D$. If $f: D \to R$ is differentiable at a, then f is continuous at a.	Analysis	True

Results

The results reported in this section are organized by two research questions concerning pre-service teachers' strategies for evaluating various conjectures.

Pre-Service Teachers' Strategies for Evaluating the Statement

Based on the theoretical framework regarding strategies for verifying conjectures, four specific categories—the example-based strategy, the mixed reasoning strategy, the naïve reasoning strategy, and the sophisticated reasoning strategy, as listed in Table 2, were proposed through the coding process to access the participants' responses to the ways they used for doing these tasks.

Table 2. Strategies for evaluating statements.

Strategies	Description
Example-Based Reasoning	● Individuals rely on numbers or diagrams to verify the statement.
Mixed Reasoning	● Individuals both use examples to identify relevant patterns and structures, and manipulate (partially) correct properties, definitions, and/or theorems to identify a reasonable example to attempt to prove or disprove the statement.
Naïve Reasoning	● Individuals manipulate partially correct properties, definitions, and/or theorems from their intuitive understanding or past experience to verify the statement.
Sophisticated Reasoning	● Individuals manipulate relevant true properties, definitions, and/or theorems to attempt to prove or disprove the statement.

Table 3 displays the distribution of strategies the pre-service teachers had for deciding each statement to be true or false. As shown in the table, the mixed reasoning strategies (33 cases out of 46) were used more often than the other three strategies (5 example-based reasoning strategies, 4 sophisticated reasoning strategies, and 4 naïve reasoning strategies) by prospective teachers when determining the validity of statements.

Table 3. Pre-service teachers' strategies for evaluating the statement.

Strategy	True								False				Frequency Count*
	Question 2 (Geom-etry)		Question 4 (Number Theory)		Question 5 (Algebra)		Question 6 (Analysis)		Question 1 (Number Theory)		Question 3 (Analysis)		
	n	%	n	%	n	%	n	%	n	%	n	%	n
Example-based	0	0	1	13	0	0	0	0	3	38	1	13	5 (4)
Mixed	8	100	7	88	6	86	5	71	5	63	2	25	33 (8)
Naïve	0	0	0	0	0	0	2	29	0	0	2	25	4 (3)
Sophisticated	0	0	0	0	1	14	0	0	0	0	3	38	4 (4)

Note. Pre-service secondary mathematics teacher 8 did not decide Questions 5 and 6 to be true or false.

*The frequency count is the number of each strategy that each participant used for evaluating the validity of six given statements.

Due to decimal rounding, the total percentage of Questions 1, 3, and 4 is greater than 100.

Data described in Table 3 also illustrates that each pre-service teacher used the mixed reasoning strategies at least once when evaluating the statement's veracity. Although the mixed reasoning strategies was the most commonly strategies used for verifying the validity of Questions 1, 2, 4, 5, and 6, substantially fewer pre-service teachers used the example-based reasoning, naïve reasoning, or sophisticated reasoning strategies for evaluating Questions 1, 3, 4, 5, and 6. Considering the statements across different domains, it is interesting to note that the false question in Number Theory was the most likely to be verified by examples-based reasoning strategies. Also, all pre-service teachers tended to employ the mixed reasoning strategies for determining the true geometric statement. The section that follows describes each strategy used by pre-service teachers when deciding the statement's veracity.

Example-based reasoning strategies. Four pre-service teachers based their determination of the statement's validity merely on numbers or diagrams related to the problem. One pre-service teacher stated what he did when determining a statement's (Question 1) veracity: "I went through a few possibilities, […], and I got up to five. […] Six times five is twenty five, so the greatest common divisor between twenty five and five is five" (PSMT1). Another participant used diagrams to evaluate a statement's (Question 3) veracity: "I was thinking like x, and it would be continuous. And then I was thinking about x^2, and it would be continuous as well" (PSMT5). Thus, these pre-service teachers tested a few numbers or diagrams to evaluate the statement's validity.

Mixed reasoning strategies. In general, every pre-service teacher used the mixed reasoning strategies at least once when determining the veracity of statements. Moreover, these teachers decided the statement's validity on the basis of examples and their (partially) correct understandings of definitions or theorems involved in the problem. One teacher labeled the figure first and said, "Angle *AMS* and angle *BMD* are equal because they are vertical angles, so [triangle *AMC* and triangle *BMD* are] congruent triangles, [so the] corresponding angles [angle *CAM* and angle *DBM*] are congruent, and they are alternative interior angles which make line *AC* and line *DB* be parallel." (PSMT3, Question 2). In explaining why he determined the statement to be true, one teacher said, "I plug in a couple of numbers just to see if the statement works. […] When I factor out n, I can see that any natural number that is a multiple of 3 means that is also divisible by 3" (PSMT2, Question 4). Still another teacher drew a picture and explained why she agreed with the statement to be true, "Continuous functions mean that you have a graph, it is a smooth line, and there is no hole on the graph. [T]he limit from the left and the limit from the right is the same that gonna be continuous" (PSMT8, Question 3). In summary, these pre-service teachers who applied the mixed reasoning strategies for verifying the validity of statements either relied on examples to identify the structure of the statement or based on their (partially) correct understandings of mathematical properties along with examples related to the problem.

Naïve reasoning strategies. With this strategy, three per-service teachers determined the statement's veracity based on their partially correct understandings of mathematical facts related to the problem. When evaluating a statement's (Question 3) veracity, one teacher stated, "[The continuous function] was defined at some point x on the function, or x_0 on the function. […] [T]he limit from the left is the same as the limit from the right, so since it's defined at that point and those limits are the same, then it has to be continuous" (PSMT1). Another teacher explained, "What I can remember, if a function is differentiable at a, then that means that the corresponding y value exists at a, so it doesn't necessarily mean the function to be continuous" (PSMT7, Question 6). Thus, these pre-service teachers seemed to possess fragile mathematical knowledge to verify the statement's validity.

Sophisticated reasoning strategies. Four pre-service teachers decided the validity of statements on the basis of true mathematical definitions, theorems, or properties. For example, one teacher explained his strategy for evaluating a statement's (Question 5) validity:

I know the definition of one-to-one function means that for exactly one, there is one x value for only one y value, so from the domain to the range. Umm, since both of those are one-to-one *functions and since, umm, the domains of each function h and g, respectively, are mutually exclusive as well as the range of mutually exclusive, the intersection is the no set, and the union between the two, umm, will still be one-to-one. (PSMT1).*

In responding to what he did when verifying the validity of a statement (Question 3), one teacher stated, “The statement is false because the definition of the continuity is the limit of the function at a point is also equal to the value of the function at that point” (PSMT2). In short, these pre-service teachers attempted to employ relevant mathematical facts logically when determining the statement’s veracity.

Pre-Service Teachers’ Strategies for Evaluating the Statement and Their Determination of its Validity

Table 4 presents the frequency count of strategies for verifying statements used by pre-service teachers along with their correct and incorrect decisions on the validity of statements.

Table 4. Frequency count of pre-service teachers’ strategies for verifying the statement and their decisions on its validity.

Strategy	True								False				Frequency Count*	
	Question 2		Question 4		Question 5		Question 6		Question 1		Question 3			
	Co	In	Co	In	Co	In	Co	In	Co	In	Co	In	Co	In
Example-based	0	0	1	0	0	0	0	0	3	0	0	1	4(3)	1(1)
Mixed	8	0	7	0	5	1	4	1	4	1	1	1	29(8)	4(3)
Naïve	0	0	0	0	0	0	1	1	0	0	0	2	1(1)	3(3)
Sophisticated	0	0	0	0	1	0	0	0	0	0	3	0	4 (4)	0(0)

Note. Pre-service secondary mathematics teacher 8 did not decide Questions 5 and 6 to be true or false.
Co indicated that the correct decision regarding a statement’s validity made by pre-service teachers.
In indicated that the incorrect decision regarding a statement’s validity made by pre-service teachers.

[a]The frequency count is the number of occurrences that a particular strategy was used. Totals may include multiple counts for a single pre-service teacher (i.e., a teacher may have used the example-based reasoning strategy for verifying the validity of more than one statement). The number of different teachers citing a particular strategy is provided in the parenthesis.

As seen in Table 4, the participants who used the sophisticated reasoning strategies accurately made a decision on the statement’s veracity. It is clear from the table that the pre-service teachers were more likely to make an incorrect determination of the statement’s validity by employing the naïve reasoning strategies. If we consider the most effective strategy used in allowing a correct decision on a statement’s validity, then the mixed reasoning strategies (29 cases out of 46) far out-numbered the other three strategies (4 sophisticated reasoning strategies, 4 example-based reasoning strategies, and 1 naïve reasoning strategy). These findings show that most pre-service teachers did not solely rely on examples when testing the proposition to be true or false. These findings also show that only a few pre-service teachers were able to employ true mathematical properties and theorems to decide the validity of statements correctly. Considering the statements across various domains, it is interesting to note that all participants accurately evaluated the geometric statement (Question 2) by using the mixed reasoning strategies. Perhaps relying on pictures serves as a helpful means for pre-service teachers to identify the structure of the given statement in Geometry.

Discussion

This study examined the processes through which pre-service secondary mathematics teachers evaluated the truth and falsity of given statements across domains. The overall findings indicate that all participants employed the mixed reasoning strategies—using both examples and their (partially) correct understandings of mathematical concepts—for determining the statement's veracity at least once. The fact that the pre-service teachers in this study overwhelmingly used examples when asked to verify the validity of statements is not surprising considering the literature on undergraduate students' tendencies to rely on example-based reasoning strategies (Gibson, 1998; Goetting, 1995; Harel & Sowder, 1998). This study, however, shows that the participants used examples to identify patterns and structures rather than conclusively determined the truth of statements. Considering the strategy used in allowing a correct decision on the statement's validity, the mixed reasoning strategies were more effective ways (29 cases out of 46) than the other three strategies (4 sophisticated reasoning strategies, 4 example-based reasoning strategies, and 1 naïve reasoning strategy). There were a few cases in which the pre-service teachers inaccurately made a decision on the validity of statements by employing the example-based reasoning strategy (1 cases out of 46), the mixed reasoning strategies (4 cases out of 46), or the naïve reasoning strategies (3 cases out of 46). These teachers seemed to use their partially correct understandings of limits, continuous functions, one-to-one functions, and the greatest common divisor to determine the statement's veracity. This result is reminiscent of Weber and Alcock's (2004) suggestion that "students' [concept] images of mathematical concepts are often inconsistent with the corresponding formal definitions" p. 232). In order to help individuals develop concept images—defined as mental pictures—to be consistent with mathematical definitions, mathematics instructors need to draw attention to pre-service teachers' misconceptions of concepts and seek ways to refine their content understanding.

Another feature of the results is that the majority of participants determined the validity of statements accurately (38 cases out of 46). Yet, the wealth of existing studies investigating pre-service secondary teachers' conceptions of proof show that many teachers have considerable difficulty understanding and producing proofs and counterexamples (e.g., Goetting, 1995; Harel & Sowder, 1998; Weber, 2001). The results of this study suggest that providing pre-service teachers with opportunities to experience determining the validity of statements across domains might be a way to enhance their development with proof and counterexample. Such practices can help pre-service teachers not only see the logic behind a statement (Tall, 1992) and explain why something is true or false, but also foster their mathematical reasoning and understanding (Buchbinder & Zaslavsky, 2007). If engaging learners in verifying the truth and falsity of statements can illuminate underlying concepts of propositions as well as promote learners' mathematical reasoning, more research is needed on what curricular tasks can better develop pre-service secondary school mathematics teachers' conceptions of proof and counterexample.

In summary, four strategies—example-based reasoning, mixed reasoning, naïve reasoning, and sophisticated reasoning—used by pre-service teachers to evaluate the validity of statements in different domains cannot make generalizations due to the small number of participants in this study. While most pre-service teachers relied on examples to test Question 1 (Algebra), the majority of participants were inclined to use numbers or diagrams along with their (partially) correct understandings of mathematical facts when verifying various statements across domains. These findings suggest a need to help pre-service teachers apply example-based and deductive reasoning strategies flexibly to evaluate the statement's veracity. These findings also suggest that designing instructional approaches by drawing from individuals' strategies for evaluating statements to help pre-service secondary mathematics teachers learn mathematics meaningfully and foster their mathematical reasoning is needed.

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MATHEMATICAL KNOWLEDGE FOR TEACHING OF STUDENT TEACHERS AND ITS ENHANCEMENT THROUGH A SPECIAL FINAL COURSE

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In this article we describe the results of a special final course, at the main teachers' college in Mexico, which had two related main objectives: one was to find out the Mathematical Knowledge for Teaching (MKT) held by student teachers (ST) at the end of their instructional preparation. The other was to discern ways to improve this knowledge and to document the changes observed. In teachers' colleges in Mexico, math contents and pedagogical ideas are taught separately, so we aimed to help student teachers to integrate these. The analysis showed that their knowledge is mainly instrumental but that through discussions and reflection about the main issues, they were able fairly quickly to attain a significant improvement in all the contents included. Moreover, they also showed changes on some of their views about math and its teaching.

Introduction and Theoretical Framework

For two decades in Mexico, great effort had been placed on improving education at the basic levels. Study programs have been changed, text books have been replaced and computers have been brought to help out, but internal and external evaluations have shown at best very small improvements in students' achievements. The most likely explanation, supported by research in math education (Adler et al., 2005) is that a very important element has been overlooked: teachers' professional development. With new principles, standards and approaches brought everyday into education, the preparation of teachers had become even more crucial.

A very important sector of teachers, which also has to be taken into account, are the future teachers being prepared in the different pedagogical schools and colleges. Ponte and Chapman (2008) give an overview of the studies carried out with student teachers (ST) about their math knowledge, their teaching knowledge and their development.

A common practice in teachers' instruction in Mexico is to separate content and pedagogy in different courses, with the assumption that the future teachers will be able to integrate them in their practice. However, this and the formal procedural orientation of their math courses, leave the ST with limited skills and inadequate conceptual understanding. Thus, in the main teachers' college in Mexico for secondary education (Escuela Normal Superior de Mexico), it was felt necessary to introduce a special course at the end of their training that would give the ST an opportunity to build on their previous knowledge and to restructure their conceptions through group discussions and reflection. Within this context we initiated a research study to find out the general knowledge held by ST at the end of their instructional preparation and to assess to what extent their pedagogical content knowledge in several topics was modified by this didactical intervention. We also recognized the importance and close relationship between conceptions, beliefs and practice (Thompson, 1984; Leder, et al. 2002) and although it won't be described in here, another portion of this project looked into these issues.

Pedagogical Content Knowledge (Shulman, 1987) refers to a complex mixture with many components like content, pedagogy, organization of topics and problems, student conceptions, models, representations, activities, curriculum, etc. Some facets of this teachers' knowledge are more closely related to the mathematical content, like knowing the structure and connections of mathematical concepts and procedures, deconstructing one's own knowledge or understanding students' methods of solution. Ball and Bass (2000) associated this special knowledge with the term: Mathematical Knowledge for Teaching (MKT). Hill and Ball (2004) continue developing the concept of MKT and describe another of its components: profound math knowledge, which is a specialized knowledge that helps teachers understand and plan their classroom activities.

Professional development programs seek, in various ways, to enhance teachers' practices and therefore students' competence. Some are centered on pedagogical ideas, others on mathematical content and some others on the mix, Pedagogical Content Knowledge (PCK). Our orientation is along this last line, with an emphasis on MKT and following the view of many researchers (Ponte and Chapman, 2006) who stress that teachers' training should be connected to their practices, using the same tasks, materials and techniques that could be used in their classrooms.

Based on different frameworks and methods of inquiry, there have been a number of research studies connected to teachers' professional development projects in different countries. Amato (2006), within a mathematics teaching course for student teachers, conducted a study to improve their relational understanding (Skemp, 1976) of fractions, by playing games. In a study investigating the Pedagogical Content Knowledge (PCK) of elementary school teachers in the topic of decimals, Chick, et al. (2006) proposed a framework with three categories for their analysis: 1. *Clearly PCK*; 2. *Content Knowledge in a Pedagogical Context* and 3. *Pedagogical Knowledge in a Content Context*. Like several other authors, Seago and Goldsmith (2006) studied the possibility of using classroom artifacts like students' work and classroom videos to assess and promote MKT. Also, in a collaborative action research, Cooper, et al. (2006) uncovered some characteristics of instructional interactions that lead to positive results in students' learning.

In an article about cognitively guided instruction, Carpenter, et al. (2000) stressed the importance of teachers' knowledge about the mathematical thinking of children. The authors identified four levels of teachers' beliefs that correlate with their mode of instruction: I. They believe that math has to be taught explicitly and therefore they show procedures and ask the students to practice them. II. They start to question this explicit mode and therefore they give to the students some opportunity to solve problems by themselves. III. They believe that students can have their own strategies so they provide problems and the students report their solutions. IV. Teaching becomes more flexible, with the teacher learning from his students' productions and adapting his instruction to this knowledge.

Since we were interested in assessing the MKT, we based our analysis on the framework given by Ball, et al. (2008), who divides this knowledge into three main domains:

A. *Specialized content knowledge* is the math knowledge and skill needed almost exclusively for teaching. The teacher requires supplementary math knowledge in the multiple activities of his practice. Among other things, he must have a profound understanding of the fundamental concepts of each of the topics, knowing not only 'how' but also 'why'. In addition, he should be able to unpack math ideas to make them more visible to others. Moreover, he should know the connections of different concepts and topics and relations with other subjects.

B. *Knowledge of content and teaching* consists of knowing about teaching math. This knowledge is required in the planning, design and instruction in the classroom work to answer questions like: What would be an appropriate sequence of teaching? What example would illustrate this or that? What classroom methods should be followed? What representations are adequate? Each of these tasks requires a combination of specific math understanding and pedagogical ideas.

C. *Knowledge of content and students* combines knowing about students and math. This knowledge is related to students' thinking, strategies, difficulties and misconceptions. It is needed to infer and evaluate what the students say or do, their methods, their solutions, etc. These tasks require a blend of a specific math understanding and a familiarity with students' thinking.

Methodology

The special course was given during the last semester of formal courses to a group of 21 students in a teacher's college in Mexico City (ENSM – Escuela Normal Superior de Mexico), preparing them for teaching at the secondary school level. Each six hour session of the 16 given was divided into two thirds of selected contents and one third of additional pedagogical elements. In this article we will describe only the selected contents section of the course.

Before the course, we asked the ST to mention math topics from their previous courses which they found difficult or they think to be a source of conflict to students. Among the most cited answers were: fractions, decimals, mental calculation, algebra, variables, functions, probability… Thus, we decided to form three sequential blocks for discussion in the course: 1) Fractions and decimals, 2) Mental calculation and estimation and 3) Variables and functions.

Professional development programs use a variety of resources to make teachers reflect on the ideas involved. For example, Borko and collaborators (2008) suggested video analysis to motivate discussions between teachers and Chamberlin (2003) employed written students' productions. In our study however, our instruments for analysis and means of motivating reflection were a series of questionnaires and further inquiries during the sessions.

All the 16 sessions were audio taped for their analysis. At the beginning of each of the three blocks and also in most of the sessions, a questionnaire or a worksheet was handed out (or previously given to answer at home) with the objective of making the ST reflect about some important ideas and concepts of that particular content. This also had the purpose of steering their thinking about the three main domains of MKT described above. From the research point of view, their written answers gave us a small window of their knowledge. Then a full class discussion took place to argue their own ideas and to hear and evaluate the others'. Because of the dual character of the course, for the most part, we let them express their ideas and encourage interaction between them. In the latter sessions of each block, we gave them notes (taken home to read) synthesizing some important ideas of the topic being covered and then the ST discussed those ideas further. This second phase gave us another window of their knowledge and of the possible changes brought about. The main sources for these notes were research papers on mathematics education in each subject. The pedagogical support for the course was taken also from the research literature, for example Schwartz, et al. (2006), Chick and Baker (2005) and McDonough and Clarke (2003).

Results

To illustrate the two ways data was collected, here we will describe 1) the results of the questionnaire applied before and after the six sessions of the block of fractions and decimals and

2) the results of the observations of the four sessions of the block of mental calculation and estimation (the other block not discussed shared similar conclusions).

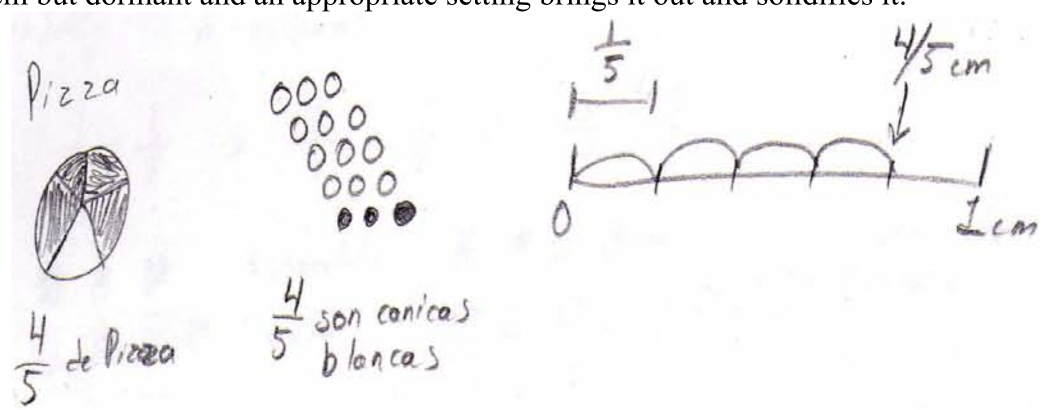
Results of the Questionnaire Applied Before and After the Block of Fractions and Decimals

The questionnaire consisted of 6 items (some with two parts). Here we show three of the six questions which seem representative (from the 21 ST, 18 took the initial questionnaire and 19 took the final).

Item 2. Consider the fraction 4/5. a) Describe some of its different meanings. b) Represent it in different ways.

Part a): *In the initial questionnaire*, 16 of the ST wrote only one or two meanings, without explanations: For example, "Four parts of five.", "Four over five.", "4/5 = .8", "Four of five.", "Four is to five. Four fifths." This exhibited a poorly established knowledge, extracted from what they vaguely remembered. *In the final questionnaire,* 15 of the ST gave much more complete answers, adding also the associated meaning. One short answer was: "As quotient → four over five; as ratio → four to five; as part-whole → four fifths…" Of course, this means that they learnt something, but the value we see in it is that, by briefly exposing them and letting them reflect on these ideas, the ST were able to demonstrate a more sound knowledge.

Part b): This is an important question that shows the amplitude and flexibility with different representations. *In the initial questionnaire*, 10 of the ST represented the fraction only as part-whole of continuous sets like rectangles and circles; 7 others used also part-whole in both types of sets (three of them include a third representation: an equivalent fraction, the numerical line or as a ratio). *In the final questionnaire,* we observed a multiplicity of answers. All of the ST employed at least three different representations like the one shown below in the figure. Five of them supported their answers with well elaborated explanations. Here we observe a richer *knowledge of content and teaching.* Again, this shows that much of this knowledge is held by them but dormant and an appropriate setting brings it out and solidifies it.



Item 4. a) Give a problem illustrating the operation: 1 ½ divided by 3. Explain its solution. b) Give a problem illustrating the operation: 1 ½ divided by ½. Explain its solution.

Part a): *In the initial questionnaire*, most of the ST gave a partitive context when posing the problem: "Juan wants to divide 1 ½ chocolate bars in equal parts for his 3 children. ¿How much would each one gets?" *In the final questionnaire,* we observed basically the same, but their symbolic solutions are more frequently supported with graphical representations and with more congruent explanations.

Part b): This obviously was more revealing than the first part. *In the initial questionnaire*, only 5 of the ST posed a correct problem, based on the interpretation "how many times it fits." For example: "I have a soda with 1 ½ liters and a glass of a ½ a liter. How many times will I be able to fill the glass without any leftover?" Another 6 ST posed an incorrect problem, either forcing the result of 3 of the original operation, not even using the numbers contained in it: "A car need gas every 50 km. How many times we have to stop for a trip of 150 km?" or posing instead the operation 1 ½ divided by 2: "If you have a chocolate bar and half of another, and we want to divide it between 2 people, how much does each one get?" The other 7 ST didn't answer. We can characterize this as a deficient *specialized content knowledge. In the final questionnaire,* more than half of the ST gave a correct problem, although we still observe the same two incorrect thinking mentioned. This seems to be a hard notion for them.

Item 6. Consider the numbers, 0.245, 0.2 and 0.1089. a) If a student says that 0.1089 is the biggest, what do you believe he is thinking? b) How would you show him which is bigger?

Part a): *In the initial questionnaire*, there were two types of similar but not equivalent answers; 11 of the ST mentioned that the decimal point was ignored and 7 others write that the student looks at the number of digits. This shows some *knowledge of content and students. The final questionnaire* showed similar results.

Part b): *In the initial questionnaire*, 7 of the ST drew a diagram showing the position value of each digit, 5 only proposed the use of the numeric line, another showed sequences of numbers without explanation and the rest didn't answer. This demonstrates a reduced *knowledge of content and teaching.* The main difference between the two questionnaires is that in the initial one, we observe mainly technical explanations based on the separation of the numbers into its digits, but in the *final questionnaire* there was an attempt to give semantic descriptions and more extended explanations. For example, one of them wrote: "In a table of positional values indicating integers, tenths, hundredths…" Shows the table and continues "We observe in the tenths, which is bigger, and see that in two numbers they are the same, so we compare the hundredths…" In general, they revealed afterwards a greater capacity for instruction.

Observations of the Sessions of the Block of Mental Calculation and Estimation

In the four sessions of mental calculation and estimation, the three issues addressed through diverse activities were: i) characteristics; ii) strategies; iii) comparison with the traditional algorithms. In the next subsections we will described the knowledge observed from the ST and their changes, according to the three main domains of the MKT proposed in the theoretical framework.

A. Specialized content knowledge. At the beginning of this block the ST were given three questions about mental calculation and then they argued their answers with the whole group. To the question "*Give two examples that show how people use mental calculation in real life."* the great majority had difficulties identifying situations. There was a clear confusion between mental calculation and estimation since most of the examples were calling for estimations like: "In the kitchen mental calculation is used to estimate portions…" "Time is a factor that we estimate constantly…" "When we go shopping we perform estimations…" When they were told that in mental calculation we expect an exact answer, they gave very vague examples related to some operation and frequently mistaken like "When you multiply by 99 you add 1 to be 100 and then you subtract the result by 1." or "Rounding quantities, for example $25.65 × 4, can be converted to 25.50." Thus we observed a very small *specialized content knowledge*. The ST claimed that "In our classes they don't develop abilities for these subjects." and pointed out that the only strategies they know, they learnt as rules of memorization.

To the next question formulated to the ST "*Which strategies do you know of mental calculation?*" their responses were only a list of names like: "counting", "rounding", "basic operations", decomposition", "chopping" and "multiples of 10". In the follow up discussion, we requested an example illustrating the strategies, but the ST claimed they didn't know how to apply them. Only two ST gave an example: One was: "Decomposition: For 35 + 48 =, first add 30 and 40 to get 70 and the 5 and 8 to get 13. To finish we make the operation 70 + 13 = 83." and the other was, according to him about "counting", but was in effect another example of decomposition. We can appreciate again a very limited knowledge of this subject, with one single strategy known (decomposition) which is the closest in form to the conventional algorithms.

Later on, we applied a worksheet with 10 simple operations where the ST were asked to describe the procedures that they could follow to solve them. More than half of the responses were mental applications of the conventional algorithms. Another 35% were answered by applying a decomposition strategy (the way to differentiate between this one and the algorithms is that in the former the value of the quantities are preserved but in the later, only the digits are handled). Another strategy appearing in approximately 10% of the cases, it is called "by steps". To give an example, for the division 400÷25, one of the ST wrote "4×25=100; 100×4=400; 4×4=16". Rounding appeared only in 4% of the responses although a few operations were designed explicitly to apply this strategy. For example, for 37+46, one ST wrote "40+46=86, 86-3=83"; for 87÷3, another ST wrote "90÷3=30, 3÷3=1, 30-1=29" and for 693÷7, yet another ST wrote "700÷7=100, 7÷7=1, 100-1=99". So not only the ST didn't have a clear set of concepts but their familiarity with strategies was very poor.

Before the second session on mental calculation, the ST were given reading materials on the subject. With this new information, they were able to bring out many strategies during the oral exercises proposed. For example, for the subtraction 56–18, one proposed: "To 18, I take away 2 and to 56 I take away 16 to get me 40. Then I take away the other 2 to give 38" (although hard to follow, this is a correct strategy, using compatible numbers); another followed rounding of the 56 with compensation: "I add 56+4 to get 60 and add 4 to 18 to get to 22. Subtracting these we get 38" and a third one also used rounding but of the 18 with compensation: "I would add 2 to both to get 58–20 = 38". In this same exercise, we observed also decomposition and "by steps" strategies. We see once again here that the ST were capable of applying very rapidly the different strategies shown in the readings and developing their knowledge with the interactions with others. So what was hindering them was a lack of knowledge. As one of them expressed, "Now we have the strategies and know how to use them. We only need to direct them to each exercise."

On to the sessions on estimation, we observed too that the ST' initial conceptions were quite vague. They describe estimation as "It is something approximated." "It is what we use to approximate large quantities." Afterwards, the ST were given a worksheet consisting of 5 exercises related to this topic (which we will refer below as I, II, III, IV and V.)

In items I and II an operation (374 + 421 + 339 + 472 and 1797.50 – 635.10) was given to the ST to estimate the result and give an appropriate context of estimation. In items III and IV, the ST were given a problem (one of addition and one of multiplication) to estimate the result and to explain their solution. In the discussion that followed, they heard others' responses and learned from them. We observed, in general, a significant growth in the variety of their strategies, applying rounding of various types (with one or two significant figures or to units, tenths…) and translations like 400×4 for the first sum.

B. Knowledge of content and teaching. Another question posed in the initial session was: "*How would you teach mental calculation?*" The ST' answers were all along these lines "With a lot of exercises so the students practice their calculations." "Through fast operations without using paper." We observed several misconceptions and inappropriate approaches to the teaching of mental calculation like repetition and memory as opposed to reasoning and the construction of strategies. Furthermore, two of the ST indicated that mental calculation is something that cannot be taught: "It is something used outside the classroom and there is no way to instruct it." In general this shows a poorly founded knowledge for teaching.

In the next session, after reading the materials given, they were requested to think of advantages and disadvantages of mental calculation and the conventional algorithms. Here we observed that their initial views changed considerably. For the algorithms they mentioned: "They are not used in higher courses because the calculator is used, not to lose time." "The numbers lose their meaning." "They are general, but we don't have a feeling if it is right or not." "About mental calculation they stated that: "An advantage is that each one looks for his strategy and applies it." "It develops a numerical sense, for their studies and their life." The progress of the ST illustrated here is in their motivation to teach mental calculation and in knowing a reasonable approach for teaching it. As one of them stated "The development of mental calculations is not only to give the steps to follow but we should base our instruction on strategies so they can decide which to follow."

After this, we questioned the ST about the differences of mental calculation and estimation. Two of their answers were: "there is only one exact answer but many reasonable estimations." "In estimation we should use much bigger numbers." We can appreciate that they showed an increased understanding of these subjects for teaching.

C. Knowledge of content and students. In exercises III and IV mentioned before, we added in each, a possible student's solution. We requested the ST that they would explain the solution and also evaluate it. For example, question III "*For the construction of a special classroom, the school has to collect $6,274 pesos. a) About how much does each of the 28 groups have to contribute? b) If a student writes: 6000÷20=300, explain and evaluate his answer."* Some of their explanations were "The student chopped. He didn't care if the last digits were 6, 7, 8 or 9." "He must take into account a smaller quantity to drop it." We appreciate here some dislike for the 'radical' way to carry out the estimation given. They showed in general very little tolerance for approximations. This is also shown in their evaluations of the estimation given: "this is correct although the error is very big." "It is right although his approximations were very far off."

Conclusions

The knowledge that a teacher should hold is very extensive and it has been characterized by what it is call MKT, which is the mathematical knowledge useful in the teacher practice, strongly conceptual in nature.

However, in many teachers' schools, math and pedagogy are separated in different compartments. ST in their last year of formal learning have a knowledge of mathematics and mathematics teaching, but our observations showed that the former is somewhat limited and forgotten, based mostly on procedural understanding. The latter is composed of conceptions formed through their own experiences.

In general, we observe important deficiencies in the three domains of the MKT. In the *specialized knowledge*, there was a lack of a variety of strategies of solution and the ability to unfold procedures. With respect to *knowledge for teaching*, they had a very small array of representations and illustrations. Their *knowledge of students* was based on their intuitions.

In this paper we described two rather different situations. In the case of fractions and decimals, the ST already received in previous courses, the same topics we covered in our sessions. In the case of mental calculation and estimation, although it is included in their programs of study, many times it was overlooked or the approach given is very mechanical without looking at strategies. In both cases, we observed a weak knowledge of the subjects. However, in the case of fractions and decimals, the six sessions given were able to bring back and solidify the main concepts and in the case of mental calculation and estimation the four sessions were able to provide a good background for further development. Furthermore, the ST learnt to ask themselves the proper questions about their practices within the three domains of MKT.

These results strongly suggest that it would be a very favorable strategy in teachers' colleges to implement a short special course at the end of the formal learning like the one described here, with the objective of ST discussing and reflecting about their strengths and weaknesses in some core subject matter. Currently in Mexico, training of ST is done in classrooms with in-service teachers. However this does not recuperate and refresh their knowledge and practices are often perpetuated by the same traditional ideas.

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PROJECT CRAFTeD: AN ADAPTED LESSON STUDY PARTNERING PRESERVICE MATHEMATICS TEACHERS WITH A MASTER TEACHER

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Reports on a research project designed to implement an adapted Lesson Study cycle whereby preservice mathematics teachers co-create a lesson with a Master Teacher and observe the Master Teacher teach the lesson. Results show the preservice teachers' shift, when designing a lesson, from a 'teacher presenting the task' perspective to a 'student engaging in the task' perspective and the importance of seeing their own *lesson taught.*

Introduction

This paper reports on a study designed to show how an adapted Lesson Study (Isoda, 2007) can enhance preservice teachers' early experiences in constructing mathematical tasks for use in the classroom. For the purposes of this study, the term Lesson Study refers to an improvement cycle in which teachers collaborate to discuss learning goals and plan one or more actual classroom lessons. Teachers engaged in lesson study assume shared responsibility for the teaching lessons they create, observe the effectiveness of these materials as they are taught, and then work to revise lessons for future use (Lewis, 2002).

Specifically we studied a community of planning and practice (Wenger, 1999) by implementing and studying a cycle of preservice teachers designing lessons which promote inquiry and learning for understanding; the implementation of those lessons by a Master Teacher (observed by the preservice teachers); and co-reflection on the lesson by the inservice and preservice teachers. We call this the CRAFTeD (Call for lesson; Referendum, Advising session; Fine-tune; Teach-Experience; Debrief) cycle. The key component of this cycle is that preservice teachers see *their own* Lesson Plan implemented by a Master Teacher which gives them a different level of investment in the lesson than if they watched exemplary lessons, and they see their Lesson Plan implemented without having the pressure of teaching the lesson themselves whereby their concentration may be such aspects as classroom management, teaching style and their interactions with the students.

In pilot studies undertaken in the past (Meagher, Edwards & Koca, 2009) we have seen the importance of exemplary field placements for the consolidation of issues about inquiry learning addressed in the university classroom. The use of video technology in our most recent work has shown how the school-university partnership has the possibility be scaled up to impact far more preservice teachers than are able to attend a field placement site.

The research questions guiding the research and the analysis are:

(a) How do preservice teachers engage in creating lessons with a Master Teacher?

(b) What is the impact on preservice teachers of seeing *their own* lesson being taught?

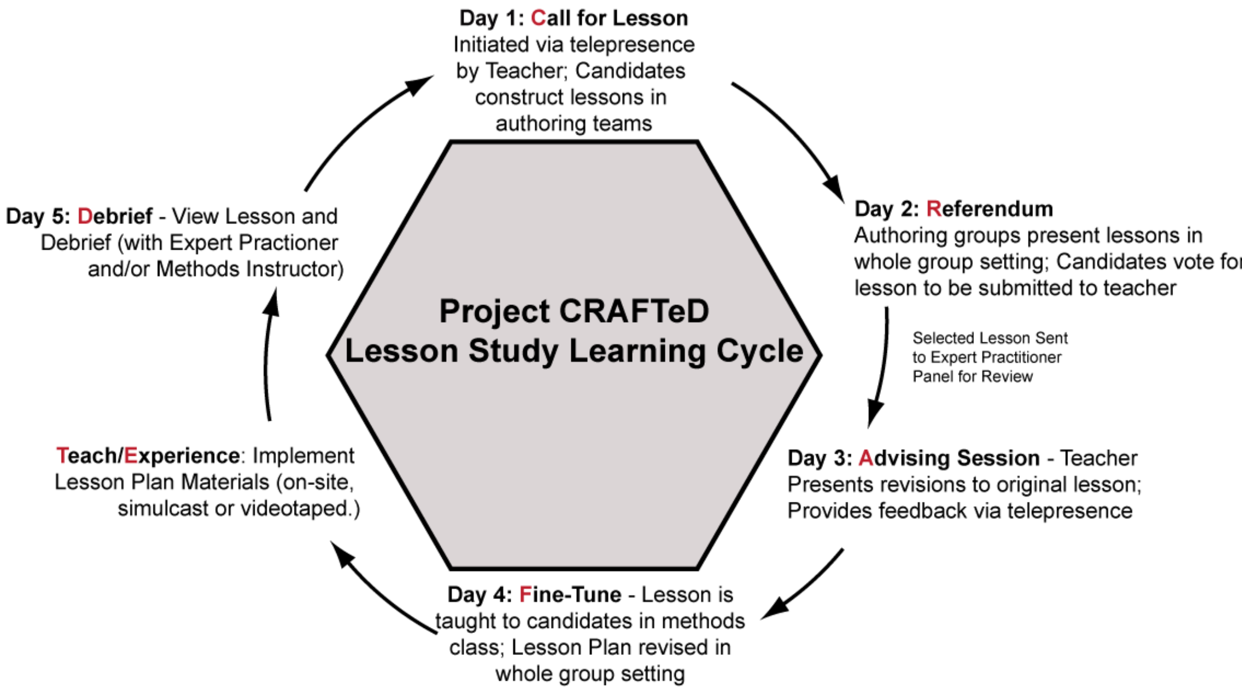
Literature Review and Relationship to Research

There is now a large body of literature (Conference on Learning Study, 2006; Fernandez, Cannon & Chokshi, 2003; Fernandez & Yoshida, 2004; Isoda, 2007; Lewis, Perry, & Murata, 2006) on the importance and effectiveness of Lesson Study approaches in improving teaching, curricular content, and instructional sequences. Hart, Alston & Murata (2009) draw on a plethora of research studies to argue that while many Professional Development models such as action research or teacher place teachers at the centre of the research, Lesson Study is unique in focus that is brought to bear on a "live lesson." They assert that "teachers notice certain aspects of teaching and learning, and this implicit and organic noticing does not happen in artificially replicated professional development settings." (p.1) Lesson Study approaches can also provide for more focused professional development than many traditional professional development models (Lewis, 2002). Furthermore, developing communities of practice (Wenger, 1999) and lesson study groups (Fernandez, 2002) can help teachers to adopt a more research-based focus in their lesson planning and to develop a shared repertoire of communal resources which can transcend individual contributions. Most research thus far has focused on inservice teachers: our research involves implementing an adaption of the Lesson Study approach for pre-service teaches.

Our research employs lessons learned from Lesson Study approaches to address the problem that the teaching methodologies advocated by methods instructors in teacher preparation programs are not readily observed in actual classroom settings, a disconnect that has become more pronounced in the age of high-stakes standardized testing. While university methods instructors laud the merits of student-led inquiry, exploration, and discovery-based teaching methods, secondary mathematics teachers in too many schools "set aside" such teaching in favor of instruction directly focused on student preparation for high-stakes, multiple choice state tests (Seeley, 2006). In an age in which testing dominates the landscape of too many classrooms, it becomes increasingly difficult to provide teachers-in-training with models of high-quality mathematics instruction in secondary school environments. The proposed study looks to explore an answer to this situation by providing preservice teachers with opportunities to collaborate with exemplary high school mathematics teachers by means of inexpensive, readily available web-based conferencing services and strategic face-to-face visits throughout the semester.

Methods and Methodologies

This research is designed to implement the following CRAFTeD cycle:



The cycle emerged from our previous work where the decisive influence of the field placement in terms of the mentorship/exemplars students experienced became apparent.

(i) A class of preservice high school teachers wrote Lesson Plans on a given topic and then worked together to develop improved lessons/short units designed often for technology-rich environments; (ii) an experienced inservice teacher reviewed the lessons/short units and presented an initial redesign; (iii) the inservice teacher taught the lesson, observed on video by the preservice teachers; (iv) the preservice teachers and inservice teachers met together to reflect on and redesign the lesson based on their experiences in the classroom

The purpose of the cycle is to examine (i) how pre-service teachers co-create a lesson and (ii) to examine the particular impact of ownership of the lesson on preservice teachers learning.

The preservice teachers

The pre-service teachers (n=15) were engaged in routine activities that comprise a mathematics teaching methods course, which met for two 75 minute sessions a week for 15 weeks, at a small Midwestern university. The course was the second in a two-course Methods sequence, these courses being the pre-service teachers' only Methods courses in the program. Prior to taking the Methods courses the pre-service teachers take Foundations and Mathematics Content courses. The course was designed specifically for pre-service secondary school mathematics teachers, with the subjects engaging in activities focused primarily on pedagogical issues (e.g. constructing lesson plans and grading rubrics, creating technology-oriented math activities) and content issues (solving mathematics problems, assessing student work). As part of the course the preservice teachers engaged in two iterations of the CRAFTeD cycle.

The inservice teachers

The inservice teacher was a Master Teacher at a local partnering school. The inservice teacher has been a co-operating teacher with the university for many years and is a teacher leader in the district with particular expertise in the use of technology in teaching mathematics.

Data Collection

The data collected during each cycle, as described above was

- A Skype-based discussion between the class and the inservice teacher initiating the cycle
- Lesson Plans created by small groups of the preservice teachers
- The Lesson Plan voted on as the best from the 8 created
- The revisions suggested by the inservice teacher
- Reflections from the inservice teacher and the preservice teachers on the revised lesson
- The final implemented Lesson Plan
- Videotape and fieldnotes from the lesson as taught by the inservice teacher
- Videotape and fieldnotes of the debriefing between the preservice and inservice teacher.
- Interviews with inservice and preservice teachers on the entire cycle.
- A revised Lesson Plan after the teaching of the Lesson.

The data coded for direct answers to the research questions with two basic codes: "co-creation" (CC) and "*their own* lesson" (TOL). The data was then re-analysed using the constant comparative method (Cresswell, 1998) to examine emerging patterns that were not revealed at the first round of coding. Sub codes such as Modifications to the Lessons, Classroom Management, Student Learning emerged at this stage. Quotes exemplifying the quotes were organised and exemplary quotes for each of the codes chosen to support the analysis. The analysis is presented in the order of the cycle.

Results

Call for Lesson

The cycle was implemented on two occasions in the semester. On each occasion there was a Skype teleconference with the experienced teacher. In the first iteration of the cycle the Call was for a lesson introducing right angle trigonometry in a manner that would allow the students experiencing the lesson to "make a connection to the 'sin' button on their calculator." For the second lesson the Call was for a lesson connecting the unit circle and the sine curve.

Referendum

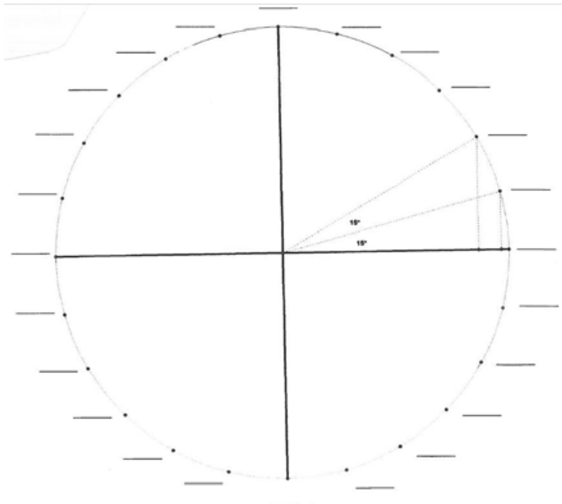
At the Referendum stage in the first iteration the preservice teachers decided to take elements from each of the group's lessons to make what they ended up calling the 'Frankenstein' lesson. The consensus was that the 'Frankenstein' lesson didn't work as well as it could have so, for the second iteration of the cycle the preservice teachers decided to pick one lesson and then work together to strengthen that lesson.

Advising Session

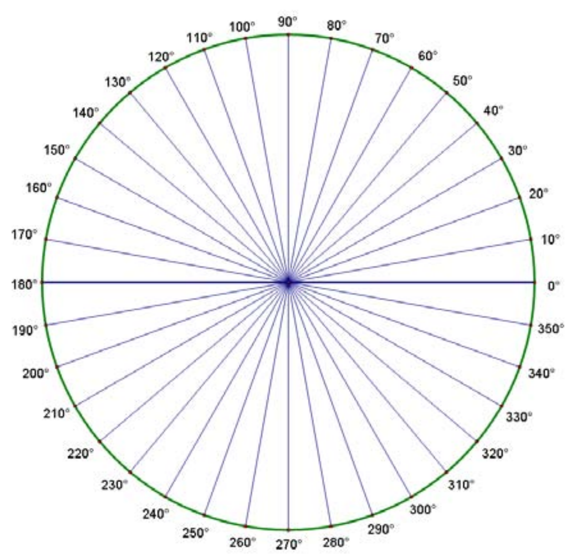
Two candidate lesson's were sent to the inservice teacher who reviewed both and then chose on commenting "I will use A's lesson. The other is good, but too algebraic. I really wanted something visual and kinesthetic, and A's fit the bill." (Inservice Teacher Interview)

The core of the Lesson chosen is a fairly standard use of pieces of dry strands spaghetti placed on a unit circle to create a physical measurement of the y-coordinate associated with a chosen x-coordinate. Students then paste these pieces onto an axis at the relevant place to create the outline of a sine curve.

The students sent a complete lesson, including worksheets to the inservice teacher who then return the lesson with several comments. It is interesting to note that many of the modifications made by the inservice teacher at this point showed his experience in thinking about the lesson from a students' point of view. To use the language of Stein and Schwab Smith (1998) he is thinking about "Tasks as implemented by students" rather than "Tasks as set up by teachers." Many of the modifications are simply practical elements which the inservice teacher recognises as crucial to allowing the students to concentrate on the substance of the task. For example, in the original lesson the students constructed a story about the bug's journey which involved taking measurements every 15 seconds which should then be equated to degrees (the journey having been set up to take 6 minutes so each degree would be a second) resulting in the following unit circle on their worksheet:



with just two indicators each showing 15 degrees. The inservice teacher judged the time issue to be a distraction and also recognised the ambiguity in this set up: Will students recognise that the two 15 degrees indicated are supposed to total to the 30 degrees required for the second measurement? Might students get confused about what happened when you get passed 90 degrees? There are some valuable mathematical discussions possible around those two questions but the inservice teacher wanted the students to concentrate on getting the correct measurements and not be distracted at his point of the lesson and so proposed the following unit circle for the worksheet:



The preservice teachers recognised this as having the advantage of focusing on the degree measurements of the unit circle and therefore being more focused on the substance of the mathematics.

Fine-tune:

The preservice teachers moved to the fine-tune part of the cycle wherein the authors of the original lesson plan formally taught the lesson to the other preservice teachers. This stage of the process turned out to be surprisingly powerful. It furthered the preservice teachers' development in taking the crucial step of thinking about the lesson from a students' perspective rather than from a teacher's perspective (students' tasks vs teachers' tasks again). Suddenly a plethora of practical issues were brought to light and modifications were made to the lesson as the preservice teachers realised that they had not thought carefully enough about many practical issues such as: "The blank graph also had a suggested location to place tape, which we discovered was problematic because the tape being placed over the x-axis made it difficult to determine the exact location of the x-axis." (Preservice teacher 14). The importance of this fine-tuning stage was discussed at length by most of the preservice teachers in their reflections resulted in statements such as "I think the debugging effort demonstrated the importance of "piloting" one's lesson. It is easy to imagine how smoothly things will be, but until you actually put yourself in the role of the student and follow the instructions and complete the activity, it is difficult to identify potential problems." (Preservice teacher 15).

As we have mentioned before this sort of realisation may read as something that should be obvious to teacher candidates but it was an important lesson for them to learn and a lesson that was only learned because their lesson was actually going to be taught not simply produced as an exercise for a class.

Teach/Experience

The Lessons were taught by the inservice teacher and videotaped so that the preservice candidates could spend adequate time watching and reflecting on the implementation of their lesson.

Debrief

The other stage of the cycle where the benefit to the students was greatest was in the debrief on the Lesson. The debrief was conducted through in class discussion and by having the preservice teachers write reflection papers.

In this part of the cycle we again saw the importance of the lesson being *their own* lesson coming through strongly. The level of investment and, therefore, the level of attention the preservice teachers paid to the lesson seemed to be a direct result of this ownership.

Modifications

The preservice teachers noticed straight away that the inservice teacher modified their lesson by including an introduction on families of graphs such as |x| and x^2. He then spent some time using a SmartBoard to animate the journey of the bug walking around the circle. He then challenged the students to think about what the graph of the height of the bug relative to the x-axis over time would look like. This introduction and conjecture setting may seem, to any experienced reader, an obvious aspect of any lesson and it may indeed seem like even novice teachers should know to do this but it is interesting to read in the student reflections how powerfully this aspect impacted them. The impact had two aspects (i) the conviction that it improved the lesson and (ii) this gave them an example, in a context, that they were familiar with and of which they had ownership, of how a teacher can draw on students' previous learning, connect to the current topic and force students to think through what the outcome may be to create a *student* investment in the outcome of the lesson. (Will the graph turn out to be a parabola as suggested by one of the students in the class?). In their reflections almost every preservice teacher wrote about these modifications and the impact they had on the lesson. As one preservice teacher reflected: "In the Spaghetti Sine lesson, the inservice teacher had to include a warm-up since we did not provide one with the lesson. The warm-up he presented was excellent in that it helped students to recall previously learned families of functions as a precursor to the introduction of a new family function, the sine curve. This modification significantly improved the lesson's effectiveness from a constructivist point of view. Students were asked to incorporate new information by layering it upon former knowledge." (Preservice Teacher 2). The idea of the conjecture also resonated strongly with the preservice as exemplified in the following comment: "Most students predicted that the graph of the bug's height above the x-axis would take the shape of a parabola. Having students predict what this graph would look like was a good idea because it forced students to really contemplate the problem before actually investigating it." (Preservice Teacher 5).

Classroom Management

As the lesson progressed the preservice teachers had an opportunity to closely observe a Master Teacher at work and see an exemplary implementation of the lesson without having to worry about classroom management let alone implementation. In fact, it gave the students an opportunity to observe how a skilled and experienced teacher used classroom management to further the goals of the lesson and engage with the students. As one preservice teacher noted "One small thing that he did to increase the quality of the lesson was going around and handing out tape to the students as they needed it. This small gesture allowed the inservice teacher to roam the room, checking student progress and answering questions while keeping the order by keeping them seated. Otherwise, the activity would be more chaotic if students were constantly getting up to get more tape. This is a management detail that I am now aware of as I begin my teaching career." (Preservice Teacher 5). The preservice teachers also observed that the inservice teacher was experienced enough to head off some potential difficulties before they could cause a problem "[He] took care of the issue we were concerned about with having the calculators in the correct mode right away at the beginning of class. By taking care of this before getting into the activity, he prevented the possible disaster that could have occurred later and disrupted the entire class." (Preservice Teacher 9).

Student Learning

The videotaping of the lesson also allowed the preservice teachers the opportunity to observe the students working on the materials they had created. The ownership here provided a real investment in student learning and the tone of excitement in the reflections as the preservice teachers wrote about how they observed student engagement and learning shows the impact of the cycle on their development as novice teachers.

> It was pretty cool to watch kids make connections as they were working. After the slope of their sine graph turned negative, many of them began to see the symmetry in the curve they were constructing. Then when the graph slipped below the x-axis many of them realized they were going to make the same curve but upside down from the first. The coolest connection students began to make though was that they could have cut out multiple pieces at the same time. We could literally hear students coming to this conclusion as we watched the video. As connections were being made we could see the students were adding meaning to what they were doing and that this lesson would be one that they would remember. (Preservice Teacher 1)

This observation of connections the students were able to make was frequently remarked upon. Others were encouraged by seeing that the kind of hands-on lesson advocated for by NCTM (2000) and most professors can work in a real class: "I found the level of student engagement heartening. We have been encouraged to find hands on activities when creating lessons. We are taught that students learn in stages and that abstraction is the last phase. At the secondary level, I believe teachers expect students to leap to abstraction far too quickly. When this happens, the students do not enjoy the process, and often feel left behind in their thinking. Lessons like the Spaghetti Sine lesson remind me to make the leap more of a step. I hope I never forget this pedagogical reality." (Preservice Teacher 2).

Discussion

This study was designed to implement the six stage CRAFTeD cycle we developed to provide preservice teachers enrolled in a Mathematics Methods class a rich and meaningful experience in writing lesson plans and to answer the following research questions: (a) How do preservice teachers engage in creating lessons with a Master Teacher? (b) What is the impact on preservice teachers of seeing *their own* lesson being taught?

The study was not on a large scale and was exploratory in nature but the data presented above provides evidence that there is a trajectory of development in understanding Lesson Plans as well as many basic and more nuanced issues of teaching when the CRAFTeD cycle is implemented. The fact of the lesson being taught provides extra motivation "These experiences have made something I wouldn't have cared about as much more worthwhile. If we had been doing lessons for this class and I knew that all they were going to be: all that was going to happen is that they were going to get graded and I was going to get them back I wouldn't be as prone to put as much time and effort into it as I am now knowing that this is actually being taught to actual students." (Preservice Teacher 6). The preservice teachers not only had an investment in what was happening but were able to follow every detail of the lesson closely. "I really appreciate having the extra time to create, get feedback, edit, and then give a final version of our lesson to him before he taught it. This made watching the lesson easier since I *knew* what was happening." (Preservice Teacher 10). The process can also encourage students to be innovative "being able to think outside the box and come up with these things and have a veteran who has outstanding classroom management skills to implement it helps you feel a little more at ease about thinking outside the box." (Preservice Teacher 8).

The investment in the lesson the preservice teachers had designed and worked on with an experienced teacher made this a very different experience from creating lessons which will never be taught and a very different experience from simply watching an exemplary lesson on video. We believe the CRAFTeD cycle has considerable potential to provide a variant model on traditional Lesson Study that can be very effective the development of preservice teachers.

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Toward Common Ground: A Framework for the INVESTIGATION of Mathematics Methods Courses

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We present an analysis of the practices of one mathematics teacher educator in the context of an elementary methods course. Our analysis of the 51 tasks used in the course revealed a content structure characterized by four components: mathematical knowledge, psychology of mathematics learning, teachers' didactic actions, and reflection. The epistemic nature of the tasks in these content areas is described. We also observed a number of structuring frameworks, largely informed by theory, that were presented to the students as tools for completing course tasks. We conclude with a general framework for the investigation of other methods courses.

Objectives

Teachers have a number of ways to check the degree to which their practice is aligned with local and national objectives set out for elementary mathematics teaching. Aside from assessing their students' understanding in the classroom, there are a number of standards documents that can assist them to stay "on track," such as the NCTM *Standards* (NCTM, 2000), which provides teachers a comprehensive vision of elementary mathematics instruction in terms of content and process objectives. The same cannot be said of mathematics teacher educators (MTEs), those who prepare future elementary teachers of mathematics in the context of teacher education programs. Few, if any, guidelines exist with respect to content, curriculum, or pedagogical approach for the MTE, and as a result, MTEs engage in widely different practices, with virtually no communication among them (Osana, Sierpinska, Bobos, & Kelecsenyi, 2010). Furthermore, while the research on the practices of MTEs has grown over the last decade (e.g., Even & Ball, 2009), it provides at the same time analyses that are at a large and small grain size. For example, cross-cultural comparisons of the mathematics components of teacher training programs are available (e.g., Pope & Mewborn, 2009) as well as psychological descriptions of individual preservice teachers' thinking (Newton, 2008; Tirosh, 2000), and detailed, narrative accounts of single MTEs engaging preservice teachers in specific tasks (see Chapman, 2009). Other studies have examined the practices of MTEs by using their syllabi as data (Taylor & Ronau, 2006), but the limitation of this approach lies in the challenge of determining the types of knowledge their tasks actually or intend to generate, resulting in a superficial or disconnected analysis that often generates more questions than answers.

Such varying perspectives and levels of analysis make it difficult to compare the practices of MTEs in their methods courses, which is the objective of our current research. Little is known about what MTEs do as they prepare future teachers of mathematics, including the reasons for what they do and the effects of their practices. Before one can even determine the effects of their specific practices, however, one needs a common language and organizing framework to analyze and document them. Thus, the objectives of the current study is twofold: (a) to present a detailed analysis of the practices of one MTE, and (b) to present a framework that can be used to document the practices of other MTEs in the context of their own methods courses.

The present study is part of a larger, ongoing project in which we are analyzing the data from five other university sites. We are currently using the framework presented in this paper, called the Mathematics Teacher Educator Framework (MTEF), to document the practices of the other five MTEs from whom we collected data. In so doing, we are refining, and may possibly expand, the framework so that it is general enough to examine the practices of any other MTE in the context of an elementary methods course. This will lead the way for more meaningful dialogue to take place among MTEs and allow for common ground on which to conduct future research in the area. While the findings presented in this paper are the beginnings of a work in progress, we nevertheless believe that our insights at this stage may prompt other MTEs, as well as scholars in the field, to think about adopting the common language we propose through the MTEF to communicate more productively about the preparation of mathematics teachers.

Theoretical Framework

The approach we take in our analysis of MTEs' practices is through an examination of the tasks they use with preservice teachers, those that are required of them in class as well as assignments that are to be completed outside of class. A theoretical framework that we have found particularly useful in our examination of tasks is derived from Chevellard's (2002) "praxeology theory." A praxeology is a theoretical model of a practice, and as such, a praxeology of a practice describes it using four dimensions: tasks, techniques, technology, and theory. One can begin talking about the existence of a "practice" in the execution of a class of tasks if (a) the *tasks* have been divided into types; (b) there are *techniques* for tackling each type of task; (c) there is a conceptual framework and *technological* tools to justify the purposes of the tasks and the validity of the techniques; and (d) there is a discourse that systematizes and brings theoretical coherence into the previous elements of the practice. The discourses in (c) and (d) are necessary for the practitioners to communicate their practice to others and to assess whether a task has been satisfactorily completed.

So far, the Chevellard's (2002) framework of "praxeology" has been usefully applied primarily in the study of practices in mathematics teaching (e.g., Barbé et al., 2005; Sierpinska, Bobos, & Knipping, 2008). We argue, however, that the same theory can be used to analyze our observations of MTEs' practices in their methods courses. In this paper, we show how we used Chevellard's model of Anthropological Theory of the Didactic (ATD) to analyze the tasks collected from one MTE in an elementary teacher-training program at a Canadian university.

Method

Participant and Settings

The mathematics teacher educator in the present study (referred to in this paper as MJ) was a full-time, tenured faculty member in education at a large university in Canada. For the 10 years prior to our data collection at this institution, MJ was responsible for teaching two required elementary mathematics methods courses, each offered during 13-week semesters. The first of these courses, offered every fall, provides the context for the present study. MJ's research background was in educational psychology, which included a minor in mathematics education.

Data Collection and Analysis

The primary source of data for the present study was 51 tasks used in MJ's methods course. We created a rubric based on the ATD model, and used it to code each task as we made a first pass through the data. The rubric contained two major categories: (a) the institutional status of the task, which contained the subcategories of in-class activity, take home assignment, and in-class test, and (b) ATD analysis, which contained the four subcategories specified by

Chevellard's (2002) ATD model, namely task type, technique, technology, and theory.

We also interviewed MJ to obtain more detail about each task and to tap into her justifications for including the tasks in her course. We conducted a total of 15 interviews with MJ over the course of one year, with the duration of each ranging from half an hour to two hours. The interviews were semi-structured, and included questions such as, "Why did you choose this task in particular? Why did you design it this way? What do you intend for the students to learn by engaging in this task? How was it implemented in the class?" Four of the interviews were audio recorded; detailed notes were taken during the remaining interviews. Finally, a third source of data used in our analyses was the collection of documents provided to students in the course, which included the syllabus, all handouts and tests, and lecture slides.

The transcriptions of the audio recorded interviews and the interview notes were analyzed using grounded theory techniques (Strauss & Corbin, 1998). These analyses were used to supplement the initial coding of the tasks to generate further categories in the rubric, which, through several passes through the interview transcripts and careful examinations of course documents, allowed us to describe the course and to create the MTEF in its current state. More specifically, when coding for task type, we coded the data for the content that was targeted by each task (e.g., psychological principles underlying children's thinking). This allowed us to address the structure of the course content. When coding for techniques, we searched for ways suggested by MJ to go about working through the task (e.g., "use your knowledge of problem types and apply it to extend the given third-grade activity"). We also coded the data for the techniques associated with each task, which directed us to search for tools suggested by MJ that justified the use of the techniques (e.g., a set of principles for analyzing a classroom lesson). Finally, we coded the data for evidence of any discourse that was intended to systematize and bring together theoretical coherence to the technologies recommended for each task.

Results and Discussion

In this section, we illustrate MJ's practice by presenting the content structure of the course (*task type*), the *techniques* suggested by MJ to tackle the tasks, the *technologies* MJ required her students to use as tools in their use of techniques, and finally, the *theories* used to justify the technologies. Our analysis of the tasks using the ATD framework allowed for an additional theme to emerge, called *epistemic actions*, which describes the kinds of knowledge that MJ intended to foster using the tasks she chose. Note that in this section, we refer to the preservice teachers in MJ's course as "students."

Task Types and Techniques

The content structure of MJ's course consisted of four content categories: (1) mathematical knowledge for teaching, (2) psychology of mathematics learning, (3) the teacher's didactic actions, and (4) reflection. We describe each of these categories below.

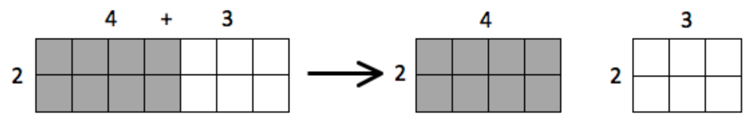
Mathematical knowledge for teaching. The first category – mathematical knowledge for teaching (MKT) – is defined here as knowledge of the concepts and principles of school mathematics (1). In MJ's course, MKT included knowledge of numbers and their properties, models of mathematical operations, and the ability to solve the problems in the elementary mathematics curriculum. There were 17 MKT tasks in MJ's course, which constituted 33% of all the tasks analyzed. A sample MKT task is presented in Figure 1.

Psychology of mathematics learning. We defined the category of Psychology of Mathematics Learning (PML) as knowledge of how children learn mathematics, which included, in MJ's course, (a) how children learn to count and understand number, and (b) the types of strategies children acquire as they learn to solve problems with whole numbers. The same number of PML tasks was assigned by MJ as MKT tasks – that is, we coded 17 PML tasks, which made up 33% of all tasks in the data set. A sample PML task is presented in Figure 2.

Is the statement below true or false? Justify.

On one of Mr. Barr's lessons on multiplication in his fifth-grade classroom, he writes the following on the board: $2 \times 7 = (2 \times 4) + (2 \times 3)$

One of his students raises his hand and asks, "Where does the extra 2 on the right hand side come from?" Mr. Barr then draws the following picture and says, "See? It was there all along!"



Statement: Mr. Barr is using a picture to illustrate the associative property.

Figure 1. Sample MKT task (adapted from Sowder, Sowder, & Nickerson, 2011).

Is the statement below true or false? Justify.

Jonathan used the following method to solve 234 ÷ 3 = □.

What is 234 divided by 3?
70 + 70 + 70 = 210
20 ÷ 3 = 6 R2
4 ÷ 3 = 1 R1
So, 234 ÷ 3 = 77 R3 = 78

Statement: This method will work for any division problem with whole numbers (divisor ≠ 0).

Figure 2. Sample PML task.

Teacher's didactic actions. There were 11 tasks (22% of all the tasks) that we placed in the category of Teacher's Didactic Actions (TDA), which we defined as the knowledge and skills needed to perform pedagogical actions, such as identifying the didactic objective of an activity, using a pictorial representation to explain an algorithm, and producing tasks intended to mobilize a specific mathematical concept. We present a sample TDA task in Figure 3.

Reflection. In the fourth category of task type, Reflection, we placed 6 of MJ's tasks, which constituted 12% of all 51 tasks. Reflection tasks were those that targeted the students' beliefs about the goals and objectives of the course in relation to their development as teachers. More specifically, MJ required her students to think about why subject matter knowledge is important for elementary mathematics teachers, how a "good" teacher uses mathematics in the classroom, and to identify a new kind of knowledge (namely, the professional body of mathematical knowledge that is "pedagogically useful," Ball & Bass, 2000) and distinguish it from the "plain old math," a term used by MJ to describe what she believed reflected her students' conceptions of the subject-matter. One such task assigned by MJ was to write a brief "journal entry" on the first day of class describing her students' thoughts about various aspects of a teacher's subject-matter knowledge, including their beliefs about the types of mathematical knowledge necessary for elementary school teachers. Subsequently, MJ required her students to read a non-technical article (Ball, Hill, & Bass, 2005) on the topic of mathematical knowledge for teaching and to compare Ball et al.'s account to the beliefs expressed in the journals of their peers, which were anonymously posted on the course's website.

Printing Pages

Book pages are printed on large sheets of paper called *forms*. Each form contains the same number of pages, so the number of pages in a book will always divide by that number with no remainder. After the forms are printed, they are cut apart so the pages can be bound.

? How many pages can be on a form? Explain your answer.

Book	Pages
1. How Come the Best Clues Are Always in the Garbage?	176
2. How Can I Be a Detective if I Have To Babysit?	160
3. Who's Got Gertie? And How Can We Get Her Back!	176
4. How Can a Frozen Detective Stay Hot on the Trail?	168
5. What's a Daring Detective Like Me Doing in the Doghouse?	192
6. How Can a Brilliant Detective Shine in the Dark?	200
7. What's a Serious Detective Like Me Doing in Such a Silly Movie?	192

As a teacher, you want to use the context of the problem to extend students' knowledge of multiplication and division – that is, to see how they think about other problem types. Write one word problem, in the context of this situational problem, that you might use with your students to meet this objective.

Figure 3. Sample TDA task (situational problem from Kestell & Small, 2004).

Across all the tasks in the four content categories, we observed MJ specifying a number of techniques for her students in the course. These techniques included using a checklist; recalling a definition; applying a principle or property; reading a text; describing observations; and comparing actions, tasks, and strategies; and were made explicit in all the tasks MJ assigned.

Technologies and Theories

Technologies can involve terminology, number facts, mathematical definitions, types of strategies used by children to solve problems, and didactic principles. Some technologies are loose collections of facts or principles, and others cohere as comprehensive frameworks. In classifying the technologies in MJ's course, we observed that they fell into two categories: (1) related to the mathematics itself (e.g., principles and properties of involving operations with whole numbers), and (2) related to teaching actions and children's learning. The majority of the technologies in MJ's course took the form of "structuring frameworks"; they were collections of principles, tightly bound together, to be used as tools for completing tasks.

To illustrate, MJ presented a number of mathematical properties, such as commutativity, associativity, and distributivity, as tools for thinking about a teacher's actions. The task provided in Figure 1 shows how MJ required her students to use their knowledge of the distributive property of multiplication over addition to interpret a teacher's actions during a lesson. In addition, a large component of MJ's course centered on frameworks of problem types and children's thinking taken from Cognitively Guided Instruction (CGI; Carpenter et al., 1996), a mathematics professional development program for elementary teachers. In several of her tasks, MJ required students to use the CGI taxonomy of children's strategies to interpret and evaluate their mathematical work. Another task MJ asked her students to complete was to use the taxonomy to evaluate the relative difficulty of two division word problems – one measurement division and one partitive division – while taking into account a specific strategy to be used in each case. As a final example, MJ required her students to use a framework presented by Hiebert et al. (2007) on the dimensions of effective learning environments in mathematics. In one task, she required her students to analyze a videoclip of a fourth-grade lesson on area and perimeter and to analyze the teacher's actions using Hiebert's framework. In the assignment, MJ states: "This is not necessarily a perfect lesson... Use the Hiebert reading to point out, where applicable, ways the teacher creates an effective environment and ways he can improve his teaching."

Our analysis revealed that MJ's technologies were informed by a deep theoretical understanding of teaching and learning in mathematics, but were left unjustified by theory. In particular, MJ noted specific reasons for not including theory in the course, such as a lack of time and pressure from the students themselves to know "the answer" and the one "right" way to teach. MJ was sensitive to the many "layers of complexity" in teaching mathematics, and as such, there were some tasks in her course that were left open to her students' interpretation. More specifically, MJ expected her students to use the structuring frameworks in completing the course tasks, but was less concerned with their analyses or creations *per se* – as long as they engaged in analytical thinking (in her case, used the frameworks consistently, applied the correct definitions, and attended to the information provided in the task), the task was left open to the students' interpretation of it. This relatively open-ended nature of many of her tasks caused, in her experience, great discomfort and uncertainty in her students. As such, MJ indicated to us that she was unwilling to add to the uncertainty by also requiring them to use theory to justify their actions in the context of the course tasks.

Epistemic Actions

We used our analysis of the 51 tasks to discern their epistemic nature – that is, from the results of our examination of the types of tasks emerged a picture of the type of knowledge MJ was intending to impart to the students in her course. Across all four content categories (MKT, PML, TDA, and Reflection), we observed seven epistemic actions intended by the tasks. These epistemic actions were: identify, produce, discuss/reflect, assess, model, solve, and explain. The actions differed according to the content with which they were associated. For MKT tasks, for example, students were required to *produce* a representation of a number with base-ten blocks; *identify* which property of an operation, among those they learned in class, was used in a teacher's didactic action or a student's solution; *identify* the type of a given problem based on a list of types of problems presented in class; and *solve* a mathematics problem. In the PML category, sample actions were to *identify* a child's strategy; *produce* an example of a given problem; use knowledge of children's thinking *reflect* on parents' false beliefs about mathematics learning and to *produce* an argument to convince them otherwise; *discuss* the relative difficulty of a problem; and *assess* the counting skills of a child. With respect to tasks of type TDA, for instance, students were asked to *produce* a problem of a given type; *explain* a standard algorithm to a child; and *identify* the didactic objective of a given curricular activity. Finally, in the Reflection category of task type, students were asked to *reflect* on why teachers need to know mathematics and how they use their knowledge during teaching; and to *identify* a new category of mathematical knowledge, namely "mathematical knowledge for teaching," of which they should have become aware by reading an article by Ball et al (2005).

The most frequently occurring action observed in the data was "identify," which accounted for 33% of all the epistemic actions found in the 51 tasks. This was nearly twice as many as the number of actions in the next frequent category, "produce," which made up 17% of all actions. "Discuss/reflect" and "assess" were actions accounted for 16% and 14% of all epistemic actions, respectively, followed by the final three (i.e., model, solve, and explain) which together constituted 20% of all actions. An additional epistemic action emerged from our analysis, and that was of justification, which was systemic throughout the course and was required of students in almost all tasks. Indeed, 18 of the 51 tasks (35% of the total) required the students to provide a written justification of their claims – oral justifications were required in almost all other cases. MJ expected the students' justifications to be systemic, or in other words, based on a system of concepts, properties, and principles in the areas of MKT, TDA, and PML, which were, in MJ's course, accepted as true. The choice of the concepts and the truth of these properties, principles, and statements was not questioned or debated.

Contributions: The Mathematics Teacher Educator Framework

Cochran-Smith (2003) argued that the responsibility of preparing qualified teachers rests primarily with teacher educators, who are becoming increasingly active in studying their own practices. Because of this, she continues, one sees, "the emergence of new terminology and new contexts for doing and making public the work of teacher education" (p. 9). We present here what we call the Mathematics Teacher Educator Framework (MTEF), an organizing model that we believe provides a common discourse to describe and communicate the work of the MTE. The MTEF emerged from our analysis of the tasks used in MJ's methods course and, while it is relatively restrictive because it is based on only one site, we present it here as a first of its kind.

The MTEF is anchored in a task analysis informed by the ATD model (i.e., task, technique, technology, and theory), through which one can describe the content structure of a methods course. Currently, the model contains four content areas (i.e., MKT, PML, TDA, and Reflection), but as our data analysis on the larger project unfolds, additions and reconfigurations to this content structure are inevitable. From the present task analysis, we were able to discern the epistemic nature of the tasks used in MJ's course, which provides a view into the knowledge that she was trying to impart to her students. Given that uncovering the knowledge objectives of MTEs is by no means straightforward, we consider this aspect of the MTEF to a be a particularly important contribution to the literature in mathematics teacher preparation.

Once the epistemic actions in any given methods course are determined, one can begin to classify them according to the degree to which they call for higher level thinking. In MJ's course, the tasks that called for "identification" were of lower level than those that called for "reflection" (Vygotsky, 1987). Indeed, in one of our interviews with MJ, we noticed that she was surprised that she had given so much weight to relatively low-level tasks. We used this opportunity to discuss with her how to increase the presence of reflection, generalization, and abstraction in her tasks. Reflective practice and adopting an inquiry stance toward teaching have been identified as key areas of growth for the developing mathematics teacher (Doerr et al., 2010; van Es & Sherin, 2008). The epistemic nature of tasks gleaned from the MTEF can itself be used as a tool for reflection, as MTEs may not be aware of the effects their own practices may have on the development of preservice teachers (cf. Torff & Sternberg, 2001).

Endnotes

1. We note that our conceptualization of MKT is more restrictive than that offered by Ball et al. (2008), who defined MKT as a combination of subject matter and pedagogical content knowledge (Shulman, 1987). We restrict our characterization of MKT to knowledge of school mathematics and the "specialized content knowledge" described by Ball et al. as the mathematical knowledge unique to teaching.

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HELPING STUDENT TEACHERS LEARN TO NOTICE: A PRELIMINARY REPORT

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Noticing is important to skilled teaching, as it enables teachers to modify instruction in response to students' mathematical thinking and understanding. Novice teachers, however, often fail to notice important student ideas. A study was conducted with two secondary-level student teachers that focused on using research-like video analysis to help them learn to notice and assess the mathematical potential of important moments during instruction. Preliminary results of this work, as well as participants' perspectives on their learning, are discussed. This use of video analysis holds promise for helping novice teachers use their classroom as a site for inquiry and learning.

Objectives and Purposes of the Study

The vision of mathematics instruction advocated in documents such as the National Council of Teachers of Mathematics' [NCTM] *Standards* (e.g., 2000)—one in which teachers listen to and make sense of student ideas in order to build on them during instruction—requires a very different set of skills, dispositions and competencies than the traditional, teacher-centered model of instruction (Feiman-Nemser, 2001). It requires that teachers use the classroom as a site for inquiry and continued learning. Given the recognition that completing a teacher education program is not sufficient to develop all of the finished competencies of an effective teacher (Feiman-Nemser, 2001; Hiebert, Morris, & Glass, 2003), it has been suggested that it may be more productive to conceptualize teacher education as a venue for helping teachers acquire tools that enable them to learn from practice over time (Ebby, 2000), rather than as a site for producing a fully-formed teacher.

Tools to learn from practice can be developed through engaging with student thinking (Feiman-Nemser, 2001; Philipp, et al., 2007), but many opportunities to do so within teacher preparation programs are not capitalized upon. Traditional field experiences, for example, have been criticized for helping prospective teachers learn what is currently being done, rather helping them rethink the possibilities for instruction (e.g., Philipp, et al., 2007). Even when placed in exemplary classrooms, prospective teachers often do not have the knowledge or dispositions to meaningfully observe classroom interactions (Masingila & Doerr, 2002). Adding to the problem is the fact that the goals of field experiences are often not well-articulated, leaving school-based mentors to develop their own goals for the field experience that often have little connection to mathematical content or students' understanding of it (Leatham & Peterson, 2010). Thus, it is becoming recognized that intentional and substantial teacher educator involvement—visiting school sites and engaging teacher candidates in discussions about what is observed—is critical to supporting meaningful learning from field experiences (Grossman, et al., 2009; Masingila & Doerr, 2002).

The use of video is another promising method of positioning prospective teachers to continue to learn from practice as it grounds learning in practice. In fact, studies have found that video cases can support the development of teachers' analytical skills by helping them learn to make sense of students' mathematical ideas and use classroom-based evidence to support analyses of teaching and learning (Sherin & van Es, 2005; Author reference). Much of the current work with video, however, has used classroom video clips that have been selected and edited by experienced teacher educators, eliminating the opportunity for teachers to determine what instances might be worthy of analysis. This is problematic because in order to learn from teaching practice, teachers first need to *notice* (Sherin & van Es, 2005) important instructional events. In fact, a major difference between expert and novice teachers' practices is their ability to notice and respond to important instances during instruction (Hogan, Rabinowitz, & Craven, 2003), with novices failing to notice or act upon many instructional instances that experienced educators intuitively recognize and productively respond to (Peterson & Leatham, 2009). Thus, to position teachers to learn from practice, teacher educators need to not only develop prospective teachers' ability to analyze classroom events, but also their ability to notice the events that are worthy of analysis because of their potential to support students' mathematical learning.

Ongoing work has led to an initial conceptualization of important mathematical moments that teachers should attend to during instruction (e.g., Leatham, Peterson, Stockero, & Van Zoest, in press) and a characterization of circumstances likely to lead to such moments (e.g., Stockero, Van Zoest, & Taylor, 2010). This work is an important first step in making such moments visible to teachers, as it provides frameworks that can be used to scaffold teacher noticing by focusing their attention on characteristics of potentially fruitful mathematical moments that could be capitalized on during instruction. The study reported here aimed to develop a preliminary understanding of how such frameworks might be used in practice. In particular, the frameworks developed as part of related projects were used to inform the development and implementation of instructional activities aimed to help two prospective secondary school mathematics teachers learn to recognize and analyze important mathematical moments in instruction during their student teaching experience. In this paper, the teacher learning activities, data collection, and analysis are described, and the results of initial data analysis are shared. The paper concludes with a discussion of the potential implications of this work for teacher education and future directions of the work.

Theoretical Perspectives

Consistent with current thinking in mathematics education (i.e., NCTM, 2000), this research is based on the view that the use of student thinking is central to effective mathematics instruction. Thus, it follows that noticing important student mathematical thinking is an essential skill of effective mathematics teachers, since teachers cannot intentionally act upon that which they do not notice (Sherin & van Es, 2005). The definition of noticing used in this work draws on that of Sherin and van Es (e.g., Sherin & van Es, 2005; van Es & Sherin, 2002), who define noticing as comprised of three interrelated skills: identifying important events during instruction, reasoning about them, and making connections between the events and broader educational principles. The conceptualization of the act of noticing during instruction underlying the research is shown in Figure 1; the first two components are the focus of the part of the work reported here.

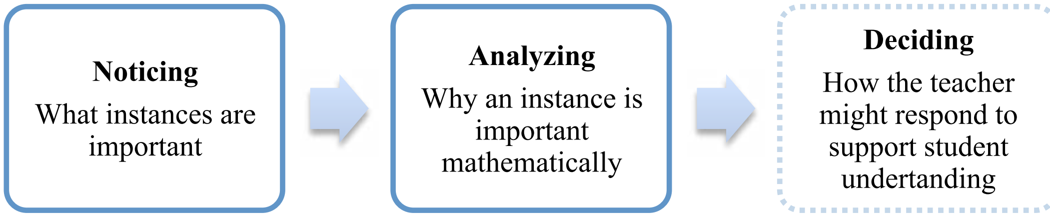

Figure 1. Components of teacher noticing.

The idea that student thinking is central to effective mathematics instruction does not imply that all student thinking is equally useful. Some student ideas, for example, are only tangentially related to the goals of the lesson or do not have the potential to add to students' understanding of important mathematical ideas. These ideas are probably not worthwhile to pursue during a lesson, as the benefit they would bring would not outweigh the time taken from the lesson. Other student ideas, however, are mathematically rich and provide an opportunity for the teacher to use the student thinking to develop important mathematical ideas (Author reference). These are the instances that were the focus of this study. The central component of noticing as used in this work was helping novice teachers learn to notice student thinking that has the potential to advance student understanding of important mathematical ideas, while not expending time and energy on that which does not.

Methodology

Participants

Two prospective secondary school teachers enrolled in the same teacher education program participated in the study during an 11-week student teaching experience. Both participants had been enrolled in a mathematics methods course taught by the researcher during the previous semester; an explicit focus of this course was on listening to and making sense of students' mathematical thinking in order to build on it during instruction. Both participants were working with two cooperating classroom teachers, one in their major content area and one in their minor; this is typical of students in the program. Mary was completing her student teaching in high school chemistry (major) and mathematics (minor) classrooms and Audrey in middle school mathematics (major) and science (biology minor) classrooms. Project activities focused only on the participants' mathematics classes. A graduate student research assistant (RA), an experienced teacher and Audrey's cooperating teacher for mathematics, assisted on the project. The RA had no previous experience with the type of research-like analysis of instruction that was the focus of the study and was explicitly asked not to discuss the project activities with Audrey outside of project meetings.

Context

To help them learn to notice and make sense of important mathematical instances during instruction, the participants used the Studiocode video analysis software (SportsTec, 1997-2011) as a tool to analyze video recordings of classroom instruction. This software was selected because it allows multiple users to individually code and annotate a video segment and then merge and compare coding among users. It was hypothesized that engaging participants in research-like video analysis, rather than in more open discussions of classroom video that have been typical in past video case research (e.g., Author reference), would enhance their ability to notice important mathematical moments during classroom instruction, as it required them to individually code the video and justify their coding before they engaged in discussions about it.

The participants coded a total of eight classroom videos during the project. The first was of the RA's instruction during the second week of the student teaching experience when the participants were just beginning to take on teaching responsibilities. The remaining videos were recorded in Mary's and Audrey's classrooms, alternating between classrooms each week. After recording, the researcher prepared each video for analysis by editing out portions where audio could not be heard or that were not related to the lesson content; this was done to reduce the participants' video analysis time. It is important to note that the researcher did not edit any portion of the video where the teacher or students were talking about mathematics. The only portions what were edited were those where the teacher was taking care of administrative tasks or where students were working quietly.

The participants were instructed to code each video for *mathematically important moments* that would be important for a teacher to notice during instruction and to annotate each coded segment to indicate its perceived mathematical importance. The phrase *mathematically important moment* was intentionally left undefined by the researcher so that the group could refine the definition during project meetings. The participants first coded and annotated each video individually, after which they sent their Studiocode coding timeline to the researcher via email. The researcher and RA also individually coded the same video and then met to discuss their coding and agree upon the mathematically important moments in the video that a teacher could capitalize on during instruction. The researcher then compared these agreed-upon instances to those coded by the participants to determine which instances would be discussed in the next project meeting. The instances that were agreed upon by the researchers and both participants were always discussed during the meetings, with an emphasis on making sense of the mathematical importance of these moments. Instances that were coded by only one participant, or only by the researchers, were sometimes chosen to help distinguish instances that were mathematically important from those that were less important, or to highlight important mathematics embedded in student thinking.

The participants and the researcher met weekly for 60 to 90 minutes to view and discuss the coded video segments; the RA attended the first four meetings. During each meeting, the researcher pushed the participants to articulate the mathematical importance of instances they had identified and to consider whether there were characteristics common among mathematically important moments during instruction. For instance, the participants came to agree that student involvement seemed central to mathematically important moments, and thus, it was unlikely for such a moment to occur if students were not actively participating in the lesson.

During the last two weeks of the project, each participant taught a lesson in which she was asked to indicate her real-time noticing during instruction. This was done via a self-mounted camera that could be activated by the participant. The researcher also recorded the entire lesson. Following the lesson, each participant engaged in an interview that focused on the mathematically important moments that she had noticed during the lesson. The interview also included questions aimed at understanding participants' perspectives on the project activities and how these activities supported their learning during the student teaching experience.

Data Collection and Analysis

Data included participants' individual and merged Studiocode coding timelines and the accompanying video, the researchers' Studiocode timelines, and video recordings of each project meeting. In addition, video snapshots from the real-time noticing experience, the corresponding complete classroom video, and a recording of the post-interview were also included in the data.

The first phase of data analysis focused on summarizing the instances in each video that participants coded as a mathematically important moment and making comparisons both between the two participants' coding and between the researchers' and the participants' coding. This analysis focused on documenting the segments that were coded as mathematically important, as well as the participants' reason for identifying them as such. In many cases, the participants did not give a complete explanation of the importance of an instance in their annotation, so were pushed for a more complete explanation during the project meeting. In these cases, information from both the Studiocode timeline and the project meeting video was used to develop a complete explanation of the perceived importance of the moment. The outcome of this phase of analysis was an account of participants' noticing and their explanations of the importance of the coded moments in each video. This gave preliminary information about how each changed over time.

The participant interviews were analyzed to understand the participants' perspectives on the learning activities themselves, the value of specific activities in terms of supporting their learning during student teaching, and what they perceived to have learned by engaging in the project activities.

Results

Evidence of Learning in Coding and Project Meetings

Preliminary data analysis indicates that there was a change in participants' abilities to notice and analyze mathematically important moments during the study. In the first video, both participants focused mainly on identifying "good teacher moves." For example, both coded instances in which the teacher used a good visual representation of a problem, asked a series of questions to ensure that students understood the rules of a probability game, and summarized the lesson at the end. In addition, Audrey noted several instances where the teacher clearly defined mathematical terms. Although these were all productive teacher moves, they were each part of the teacher's plan for the lesson, and thus, are not the types of instances that are important for a teacher to notice during instruction. During the project meetings, instances such as these were used to discuss the difference between good planned teacher moves and moments that require a teacher response during a lesson.

On the other hand, in the same video, the researchers noted a mathematically important moment while students were playing a dice-rolling game related to probability that the participants did not. In the game, students placed 18 chips in columns numbered zero through six and then removed a chip each time the difference of two rolled dice was equal to the column number; the winner was the first to remove all 18 chips. After playing the game and recording the differences of the dice on each roll in a frequency table, the teacher told his students that they would play again. A number of students placed their chips by copying the outcomes in the frequency table in hopes of winning the game the next time, not realizing that the same outcome was unlikely. Although this moment had the potential to be used to highlight important ideas about theoretical and experimental probability, neither participant noticed its mathematical importance.

By the eighth video, both participants had shifted their primary focus from the teacher's actions to the thinking of students in the classroom, although Mary was generally better able to discern important student thinking from that which was less important than was Audrey. For example, both participants coded an instance in which a student used the distance formula to find the distance between some pairs of coordinate points, but used simple subtraction of coordinates for other pairs that were located vertically or horizontally from one another on the coordinate plane. Most students in the class had used the distance formula for all pairs of points. Both participants recognized this as an opportunity to highlight when the distance formula needed to be used, and when other methods were appropriate. Similarly, both coded an instance in which a student had included an unnecessary step in a geometric proof and another student explained why the step was not needed. Audrey, however, also coded an instance in which students were guessing which angles in a figure were supplementary to one another. Although supplementary angles are an important topic in geometry, there was no mathematical basis for their guesses, so it would be difficult to use the guessing to support student understanding.

Participant Perspectives on Learning

In the post-interviews, both participants agreed that the project activities were a valuable addition to their student teaching experience, despite the extra time they required. Both stated that coding the videos and then using the merged Studiocode timelines as a basis for project meeting discussions helped them really think about what moments during the lesson were mathematically important; instances on which they agreed confirmed their thinking, and those about which they disagreed pushed them to explain their thinking and continually refine their definition of what moments were mathematically important. Both participants also agreed that leaving *mathematically important moment* ill-defined at the start and then developing and refining the definition together made them think hard about what moments might meet the mathematically important criteria and which might not. They both stated that introducing the researcher's conceptualization of the construct via a reading (Author reference) mid-way through the project was helpful in that it gave them a framework to use to make sense of the work they had already done.

When asked to discuss how they felt the project activities affected their teaching, both participants noted that it shifted their focus from themselves to their students. Audrey said that, "It was good because it helped me key in onto the student a little bit more than I would have otherwise. Cause I would have been just trying to get, like, the content out there." Mary said that analyzing the videos helped her become more objective about her teaching, noting that it was important that the focus wasn't on "am I a good teacher, am I a bad teacher" but was on the mathematically important moments and how they could be used in the lesson. This, she said, helped her focus "on what [students] are learning rather than on 'Oh shoot, I should have done that.'" Mary also noted that she often thought about how to set up mathematically important moments when planning lessons. She said that she focuses on "trying get them to come up with things and asking them why more. I want to put a lot on them. And like, I already wanted to do that as a teacher, but I think this helped me think about it more, like, when I'm in class." Later in the interview she added, "You can, like, set [a moment] up and it might not happen, but it's more likely to happen if you if you set it up than if you don't."

Interestingly, both participants also noted that the activities helped them to think about the other non-mathematics classes they were teaching. Mary said that it helped her ask better questions in her chemistry class, where she noted that she had started out asking mainly lower-level questions. Audrey said, "I think [coding and discussing the video] helped me respond to those moments [in math], but then I feel like, why aren't we doing this in science, cause I don't know what those moments are in science. You know what I mean, the scientifically important moments. It's like, in science, I'm like what, I think I'm kind of struggling a bit with what is the big picture." Although not an expected outcome of the activities, this type of transfer is particularly encouraging to improving instruction more broadly.

Discussion and Conclusion

Preliminary findings suggest the value of engaging prospective teachers in detailed, focused analysis of practice as part of the student teaching experience. Both the participants' video coding and their interviews indicate that engaging in the project activities helped keep them focused on what matters the most during instruction—listening to student thinking and assessing the mathematical potential of that thinking in order to determine whether it might be worthwhile to incorporate. This is a very different focus than what has been documented in many student teaching experiences, where prospective teachers' primary focus is on issues, such as classroom management, that have little connection to mathematical content or to students' understanding of it (Leatham & Peterson, 2010).

The use of video analysis software seemed to support teacher learning by forcing them to identify specific moments and their reasons for selecting them as mathematically important prior to discussing them with others. Although the coding could have been done in other ways—for instance, on paper by noting video time stamps—Studiocode allowed the researcher to merge the participants' timelines in order to display their coding together and also to quickly select a coded moment and replay the instance in the video during a project meeting. These features of the software allowed for focused discussion of particular coded moments in ways that other modes of coding and reflection could not.

The discussions in which the researcher and participants engaged seemed critical to helping the participants learn to differentiate between moments that had significant mathematical potential and those that did not. In the discussions, the researcher pushed the participants to clearly articulate the mathematical importance of coded moments; when they could not, they were engaged in discussions about whether the moment was, in fact, mathematically important. Discussing instances that the participants had not coded provided an opportunity to help them better understand secondary school mathematics and to begin to make connections among mathematical ideas. Thus, although time-consuming, the sustained interactions with the researcher/teacher educator seemed critical to supporting participant learning. This study suggests, as have others (e.g., Grossman, et al., 2009), that teacher educators need to find ways to keep field experiences, particularly extended experiences such as student teaching, focused on students' learning of mathematics. Understanding what facilitator moves and tools are most productive in doing so will help teacher educators plan activities that most effectively and efficiently advance teacher learning.

Although the results of this work appear promising, there is much work to be done. Thus far, the data indicates that the activities supported learning, but does not illuminate the specifics of how that learning occurred. Subsequent data analysis will aim to draw connections between changes in participants' noticing and specific learning activities. This will entail detailed analysis of the questions the researcher asked to push participant thinking during project meetings, questions the participants raised during the meetings, and how specific tools and frameworks may have supported participant noticing. It is expected that this analysis will lead to hypotheses about teachers' learning-to-notice trajectories and specific learning events that might trigger advances in teachers' noticing. In future projects, the teacher education model will be refined and tested with larger groups of student teachers in a range of mathematics classrooms, and additional data will be collected to understand how the noticing skills that teachers develop affect their classroom instruction.

The preliminary results of this study provide support for including research-like video analysis in teacher education programs as a means of advancing teachers' ability to learn from practice. Developing teachers' abilities to notice and capitalize on important instructional events holds great promise in positioning teachers to become life-long learners in their own classrooms, and thus, improve their instruction over time.

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PRESERVICE ELEMENTARY TEACHERS' LEARNING FROM VIDEOCASES: RESULTS FROM THE VPEM PROJECT

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While understanding children's mathematical thinking is an important part of what teachers need to know in order to be effective in the classroom, preservice coursework often fails to provide learning opportunities focused on this aspect of teaching. Using a videocase curriculum focused on children's mathematical thinking, the authors examined changes in preservice elementary teachers' knowledge of, beliefs about, and ability to analyze children's thinking. Results from a quasi-experimental study indicate that the videocase curriculum had little effect on participants' knowledge and ability to analyze children's thinking, yet a moderate effect on participants' beliefs about mathematics teaching and learning.

Research points to the importance of attending to children's mathematical thinking as an important aspect of what teachers need to know (Carpenter et al., 1989). Despite the centrality of children's thinking to teachers' mathematical knowledge, there exists little research involving preservice teachers' (PSTs) learning about children's mathematical thinking, with few exceptions (e.g., Philipp et al., 2007). Such research provides evidence that PSTs, if given opportunities to closely study children's mathematical thinking in the form of video clips, can further their own mathematical knowledge in relation to children's thinking (e.g., Philipp et al., 2007).

This study builds on and extends the extant research by examining how the design and implementation of a videocase curriculum supported PSTs' ability to analyze children's mathematical thinking. Specifically, the authors discuss findings from a quasi-experimental study of PSTs' engagement with the videocases in a required mathematics content course across two semesters, focusing on changes in PSTs' knowledge of, beliefs about, and ability to analyze children's mathematical thinking. The authors administered in both semesters a pre- and post-test using a mathematics content knowledge assessment instrument developed to measure mathematical knowledge for teaching, a pre- and post-test using the Integrating Mathematics and Pedagogy (IMAP) Beliefs survey, and a pre- and post-test video activity developed by the authors.

Theoretical Framework

Although most scholars and educators agree that mathematics teachers at all levels need to have a thorough knowledge of the content they teach (e.g., Kilpatrick et al., 2001), there is less agreement about the precise nature of the mathematics content that teachers should learn in preservice education programs. Some researchers have reconceptualized mathematics content knowledge, arguing that teachers need to not only know mathematics content, or common content knowledge, but that they need to know mathematics in ways needed for teaching, or specialized content knowledge (Ball et al., 2008). While common content knowledge refers to the knowledge that bankers or retailers, for example, have to know (e.g., computing percentages), specialized content knowledge refers to the mathematics knowledge that is specific to teaching (e.g., analyzing children's errors), and more closely resembles what teachers have to know and do with children in the classroom. If they are expected to support children as they investigate mathematical concepts, PSTs need to have a strong understanding of mathematical knowledge for teaching, which includes both common and specialized content knowledge.

Preservice mathematics coursework, however, often fails to adequately prepare PSTs for the work of teaching, as such courses focus solely on the learning of content with limited attention given to how such knowledge is used in actual teaching practice (RAND, 2003). In addition to knowing the content of the mathematics problems they use with children, teachers also have to analyze unusual solution methods that children may pose, appraise children's explanations, and ask mathematical questions that further children's thinking, teaching tasks for which PSTs are often ill prepared. These kinds of mathematical tasks of teaching (Ball et al., 2008), however, often receive scant attention in preservice mathematics coursework. Given what we know about the mathematics knowledge teachers need to be effective in the classroom (Hill, Rowan & Ball, 2005), mathematics content courses need to include opportunities for PSTs to develop mathematical knowledge for teaching.

Recent research has demonstrated that preservice coursework can provide opportunities for PSTs to develop their abilities to analyze children's thinking in ways needed for teaching. For example, in their experimental study involving PSTs in content courses, Philipp et al. (2007) argue that PSTs' content knowledge and beliefs about mathematics teaching practice can be enhanced if they have opportunities to learn about children's thinking while simultaneously learning mathematics needed for teaching. Assigning PSTs to four different treatment groups, the authors found that PSTs in the two treatment groups that involved the close study of children's mathematical thinking through the use of videos and interviews demonstrated more changes in their beliefs about mathematics teaching and learning as compared to the PSTs in the other treatment groups that involved more traditional field experiences. These findings are particularly important because they demonstrate that although PSTs may begin a content course with beliefs that stand in contrast to reform-oriented mathematics practice, content courses that connect content knowledge with the analysis of children's thinking can foster changes in PSTs' beliefs.

Project Background

Content Course

The content course in question is the first of two required content courses PSTs take during their freshman or sophomore years at the university, and precedes the mathematics methods course that PSTs typically take during their senior year. The content course is designed around developing PSTs' mathematical knowledge for teaching, and includes a focus on whole and rational numbers and operations, place value, proportional reasoning and aspects of number theory. The course also provides PSTs with opportunities to develop their abilities to engage in explaining, representing, and understanding and reacting to mathematical thinking that is different from their own.

Videocase Curriculum

As part of the Videocases for Preservice Elementary Mathematics (VPEM) Project, the authors developed 9 videocases and accompanying facilitator guides to be used in mathematics content courses for PSTs. These materials were designed to support the development of PSTs' understanding of the mathematical knowledge needed for teaching whole and rational number concepts and proportional reasoning by providing PSTs with opportunities to examine children's mathematical thinking in the context of actual classroom lessons. The videocases were implemented at regular intervals across the semester. Each videocase included a video clip(s) - focused on children's discussions of certain content and taken from actual classrooms - and a facilitator's guide - designed to support instructors in facilitating PSTs' discussion of the videocase, including focus questions. During the content course, PSTs first worked on the mathematics problems discussed in the video, and then viewed the videos through the lens of the focus questions. Table 1 below includes descriptions of the 9 videocases, including mathematical topic and mathematical tasks of teaching foci.

Videocase Title	Mathematics Topics	Mathematical Tasks of Teaching
Finding Patterns	Algebra	Evaluating the plausibility of children's claims Asking productive mathematical questions
Counting Strategies	Meaning of addition, subtraction; Addition, subtraction problem types	Evaluating children's mathematical explanations Asking productive mathematical questions
Understanding Place Value	Counting; Place value	Analyzing children's thinking
Modeling Double-Digit Subtraction	Subtraction algorithm	Linking representations to underlying mathematical ideas
Debating Remainders	Division; Remainder interpretation	Evaluating children's mathematical arguments Asking productive mathematical questions
Student Errors with Multiplication & Division Algorithms	Place value; Standard, alternative multiplication & division algorithms	Analyzing children's errors
Pattern Block Fractions	Relationship between improper fractions and mixed numbers	Analyzing children's thinking Analyzing children's errors
Why is 1/20 not equal to 20%?	Relationship between percentages and decimals	Analyzing children's errors Linking representations to underlying mathematical ideas
Making Predictions Using Multiple Solution Strategies	Fractions; Liner functions	Evaluating children's mathematical explanations

Table 1. Videocase design overview

Methods

The authors employed a quasi-experimental design to the study of changes in PSTs' knowledge of, beliefs about, and ability to analyze children's mathematical thinking to understand the overall effectiveness of the videocase curriculum.

Participants

Participants were recruited from PSTs who enrolled in two sections of a required mathematics content course for PSTs at a large midwestern university during the Spring 2010 and Fall 2010 semesters. PSTs from the sections of the course taught by the first author served as the treatment group (i.e., videocases), while PSTs from the other section of the course served as the control group (i.e., no videocases). 22 PSTs were in the treatment group and 17 PSTs were in the control group during the Spring 2010 semester; 30 PSTs were in the treatment group and 24 PSTs were in the control group during the Fall 2010 semester.

For both semesters, in the treatment course, PSTs viewed the videocases, answered focus questions related to the videocases, engaged in small group and then whole group discussions of the focus questions, all facilitated by the instructor. For both semesters, in the control course, PSTs worked on tasks similar in content focus to the tasks in the videocases in lieu of watching the videocases. However, in the Fall 2010 semester, the implementation of the videocases was modified in order to improve the quality and duration of videocase discussions, and PSTs' responses to the focus questions. These changes included the following: 1) PSTs watched videocases from the course website as part of homework assignments; 2) focus questions were revised to be more succinct; 3) guidelines for responding to the focus questions and a rubric for evaluating responses were developed to scaffold PSTs' written responses to the focus questions; and 4) PSTs' written responses were graded using the rubric in order to provide feedback for improvement.

Measures

As the study focused on changes in PSTs' knowledge of, beliefs about, and ability to analyze children's mathematical thinking, we administered the following measures pre-post in both semesters. First, we used a mathematics content knowledge assessment instrument developed by the Learning Mathematics for Teaching Project. Items not only capture whether teachers can answer the problems they use with children, but also how teachers solve the special mathematical tasks that arise during teaching (e.g., given 3 different multiplication strategies, determine which method can be used to multiply any two numbers). Second, as teachers can have differing conceptions of mathematics and mathematics learning which may or may not align with ideas about teaching and learning underlying recent reform efforts, we administered the IMAP Beliefs survey that was specifically designed to capture the characteristics of beliefs most closely related to understanding children's mathematical thinking: beliefs about mathematics as a discipline, beliefs about learning and knowing mathematics, and beliefs about children's learning and doing mathematics. Finally, we administered a video activity developed by the authors designed to analyze PSTs' ability to analyze children's thinking as depicted in video format.

Results

Mathematics Content Knowledge Assessment

Identical content knowledge assessments were administered in the first and last weeks of classes in both the treatment and control courses. The authors conducted a one-sample T-test to examine the gains from pre- to post-course on the assessment. Table 2 shows the results of a one-sample T-test for each group.

Semester	Group	N	Mean	SD	T	Df	Sig. (2-tailed)	Cohen
Spring	Treatment	22	.60	.58	4.82	21	.000	1.03
2010	Control	15	.70	.67	4.07	14	.001	1.05
Fall	Treatment	28	.85	.70	6.38	27	.000	1.20
2010	Control	25	.91	.58	7.89	24	.000	1.58

Table 2. One-sample T-test on content assessment gain scores by group

These results indicate that for the treatment group in Spring 2010, for example, the equated average IRT change score or the gain score from pre- to post-test was 0.60 logits with SD = 0.67. A one-sample T-test indicates that PSTs' content knowledge increased significantly during this semester, $t(21) = 4.82$, $p < 0.05$, $d = 1.03$. As displayed in Table 2, both treatment and control groups in both semesters demonstrated significant development in their mathematical knowledge for teaching. An independent T-test was conducted to examine the differences between treatment and control groups for each semester. However, we found no significant difference between treatment and control groups. One reason for no difference between treatment and control groups may be that the overall content course design and materials used in both courses were well aligned with the goals of learning mathematical knowledge for teaching. Indeed, both instructors were members of a planning group around the two content courses who collaboratively plan and design the course (Castro Superfine, 2010), thus making the two courses almost identical with the exception of the videocases.

IMAP Beliefs Survey

Identical online IMAP belief surveys were assigned as homework assignments for the first and the last weeks of classes in both courses. All PST responses were collected with the exception of the pre-test IMAP survey from the control group in Spring 2010. All responses were double coded blind using the IMAP belief rubrics in Ambrose et al. (2004). The survey data were then analyzed using the methods reported in Philipp et al's. (2007) work. Table 3 shows the distribution of beliefs score changes for the treatment and control groups in Spring and Fall 2010 by belief. Table 4 shows the percentage of PSTs in the treatment and control groups that demonstrated large increases, small increases, or no increase in terms of changes in their average belief scores across the seven beliefs.

Group		Large Increase	Small Increase	No Change or Decrease
1. Mathematics is a web of interrelated concepts and procedures (and school mathematics should be too).				
Spring	Treatment (n=22)	32%	18%	50%
Fall	Treatment (n=30)	17%	37%	47%
	Control (n=24)	21%	29%	50%
2. One's knowledge of how to apply mathematical procedures does not necessarily go with understanding of the underlying concepts.				
Spring	Treatment (n=22)	0%	19%	81%
Fall	Treatment (n=30)	7%	7%	87%
	Control (n=24)	4%	21%	75%
3. Understanding mathematical concepts is more powerful and more generative than remembering mathematical procedures.				
Spring	Treatment (n=22)	18%	27%	54%
Fall	Treatment (n=30)	33%	20%	47%
	Control (n=24)	25%	29%	46%
4. If students learn mathematical concepts before they learn procedures, they are more likely to understand the procedures when they learn them. If they learn the procedures first, they are less likely ever to learn the concepts.				
Spring	Treatment (n=21)	27%	27%	46%
Fall	Treatment (n=30)	13%	23%	63%
	Control (n=24)	17%	21%	63%
5. Children can solve problems in novel ways before being taught ho to solve such problems. Children in primary grades generally understand more mathematics and have more flexible solution strategies than adults expected.				
Spring	Treatment (n=21)	14%	23%	64%
Fall	Treatment (n=30)	14%	31%	55%
	Control (n=24)	13%	13%	75%
6. The ways children think about mathematics are generally different from the ways adults would expect them to think about mathematics. For example, real-world contexts support children's initial thinking whereas symbols do not.				
Spring	Treatment (n=21)	18%	14%	68%
Fall	Treatment (n=30)	27%	17%	57%
	Control (n=24)	13%	38%	50%
7. During interactions related to the learning of mathematics, the teacher should allow the children to do as much of the thinking as possible.				
Spring	Treatment (n=21)	18%	14%	68%
Fall	Treatment (n=30)	37%	27%	37%
	Control (n=24)	17%	8%	75%

Table 3. Beliefs-score-change percentages by belief, group and score-change category

We found the following patterns from the distribution shown in Table 3. First, treatment groups, in both Spring and Fall 2010, had the greatest percentage of PSTs with large increases on every belief. Second, for Beliefs 2, 3, 6, and 7, the treatment group in Fall 2010 had a large percentage of PSTs with large increases as compared to the treatment group in Spring 2010. Finally, for Belief 7, there is a relatively large percentage (i.e., 63%) of PSTs with large and small increases in the Fall 2010 treatment group, compared to PSTs (i.e., 25%) in the control group for the same semester.

	Spring	Fall	
	Treatment	Treatment	Control
Large Increase	18%	21%	16%
Small Increase	20%	23%	23%
No Change or Decrease	62%	56%	62%
Ratio Large/No	29%	38%	26%

Table 4. Average percentages of PSTs in each beliefs change-score category (with ratio of percentage with large change to percentage with no change) by group

As Table 4 indicates, the treatment group in Fall 2010 had the largest ratio of large increase to no increase in the average belief scores. Taken together, the IMAP beliefs survey data seem to indicate that engagement with the videocases influence PSTs' beliefs about mathematics teaching and learning, and reflect findings from previous research (Philipp et al., 2007).

Video Activity

A video activity assessment based on a video clip from an actual classroom was designed to assess PSTs' ability to attend to and analyze children's mathematical thinking in video format. As part of the video activity, PSTs were asked to watch a video clip and then to respond, in writing, to prompts about children's thinking: (1) What do you notice about student's mathematical thinking in the video? (2) Identify the different strategies used by students in the video, and (3) How are students' strategies different? Identical pre- and post-video activities were administrated in the first and the last weeks of class in both courses. All responses were double coded blind using a rubric developed by the authors, and focused on differentiating between the level of sophistication of PSTs' analysis of children's thinking. Level 1 indicates a general statement about the strategies was given (e.g., the child used addition). Level 2 indicates that a response included a mathematical focus, but yet was not specific enough to explore the underlying concepts. Level 3 indicates that a response included a mathematical focus and specifically unpacked the mathematics in the different strategies. Level 4 indicates that, in addition to the criteria for Level 3, the response also included the use of critical words (e.g., repeated addition, repeated subtraction, evenly distributed). Codes of G, U, and B denote responses focused on general aspects of the video, children's understanding, or children's behavior, respectively.

		Fall				Spring			
		Treatment		Control		Treatment		Control	
		Pre	Post	Pre	Post	Pre	Post	Pre	Post
Question 1	3SS-1	11	7	16	6	9	7	10	8
	3SS-2	8	11	5	5	9	9	1	0
	3SS-3	5	5	2	2	2	3	0	1
	3SS-4	0	0	1	0	0	0	0	0
	3S-G	10	9	8	15	5	9	6	3
	3S-U	6	8	0	1	6	5	6	10
	3S-B	2	0	1	0	0	0	0	0
Question 2	6-1	0	1	0	0	1	1	0	0
	6-2	2	1	0	3	2	3	1	1
	6-3	6	5	5	3	2	6	7	7
	6-4	20	23	23	17	17	11	9	9
Question 3	7-1	4	3	3	5	3	8	5	5
	7-2	19	20	23	17	15	11	7	9
	7-3	5	7	2	1	4	3	4	2

Table 5. Frequencies of PSTs' video activity responses by question and by level

As Table 5 indicates, while there were small shifts from pre- to post-course on certain questions, overall there were minimal differences between the treatment and control groups for the three questions on the pre-post video activity.

Conclusion

A primary aim of this study is to explore the potential of a videocase curriculum for supporting PSTs' learning about children's mathematical thinking as part of a required mathematics content course. Overall, results of the study indicate that, besides differences on the pre-post belief survey, there were no marked differences between the treatment and control groups. In other words, use of the videocase curriculum seemed to have a minimal effect on PSTs' knowledge of and ability to analyze children's mathematical thinking, yet the videocases did have a moderate effect on PSTs' beliefs about mathematics teaching and learning. We posit that one reason for such results may be that the content courses in the study were collaboratively designed and implemented by a group of mathematics educators and mathematicians around the idea of developing PSTs' mathematical knowledge for teaching (see Castro Superfine, 2010). Thus, the treatment and control courses were essentially the same course using the same tasks, in-class activities, homework assignments, and exams, with the exception of the videocases. In this particular course context, one would not expect the videocases to have a significant effect. As the videocases did have an effect on PSTs' beliefs about mathematics teaching and learning, there seems to be some potential for effecting change. However, the collection of videocases needs to be implemented in different course contexts in order to fully test the potential of the videocase curriculum.

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A MULTIPLE-CASE STUDY OF ELEMENTARY PROSPECTIVE TEACHERS' EXPERIENCES IN DISTINCT MATHEMATICS CONTENT COURSES

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This multiple-case study explored the experiences of two groups of elementary prospective teachers (n=12) completing distinct mathematics content courses. Individual interviews revealed perspectives on knowing, learning, and teaching mathematics as experienced by the two groups; the quantitative findings indicated differences in mathematical beliefs. One group characterized mathematics as a record of knowledge, difficult to understand and lacking in relevance; learning occurred through rote memorization and via external expertise, with teaching typified as explaining. The other group portrayed mathematics as process-focused, internally constructed, and relevant; learning took place through a focus on children's thinking, with teaching characterized as guiding and questioning.

The context for this study was an elementary teacher preparation program that was evolving in response to university system mandates requiring more courses in mathematics content for elementary teachers. Specifically, programs had to include a particular 3-hour course in elementary mathematics during the sophomore year in addition to 9 hours of upper-division mathematics courses. Throughout this period, mathematics department faculty members developed and taught the required courses for elementary teachers in Number and Operations, Algebra, Geometry, and Statistics.

Many prospective teachers failed or withdrew from these courses and consequently had to defer student teaching. Over one 4-semester period (Fall 2004-Spring 2006), 24.8% of prospective teachers did not complete or pass one or more of these mathematics courses. In response to this troubling trend, an experimental group of prospective teachers enrolled as a cohort in a one-time sequence of four content courses having specific foci on the perspectives of the National Council of Teachers of Mathematics (NCTM, 2000) and the development of the *specialized content knowledge* (SCK) needed for teaching elementary mathematics. This experimental sequence is referred to here as the "alternative courses" and was taught by an instructor in the elementary education department (grades PreK-5). The other group of prospective teachers in this study participated in what is referred to here as the "traditional courses" taught by instructors in the mathematics department.

This study explored these two groups of teacher candidates' (n=12) perspectives on knowing, learning, and teaching mathematics in the context of these experiences, as well as their mathematical beliefs and affect. More specifically, this study used a multiple-case design that applied mixed methods to explore the following research questions: (1) What are the perspectives on knowing, teaching, and learning mathematics of two groups of elementary prospective teachers' in the context of distinct mathematics courses? and (2) What are the mathematical beliefs and affect of these prospective teachers?

Theoretical Perspectives and Related Research

Perspectives on Teacher Knowledge in Mathematics

A pressing concern for elementary teacher preparation programs is the development of adequate and appropriate mathematical content knowledge. The precise nature of this knowledge has prompted significant debate in the mathematics education community (Ball, Hill, & Bass, 2005; Hill, 2010; Rowland, Huckstep, & Thwaites, 2005). In recent years, researchers (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill 2010) have proposed a *specialized content knowledge* (SCK) described as "the mathematical knowledge 'entailed by teaching' - in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). Examples of SCK include teachers' abilities to: (a) analyze and interpret students' mathematical thinking and ideas, (b) use multiple representations of mathematical concepts, and (c) define terms in mathematically correct and accessible ways.

Teacher Beliefs and Affect

Prospective teachers come to the teaching profession with deep-rooted mathematical beliefs and affect formed during their seminal years as students in K-12 classrooms (Lortie, 1975); they resist changing these beliefs and affect as they move through teacher preparation programs (Phillip, 2007). Three teacher belief constructs are relevant to this study including: efficacy beliefs, pedagogical beliefs, and beliefs about the nature of mathematics. The first belief construct, teacher efficacy, has been defined as teachers' beliefs in their skills and abilities to teach mathematics effectively and influence student learning (Hoy, 2004). The second belief construct, pedagogical beliefs, includes teachers' beliefs related to how they should teach and how students should learn; these beliefs can be viewed as grounded in theories of learning. Lastly, beliefs about the nature of mathematics include what mathematics is as a subject or what it means to know and do mathematics, ranging from mathematics as unrelated facts, rules, and skills to mathematics as problem solving, fluid and expanding in nature (Handal, 2003; Wilkins, 2008).

Elementary Prospective Teachers' Experiences in Mathematics Content Courses

Studies of elementary prospective teachers' experiences in mathematics content courses have focused on efforts to align the curriculum and instructors' practices with current reform recommendations (Lubinski & Otto, 2004; Philipp et al., 2007; Royster, Harris, & Schoeps, 1999). Royster et al. found that elementary education majors showed the greatest positive changes in mathematical dispositions when compared to other majors upon completion of a mathematics course that had been revised for congruence with reform recommendations. In another study, Phillip et al. concluded that prospective teachers in a mathematics course studying children's mathematical thinking developed more sophisticated beliefs about mathematics and improved their mathematical content knowledge than prospective teachers who did not have children's thinking as a focus.

Studies have also examined college students' perceptions of effective mathematics teaching and learning (Harkness, D'Ambrosio, & Morrone, 2006; Powell-Mikle, 2003; Schulze & Tomal, 2006). For example, Powell-Mikle reported certain classroom characteristics that support student learning in mathematics: adequate instructor availability, clear instructor explanations, prevalent classroom discourse, and a caring classroom environment. Schulze and Tomal's sizeable study of 2,042 college students examined factors contributing to a "chilly" mathematics classroom climate, which include: difficulty level of course content, teaching style/personality of the professor, and personality styles of classmates.

Methodology

The design of this study included a descriptive, holistic multiple-case approach (Yin, 2003). A case study methodology was applied as it was "impossible to separate the phenomenon's variables from the context," (Merriam, 1998, p. 29) and two bounded units were investigated. The two cases, which were the units of study and analyses, were the distinct mathematics course experiences, and the purpose of the study was to provide a "thick description" (Merriam, 1998, p. 29) of each. Further, this rich description provided opportunities to compare and contrast across the two mathematics course experiences in the interpretation of the findings.

Within this multiple-case design, mixed methods were applied. More specifically, a "concurrent triangulation" (Creswell, Clark, Gutmann, & Hanson, 2003, p. 224) approach to mixed methods was used, with data collection occurring via individual interviews and surveys. In this present study, the concurrent triangulation approach implies: (a) quantitative and qualitative data were collected concurrently, (b) qualitative data were given priority, and (c) integration occurred in the interpretation phase.

Participants and Setting

The participants were 12 randomly selected prospective teachers enrolled in an elementary teacher preparation program at a large, urban university in the southeastern U.S. The gender of the participants included 11 females and 1 male. At the time of this study, the participants were in the student teaching semester of the program. The program was two years in duration and included three semesters of courses with concurrent two-day-per-week field placements, followed by a full semester of student teaching.

The participants had completed one mathematics methods course and the four required mathematics content courses for elementary teachers. The mathematics methods course was taken during the second semester of the program and taught by instructors in the elementary education department. The content courses were Number and Operations (lower level), Geometry, Algebra, and Statistics (all upper level). As an admittance requirement, all 12 of the prospective teachers completed the Number and Operations course prior to entering the teacher preparation program. The other content courses were completed at different times during the program but were finished prior to the student teaching semester.

Six of the prospective teachers experienced the "traditional courses" (see introduction) taught by various instructors in the mathematics department, and six prospective teachers experienced the "alternative courses" taught by an instructor from the elementary education department. The nature of the course experiences is best viewed from the responses of the participants who experienced the courses firsthand; however, a syllabi analysis revealed that in general the two sets of courses focused on the same mathematics content with considerable differences in the nature of this knowledge as well as the ways in which this content was taught and learned. One major difference in these approaches was that the alternative courses included a significant amount of time studying children's development of mathematical thinking, problem solving, and understanding, whereas the traditional courses focused primarily on memorization of formal definitions and proficiency with standard algorithms.

Instruments and Data Collection

This mixed methods study included qualitative data collected via individual interviews and quantitative data collected via two belief surveys. Data collection occurred during the student teaching semester of the teacher preparation program.

The interview protocol included six multi-part questions such as: (a) What are your overall impressions of the math courses? What was easy and hard? What did you like and dislike? (b) After taking the math courses, do you feel confident that your content knowledge is sufficient to understand PreK-5 math? Why or why not? and (c) After taking the math courses, do you feel prepared to analyze children's math strategies in grades PreK-5? Why or why not? The semi-structured interviews were conducted at the convenience of the prospective teachers at the researchers' offices or the student teaching schools.

Two belief surveys were completed by the prospective teachers on campus: the Mathematics Beliefs Instrument (MBI) and the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI). The MBI is a 48-item Likert scale instrument designed to assess teachers' beliefs about the teaching and learning of mathematics and the degree to which these beliefs are cognitively aligned (Peterson, Fennema, Carpenter, & Loef, 1989, as modified by the Cognitively Guided Instruction Project). The three subscales include: (b) role of the learner (Learner), (b) relationship between skills and understanding (Curriculum), and (c) role of the teacher (Teacher). The Learner subscale contains 15 items that assess the degree to which teachers believe that children can construct their own mathematical knowledge. The 16-item Curriculum subscale examines the degree to which teachers believe that mathematics skills should be taught in relation to understanding and problem solving. The 17 items on the Teacher subscale address the extent to which teachers believe that mathematics instruction should be organized to facilitate children's construction of knowledge. The instrument uses a Likert scale with five response categories (strongly agree, agree, uncertain, disagree, and strongly disagree), with higher scores indicating beliefs that are more cognitively aligned. These subscales have high reliability (Chronbach's alpha =.89 for Learner, .80 for Curriculum, and .90 for Teacher) and represent independent constructs based on confirmatory factor analysis.

The MTEBI consists of 21 items, 13 on the Personal Mathematics Teaching Efficacy (PMTE) subscale and 8 on the Mathematics Teaching Outcome Expectancy (MTOE) subscale (Enochs, Smith, & Huinker, 2000). The two subscales are consistent with the two-dimensional aspect of teacher efficacy. The PMTE subscale addresses the prospective teachers' beliefs in their individual capabilities to be effective mathematics teachers. The MTOE subscale addresses the prospective teachers' beliefs that effective teaching of mathematics can bring about student learning regardless of external factors. The instrument uses a Likert scale with five response categories, with higher scores indicating greater teaching efficacy. Possible scores on the PMTE subscale range from 13 to 65; MTOE subscale scores range from 8 to 40. These subscales have high reliability (Chronbach's alpha = .88 for PMTE and .81 for MTOE) and represent independent constructs based on confirmatory analysis.

Data Analysis

This multiple-case design included analysis of the data within each case. Audiotapes of the interviews were transcribed and analysis of the data began by applying the *a priori* codes of knowing, learning, and teaching mathematics as experienced by the two groups of prospective teachers in the mathematics courses. Through this process an additional code emerged: beliefs and affect. Researchers used constant comparative methods (Lincoln & Guba, 1985) to generate more refined categories within these codes. Specifically, researchers individually analyzed the qualitative data through open coding, which generated numerous categories and subcategories that represented observed phenomenon found in the data (Strauss & Corbin, 1998). Researchers periodically met and discussed the subcategories to reach consensus on their meanings related to the categories. This recursive process of discussion and analysis of all interview data initiated development of a coding manual that was used in subsequent analyses. The researchers then engaged in data reduction by recoding data using the coding manual for guidance in comparing and refining categories. Coded categories were collapsed and renamed related to the themes of knowing, teaching, and learning mathematics, as well as beliefs and affect. Data from the belief surveys were considered at the case level by subscale and overall scale. The quantitative data were used for descriptive purposes.

Results

Quantitative Findings

Mean scores, including differences in mean scores, and standard deviations for the two groups of prospective teachers on the MBI and MTEBI (subscales and overall scale) are shown in Table 1. When comparing the two sets of scores, all subscale and overall mean scores have at least half-point differences in the Likert scale value. These findings suggest the prospective teachers in the alternative courses had stronger mathematics teaching efficacy beliefs and pedagogical beliefs that were more cognitively aligned. Two subscales, Teacher and Learner, evidenced the largest differences in mean scores, .77 and .76 respectively. It seems the prospective teachers in the alternative courses, more so than those in the traditional courses, believed that children can construct their own mathematical knowledge and that instruction in mathematics should be organized to facilitate this construction. Interesting, the subscale that revealed the next largest difference in mean score (.70) was the MTOE. When comparing the two groups of prospective teachers, those completing the alternative courses seem to have stronger beliefs that their teaching of mathematics positively influences student learning.

Table 1

Means and Standard Deviations for Mathematics Pedagogical and Teaching Efficacy Beliefs as Likert Scale Values

	Traditional Courses		Alternative Courses		Both Courses
Subscale and Overall Scores*	Means	Standard Deviations	Means	Standard Deviations	Differences in Mean Scores
Learner	3.70	.60	4.46	.49	.76
Curriculum	3.61	.55	4.15	.47	.54
Teacher	3.78	.49	4.55	.32	.77
Overall MBI	3.70	.50	4.39	.31	.69
PMTE	4.08	.23	4.68	.34	.60
MTOE	3.61	.28	4.31	.74	.70
Overall MTEBI	3.90	.13	4.54	.44	.64

*MBI = Mathematics Beliefs Instrument; PMTE = Personal Mathematics Teaching Efficacy; MTOE = Mathematics Teaching Outcome Expectancy; MTEBI = Mathematics Teaching Efficacy Beliefs Instrument.

Qualitative Findings

Traditional courses. The data analysis reveals the prospective teachers characterized the mathematics in the traditional courses as: procedural knowledge, lacking relevance, and difficult. More specifically, mathematics as procedural knowledge included descriptors such as "formulas," "step-by-step," "right and wrong," "abstract," "information," and "definitions," with little attention to processes in mathematics. Mathematics was typified as a *record of knowledge.* Further, the prospective teachers frequently spoke of the irrelevance of the mathematics, describing the mathematics as "high school" or "college" level, with little connection to the elementary classroom. Interestingly, before completing the courses, the prospective teachers believed they already had the mathematics knowledge needed for the elementary classroom; the courses did not challenge the prospective teachers' paradigm about the SCK needed for teaching elementary mathematics. Additionally, the prospective teachers described the difficulty of the mathematics in the courses, particularly as "hard" and "unattainable."

In considering the learning and teaching of mathematics in the context of the traditional courses, it is noteworthy there was little mention of the learning and teaching of mathematics for elementary students. Learning mathematics was characterized as: rote memorization, a process that occurs via experts, and "passing the course." The prospective teachers described their learning through rote memorization as "time-consuming, extensive practice," "note-taking", "homework," and "repetition and regurgitation." Further, this learning took place via receipt from external expert sources, and this expertise included the course instructors, tutors, textbooks, and class notes. Learning was also typified as "passing the course." The prospective teachers spoke of "passing the test" and "getting in and getting out."

The teaching of mathematics in the traditional courses was typified as explaining, and the prospective teachers experienced teaching as "lecturing," "showing," "step-by-step explanations," "Power Points," and "covering content." Further, teaching in the courses was characterized as teacher-centered and content-centered rather than attentive to the needs of the prospective teachers. The teaching was frequently described as "fast-paced," and instructor dispositions, such as differing levels of helpfulness, responsiveness, and availability, were a factor.

The prospective teachers' experiences with knowing, teaching, and learning mathematics were linked with their affective responses in the courses. They described emotional reactions to the courses, including "terrifying" and "frustrating." They also portrayed their experiences as having negative influences on their mathematics teaching efficacy (i.e., beliefs in their capabilities to be an effective mathematics teacher and influence student learning) and mathematics self-efficacy (i.e., beliefs in their capabilities to do mathematics).

Alternative courses. The analysis of the data indicates the prospective teachers portrayed mathematics in the alternative courses as process-focused, useful, challenging, and internally constructed. The process focus included an emphasis on "problem solving" and "understanding," which contributed to flexibility in their mathematical knowledge. The mathematics was also described as "useful" or "relevant," with explicit connections to the mathematics in the elementary classroom. Further, the mathematics was typified as "challenging;" it was a "struggle" for the prospective teachers to "unlearn" mathematics as being simply procedures. Mathematics in the courses was also portrayed as internally constructed (i.e., "in my head") rather than received from other external expert sources. Further, in describing the mathematics in the courses, the prospective teachers frequently contrasted it with the mathematics learned in other content courses.

The learning of mathematics in the courses was typified in several ways by the prospective teachers. They described learning as occurring through a community of learners, with an emphasis on discourse. Further, learning took place through mathematical processes such as "problem solving," which were portrayed as "engaging" and perceived as "okay to be wrong." Learning mathematics also took place through a focus on children's learning and thinking. The courses built the mathematical knowledge of the prospective teachers by studying how children think about mathematical concepts and ideas; this focus and course assignments led to learning as being perceived as directly applicable to the elementary classroom.

Teaching in the context of the alternative courses was typified by the prospective teachers as "guiding" and "questioning." The instructor promoted discourse, created a safe learning environment, and used tools (e.g., manipulatives) relevant to the elementary classroom. Interestingly, the prospective teachers described a "struggle" or tension in connecting what they were learning in the courses to their teaching in their field placement classrooms, which were often characterized as "traditional." Further, the dispositions of the instructor, such as "helpfulness" and "accessibility," were described as important to teaching in the course.

The prospective teachers' characterizations of knowing, learning, and teaching mathematics were linked with their affective responses in the courses. The prospective teachers described the courses as positively influencing their mathematics teaching efficacy and mathematics self-efficacy.

Concluding Thoughts

More than two decades have passed since Ball and Wilson (1990) challenged the assumptions that: (a) the development of elementary prospective teachers' mathematical content knowledge should occur within the context of traditional undergraduate mathematics courses, and (b) that content knowledge is the only professional knowledge necessary for teaching. These beliefs continue to be espoused by some policy makers and faculty members at institutions of higher education, who believe that prospective teachers need only take additional advanced mathematics courses to acquire content knowledge in mathematics, while disregarding SCK (Sowder, 2007). The prospective teachers' experiences related to knowing, learning, and teaching mathematics, as well as the differential outcomes in beliefs, suggest several benefits of alternative ways of thinking about elementary teacher preparation. Further, the findings provide insights into the issues and challenges of building the SCK needed for teaching elementary mathematics.

It is evident the prospective teachers in this study learned what they had the *opportunity to learn* (OTL) (Hiebert & Grouws, 2007), as revealed in their characterizations of knowing, learning, and teaching mathematics in the distinct course experiences. The Introduction to the original National Council of Teachers of Mathematics (NCTM) *Standards* (1989) states "*what* a student learns depends to a great degree on *how* he or she has learned it" (p. 5, italics in original). OTL is considered a complex process and product of both the curricular emphasis and the quality of instruction (Hiebert & Grouws, 2007). Teacher competencies developed in teacher preparation programs, including SCK needed for teaching mathematics, clearly depend upon how this knowledge is acquired.

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STATICICAL PREPARATION OF TEACHERS: PRESERVICE ELEMENTARY TEACHERS (PSTS) CONCEPTIONS OF DISTRIBUTIONS OF DATA – THINKING ABOUT MEASURE OF CENTER AND VARIAIBLITY

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Prior research investigating PSTs' statistical thinking is sparse, yet teacher educators need to know how best to prepare future teachers for their work and given the increasing importance of STEM education we need teachers who are capable of preparing students at an early age to think about data. In this paper we share the results from a study investigating pre-service teachers' (PSTs') initial thinking about distributions of data in four different contexts. In particular we investigated how PSTs reasoned about different distributions of data - including how they consider measures of center and measures of variability. Data was collected through surveys and interviews conducted prior to their beginning a statistical unit in their elementary teachers mathematics content course.

Without basic statistical literacy adults are unlikely to have the knowledge base to make informed personal decisions and, on a professional level, the door to many higher paying jobs closes. (Cobb & Moore, 1997; Garfield & Ben-Zvi, 2008).

Why Focus on PSTs' Conceptions?

Recent research on teacher knowledge has shown that the knowledge needed for teaching mathematics/statistics is complex and multifaceted (i.e. Hill, Ball, & Schilling, 2008; Ma, 1999; Shulman, 1986) and goes beyond content knowledge including such aspects as knowledge of how students think about the mathematical/statistical content. Hill, Ball, and Schilling (2008) introduced a framework for *mathematical knowledge for teaching* listing six different types of knowledge needed to teach mathematics, three in the realm of subject matter knowledge and three in the realm of pedagogical content knowledge (see Figure 1). In addition, teachers' mathematical knowledge for teaching has been empirically linked to instruction (Borko & et al., 1992; Fennema & Franke, 1992); and student achievement gains (Hill, Rowan, & Ball, 2005).

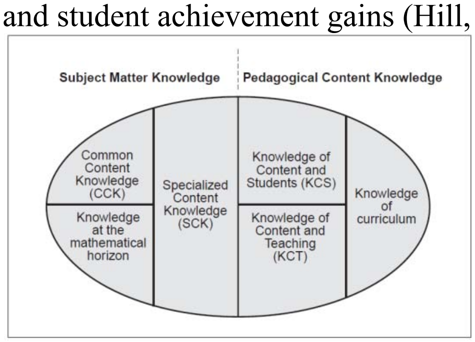

Figure 1: Domain map of mathematical knowledge for teaching (Hill et al., 2008).

New mathematical ideas are built on currently held conceptions. In order to help PSTs

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develop such multifaceted knowledge of mathematics/statistics for teaching, teacher educators need to build on PSTs incoming conceptions. The conceptions we examine in this paper fall beyond common content knowledge (i.e. how to calculate the mean for example) and are situated in the realm of specialized content knowledge (i.e. knowledge of center and distribution specifically used for teaching). Mathematics teacher educators need to understand PSTs' initial conceptions when they enter our classrooms (Bransford, Brown, & Cocking, 1999) as these types of understandings underpin and inform the pedagogical decisions made when teaching.

Why Focus on PSTs' Conceptions of Statistics?

The importance of statistics for all citizens living in a democratic society is reflected by the Guidelines and Assessment in Statistics Education (GAISE) report (Franklin et al., 2007). GAISE highlights the important role of statistics education in preparing students to navigate our current society (reading newspapers, understanding political implications etc.). In particular, understanding distributions of data is a unifying theme in the study of statistics and one of the foci of the elementary math curriculum as laid out by NCTM and Common Core Standards (National Governors Association & Council of Chief State School Officers, 2010). Bakker and Gravemeijer (2004) suggest that distribution is an "organizing structure or conceptual entity" for looking at data (p. 149) and they argue that without a notion of distribution it is not possible to reasonably summarize a data set and make appropriate choices between different measures of center. While substantial efforts have been made to better understand students' statistical reasoning and how they think about distributions of data, there is a paucity of empirical research investigating teachers' statistical reasoning (Shaughnessy, 2007b). What little empirical research has been conducted (Groth & Bergner, 2006; Jaccobe, under review; Leavy & O'Loughlin, 2006) indicates that PSTs do not have the knowledge needed to teach statistics at the level endorsed by GAISE. It is not sufficient to know that teachers do not have adequate statistical knowledge for teaching statistics, teacher educators need to better understand teachers' statistical knowledge when they enter our classrooms so we can build on those conceptions when working with teachers.

In this study we examined PSTs' conceptions of distributions of data in four different contexts. In particular, we investigated: (a) how PSTs might compare two different distributions of data that had the same mean and median, but differed in variability; (b) the ways in which PSTs' construct a data set for a given mean (with a particular focus on the range of the data sets they created); (c) whether PSTs recognize the mean as an appropriate measure to compare two data sets of different size; and, (d) the ways in which PSTs' think about the median through a task that focuses on possible student interpretations of median. The results of this study provide a glimpse into U.S. PSTs' initial knowledge of distributions of data. We selected these particular tasks for two reasons. First, we wanted to build off the small body of prior research (see Leavy & O'Loughlin, 2006) and look for possible comparisons between U.S. PSTs and PSTs in Ireland. Second, the tasks used in this study align with Level A in the GAISE document and highlight components of statistical knowledge that PSTs need to be successful in their work. Thus, the strategies used by PSTs' when working on these tasks provides valuable insights into their specialized content knowledge; these insights inform the decisions that teacher educators make regarding the types of experiences we provide in our pedagogy of mathematics courses.

Methods

Twenty-seven PSTs enrolled in the 2nd of 3 mathematics-for-elementary-school-teachers courses completed a written survey followed by a brief interview. The survey items were drawn from various previous research studies (as cited below) to give an initial insight into PSTs' knowledge of distributions of data. We introduce the tasks in the results section. Data analysis was conducted by the first two authors and focused on the survey results. The authors independently categorized student responses to each item and then met to discuss the categories. Once common categories were established the data was coded using those common categories. Disagreements were resolved through discussion.

Results

What Measures Do PSTs Use When Comparing Distributions of Data?

To examine whether PSTs pay attention to distribution in a given data set we asked them to respond to the Movie Wait Time Task (Shaughnessy, 2007a). On this task PSTs are presented with a scenario involving the comparison of two distributions of data. In response to the task, 19 PSTs disagreed with Eddie, 4 PSTs agreed with Eddie; and 4 did not express agreement or disagreement. 25 of the PSTs mentioned the range or the variability of the wait times in their response. The 4 PSTs who agreed with Eddie agreed that the average is 10 but all 4 mentioned the variability in wait times. The two PSTs who did not mention variability performed calculations to find the mean. Thus at least in this task almost all PSTs recognized variability as a measure to take into account when comparing distributions of data.

Movie theaters show commercials along with previews before the movie begins. The *wait-time* for a movie is the difference between the ADVERTISED start time (like in the paper) and the ACTUAL start time for the movie. A class of 21 students investigated the wait-times at two popular movie theater chains: Maximum Movie Theaters and Royal Movie Theaters. Each student attended two movies, a different movie in each theater. The class's results are shown in the chart below. (Times were rounded to the nearest half-minute.) The students in the class found the median wait-time for both of the theaters to be 10 minutes. The students also calculated the mean wait-time for each theater to be 10 minutes.

Here's what Eddy, a 4th grader, concluded: There is no difference in wait-times for the theaters because they both were about 10 minutes.

Do you agree or disagree with Eddy? Explain all of your reasoning:

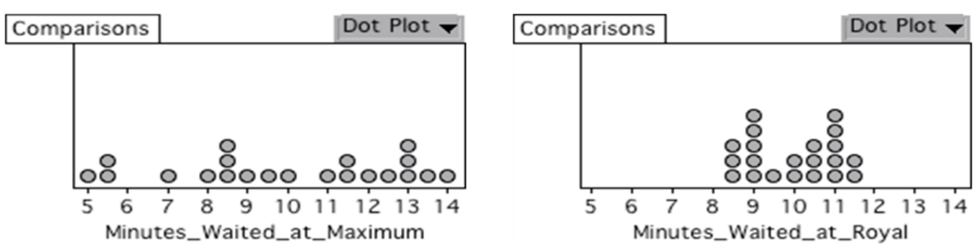


Figure 2: Movie Wait Time Task (Shaughnessy, 2007a)

Can PSTs Construct a Distribution of Data to Reflect a Given Mean? How Does Variability Factor in their Decisions?

To examine whether PSTs' can construct a distribution to reflect a given "average" the PSTs were asked to label prices on individual potato chip bags given that the average price is 27 cents. PSTs were asked to come up with a second labeling excluding 27 as a value (see Figure 3). In our analysis we focused on the PSTs' strategies and the range of the data set they provided.

Potato Chip Task A: The average price of a bag of chips is 27cents. We have seven bags of chips each of which has an empty price tag. Place prices on each of the bags so that the average price is 27 cents. □□□□□□□	*Potato Chip Task B:* The average price of a bag of chips is 27cents. We have seven bags of chips each of which has an empty price tag. Place prices on each of the bags so that the average price is 27 cents. *However, none of the bags of chips can cost 27 cents.* □□□□□□□

Figure 3: Potato Chip Tasks (see Leavy & O'Loughlin, 2006; Mokros & Russell, 1995)

For Part (A) of the Potato Chip Task our results showed that 14 PSTs (about 50% of the PSTs as opposed to 25% of PSTs in Leavy & O'Loughlin's study) put a 27 into each of the boxes on the first task (all listed results can be seen in Table 1). Seven PSTs (about 25% of the PSTs, similar to Leavy and O'Loughlin's study) started with 27 in the middle and then varied the numbers by +1/-1 as they moved outward resulting in the following prices: 24, 25, 26, 27, 28, 29, 30. Three PSTs used the mean algorithm, one selected numbers across a wide range summing to 189, two iteratively subtracted numbers from 189. One PST entered some 27s and then used balancing (i.e., adding and subtracting the same number from 27 in order to "balance" out to 27) to find the other prices in the data set. Two PSTs used what we call reasonable responses – using some type of reasoning that the bags combined have to be $1.89 the average price would have to be 27 thus the values would be *around* 27.

As expected, it seemed most PSTs understood the "average" as the "mean" rather than another measure of center. One PST used the mode in their reasoning for all 27s. Twenty-three PSTs chose numbers for the prices that stayed within a close range [24,30]. Four PSTs chose wide ranges for their data such as (100, 20, 20, 10, 10, 10, 19). These PSTs either gave no justification or used the mean algorithm (summing up to $1.89) as their explanation.

For Part (B) of the Potato Chip task all PSTs interpreted "average" as mean. Twenty-one PSTs gave values in between 20 and 35, while six PSTs gave wide ranges in value. Most PSTs used the mean algorithm some with guess and check, some with some basic balancing. Only four PSTs used a conceptual approach using balancing (pairs and one set of three) or a reasonable approach. Initially many of the responses in Part (A) looked like balancing approaches (i.e., balanced around the 27 like this response: 24, 25, 26, 27, 28, 29, 30), however, in Part B these PSTs seemed to focus more on the algorithm for finding mean. This indicated to us that just because their work "looks like" a balance approach does not mean it is. In fact, PSTs may be drawing on procedural rather than conceptual knowledge and using the mean algorithm to find pairs that average to 27.

Table 1: PSTs' responses to the Potato Chip Tasks

Responses to Part (A)	Reasoning for Part (A)	Counts
All 27s	No Justification	9
	Mean Algorithm	6
	Mode	1
27 & Values close to 27 (e.g., 26, 26, 27, 27, 27, 28, 28)	No Justification	3
	Mean Algorithm	1
	Balance	1
24, 25, 26, 27, 28, 29, 30	No Justification	1
	Mean Algorithm	1
	Mean Algorithm & Basic Balance	3
	Balance	2
Wide Ranges (e.g., 100, 20, 20, 10, 10, 10, 19)	No Justification	1
	Mean Algorithm	3

Responses to part (B)	Reasoning to Part (B) & Interview follow-up	Counts*
Range 20-35	Mean Algorithm / Guess & Check	6
	Mean Algorithm	3
	Mean Algorithm & Basic Balance (few over/few under)	8
	Algorithm & Balancing (pairs or triples)	1
	Reasonable	2
	Balance	1
Wide Ranges	Mean Algorithm	3
	Mean Algorithm / Guess & Check	3

Responses to Part (A)	Reasoning for Part (A)	Counts	Reasoning Part (B)
24, 25, 26, 27, 28, 29, 30	No Justification	1	Balance
	Mean Algorithm	1	Mean Algorithm
	Mean Algorithm & Basic Balance	3	Mean Algorithm (2) Mean Algorithm /Guess and check (1)
	Balance	2	Algorithm

All of the wide range responses appeared to use only the mean algorithm in order to justify their solutions – no evidence of average as balance or reasonable. Those PSTs who displayed balance type reasoning did so via general sense (some over, some under) or pair-balance. Only one PST displayed explicit knowledge of balancing three values; we argue that this may be an important first step in developing a more conceptual understanding of mean as point of balance.

Combining the results of the Movie Wait Time Task and the Potato Chip Task it seems that PSTs can recognize variability in the data but when asked to create a set of data may stay close to the mean value. Although our sample size is small and we only have evidence of this from two tasks, this finding is consistent with work done by Rubin, Bruce and Tenney (1991) with middle and high school students who seemed to recognize variability in some data contexts, but not in others. In particular, it seems that in tasks where students are asked to provide the data they may gravitate toward the expected value and not consider variability in the data; yet, in other contexts where students see different sets of data they may be more likely to focus on variability within the data or between data sets. However context is always an important consideration when dealing with data. In this case the actual context of the potato chip task (i.e. pricing bags of chips) may impact the variability of the distributions constructed by PSTs. Even though PSTs may recognize the 'possible/theoretical' variability in the distribution of prices, the context may limit the variability of the values they choose to represent the distribution of data. This raises an interesting issue for further research, would PSTs construct data sets with a wider variability if the context was not restrictive?

Do PSTs Recognize the Mean as a Way to Compare to Data Sets of Different Size?

A third task (see Figure 4) was given to examine whether PSTs would recognize the mean as a suitable measure to compare the data sets (Leavy & O'Loughlin, 2006). This recognition of the mean as a comparison measure falls into the realm of specialized content knowledge (SCK) and is an important component of the knowledge needed for teaching.

Basketball Task: Coach Andrews is selecting students to play on the All Star Team. He has decided to look at the scoring of each player during the last three weeks of the season. Below are the points scored by Bob and Deon. If Coach Andres can only select one of the two players, who would you recommend and why? [Note: The coach has not received scores for Deon's last two games played]

Bob	**21**	**16**	**23**	**21**	**20**	**17**	**16**	**22**
Deon	**24**	**18**	**21**	**25**	**22**	**28**		

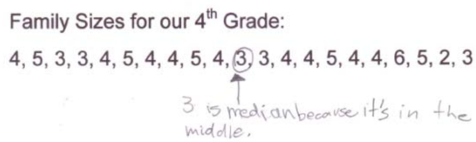
Figure 4: Basketball Task (developed by McGatha et al., 2002 and also used by Leavy & O'Loughlin, 2006)

In this task 15 PSTs (a little more than half of the PSTs in the US vs. 70% of the PSTs in Ireland) recognized the mean as an appropriate measure to compare scores. Thirteen PSTs compared group means while 2 PSTs made equal sized groups (putting in 0s or 23s for Deon's last scores – 23 is the average of Deon's first 6 scores). Eleven PSTs compared the scores without comparing the means. Seven PSTs compared pairs of games, one PST, for example, stated: "I would choose Deon because he has done better than Bob 5 out of the 6 games. Deon's avg. is higher than Bobs." One PST compared total points scored thus far as well as the differences in the first 6 games, one PST looked at consistently scoring above 18, one PST gave no response, and two PSTs wanted to await the last two scores before making a decision. This task illuminates that only about half of the PSTs recognized the appropriateness of the mean to compare data sets of different sizes.

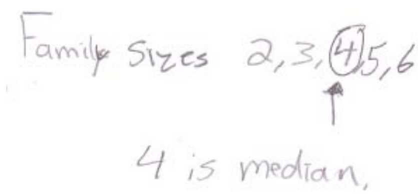
How Do PSTs' Think about the Median in Response to Children's Incorrect Interpretation of the Median?

A fourth task was constructed to examine whether PSTs would recognize children's misconceptions when responding to a task about the median (see Figure 5).

Family Size Problem: In a 4th grade class the students collected data on their family sizes (how many people are in their immediate family). The teacher put the data on the board and asked the students to find the median of the data. The students copied down the numbers and wrote their answers (See Figures 3a and 3b for Anna and Lisa's solutions). Please write a reaction to each of the solutions. In your reaction please include (a) whether you think the answer is correct/incorrect, and (b) what you might do next with that child if you were the teacher.



Anna's solution to identifying the median in the family size problem.



Lisa's solution to identifying the median in the family size problem

Figure 5: Two children's solution to a Family Size Problem**.**

The problem was designed to highlight two potential conceptual difficulties students may have understanding the median, the order of the data and the number of times each data point appears. Thus, the task was designed so that one student chose the middle number of the unordered data set, another the middle number of the various family sizes. The first of these solutions results in an incorrect answer the second in a correct answer, but based on an incorrect argument. Agreement with Anna's solution indicates a conception of median that may not include the notion of the importance of ordered data. Agreement with Lisa's solution, indicates that the frequencies of each family size do not matter. At Level A of the GAISE document students "should understand that the median describes the center of a numerical set in terms of how many data points are above it and below it" (p. 29). All but one PST identified the misconception in Anna's solution (i.e. that the data set needs to be ordered), however only 15 of 27 PSTs recognized that Lisa's argument was incorrect (see Table 2). This may be due to the fact that it resulted in the correct answer or due to the fact that the PSTs did not pay attention to the fact that each data point needs to be accounted for when calculating the median. One PST argued that Lisa's solution is correct because "each number only needs to be represented once to find the median" and another that Ana's is incorrect "Anna is … confused what the middle is in this situation… she should be considering the range of the numbers given (2-6) not the order in which they were written." Thus while most of the PSTs recognized that order mattered when all the data was presented about 44% of the PSTs did not recognize the need for all data points when they were not presented in the solution.

Table 2: PSTs responses to Family Size Problem:

	Identified misconception	Did not identify misconception
Anna's solution	26	1
Lisa's solution	15	12

Conclusions

The PSTs responses to these four tasks give a glimpse into U.S. PSTs' initial thinking about distributions of data. All PSTs, except one, displayed knowledge of the mean algorithm (Potato Chip Task), however, it can often be difficult to decipher procedural averaging of pairs from conceptual averaging through balancing (Potato Chip Task). What may look like balancing may be grounded in the algorithm. All but one of the PSTs interpreted average as "mean" – this is not surprising as the terms are often used interchangeably and PSTs often encounter the notion of average referring to all three measures of center for the first time in their mathematics/statistics for preservice teachers course. When given tasks like comparing two data sets of unequal size (Basketball task) only about half the PSTs recognized the mean as an appropriate measure of center and when confronted with a child's solution to the Family Size Problem about half the PSTs did not recognize the need for all data points to calculate the median. This shows that these PSTs conceptions of mean and median need to be further developed to be at the level recommended in GAISE. When we relate these findings back to the knowledge needed to teach mathematics, we can conclude that most PSTs possess common content knowledge (CCK) of the measures that index distribution, however performance on the tasks reveal problems with specialized content knowledge (SCK) of center and variability and understandings that extend beyond the ability to carry out procedures and manipulation of quantities. It is this specialized content knowledge of statistics that supports thinking and reasoning about distributions and that underpins the instructional decisions PSTs may make in classrooms when providing explanations for procedures, when selecting tasks and in recognizing sources of students errors.

PSTs seem to recognize variability in the data when presented with a a (Movie Wait Time Task) or already constructed data sets, but seem to provide responses close to the mean value when asked to construct data from a given mean (Potato Chip Task). Given that distributional reasoning entails the ability to simultaneously consider measures of center and measures of variability in data, more work needs to be done to understand how PSTs consider distributions of data and how they coordinate measures of center and variability when reasoning about data. Furthermore, we cannot ignore the potential influence of context on the decisions PSTs make when choosing values to represent a distribution of data. This area needs to be explored in future research.

To help PSTs develop their ability to reason coherently about distributions of data, we may need to explicate the relationship between the various measures of center and the data points from which they are created, as well as the role variability plays in statistics. In particular to help PSTs develop a conceptual understanding of the mean we may need to begin with balancing two numbers (pair) or fair share, then move to balancing 3 numbers or fair share among three, and then move to balancing more numbers or fair share. In addition we need to focus PSTs' attention to the range of values in a data set. The fact that one mean can originate from many different data sets is something that PSTs need to come to understand and they must begin to negotiate how context may play a role in the type of variability they expect to see in data.

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ELEMENTARY PRESERVICE TEACHERS' CRITIQUES, COMPARISONS, AND PREFERENCES IN EXAMINING STANDARDS-BASED CURRICULAR MATERIALS

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Shulman's (1986) curricular knowledge includes knowledge of the variety of curricular materials and the ability to evaluate, compare and justify curricular choices. To meet this need we have engaged our pre-service teachers (PSTs) in a comparison activity of two Standards-based lessons. We present PSTs' perceptions of the affordances and constraints within each lesson, interpret these results and present implications for further research.

Because *Standards*-based curriculum materials are becoming ubiquitous in schools (Archer, 2005; Remillard, 2005), there is a need for teacher education programs to prepare PSTs to be able to perceive the intended meaning of, mobilize the potential of, and continually develop their pedagogy through curricular resources. Our overarching research concerns the design of research-based activities for the elementary mathematics methods course that will allow pre-service teachers (PSTs) to learn about- and from- *Standards*-based curricular materials. We posit these types of activities can enable PSTs to move along a trajectory towards expert curriculum use. Our conceptualization of expert curriculum use can be associated with Taylor's (2010) notion of the "curriculum-proof teacher"; a teacher "who can use any given curriculum in highly-effective ways" (p. 152). This paper presents our findings related to PSTs' critiques and perceptions of two *Standards*-based lessons.

Theoretical Frame

Framing our examination of PSTs' perceptions and preferences in working with materials from *Everyday Mathematics* and *Investigations* is the research around teacher learning about and from curriculum materials and PSTs' capacities to evaluate curriculum materials. We also considered research that provides some insight into the relationship between teacher capacity, curricular use, and instruction involving *Standards*-based curriculum materials.

Learning about and from Curriculum Materials

To frame our use of curricular materials and to inform our examination of what PSTs learn about and from *Standards*-based curricular materials, we employ Shulman's (1986) construct of curricular knowledge:

> The curriculum is represented by the full range of programs designed for the teaching of particular subjects and topics at, a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances. (p. 10)

The learning activity reported on in this study addresses PSTs' curricular knowledge by introducing examples of available materials and asking PSTs to evaluate, compare, consider and justify a curricular choice.

Several studies have examined how teachers use *Standards*-based curriculum materials and the impact those materials have had on teacher learning (e.g. Collopy, 2003; Nicol & Crespo, 2006; Remillard & Bryans, 2004). Some of those studies (e.g. Collopy, 2003; Remillard &

Bryans, 2004) have shown that external resources (e.g., curriculum materials, professional development) that have prompted some teachers to use and learn from curriculum materials in reform-oriented ways do not prompt *all* teachers with those same resources to learn or teach in similar ways. For example, Collopy (2003) reported the stark contrast between two teachers in their learning from the use of the same curriculum series with the same professional development opportunities. One teacher (Ms. Ross) developed a new teaching practice by using the materials as her primary source of professional development. Ms. Clark, however, did not change her practice and continued to emphasize memorization and the use of standard algorithms. Collopy (2003) contends that the two teachers differed in their "opportunities to learn" because of the ways in which they read and enacted the curriculum along with how they used the materials when collaborating with colleagues. We posit Collopy's (2003) findings on how teachers read curriculum may allow us to explain the PSTs' critiques and perceptions of the two lessons.

PSTs' Capacity to Evaluate Curriculum Materials

Studies in science and mathematics education (e.g. Beyer & Davis, 2009; Davis, 2006; Lloyd & Behm, 2005, and Nicol& Crepo, 2006) have found that 1) PSTs need continuous supports available in order to evaluate curriculum materials in reform-oriented ways, and that 2) PSTs can misinterpret lessons when they look for aspects of lessons that are familiar to them. In one study, PSTs were asked to complete three science lesson plan analyses using a narrative that provided a description and rationale for an important "principle of practice" (Beyer & Davis, 2009, p. 6). The majority of PSTs used the educative support in their analysis and made adaptations that better supported key principles in science teaching, but did not make similar analyses and adaptations when the support was not available (Beyer & Davis, 2009). Lloyd & Behm (2005) investigated the ways in which PSTs compared and contrasted two textbook lessons (one traditional and one reform-oriented). The researchers found that PSTs looked for aspects of the lessons that were familiar to them. Thus, the lessons they preferred were more traditional. Furthermore, the researchers concluded that PSTs' fondness for traditional lessons led the PSTs to misinterpret those lessons.

Capacity and Mobilization of Standards-based Curriculum

Stein and Kaufman (2010) investigated how teacher capacity and teachers' mobilizations of curriculum materials influenced instruction. They found that teachers implementing *Investigations in Number, Data, and Space* (*Investigations*) had higher-quality lessons (measured by maintaining high levels of cognitive demand, attending to student thinking and vesting intellectual authority in mathematical reasoning) than those implementing *Everyday Mathematics* (*EM*). The researchers attribute this finding to that fact that *Investigations* (TERC, 2008) provided more support to teachers for "locating and understanding the big mathematical ideas within lessons compared to *Everyday Mathematics*" (Stein & Kaufman, 2010, p. 663). After examining levels of cognitive demand and supports for teachers in both series however, the researchers branded *Everyday Mathematics* as a low demand/low support curriculum and *Investigations* as high demand/high support.

Methods

The Developing Addition Strategies module asks PSTs to examine and engage in two addition lessons from *Standards*-based curricula. "Addition Starter Sentences" is a third grade lesson from *Investigations* (TERC, 2008) and "A Shopping Activity" is a second grade lesson from *EM* (USCMP, 2007). Both lessons are designed to provide opportunities for children to develop alternate addition strategies for multi-digit addition. Over the course of two 75-minute class sessions, the first author used the Developing Addition Strategies Module within two sections of an elementary mathematics methods course. He engaged PSTs in the examination of the curricular materials and also in completing the learning tasks from each lesson. The researchers had several goals for this module, including affording PSTs an opportunity to develop knowledge of alternative addition strategies as well as how children develop, make sense of, and use alternative addition strategies (Tyminski, et al., 2010). Another aim, and the focus of this paper, was to give PSTs an opportunity to critique and compare the two lessons, culminating with the PSTs choosing and defending their preference for one of the two lessons. In order to support PSTs in critically examining curricular materials, the first author had introduced, implemented and discussed constructs and tools for analyzing tasks earlier in the semester. These supports included activities involving: levels of cognitive demand (Schwan-Smith & Stein, 1998); availability of multiple access points; whether solution strategies are teacher/textbook- or student-generated; the degree to which lessons support teaching through problem solving; and whether/how lessons address the essential skills and content in state Standards. The comparison activity was posed as follows:

> First, describe, in a bulleted list the strengths and weaknesses of these two lessons as written. Be sure to comment on:
>
> - What you think students will know or understand by the end of the lesson
> - Cognitive demand of the tasks
> - Whether strategies are teacher/textbook-generated or student-generated
> - Whether the tasks include multiple access points for different students
> - Whether/how this lesson reflects teaching through problem-solving
> - Whether/how this lesson addresses essential skills/content in your state Standards
> - Your overall impression of the lesson's strengths and weaknesses
>
> Second, write a short paragraph (4-6 sentences) telling which lesson materials you would rather teach from, why you feel that way, and what goals you would have for your students in using this lesson. Please use evidence from the materials and/or from your own experiences to support your choice.

This paper focuses on the results of the final paragraph in which PSTs selected the lesson they would prefer to teach from and justified their choice based on their perceived strengths and weaknesses of the curricular materials. We collected responses from 45 PSTs at a large, Midwestern university during the spring semester of 2010.

Addition Starter Sentences (TERC, 2008) begins by asking children to decide which of the following "addition starter problems" is easier for them to solve and why: 100 + 200 = ______, 136 + 200 = ______, 136 + 4 = ______. Next, children are asked to choose a starter sentence to solve 136 + 227 and explain why that would be a good start. After children solve 136 + 227, they share their solution paths. Children then complete a worksheet with five more "sets" of tasks, three starter problems matched to a final problem. In the methods course, we enact the lesson as written with PSTs participating as students. We then ask PSTs to read the curriculum materials and discuss the educative features (Davis & Krajcik, 2005). One educative feature we focus on is the information about alternate addition strategies that align with the starter problems: breaking apart by place; adding one number in parts; and changing a number, then adjusting (TERC, 2008). We then ask PSTs to analyze ten examples of children's mathematical thinking for the problem 249 + 175. The PSTs are to make sense of the strategies, determine if the approaches are mathematically valid, and categorize them according to the three strategies. We finish the module with a discussion of how teachers can support students in using these three strategies (see Tyminski, et al., 2010).

A Shopping Activity (UCSMP, 2007) also allows opportunity for students to develop invented strategies for multi-digit addition. According to the curriculum materials, “The main objective of this lesson is to develop and practice strategies for mental addition of 2-digit numbers” (UCSMP, 2007, p. 254). The lesson begins with some mental math exercises and a contextual task designed for students to use the “count up” strategy for subtraction. The lesson then moves into an activity in which the teacher selects two items from a list of eight, to “purchase”. Each of the prices for the items is less than $50. The teacher selects two items, for example a telephone ($46) and a toaster ($29) and asks the children how they might find the total cost. In discussing the students’ solutions the lesson reads:

> You or the children might suggest the following strategies:
> **Strategy 1:** Start with the larger addend, 46. To add 29, note that there are 2 tens in 29. Count up by 10s. Then add 9. The total cost is $75.
> **Strategy 2:** Think of $10 bills and $1 bills. Add the $10 bills. Add the $1 bills. Add the tens and the ones.
> **Strategy 3:** 29 is 1 less than 30. Add 30 to 46. Then subtract 1 to make up for the extra 1.
> **Strategy 4:** 29 is 1 less than 30 and 46 is 4 less than 50. Add 30 and 50. Then subtract the extra 1 and the extra 4. (UCSMP, 2007, p.253)

The lesson asks the teacher to pose more examples and discuss students’ solutions. It explicitly tells teachers not to introduce a traditional pencil and paper algorithm for addition at this point. The students are then put into pairs to complete the next activity, “Playing Shopping”, which uses the same eight items and prices. One child is the clerk and the other child the customer. The customer selects two items randomly and finds the total amount of the two items using a “part-part-total diagram”, but without using a calculator. The clerk then checks the total amount using a calculator. The children then switch roles. The lesson concludes with children completing a handout of shopping problems similar to the lesson tasks. In a similar manner to the Starter Sentences Lesson, we engage the PSTs in the main learning tasks as if they were students in the elementary classroom, followed by a reading of the materials and a discussion of its educative features (Davis & Krajcik, 2005).

Both lessons afford students opportunity to develop alternative addition strategies. The three strategies in both curricula are research-based student approaches to these types of problems. What differentiates the two lessons for us is in the amount of support given to students and the teacher in this process. The starter sentences are presented as a potential first step in a solution path and suggest one of the three solution strategies. The students however choose how to use the starter sentence to complete the problem and there are a myriad of ways to apply the strategies, especially within the “change the numbers” approach. Further, the Starter Sentences teaching materials include supports for the teacher in terms of student strategies (“Often the numbers in a problem will suggest one strategy more than another; for example, students may be more likely to add on to the next hundred if one number is close to a multiple of 100, such as 199” (TERC, 2008, p. 92)); questioning (“How did the Starter Problem you chose help you know what to do next?” (p. 92)); and for observation (“Do students choose one of the starts to solve the final problem? Which one? Can they follow though with the solution and keep track of the steps?” (p. 92).

Although the solutions included within the *EM* lesson are descriptions of the three strategies in *Investigations* (Strategy 1 is “add one number in parts”, Strategy 2 is “break apart by place”, Strategy 3 and 4 involve "changing the numbers"), similar scaffolds are not found within the *EM* lesson, leaving it up to the student to decide how to begin the task. In terms of teacher supports, the inclusion of the phrase "You or the students might suggest the following strategies" (UCSMP, 2007, p. 253) concerned us as it implicitly suggests to the teacher that it is acceptable to directly teach or introduce the strategies. We also identified a lack of teacher support in helping students begin to engage with the tasks. *EM* does include some instructions for supporting Strategy 2 with physical models, ("Use play money to illustrate Strategy 2. Put four \$10 bills and six \$1 bills on one stack and two \$10 bills and nine \$1 bills in a second stack. Combine the bills as indicated" (p. 253). This scaffold is more of a demonstration than an approach to help students think about the strategies, but at least it offers some pedagogical support. The lack of suggestions for the other strategies leaves it on the shoulders of the teacher to develop scaffolds for students.

Data Analysis

From two methods course sections, we collected 45 responses. These were first sorted according to their final curricular preference. Next, each justification paragraph was analyzed independently at the two university sites through a process of open and emergent coding (Strauss & Corbin, 1998), using a framework of PSTs' perceived affordances and constraints as a lens. Through an inductive analysis process, a series of codes for four categories: student affordance, student constraint, teacher affordance, teacher constraint, emerged from the data. As codes emerged from the PSTs' perceptions of the strengths and weaknesses of the materials, they were shared across university sites and refined. There were 18 codes established for student affordance, 5 for student constraint, 5 for teacher affordance, and 3 for teacher constraint.

Results

Of the 45 responses, 30 PSTs stated their preference for the *Everyday Mathematics* materials, while 13 PSTs selected *Investigations*. One PST did not include a final paragraph, and the other suggested using parts of both lessons in her teaching. We present data from the 43 responses in this section, broken down by the viewpoint of the student and the teacher.

Student Affordances and Constraints

In justifying their choice of lesson, many PSTs commented on features pertaining to the interactions and opportunities students would have in learning from the materials. A vast majority of the PSTs who preferred "A Shopping Activity" justified their choice with examples of student affordances (SA) of the materials. Twenty-seven of the 30 PSTs included an SA in their justification, and many students (22 of 27) included more than one SA in their justification. In all, 70 comments were coded as SA-EM (student affordance within *EM*). Table 1 presents the affordances that appeared in at least 15% of the PSTs' responses (N=30). None of the 30 responses included text coded as a student constraint (SC-EM) of the *EM* lesson. Three PSTs focused on student constraints within *Investigations* (SC-IN) as opposed to the affordances of *EM*. One PST felt that the *Investigations* lesson lacked student autonomy; two others described the solution strategies as being teacher/textbook generated. One of the PSTs who selected *EM* nonetheless mentioned that the *Investigations* lesson afforded students opportunities to make connections.

The majority of PSTs who had a preference for *EM* viewed "A Shopping Activity" as a familiar context, hands-on, and fun. These responses seem to focus on the interest of the activity rather than the interesting mathematics. Other, less reported reasons for preferring *EM* focused more on the mathematics – opportunities for practice, student-generated solutions, multiple solution strategies, student autonomy, and learning from peers.

Of the 13 PSTs who selected “Addition Starter Sentences” from *Investigations*, 7 included student affordances in their justification. Across these seven responses, twenty-two comments were coded as SA-IN. Table 2 presents the affordances appearing in at least 2 of the PSTs’ responses (N=13). One of the 13 PSTs who selected the *Investigations* lesson cited the lack of contextual tasks as a weakness of the materials; we coded this as a student constraint of *Investigations* (SC-IN). Three of the 13 PSTs focused on the student constraints of *EM*, two on “lack of student autonomy”, the other on a perceived “lack of goal”. Comparatively, the reasons PSTs gave for preferring *Investigations* tended to focus more on the mathematics: student autonomy, student generated solutions, and multiple solution paths.

SA Code	#	%
Contextual tasks	12	40%
Hands on tasks	9	30%
Fun	8	27%
Opportunities for practice	7	23%
Student generated solutions	7	23%
Multiple solution strategies	6	20%
Student autonomy	5	17%
Learn from peers	5	17%

Table 1: PSTs’ Perceptions of Student Affordances in EM Lesson

SA Code	#	%
Student autonomy	7	54%
Student generated solutions	4	31%
Multiple student strategies	2	15%
Scaffolding	2	15%

Table 2: PSTs’ Perceptions of Student Affordances in Investigations Lesson

Teacher Affordances and Constraints

PSTs also commented on their analysis of the curricular materials from a teacher’s perspective. Of the 30 who preferred the *EM* lesson, 13 used justifications we coded as TA-EM (teacher affordance within *EM*), 5 of these included more than one affordance for a total of 18 comments: 8 PSTs cited the materials for including “differentiation”; 7 comments indicated “ease of use (implementation)”, 2 PSTs noticed “possible student solutions” included in the materials; and 1 viewed the materials as giving the “teacher control” of the lesson. Only 1 PST who selected the *EM* lesson included a comment coded as TC-EM (teacher constraint within *EM*), it was coded as “difficult to understand”. We take this comment to mean the PST had difficulty in making sense of the organization of the curricular materials. A total of three PSTs included commentary from a teaching perspective on the *Investigations* materials within their *EM* justification. One commented on the “ease of use” of the *Investigations* materials (TA-IN: teacher affordance within *Investigations*). Two PSTs’ comments were coded TC-IN (teacher constraint within *Investigations*): one was coded “difficult to understand”; the other was coded as “difficult to use (implement)”. We specifically differentiated between difficulties PSTs perceived in *reading* the materials and *using* the materials, although we posit the two ideas are strongly connected.

Seven of the 13 PSTs’ justifications that selected *Investigations* materials included comments coded as TA-IN (teacher affordance within *Investigations*); only 2 made more than one such comment in their paragraph. There were a total of 11 comments coded as teacher affordances: 7 indicated “ease of use”; 2 indicated the inclusion of “differentiation” for students; 1 PST cited the inclusion of “examples of teacher talk”; and 1 noticed “possible student solutions” included in the materials. Two PSTs indicated teacher constraints within the *Investigations* materials (TC-IN): 1 response was coded as “needs to be adapted”, the other as “difficult to use (implement)”. One PST, who commented on the ease of use of the *Investigations* lesson, also cited the *EM* lesson as “difficult to use” (TC-EM).

We were surprised by the marked difference in the number of comments coded as student affordance or constraint and the number of comments coded as teacher affordances or constraints. There were 99 comments that addressed the materials from a student point of view and 38 from a teaching point of view. One interpretation of these results is that PSTs were more attuned to the potential experience of learning from these materials, rather than teaching with the materials. The comments that were made however, provided interesting evidence of PSTs’ critiques of the lessons. The theme of differentiation was evident within many of the PSTs’ comments on both sets of curricular materials, demonstrating to us they were aware of the importance of addressing the needs of individual learners, and noticing the supports within the materials designed to aid teachers in doing so. What was interesting to us about these results is the assignment did not specifically ask them to comment on this facet of the lessons. Fourteen PST commented on the ease of use of the materials, 7 for *EM* and 7 for *Investigations*. This data raised a new question for us to consider as researchers, what was it specifically about the materials that PSTs were referring to when citing their ease of use?

Conclusions and Implications

We agree with Stein and Kaufman’s (2010) assessment that *Investigations* is stronger than *EM* in terms of cognitive demand, and in the amount and quality of teacher support. Yet, our PSTs overwhelmingly picked *EM* as the curricular materials they would rather teach from. Why was what was clear to us, not evident to the PSTs? We conclude with three possible explanations and implications for future research.

First, we agree with Collopy’s (2003) notion that the manner in which PSTs read the materials, with a viewpoint of a learner, may have caused them to miss out on some of the opportunities to learn *about* the curricular materials. We base this upon their comparative lack of attention to teacher affordances and supports; particularly lack of attention to supports within both curricular materials in terms of presenting possible student solutions. Our future research needs to specifically address how PSTs view teacher supports. Do PSTs believe they can learn from the curricular materials? Should we include a non-*Standards*-based example lesson in the comparison in order to help them notice when supports are not available?

A second explanation also pertains to PSTs’ tendency to evaluate the materials from a student’s point of view. By selecting the *EM* lesson and focusing on aspects of the lesson such as hands on materials, real world contexts and being “fun”, PSTs are trying to ensure that the *activity* involved in the lesson is interesting. From a teaching point of view, we would hope they would try to ensure that the *mathematics* involved in the lesson is interesting. Most PSTs perceive mathematics as a discipline of discrete facts and procedures to be memorized rather than a discipline of interconnected concepts (Thompson, 1992). In other words, mathematics is not meant to be interesting. To make mathematics interesting, the activity needs to be interesting. Thus, the “shopping activity” is preferred.

A third explanation is the PSTs were drawn to *EM* as a result of their teacher capacity. Teacher capacity, including a teacher’s education, experience, and mathematical knowledge is a measure of ability needed to implement curriculum (Stein & Kaufman, 2010). At the PSTs’ current stage of development (first semester seniors), it is fair to classify them as having comparably low teaching capacity in terms of education and experience. Although they did not clearly state this in their paragraphs, perhaps given their experience and knowledge, the PSTs perceived that *Investigations* could be difficult to teach from, leading them to prefer the lower demand in *EM*. Stein and Kaufman (2010) talked about this notion – some believe that "even among standards-based curricula, some are more difficult for teachers to implement than are others" (p. 664). Our PSTs, and others, are not noticing the supports provided in *Investigations* to help implement the high-demand tasks.

Finally, it is evident that through this activity, PSTs are learning about the curricular materials, developing curricular knowledge, and are able to use educative supports in their critique and analysis of *Standards*-based curricular materials. While many PSTs did not view the materials in the same manner an expert might, we are encouraged that with further experiences, PSTs will continue to develop in their ability to do so.

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PROSPECTIVE ELEMENTARY TEACHERS DEVELOP IMPROVED NUMBER SENSE IN REASONING ABOUT FRACTION MAGNITUDE

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We report on results of a classroom teaching experiment in a mathematics content course for prospective elementary teachers. A local instruction theory for the development of number sense, which was previously applied to whole-number mental computation, was extended to inform instruction concerning reasoning about fraction magnitude. We found that students' reasoning in fraction comparison tasks improved in several ways. Their performance improved, they became more flexible in their reasoning, and they came to use less conventional and more sophisticated strategies. These changes parallel those that we previously saw around mental computation.

We report on results of the implementation of a local instruction theory (LIT) for number sense development in a mathematics content course for prospective elementary teachers. Our previous research showed that students involved in an earlier teaching experiment developed improved number sense, particularly in terms of flexible mental computation (Whitacre, 2007). The previous research was informed by a conjectured local instruction theory and informed the refinement and elaboration of that local instruction theory (Nickerson & Whitacre, 2010). In a recent iteration of the classroom teaching experiment (CTE), the local instruction theory was extended from the whole-number portion of the course to the rational-number portion. We found that interview participants' reasoning about fraction magnitude improved over the course of the semester. Their performance improved, they became more flexible in their reasoning, and they came to use less conventional and more sophisticated strategies. These changes parallel those that we previously saw around mental computation.

Background

This study represents the latest phase in an ongoing design research effort (Gravemeijer, 1999) concerning the number sense of prospective elementary teachers. Our previous research involved the design and elaboration of a local instruction theory for students' development of number sense with a focus on whole-number mental computation (Nickerson & Whitacre, 2010). In the recent CTE, the local instruction theory was applied to reasoning about fraction magnitude. Our research involves both documenting collective classroom activity and analyzing student learning. Our focus in this paper is on the analysis of interview data.

Previous Research

One area of focus in the number sense literature has been the computational strategies that students use. Good number sense is associated with flexibility, which is exhibited in the use of a variety of computational strategies. In mental computation, inflexibility often manifests in the use of the mental analogues of the standard paper-and-pencil algorithms. While the standard algorithms can be useful, skilled mental calculators tend to select a strategy for an operation based on the particular numbers at hand. Furthermore, the strategies that these individuals use often stray far from standard, as in reformulating computations or rounding and compensating (Carraher, Carraher, & Schlieman, 1987; Greeno, 1991; Hierdsfield & Cooper, 2002; 2004;

Hope & Sherrill, 1987; Markovits & Sowder, 1994; Reys, Rybolt, Bestgen, & Wyatt, 1982; Reys, Reys, Nohda, & Emori, 1995; Sowder, 1992; Yang, Reys, & Reys, 2009).

In order to teach mathematics effectively, elementary teachers need to understand elementary mathematics deeply (Ball, 1990). However, prospective and practicing elementary teachers often know the procedures of elementary mathematics, but do not understand the material conceptually (Ball, 1990; Ma, 1999). Studies of preservice elementary teachers have found that this population tends to exhibit poor number sense, even after having completed their required college mathematics courses (Tsao, 2005; Yang, 2007; Yang, Reys, & Reys, 2009). In light of these findings, we have focused on improving the number sense of prospective elementary teachers.

In 2005, we conducted a CTE in a mathematics course for prospective elementary teachers. In that study, we focused on mental computation as a microcosm of number sense. Thirteen students participated in pre/post interviews in which they were given story problems to be solved mentally. In addition to coding for the particular mental computation strategies that participants used, we coded these as belonging to more general categories of strategies. We used a scheme of Markovits and Sowder (1994) to categorize participants' strategies as *Standard*, *Transition*, *Nonstandard without Reformulation*, and *Nonstandard with Reformulation*. The essential criterion in this scheme is the extent to which the person's approach is tied to (or departs from) the standard algorithm for the operation. Nonstandard strategies are those that diverge substantially from the standard algorithms. The use of such strategies suggests an understanding of the operation that is not bound to any particular algorithm; these nonstandard strategies are associated with number sense (Markovits & Sowder, 1994; Yang, Reys, & Reys, 2009).

The Standard-to-Nonstandard framework revealed a rather dramatic shift in the strategies used by the 13 interview participants. Participants shifted from using the most standard to the least standard strategies, which suggests that their understanding of the operations moved from being bound to the standard algorithms to being unconstrained by these (Whitacre, 2007; Whitacre & Nickerson, 2006). These results were encouraging and led us to pursue further research concerning the number sense development of prospective elementary teachers.

Local Instruction Theory for Number Sense Development as Applied to Fraction Magnitude

The previous teaching experiment was reflexively related to the development of a local instruction theory. A *local instruction theory* (LIT) consists of "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer, 2004, p. 107). We have described elsewhere our local instruction theory for the development of number sense (Nickerson & Whitacre, 2010). Here, we briefly list the three major goals around which this LIT is organized: (1) Students capitalize on opportunities to use number-sensible strategies; (2) Students develop a repertoire of number-sensible strategies; (3) Students develop the ability to reason with models.

We sought to extend the LIT to the rational number domain, with a focus on reasoning about fraction magnitude. This area relates to mental computation in that a variety of strategies can be used, including traditional procedures, as well as nonstandard strategies. Behr, Wachsmuth, Post, and Lesh (1984) touted the importance of reasoning about fraction magnitude as a prerequisite to reasoning meaningfully about operations involving fractions. Fraction estimation and comparison tasks have been used in assessments of students' number sense (Hsu, Yang, & Li, 2001; Reys & Yang, 1998; Yang, 2007).

Our thinking concerning reasoning about fraction magnitude is informed by the framework of Smith (1995). Smith groups fraction comparison strategies into four categories, which he calls *perspectives*. These perspectives serve not only to categorize strategies but also to highlight commonalities in reasoning across groups of strategies. Strategies such as converting to a common denominator or converting to a decimal belong to the *Transform* perspective. They involve transforming one or both fractions in some way in order to facilitate the comparison. One can also compare fractions without performing any sort of transformation. One way to do this is to apply the *Parts* perspective, wherein the fractions are interpreted in terms of parts of a whole. This perspective alone is sufficient for relatively simple cases, such as comparing fractions that have the same numerator or same denominator.

The *Reference Point* perspective involves reasoning about fraction size on the basis of proximity to reference numbers, or *benchmarks* (Parker & Leinhardt, 1995). For example, using the *residual strategy* for comparing fractions, one compares the difference of each fraction from a common benchmark number, typically 1: To compare 7/8 and 6/7, we can notice that 7/8 is 1/8 away from 1, whereas 6/7 is 1/7 away from 1. Since 1/8 is less than 1/7, 7/8 is closer to 1, and therefore larger (Yang, 2007). (Smith refers to this as the *reference point* strategy, as do Behr, et al., 1984.) The *Components* perspective involves making comparisons within or between two fractions, as in coordinating multiplicative comparisons of numerators and denominators. For example, in order to compare 13/60 and 3/16, we can notice that 13 x 5 = 65 > 60, whereas 3 x 5 = 15 < 16. It follows that 13/60 is greater since its numerator-denominator ratio is less extreme.

In designing instruction, these perspectives informed our decisions relative to tasks, number choices, and anticipated student reasoning. We mapped out the envisioned learning routes described in our LIT in terms of the evolution of these perspectives and of particular strategies within each category.

Although Smith (1995) does not describe the perspectives or particular strategies belonging to his framework in a hierarchical way, we view the Reference Point and Components perspectives as generally more sophisticated categories of reasoning about fraction size. There is support for this in the literature. For example, Yang (2007) considers the residual strategy to be Number sense-based, as opposed to Rule-based. We posit that there is a general correspondence between Smith’s perspectives and the Standard-to-Nonstandard framework, described earlier. In particular, the Transform and Parts perspectives correspond more or less to the Standard and Transition categories of strategies, while the Reference Point and Components perspectives correspond to Nonstandard strategies (with or without reformulation). We do not intend by this a one-to-one mapping of categories, but a more general grouping into Standard (including Transition) and Nonstandard.

The general number sense literature, as well specific studies of researchers such as Yang (2007) and Newton (2008), suggested that prospective elementary teachers would come to the first course with limited number sense and would tend to apply standard algorithms for comparing fractions. Pilot interviews that we conducted with preservice elementary teachers who had completed their mathematics content courses confirmed this expectation. In our instructional sequence, we aimed for the more sophisticated strategies to eventually be used by students and established for the class by mathematical argumentation. In particular, we sought to engage students in reasoning about fraction size from Smith’s Reference Point and Components perspectives. Tasks were designed and sequenced so as to begin with students’ current ways of reasoning and to provide opportunities for reasoning about fraction size in new ways.

Instruction

Topics in the curriculum include quantitative reasoning, place value, meanings for operations, children’s thinking, standard and alternative algorithms, representations of rational numbers, and operations involving fractions. Our work in the CTE involved identifying in the curriculum particular opportunities to engage students in authentic mental computation and reasoning about fraction size, and to share and justify their strategies. Over time, a shared set of strategies was established via mathematical argumentation. These strategies were given agreed-upon names, and the class maintained a list of strategies with examples of each. Students came to the course with a Parts conception of fractions, and Parts reasoning served as a foundation upon which more sophisticated strategies came to be established. Students also engaged in activities involving placing fraction markers on a string representing a number line, and distance along the number line often featured prominently in students' arguments concerning fraction comparisons.

Methods

This study took place at a large, urban university in the Southwestern United States. The participants in the study were students enrolled in a first mathematics content course of a four-course sequence. There are multiple sections of the course, and a common final exam is used. The second author taught the section of the course in which data was collected.

Seven of the students participated in pre/post interviews concerning their rational number sense. The interview participants were female undergraduates. Participants were asked to evaluate the relative sizes of pairs of fractions, amongst other tasks. Nine pairs of fractions were presented, one at a time. The particular pairs of fractions that participants were asked to compare appear in Figure 1. The same pairs of fractions were used in both interviews. The fractions were presented visually, and participants were asked to read them aloud. Participants solved the tasks mentally and explained their reasoning verbally, as shown in Figure 2. Participants were not allowed to do any written work for this portion of the interviews.

2/8	3/8
3/4	3/5
6/7	7/8
14/13	13/12
8/24	13/39
13/60	3/16
7/28	13/50
2/7	12/43
35/832	37/834

Figure 1. Fraction comparison tasks

Figure 2. Maricela comparing 13/60 with 3/16 in her second interview

We modified the framework of Smith (1995) for use as a coding scheme in order to account for differences in student population, the particular set of tasks that we used, and the foci of our research. Certain strategies were removed from the scheme because they never occurred in our data set, and there were pairs of strategy codes that we collapsed into a single code. The refined scheme was used to code the pre/post data.

In addition to coding for strategies and perspectives, we coded responses as Valid or Invalid. This is a researcher's assessment of the mathematical validity of a participants' strategy. Fraction comparisons are multiple-choice tasks with only three options (one fraction is larger, the other fraction is larger, or the two are equal). As a result, it is not uncommon for students to give correct answers on the basis of invalid or unclear approaches. We were interested in identifying valid strategies that led to correct responses. Thus, we also coded responses for correctness. In analyzing the pre/post data, we focused on those responses that were Valid and Correct (VC). We also coded students' strategies as Standard or Nonstandard, as discussed earlier.

As an assessment of change in performance, we compared the number of VC responses pre and post for each student. To assess change in flexibility, we compared the number of distinct VC strategies used by each student pre and post. Finally, we compared the numbers of Standard and Nonstandard VC responses pre and post.

Results

A comparison of interview participants' pre/post responses reveals improved performance and increase flexibility. Furthermore, participants' reasoning shifted from predominantly employing Standard strategies to a more balanced range of Standard and Nonstandard strategies. We discuss the case of one particular participant in more detail.

For six of the seven interview participants, the numbers of VC responses increased from the first to the second interview. The mean VC score increased from 5.86 to 7.7 (of a total of 9 responses). The additional VC responses were often the result of new strategies used by the participants. The number of distinct VC strategies used also increased from the first to the second interview for six of the seven participants. The mean number of distinct strategies used increased from 4.86 to 7.57. Thus, the participants used VC strategies more often in the second interview, *and* they used a wider variety of VC strategies. These data appear in the tables below.

Student	**VC Pre**	**VC Post**
Angela	4	8
Brandy	4	7
Maricela	7	8
Nancy	8	9
Trina	7	8
Valerie	5	8
Zelda	6	6
Mean	*5.86*	*7.7*

Table 1. Counts of Valid-Correct Responses Pre and Post

Student	**Distinct VC Pre**	**Distinct VC Post**
Angela	4	7
Brandy	4	7
Maricela	3	8
Nancy	6	9
Trina	6	8
Valerie	5	8
Zelda	6	6
Mean	*4.86*	*7.57*

Table 2. Numbers of Distinct Valid-Correct Strategies Pre and Post

Maricela's Transformation from Inflexible to Flexible

Maricela was particularly inflexible in her reasoning about fraction magnitude in the first interview. For seven of the nine comparisons, she attempted to solve by finding a common denominator. In some cases, this approach was manageable and led her to the correct answer. However, she also attempted this approach for comparisons for which it was extremely unwieldy, including to compare 35/832 and 37/834. Such conspicuous inflexibility suggests that her reasoning about fraction magnitude was bound to the standard algorithms for comparing fractions. Even though pencil and paper were unavailable to her, she entertained no alternative other than attempting to mentally compute and retain multiple multi-digit products.

In her second interview, Maricela used a different strategy for each of the nine comparison tasks, and eight of these were VC responses. In her first interview, she had compared 13/60 with 3/16 by converting to a common denominator. In her second interview, she compared these by comparing their distance from 1/4. We coded her primary strategy as *Distance Below* under the Reference Point perspective. She identified a benchmark fraction that would be useful for comparison. This required an initial estimation of the size of the fractions as being roughly close to that benchmark of 1/4. She then identified fractions equivalent to 1/4 but with denominators of 60 and 16, which facilitated her distance comparison by making it easy to find differences. She identified the distances from 1/4 as 2/60 and 1/16. She simplified 2/60 to a unit fraction, which then enabled her to compare the "gaps" of 1/16 and 1/30. She stated that 1/16 was the larger of these. She then correctly concluded that 13/60 was larger than 3/16 because it was closer to 1/4. The reasoning that Maricela displayed here contrasts starkly with her very procedural approach to this and other fraction comparison tasks in her first interview.

Shift from Standard to Nonstandard

Maricela's contrasting pre/post responses suggest a shift in perspectives. Her responses to eight of the nine comparison tasks in the first interview reflected a Transform perspective: She approached the tasks by attempting to apply a procedure for converting one or both fractions to an equivalent form in order to make the comparison. In her second interview, Maricela's responses reflected a range of perspectives. Four of her nine responses were coded as Parts, two as Reference Point, two as Components, and only one as Transform.

This shift in perspectives was also a trend across the interview participants. In the first interview, 30 of the 41 VC responses involved a Standard strategy, such as converting to a common denominator. In the second interview, there were 54 VC responses, and 30 of these involved Nonstandard strategies. (See Figure 3.) We note that the responses in the first interview were weighted largely toward Standard, which reflects a lack of flexibility. In the second interview, Nonstandard responses outnumbered Standard ones, but by a relatively small margin. We view this more balanced set of strategies used as a desirable picture. There is nothing wrong with the Standard strategies in principle. In fact, four of the nine comparison tasks that we used lend themselves well to Standard approaches. Thus, we would expect flexible, skilled individuals to use such approaches approximately as frequently as the interview participants did for the given set of tasks. In fact, exactly 24 of 54, or 4/9, of the VC responses were Standard.

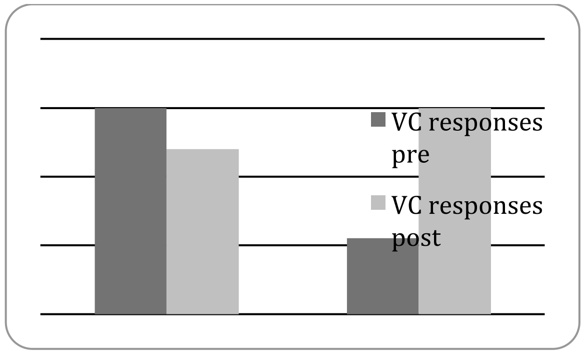

Figure 3. Shift from Standard to Nonstandard strategies

Figure 3 actually provides a concise summary of the interview results: We see the shift from Standard to Nonstandard, the increase in VC responses demonstrates improved performance, and the increase in Nonstandard VC responses coincides with improved flexibility.

Conclusion

We began this study with the idea of extending a local instruction theory for the development of number sense to the domain of rational numbers. Our focus shifted from whole-number mental computation to fraction comparison tasks. Smith's framework informed our thinking about strategies and perspectives involved in comparing fractions. The research literature, together with our pilot interviews, enabled us to anticipate prospective elementary teachers' initial reasoning about fraction magnitude. Our pre-instruction interviews enabled us to refine our expectations. We designed the instructional sequence in terms of students moving from the Transform and Parts perspectives toward the Reference Point and Components perspectives. Specific instructional activities were crafted with this envisioned learning route in mind.

Results from pre/post interviews with seven of the students show that the participants' performance and flexibility in comparing fractions improved, and that they shifted from using predominantly Standard strategies to a balance of both Standard and Nonstandard. These results provide evidence that the prospective elementary teachers developed improved rational number sense. The results also parallel those that we saw in a previous study in which the LIT was applied to mental computation. Taken together, these findings suggest to us that the LIT does serve its purpose in shaping instruction design in such a way that substantial number sense development occurs. The fact that these results have been achieved with prospective elementary teachers is especially significant since the literature tells us that this student population tends to have poor number sense, even after having completed their mathematics content courses.

Having found further evidence that the implementation of our local instruction theory supports number sense development, the current phase in our ongoing design research seeks to better understand *how* that development occurs by analyzing collective classroom activity.

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ON 'MATHEMATICS FOR ELEMENTARY TEACHERS' COURSES

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A Mathematics course for elementary school teachers (MFET) is required in North America in most teacher education programs. Our study investigates the perceptions of prospective elementary school teachers with respect to the contributions of such a course to their teaching. The results show that acquiring an understanding of concepts from the elementary school curriculum is the main contribution. We conclude with two perspectives – a pessimistic one and an optimistic one – on this finding.

In many teacher education programs in North America a Mathematics course (or a sequence of Mathematics courses) for elementary teachers is either a requirement of, or a prerequisite for, entry to a teacher education program. Despite repeated calls for a "thorough rethinking of mathematics courses for prospective teachers of all grade levels" (CBMS, 2001, p.6), and an agreement about the need for integration of mathematics and pedagogy at the elementary level in order to develop profound knowledge of mathematics for teaching (Ball & Bass, 2000), in many programs there is still the traditional separation between the Mathematics-content courses and the Mathematics-methods courses for prospective elementary school teachers. This study investigates prospective teachers' views of the contributions, both actual and potential, of the Mathematics-content course, referred to as MFET – Mathematics for Elementary Teachers – to their teaching.

On Teachers' Knowledge

With the extensive emphasis on teacher education in recent mathematics education research, the primary foci have been on assessing the knowledge that teachers have and exploring what knowledge teachers should have (e.g. Hill, Sleep, Lewis & Ball, 2007; Davis & Simmt, 2006). Acknowledging that teachers' knowledge is multi-faceted, different attempt were made to categorize the components of such knowledge. Shulman's (1986) classical categories refer to subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curricular knowledge. An extended categorization of teachers' knowledge was introduced by Deborah Ball and colleagues (Hill, Ball and Schilling, 2008). It was referred to as "mathematical knowledge for teaching" (MKT) and presented as an extension of Shulman's categorizations. The PCK refinement included Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Curriculum. The SMK refinement contained the categories of Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Knowledge at the Mathematical Horizon. As knowledge acquired in a Mathematics-content course is of interest in this study, we note that CCK was described as shared among individuals who use mathematics, while SCK was considered as the domain of teachers that allows them "to engage in particular teaching tasks" (ibid, p. 377).
In contrast to this categorization, Davis and Simmt (2006) argued against the traditional separation of content and pedagogy, and claimed that "mathematics-for-teaching" can be considered as a distinct branch of the discipline of mathematics. Extending the research of Davis and Simmt (2006) on what teachers need to know, Askew (2008) examined the research evidence for the mathematics discipline knowledge that primary teachers might need in order to

teach effectively. Askew suggested that attention of mathematics educators should be shifted from *what* mathematics primary teachers should know to *why* they should know this mathematics. He also supported the view that distinction between content knowledge and pedagogical content knowledge may no longer be helpful. However, despite the repeated claims against separation of content and pedagogy in teacher education and research, this long-established separation still exists within the coursework towards teachers' certification.

Our prior research investigated the perceptions of secondary school mathematics teachers on the use of their knowledge of 'advanced' mathematics – knowledge they acquired during their studies at colleges and universities – in their teaching practice (Zazkis & Leikin, 2010). The results varied significantly: some teachers claimed that they have never used what they learned in their university courses, while others claimed that they used it "all the time", but had difficulty in providing specific examples of this usage. The current focus on elementary school teachers, the mathematics they study in their University degrees and its perceived usefulness for teaching is a natural follow up.

Mathematics for Elementary Teachers (MFET) Course

While certification at the secondary level requires teachers to acquire a significant background in the subject matter, usually a degree or at least a minor, for elementary school teachers, as generalists, the mathematical subject matter requirements are limited. If such a requirement exists, it is usually for one or several mathematics-content courses designed specifically for this population.

A typical MFET course – and we infer what is 'typical' from a variety of textbooks with a similar tables of contents (e.g., Bassarear, 2007; Billstein, Libeskind & Lott, 2009; Musser, Burger & Paterson, 2006; Sowder, Sowder & Nickerson, 2010) and a variety of course outlines or course syllabi posted on the web – provides an overview of the underlying concepts of elementary mathematics. Typical topics include number systems and algorithms, patterns and introductory number theory, measurement and geometry, probability and data analysis. Different authors and publishers, in an attempt to satisfy the market, chose different perspectives on concepts and topics, such as problem solving, mathematical reasoning, the use of manipulatives, connection to the Standards, or the use of technology. The degree to which a certain perspective is implemented depends on the instructor's choice; however, the core topics remain the same.

The Study

Participants in this study were prospective elementary school teachers (PTs) enrolled in a one-year teacher certification program. All the PTs had some teaching experience in elementary school, having completed a 'practicum' of either 6 weeks or 5 months. All of the PTs had taken a MFET course, similar to a 'typical' course described above, as it is a required prerequisite in their teacher education program. However, they completed this course at various times, with various instructors and at various colleges or universities. As such, our study concerns the course, rather than its specific implementation.

Our study attempted to address the following question: How do prospective teachers describe the contribution of their MFET course to their teaching? Or, stated differently: What have prospective teachers learnt in their MFET course that they perceive as useful for their teaching?

Data Collection and Analysis

The data collection included a written response task and a clinical interview.

A written response task was administered to a group of 25 PTs. Initially, they were asked to provide examples of several teaching situations in which their mathematical knowledge from the MFET course could have been useful. The teaching situation could have been actual or imaginary. However, several students claimed that they "just knew things" and had difficulty identifying the source of their knowledge. Acknowledging this difficulty the task was modified. The task sought examples of usage, either actual or potential, of mathematical knowledge beyond the topic that was being taught.

The interviews were conducted with 14 PTs from a different group. The interviews initially attempted to solicit explicit examples of mathematical knowledge usage in teaching elementary school mathematics. However, the flexible structure of the interviews turned in part into a conversation about the MFET course and its contributions to teaching.

The data analysis was ongoing, using a qualitative approach based on grounded theory procedures and techniques (Strauss & Corbin, 1990). In the written responses that explicitly addressed mathematical knowledge from the MFET courses we identified the mathematical topic in each example and the setting in which the example was presented. We then identified several recurring themes in the provided responses. Then, in the analysis of the interviews we identified additional emerging themes. The themes that were identified in participants' responses serve as an organizing structure for the subsequent results and analysis.

Results and Analysis

To foreshadow our main observation, we start with an illustrative comment from Linda:

> *Linda*: I likely learned about […] in grade school. It was not until MFET course that I learned <u>the reasoning behind why this works</u> and fully <u>understood</u> […]. This was very helpful when teaching […] during my practicum.

Please note that we intentionally deleted the mathematical content that Linda mentioned, as similar comments were provided with respect to different topics and procedures. We return to the ideas of 'understanding' and "reasoning behind why this works" in our subsequent analysis. In what follows we attend to the recurring themes in the participants' responses as well as to their particular examples of knowledge usage.

Initial Hesitation

In the clinical interviews, among the initial 'warm-up' questions, the participants were asked about their MFET course. The questions were of general nature, such as, where did you take the course, how long ago was it, what did you think of it? A typical response is presented below.

> *Rita*: Actually when I enrolled in it I was so <u>freaked out</u> the first couple of days, because I'm so <u>terrified</u> of math that I was going to transfer out, but I really wanted my elementary school pre-req's. And so I was in the middle of transferring out and then I just really <u>started enjoying</u> it and I stuck with it and I ended up getting an A, so I was like, this wasn't so bad.

The words anxious, nervous, terrified, freaked out, intimidated, panicky and alike were common in participants' descriptions of their entry point to the course. However, many students reported a degree of confidence and satisfaction towards completing the course, as well as some joy and excitement. While a level of confidence with the mathematical content taught is essential for teaching, we were further interested in specific examples of what PTs believe they learned that were helpful for teaching.

Examples of Usage

In this section we summarize examples of usage from the written response task. The particular examples that were provided in the interviews appear further on, as illustrations of the recurring themes.

Out of 25 PTs who completed the written response task, 17 explicitly referred to the knowledge acquired in their MFET course, generating 42 examples of knowledge usage. Table 1 presents a distribution of the 42 examples by topics. Table 2 presents a distribution of the 42 examples by the intended usage.

Mathematical content	Number of examples
Computation (shortcut tricks, estimation, order of operations)	9
Elementary Number Theory (divisibility, prime numbers)	8
Fractions and Decimals	8
Geometry (area, perimeter, angles, π)	8
Algebra	5
Other (e.g., different bases, division by zero)	3
TOTAL	42

Table 1: Distribution of examples by topic

Method of usage	Number of examples
Evaluating correctness of a student's response	24
Helping a student, responding to a question	8
Creating examples/tasks/activities for students	5
Classroom management/grouping of students	5
TOTAL	42

Table 2: Distribution of examples by the intended usage

Computation

The most frequent category is that of computation, however the examples that we clustered there are very different. They include computational shortcuts and tricks, such as how to multiply a 2-digit number by 11, attending to compatible numbers by performing computation mentally while students learn algorithms, evaluating the potential correctness of a student's answer by estimation, and identifying a source of a student's error, such as disregard of the order of operations. In most cases the knowledge from MFET course was used in order to evaluate a response from a student by a method different from the one used by the student. Awareness and appreciation of different approaches is elaborated upon further in the analysis of the interviews.

Elementary Number Theory

As "Elementary Number Theory" we clustered examples that referred to divisibility rules, prime numbers and prime decomposition. Familiarity with divisibility rules served teachers in recognizing an error in a student's answer, such as when the difference of two odd numbers was not even, or when the sum of two numbers divisible by 9 was not divisible by 9. It further served when designing examples for student work, for example, when students are learning the long division algorithm, divisibility rules help the teacher to present students with exercises where division is without remainder, without carrying out the division. Moreover, some PTs reported that divisibility rules helped them when planning an activity that involved student work in small groups, that is, deciding on whether equal size groups were possible with a given student attendance.

Fractions and Decimals

Included in this category were examples related to introducing students to different models of representing fractions and equivalent fractions, to converting improper fractions to mixed numbers, comparing fractions and performing operations. These examples referred mostly to designing pedagogy for student understanding and, again, evaluating student answers, achieved by applying a standard procedure, by different means, such as comparing fractions by attending to units and not to a common denominator.

Geometry

Examples related to geometry mentioned the sum of the angles in a triangle and in a quadrilateral, the concepts of area and perimeter, and the meaning of π. Knowledge from the MFET course was helpful in assessing students' work, for example, when students measured all the angles of a quadrilateral and calculated their sum to 274 degrees, the teacher immediately recognized an error either in measurement or in addition. Examples in this category also referred to designing activities to introduce concepts. One such example replicated with elementary school students an activity previously experienced in the MFET course: It involved measuring the diameters and circumferences of different circular objects in order to introduce π as a ratio and not "as a strange appearance in some formulas".

Algebra

In this category PTs' examples mentioned their knowledge of solving equations in order to obtain the answer "quickly" and evaluating the work of students performed without algebraic means.

The repeating themes identified in the written responses of PTs were those of understanding and awareness of different ways to perform a mathematical task. We elaborate on these themes in further detail, and add other themes, as we turn now to the analysis of clinical interviews.

Understanding and Explaining

The most robust theme, which was mentioned in the written responses and which appeared in all of the interviews, was that of understanding. Only two participants mentioned that their MFET course helped with "revisiting" or "refreshing basic skills". The majority maintained that in their MFET course they understood mathematical ideas, in some cases for the first time, and this personal understanding was ultimately related to the ability to explain, that is, to teach. While Tanya (below) makes an explicit connection between understanding and teaching, Betty elaborates further, starting with a severe criticism of her elementary school experience and her desire to find out why certain rules exist.

> *Tanya*: I always struggled in math for myself and taking this course helped me <u>understand math better</u>. You definitely <u>need to understand</u> that to be able to teach that.

> *Betty*: She explained all of the things that we just were taught in elementary school as this is the way it is, like this is the rule. She'd explain why. I remember when I was a kid, I was like why? Why is this the rule? And then they'd be like, because it is. And I'm like well, OK, that doesn't help. So she explained why those rules were and I was just like, finally, I can understand it because it's true, if you understand why the rules exist your application of them will be more accurate but also you can figure out other rules. <u>You need to really understand why</u> the rules exist <u>in order for you to teach</u> this.

Once the connection between understanding and teaching was mentioned, the interviewer invited specific examples of concepts or ideas that were "really understood" or "understood better" as a result of the MFET course experience. The mathematical topics mentioned in the interviews echoed those from the written response task. These included measurement, algorithms, fractions, focusing on division of fractions, and multiplication tables. For limitations of space, we illustrate only the last topic in the following excerpt:

> *Lisa*: Multiplication tables – I thought that has to be a skill that's just memorization and drills and, actually, you can do a lot with understanding around that, like patterns and stuff.

Multiple Ways, Questions and Connections

Closely related to the theme of understanding is the theme of acquiring different ways to approach a mathematical task. Lisa makes this connection explicit, while Betty connects availability of different methods to understanding her students' thinking:

> *Lisa*: I have a deeper understanding for it and then I can see lots of different ways of doing it.
>
> *Betty*: I think that I've learned, and this is sort of intimidates me, is that I have to open my mind to understand how other people think […] to not underestimate the students and to try really hard to understand how they came up with, to say show me…

The connection here is obvious: personal awareness of multiple strategies is helpful in trying to understand students' strategies and approaches that may be different from the conventional ones (Leikin & Levav-Waynberg, 2009). Moreover, the ability to acknowledge different approaches in students' work was also connected to the ability to deal with students' questions.

> *Tanya*: I also think it helps when they have questions, you'll have other knowledge to draw from for questions they may have.
>
> *Betty*: I don't want to be the teacher who says that's the way it is. […] And I remember she told us why that works, I remember she did bring that up and I was like, oh, well that totally makes sense […] If they ask I'd like to be able to provide an answer, at least guide them in the direction of finding the answer, rather than just saying this is the way.

Note that Betty makes explicit reference to her Mathematics instructor, "she told us why that works", whereas she herself balances the ability "to provide an answer" with her pedagogical belief in the need to "guide them in the direction of finding the answer".

"Math today is different"

The idea that "Math today is different", that is, different from the mathematics they experienced as learners, was the second most prominent and unsolicited theme in the interviews, mentioned by nine participants. Several PTs mentioned that the course opened their minds, extended their horizons and influenced their views on what mathematics is about. This difference is contrasted with a description of prior experience, which is common among the participants.

> *Cara*: When I was going through math, it was just the numbers, not the problems.
>
> *Anne*: Math is completely different today. I asked my teacher, I don't understand this and he's like well can you do it, that's how I remember most of my math being, all I learned was equations and if you could do the equation, if you could use the equation then you didn't need to know anything else.

At first we considered the theme of mathematics being different, and the presented prior experiences of frustration, as not explicitly related to our research questions. However, on a second look, acknowledging and appreciating this difference – between mathematics of yesterday and that of today or tomorrow – can be considered as the main impact of a teacher education program in general and a MFET courses in particular. Participants implicitly or explicitly contrasted their experience of "doing" and being shown how to do, with the desired explanation and understanding, which was considered essential for teaching.

Discussion

Acquiring understanding is a declared purpose of the MFET courses. For example, Musser, Burger and Peterson (2006), authors of one of the popular textbooks for such a course, state explicitly in their introduction:

> "This book encourages prospective teachers to gain an understanding of the underlying concepts of elementary mathematics while maintaining an appropriate level of mathematical precision" (p. xi).

In the "Message to prospective and practicing teachers" on the first pages of their book, Sowder, Sowder and Nickerson (2010) mention different perspectives and contributions to teaching:

> "Some mathematics may be familiar to you, but you will explore it from new perspectives. […] Though the course is about mathematics rather than about methods of teaching mathematics, you will learn a great deal that will be helpful to you when you start teaching" (p. xiv).

As such, our findings suggest that the MFET course, or at least the offerings of the course that our participants were enrolled in at different times and at different places, achieved the set goal, at least from the perspective of participants in this study. However, this personal perspective of participants needs to be investigated further. It cannot be concluded from the participants' testimonies that they have indeed acquired a desirable level of what researchers referred to as PUFM - profound understanding of fundamental mathematics (Ma, 1999) or KDU – key developmental understanding (Simon, 2006), that is deemed as a prerequisite for MKT – mathematical knowledge for teaching (Silverman & Thompson, 2008). We further also note that the MFET course is only one step in the mathematics education of prospective teachers, and it is likely that ideas that developed in this course are reinforced and reexamined in courses that attend to "methods" of instruction, that is, to pedagogy and curriculum.

Taking a pessimistic view on our results, we note that the majority of participants entering a teacher education program for certification at the elementary level acknowledged that they did not sufficiently understand the concepts and procedures of elementary school mathematics. While this is consistent with prior research, our specific contribution, however, is in basing this finding on participants' testimonials related to their understanding (or lack of it) within particular concepts and topics of elementary school mathematics, rather than on researchers' observations.

Taking an optimistic perspective, we note that following a MFET course PTs reported that they "really understood it", whereas "it" referred to various concepts, such as place value or fractions, or algorithms, such as column addition or division by a fraction. However, what does it mean to "really understand" something? Betty summarized this as "knowing the reasons behind all the things that you teach the kids". Moreover, according to our participants, several related and further elaborated answers can be offered: For some PTs this means to know *why* and not only to know *how*, for others it means being able to provide an explanation to a student, and, furthermore, to be equipped with several different explanations. These views are in accord with the shift from *what* mathematics teachers should know to *why* they should know this mathematics, suggested by Askew (2008). The teachers' (partial) answer to "why" is the implementation of knowledge in teaching.

We mentioned above that self-report of acquired understanding does not necessarily mean that an appropriate level of understanding was achieved. Nevertheless, the personal acknowledgement of the importance of understanding and the ability to explain mathematics to students, rather than provide rules, is a valuable contributor to a teacher's comfort zone (Borba & Zulatto, 2010). Furthermore, the awareness that "math today is different", that is, that the desirable way of learning mathematics is different from the personal experience of participants, is an important step towards "teaching differently" or "teaching for understanding". We also suggest that the repeated reference to "different mathematics" may also signify a change in personal beliefs of what mathematics is about. That is to say, not that mathematics has changed, but rather the PTs' view of mathematics.

What can this imply for the teaching of mathematics? Anne's opinion is embedded in her question:

> *Anne*: And there were all those things that you learn that you're like, why didn't we just learn it like that from the beginning, because it would have helped me so much more.

In our optimistic perspective we would like to conclude with the hope that Anne's students will "learn it like that from the beginning".

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PROSPECTIVE ELEMENTARY TEACHERS' SUSPENSION OF SENSE-MAKING WHEN SOLVING PROBLEMATIC WORD PROBLEMS

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This study investigates the extent to which pre-service elementary teachers (PETs) use their real-world knowledge to solve problems for which the result of the arithmetic operation is problematic, if one takes into consideration the reality of the context. A paper-and-pencil test was administered to 566 PETs enrolled in mathematics content courses. The test included 8 experimental items and 4 buffer items. The findings for a sample of 68 PETs are not very encouraging. The total number of realistic responses varied from 5 to 58 (out of 68 possible for each problem).

Word arithmetic word problems play an important role in learning mathematics at the elementary school level. There are several practical and theoretical reasons of the inclusion of arithmetic word problems in the elementary curriculum. First, they provide contexts in which students can use their mathematical knowledge so they can develop their problem-solving abilities, an important goal of learning mathematics. Second, word problems provide practice so students can develop their abilities to use their knowledge in real-life situations. Third, word problems serve as motivators so students can see the relevance of the procedures and algorithms learned in school. Fourth, word problems have the potential to provide students with rich contexts and realistic activities in which to ground mathematical concepts and, thus, facilitate the learning of more complex concepts. Finally, word problems provide students with experiences to learn how to use mathematical tools to model aspects of reality, that is, to describe, analyze, and predict the behavior of systems in the real world (Burkhardt, 1994; De Corte, Greer, & Verschaffel, 1996; Verschaffel, Greer, & De Corte, 2000; Verschaffel & De Corte, 1997).

Some critiques (e.g., Gerofsky, 1996; Lave, 1992; Nesher, 1980) argue, however, that the mathematics curriculum fails to achieve these lofty goals because traditional instructional tasks tend to focus on a straightforward application of procedures and computations to solve artificial problems unrelated to the real world. As a result, students tend to approach word problems, more often than desirable, in a superficial and mindless way with little, if any, disposition, to modeling and realistic interpretation. Several pieces of research provide empirical evidence to these claims

(Davis, 1989; De Corte & Verschaffel, 1989; Greer, 1993, 1997; Reusser, 1988; Reusser & Stebler, 1997; Schoenfeld, 1991; Silver, Shapiro, & Deutsch, 1993; Verschaffel, 1999; Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Lasure, 1994).

Purpose of the Study

The purpose of the study was to examine prospective elementary teachers' (PETs) reactions and responses to problematic arithmetic word problems for which the solution is not the result of application of the most obvious arithmetic operation suggested by the two numbers given in the problem statement.

As suggested by the research literature, elementary school children tend to ignore the realistic constrains of the context embedded in the statement of the problem, a phenomenon that Schoenfeld (1991) coined "suspension of sense-making." Several critics and researchers argue that children' suspension of sense-making is the result of school practices (Davis, 1989; Greer, 1993; Nesher, 1980; Schoenfeld, 1991; Silver, Shapiro, & Deutsch, 1993). To develop children' disposition to realistic modeling, we must change curriculum and instructional tasks. Since the teacher has an important role in the construction or selection of learning tasks and opportunities, one may argue that researchers and curriculum developers need to understand teachers' reactions and responses to problematic problems.

Theoretical and Empirical Background

Mathematical modeling is the process of representing aspects of reality by mathematical means. In particular, the solution of some physical or real-world problems requires some form of mathematization. That is, the construction of a mathematical model. The complexity of the process of mathematization depends, of course, on the nature of the problem. There are several proposed models of representing reality by mathematical means (e.g., Silver, Shapiro, & Deutsch, 1993; Verschaffel, Greer, & De Corte, 2000), but Silver et al's model (Fig. 1) suffices for our purposes.

According to Silver, Shapiro, and Deutsch's model, a simplified version of the process of mathematical modeling consists of four different stages: understanding of the problem, construction of a model or selection of a mathematical procedure, the execution of the procedure, and the interpretation or evaluation of the outcomes of the procedure.

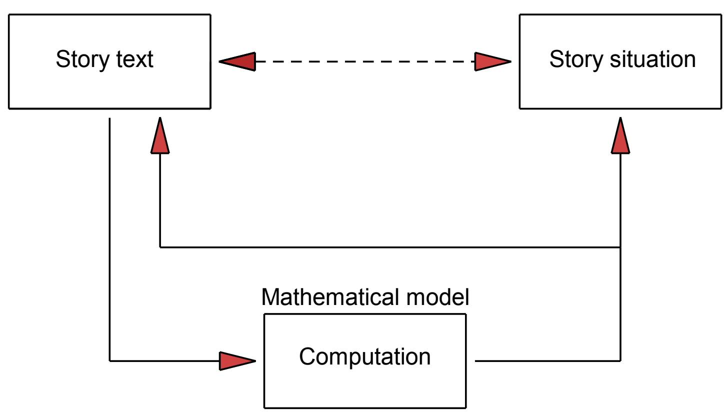

Fig. 1: Silver et al.'s (1993) referential-and-semantic-processing model for successful solutions

The first stage of the process of mathematical modeling involves understanding the problem

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

situation embedded in the story text. That is, we need to understand the given or known facts, the unknown information, the superfluous data, and missing information. The second phase involves the construction of a mathematical model or selection of a suitable procedure, operation, or algorithm whose outcome will lead us to the solution of the problem. To perform the second stage of the modeling process successfully, we must understand the mathematical structure of the problem. That is, we must understand the interconnections or relationships among the different types of information related to the solution of the word problem. The third stage of the problem involves mainly performing the computation, procedure, or algorithm either with paper and pencil or using a computational device. Finally, we should interpret and assess the outcome of the mathematical procedure in terms of the realistic context embedded in the story text of the word problem or in terms of the real-world story situation. It is during this step that we need to focus on the meaning of the result of the mathematical model so we can establish the connection between the outcome of the computation and the solution to the real-world story problem. It is during this stage that we need to assess whether our modeling assumptions are realistic or reasonable.

Silver, Shapiro, and Deutsch's model implies that there are three main potential sources of error when solving word problems: lack of understanding of the mathematical structure of the problem, which leads students to select an inappropriate procedure, executing the procedure incorrectly, and failing to interpret or assess the result of the procedure in terms of commonsense or everyday-life knowledge. Silver, Shapiro, and Deutsch (1993) examined 195 middle grade students' solution processes and their interpretation of solutions to the following problem:

The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game?

Their analysis revealed that 91% of the students selected an appropriate procedure (e.g., long division, repeated multiples, repeated additions, etc.), but only 61% of these students performed it flawlessly (about 56% of the total number of students). Overall, the researchers found that only 43% of the total number of students gave the correct answer (14) to the problem. However, some of these students provided inappropriate interpretations or justifications. For example, one student wrote "14 buses because there's leftover people and if you add a zero you will get 130 buses so you sort of had to estimate. Are we allowed to add zeros?" (p. 124-125). The researchers also reported that about 55% of the students did not get the correct answer because either they did not properly interpret the outcome of the computation or executed incorrectly the procedure. These computational mistakes could have been prevented if students had interpreted their solutions appropriately. Silver, Shapiro, and Deutsch proposed the model displayed in Figure 2 as a graphical representation of unsuccessful solutions that are due to a failure to connect the outcome of the procedure to the real-world context embedded in the story problem.

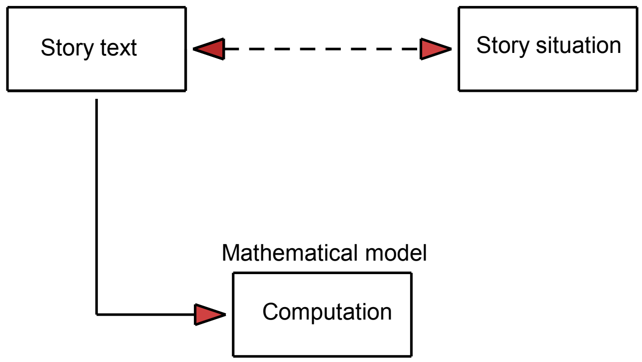

Figure 2: Silver et al.'s (1993) referential-and-semantic-processing model for unsuccessful solutions

Other pieces of research have amply documented elementary school children' improper modeling assumptions when solving arithmetic word problems. Some further examples of the word problems that students have been asked to solve are the following:

1) What will be the temperature of water in a container if you pour 1 liter of water at 80° and 1 liter of water of 40° into it? (Nesher, 1980)
2) John's best time to run 100 m is 17 sec. How long will it take to run 1 km? (Greer, 1993)
3) Lida is making muffins that require 3/8 of a cup of flour each. If she has 10 cups of flour, how many muffins can Lida make? (Contreras & Martínez, 2001)
4) In September 1995 the city's youth orchestra had its first concert. In what year will the orchestra have its fifth concert if it holds one concert every year? (Verschaffel, De Corte, & Vierstraete, 1999)

In their study with 75 fifth graders in Flanders, Verschaffel, De Corte, and Lasure (1994) reported that only 7 (9%) students provided a realistic and correct response to the temperature problem. Similarly, in the same study, these researchers found that only 2 (3%) responses included realistic answers or reactions to the running problem. In another study, Contreras and Martínez (2001) focused on prospective elementary teachers' solution processes and realistic reactions to the third problem. Their analysis revealed that only 19 (28%) of the participants' responses contained a realistic solution to the problem, but none of the participants made any comments about the problematic nature of the problem.

Verschaffel, De Corte, and Vierstraete (1999) addressed upper elementary school children' difficulties in modeling and solving nonstandard additive word problems involving ordinal numbers. The participants were administered a paper-and-pencil test consisting of 17 word problems, 9 of which were experimental items and 8 buffer items. The result of the straightforward arithmetic operation yields the correct answer for three of the nine experimental items. An example of such a problem is "In January 1995 a youth orchestra was set up in our city. In what year will the orchestra have its fifth anniversary? However, the solution of the remaining six experimental items is either 1 more or 1 less that the result of the straightforward arithmetic operation of the two given numbers. An example of such a problem is problem 4 stated above. Overall, the researchers found that the percentage of correct responses for each of the six problematic items was less that 25%. An error analysis revealed that 83% of the errors made on these problems were ± errors. In other words, most of the children' errors were due to their interpretation that the result of the addition or subtraction of the two given numbers yielded the correct answer.

Although research has convincingly documented elementary school children' strong tendency to model problematic problem unrealistically, the generalizability of the findings to more mature students, such as prospective elementary teachers, has not been established empirically. On one hand, since PETs have had even more experiences with traditional school problems, we may argue that there is no reason to expect that prospective elementary teachers would use their real-world knowledge and realistic considerations in their solution processes of problematic word problems. On the other hand, we may claim that PETs may have faced real-world problem situations outside school more often than young children and, having a more developed mathematical knowledge, have a stronger disposition to activate their real-world knowledge when confronted with problematic problems whose realistic solutions require taking into consideration contextual real-world knowledge. In the present study we focus on the extent to which the findings from previous research with pupils are generalizable to prospective elementary teachers.

Methods and Sources of Evidence

The total sample of participants consists of 566 PETs enrolled in different sections of mathematics content courses for elementary teachers at a Southern University in the United States. The present paper reports the results of two groups (68 PETs) for which the analysis has been completed. The PETs had not been previously engaged in any intentional or systematic modeling activities or tasks.

A paper-and-pencil test consisting of 8 experimental items and 4 buffer items was administered to the PETs during regular class instruction. The 8 experimental items (Table 1) were problematic in the sense that the outcomes of the arithmetic operations performed with the given numbers in the problem story does not provide the answer to the problem, if one takes into consideration the real-world situation embedded in the contextual problem story. The buffer items, on the other hand, were standard routine problems whose solution is the straightforward result of the operation performed with the given numbers. The experimental items were adapted from Verschaffel and De Corte's (1997) study. An example of a buffer item is "Joel is building a collection of 175 different stamps. He has already collected 107 different stamps. How many more stamps does he need to complete the collection?"

Table 1: The eight experimental items

1. 1175 supporters must be bused to the soccer stadium. Each bus can hold 40 supporters. How many buses are needed? (Carpenter, Lindquist, Matthews & Silver, 1983).
2. 228 tourists want to enjoy a panoramic view from the top of a high building that can be accessed by elevator only. The building has only one elevator with a maximum capacity of 16 persons. How many times must the elevator ascend to get all the tourists on the top of the building? Verschaffel, 1995)
3. At the end of the school year, 50 elementary school children try to obtain their athletics diploma. To receive the athletic diploma they have to succeed in two tests: running 400 m in less than 2 minutes and jumping 0.5 m high. All the children participated in both tests. Nine children failed the running test and 12 failed the jumping test. How many children did not receive their diplomas? (Verschaffel, 1995)
4. Carl and George are classmates. Carl has 9 friends that he wants to invite to his birthday party. On the other side, George has 12 friends that he wants to invite to his birthday party. Since Carl and George have the same birthday, they decide to give a party together. They invite all of their friends. All their friends come to the party. How many friends are there at the party? (Nelissen, 1987)
5. A man wants to have a rope long enough to stretch between two poles 12 m apart, but he has only pieces of rope 1.5 m long. How many of these pieces would he need to tie together to stretch between the poles? (Greer, 1993)
6. Steve has bought 12 planks of 2.5m each. How many 1 m planks can he saw out of these planks? (Kaalen, 1992)
7. Sven's best time to swim the 50 m breaststroke is 54 seconds. How long will it take him to swim the 200 m breaststroke? (Greer, 1993)
8. The flask is being filled from a tap at a constant rate. If the water is 4 cm deep after 10 seconds, how deep will it be after 30 seconds? (This problem was accompanied by a picture of a cone-shaped flask) (Greer, 1993)

After each problem, we have indicated its original source; however, in some cases the numbers were replaced by others.

Students' written responses to the problems were the source of data. Written directions asked students to show all their work to support each of their answers and to write down any questions or concerns they may have about each problem**.** We recognize that written responses have some intrinsic limitations when compared to verbal protocols. However, written protocols allow researchers to collect data from large samples. Moreover, some researchers (Hall, Kibler, Wenger, & Truxaw, 1989) have validated the use of written responses to infer cognitive processes.

Analysis and Results

Each response to problems 1 and 2 was coded as correct or incorrect. Each response to problems 3-8 was coded as correct, partially correct, or incorrect. Two independent raters judged every response. A response was judged as correct if it included a realistic numerical answer that estimated or indicated the range of possible solutions and took into account the contextual restraints of the real-world problem situation. A response was judged partially correct if it was incomplete or wrong but included a realistic comment suggesting that the student displayed awareness of the contextual restraints of the real-world problem situation. A response was judged incorrect when it did not suggest any awareness of the contextual restraints of the real-world problem situation. The inter-rater agreement was 99.6%. Table 2 summarizes the results of the analysis.

Table 2: The number and percentage of correct, partially correct, and correct responses for the 8 experimental items.

Problem	Number and percent of correct responses	Number and percent of partially correct responses	Number and percent of incorrect responses
1	55 (81%)	0 (0%)	13 (19%)
2	58 (85.5%)	0 (0%)	10 (14.5%)
3	3 (4.5%)	13 (19%)	52 (76.5%)
4	3 (4.5%	15 (22%)	50 (73.5%)
5	2 (3%)	4 (6%)	62 (91%)
6	14 (20.5%)	1 (1.5%)	53 (78%)
7	1 (1.5%)	4 (6%)	63 (92.5%)
8	0 (0%)	5 (7.5%)	63 (92.5%)

As shown in Table 2, PETs' performance on most items was alarmingly poor: The percentage of incorrect responses ranged from a high 92.5% for items 7 and 8 to 14.5% for item 2. Overall, the percentage of realistic responses (correct responses and partially correct responses) on the 8 problematic items was only 33%. We should notice, however, that the number of realistic responses was considerable greater for the division problems involving remainders, problems 1 and 2). If we exclude these two problems from our analysis, then the percentage of realistic responses for the remaining 6 problems is only 16%.

Discussion and Conclusion

The purpose of the present study was to collect systematically empirical data about the extent to which prospective elementary teachers activate their real-world knowledge when solving problems whose solution in not the direct result of an arithmetic operation. Using similar problems and methodology as previous studies (e.g., Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Lasure, 1994), a test consisting of 8 problematic items and 4 standard problems was administered to a sample of 566 PETs. The analysis has been completed for 68 PETs (2 groups) and it is reported in the present article.

Although previous studies have convincingly demonstrated children' strong tendency to ignore the contextual realities embedded in the story of the problem situation, we were hoping that our findings with prospective elementary teachers would be much more encouraging. After all, prospective elementary teachers are part of a more mature and experienced population and it is reasonable to assume that they have an understanding of the contextual knowledge needed to realistically solve the problems. Therefore, the question of PETs' failure to activate this knowledge needs to be further discussed and investigated. We offer several tentative hypotheses to explain PETs' lack of disposition to model contextual word problems realistically.

First, children and PETs' lack of activation of their real-world knowledge may be due to their constant exposure to traditional and stereotypical school word problems. If this is the case, then this tendency may remain constant or get stronger with additional years of immersion in the mathematical culture of traditional classrooms. Future research is needed to better understand the effects of traditional learning environments on students', including PETs, failure to activate their real-world knowledge to solve problematic problems.

A second possible explanation to understand PETs' tendency to ignore the contextual realities of the situation embedded in the problem story is that they lack enough world-real knowledge of the situational context of the problematic problems. Even though this seems unlikely, follow-up studies should provide empirical data to confirm or refute this hypothesis.

A third explanation may be that PETs approached the problematic problems in an automatic way thinking that they were standard mathematical problems without reflecting on the contextual realities of the problem. Further research is needed to better understand PETs' suspension of sense-making when solving these types of problems.

In conclusion, this study provides, at the very least, some empirical evidence that PETs lack an initial disposition or reaction to consider the contextual restraints of problems grounded in the real world. Further research is needed to better understand PETs' apparent suspension of sense-making when engaged in solving problems that require realistic interpretations.

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THE EFFECTS OF TEACHERS' UNDERSTANDING OF ADDITION AND SUBTRACTION WORD PROBLEMS ON STUDENT UNDERSTANDING

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This study investigates the influence of teacher understanding on student understanding through teacher practice. Three elementary school teachers participated in a university course that discussed mathematical and pedagogical knowledge regarding addition and subtraction word problems. The data from this study were analyzed qualitatively to describe the nature of the teachers' understandings, the ways teachers used their understandings in their practice, and the nature of their students' understandings. I hypothesize that teachers used their understanding to create and implement tasks at a high level of cognitive demand, maintaining that demand over time, which affected student understanding.

There is quantitative evidence that content knowledge and pedagogical content knowledge influence student understanding (Hill, Rowan, & Ball, 2005; Fennema & Franke, 1992), but how teacher knowledge affects student achievement needs to be explored. In this study I begin to uncover how teacher knowledge affects student understanding using a qualitative perspective. If we assume that teacher understanding affects practice and that teacher practice affects student understanding (Hurry, Nunes, Bryant, & Pretzlik, 2005; Smith & Baker, 2001), then we can begin investigating these relationships.

Objectives

In three cases, I describe the nature of a teacher's understanding of a particular mathematical topic and investigate how that understanding affects the teacher's practice while teaching that topic. In addition, I examine the link between teacher understanding and student understanding in order to hypothesize about the relationships between them.

My study is exploratory in nature and seeks out possible connections that will need to be further developed. Since teacher knowledge is so expansive and varied, I limited this study to investigating one mathematical topic in light of only a few aspects of teacher knowledge. In this study I examine the understanding of three teachers regarding the content knowledge and pedagogical content knowledge shown to be integral to teaching addition and subtraction word problems, according to the Cognitively Guided Instruction (CGI) program (Carpenter, Fennema, Franke, Levi, & Empson, 1999). I investigate how a teacher's understandings of the structure of addition and subtraction word problems and of how students typically solve addition and subtraction word problems affect her practice, and how her practices affect student understanding of addition and subtraction.

Framework

My research uses Deborah Ball's construct of the mathematical knowledge needed for teaching (Ball, Phelps, & Thames, 2008). According to Ball, Specialized Content Knowledge (SCK) is an important aspect of subject matter knowledge and Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) are vital to pedagogical content knowledge. I have chosen to examine teacher and student understanding of addition and subtraction word problems due to the extensive research that has identified the content knowledge and pedagogical content knowledge for teachers in this area. I examine qualitatively how a teacher's understandings of the structure of addition and subtraction word problems, a substantial part of their SCK, affects students' understandings of addition and subtraction through its effects on teacher practice. I also examine how a teacher's understanding of the strategies students use to solve addition and subtraction word problems, part of their KCS as well as KCT, affects students' understandings of addition and subtraction through teacher practice.

I relied on research by Wiggins & McTighe and Sierpinska to establish when participants were exhibiting understanding. Understanding is "a family of interrelated abilities" that develops over time in an individual and involves "sophisticated insights and abilities, reflected in varied performances and contexts" (Wiggins & McTighe, 1998, p. 5). Wiggins and McTighe (1998) describe six indicators of understanding: *explanation, interpretation, application, perspective, empathy, and self-knowledge*. Sierpinska (1994) also identifies four indicators, which she calls mental operations, involved in understanding: *identification, discrimination, generalization,* and *synthesis.* There was some overlap in the two bodies of research, but each offered some valuable insight into indications of understanding. I referred to both Sierpinska's work and the work of Wiggins and McTighe to qualitatively describe the understandings of my participants.

Methodology

A pilot study indicated that the teachers in the local school district, unfortunately, did not have the type of content and pedagogical content knowledge I wanted to explore. Therefore, it was necessary to offer a course taught at a large university in the Mid-Atlantic region. The mathematics coordinator for the school district invited first-, second-, and third-grade teachers in the district to participate. Two first-grade and two second-grade teachers responded. The teachers in the study were selected from this course.

The course functioned as a discussion group with all four elementary school teachers, the mathematics coordinator, and myself. We met 10 times for a total of over 18 hours. The purpose of the course was to both improve and evaluate the teachers' understandings of how students think about addition and subtraction word problems, the classification of problems, the direct modeling of problems, the counting strategies used for solving problems, and the relationships among these topics, as researched by CGI. During the first half of the course I instructed the teachers. During the second half of the course the teachers decided how this knowledge could be used to change and enhance their own teaching. Throughout the last half of the course teachers were given the opportunity to evaluate their lesson plans that involved addition and subtraction word problems. The curriculum they used contained almost solely Join and Separate (Result Unknown) problems. They rewrote almost all the word problems in the curriculum, using each of the structure types identified by CGI.

Three teachers, Carmen, Pam, and Julie (all names used are pseudonyms), clearly exhibited understandings of the content and pedagogical content knowledge as described by current research, to observe in their classroom. I observed any class time during which the teachers taught addition and subtraction word problems. I audio-recorded the teaching sessions and took field notes, which included verbatim teacher and student interactions, writing on the board, or other relevant data not captured by the audio-recording. Copies of lesson plans and classwork were collected. The field notes, lesson plans, other additional documents such as student work, and transcribed audio-recordings were used to analyze the understandings teachers had, how this understanding affected the teachers' decision making, and how this understanding influenced student understanding. I also examined the students' understandings of the mathematics by listening to class discussions, teacher and student discussions, and by accessing student work.

During interviews, the elementary school teachers in the study were asked about why they made specific decisions before and during class time. Events that merited attention during interviews included the teacher making changes to lesson plans, claims she made about student understanding during class, the teacher intervening in student work, assessments she made about students' solution strategies, or just conversations the teacher had with a student. I concentrated on observing four or five students in each classroom. Students were interviewed using mathematics tasks similar to those they encountered during their regular class work. To evaluate teacher and student understanding, I investigated participants' ability to *identify, explain, interpret, discriminate, apply,* and *generalize* within the context of understanding addition and subtraction word problems.

The analysis was conducted qualitatively using the data I collected in classroom observations, field notes, documents from the teachers, and interviews. The manner in which teachers used the curriculum provided me with an important avenue for investigating their understandings and how they utilized their understandings in practices such as planning lessons, choosing tasks, and anticipating student responses. I reviewed the data points searching for indicators of understanding in which I found multiple pieces of evidence. I also searched the data for evidence that teachers or students did not exhibit a specific understanding to make sure the assertions were made with as little bias as possible. Once I gained a better understanding of the understandings, I organized the categories into subcategories, based on types of understanding repeated in the data.

Next, I determined how the teacher used her understandings and how those understandings may have influenced the students' understandings. I usually worked backwards from a particular student understanding, investigating the data to find possible instances when the teacher may have influenced that understanding through her practice. In addition, I hypothesized how the teacher's knowledge could have affected the teacher's practice. Multiple instances of a particular student understanding, coupled with multiple instances of a particular practice influencing that understanding, merited a connection worth noting.

Results

The following paragraphs suggest some aspects of student understanding influenced by teacher understanding as well as aspects of teacher practice that influenced student understanding.

Aspects of Student Understanding Influenced by Teacher Understanding

Student understanding of their written representations was one of the most notable items influenced by the teachers' understandings. Students understood that addition and subtraction word problems could be represented with devices such as equations, pictures, diagrams, or tally marks. Teachers influenced how the students were able to interpret the representations and how they used the representations to explain and interpret the problems given to them. For example, Pam asked students to write an equation to represent each problem they were asked to solve. She used equation writing as a tool for students to observe and articulate the differences in structures. Her consistent emphasis on where the question mark was in the equation enabled students to show what parts of the problem were given and what part of the problem was missing. Similarly, the teachers influenced the quality of students' verbal explanations of their thinking. All three teachers used written and verbal representations that described, as literally as possible, the thoughts and actions of a typical student when solving the problems (as described by CGI), which may have contributed to why the practices were relatively easily incorporated by their students.

There was an extensive amount of evidence in all three classrooms that teacher understanding can lead to an expanded student understanding of mathematical operations. I believe that the teachers' flexible understandings enabled the students to interpret the mathematical operation as a strategy used to solve the problem instead of a defining characteristic of the problem. The students could focus on the problems' structure, not relying on an operation to interpret problem. The operations were possible avenues for solving the problems. The teachers asked the students to solve problems in multiple ways, including by the use of different operations. Students showed an ability to interpret the operations as "opposites" of each other. They explained that the opposite operation could be used to check their work or to solve a problem using a second method.

Students in this study were able to learn about concepts such as recognizing invariance, generalizing, justifying, and focusing on the structure of the problem instead of just how to solve a problem. For example, when one student, Niki, solved a Join (Change Unknown) she wrote the equation 32 + ? = 50. Niki then wrote 30 + 20 = 50 and then 32 + 18 = 50. Her teacher asked her about her thinking. The student responded:

Niki: Because this [the number in the problem] is 32, this [the number in her equation] has to be 32. So I added 2 more. Then I took 2 less from the 20 because if I added 2 more to here [the first addend] I have to take away from there [the second addend].

Pam: How did you know the 50 would not change?

Niki: Because I added 2 more and took away 2 more so it would be the same thing.

Niki showed surprising mathematical insight with her generalization and her nascent understanding that she could add to one addend and subtract from the other addend and maintain the same sum. These young students were developing the building blocks for identifying and explaining these important mathematical practices and powerful mathematical concepts.

The ability of students to progress to more sophisticated levels of problem solving was also influenced by the teachers' mathematical and pedagogical understandings. The teachers understood the students' current ability levels and they also understood the more sophisticated strategies, so they developed tasks to build on the students' understanding. The teachers understood the importance of students developing their individual interpretations of the mathematical topic. Practices such as refraining from teaching a standard algorithm until the students were capable of understanding the algorithm allowed students to develop their own strategies as shortcuts for direct modeling and counting strategies. All three teachers encouraged their students to use less sophisticated methods for solving problems when they believed a student lacked the ability to interpret the nature of the problem, which the teachers hoped would lead to a deeper mathematical understanding for the student.

Aspects of Teacher Practice That Influenced Student Understanding

The first and most drastic change in the teacher's practice was the complete revamping of the problems given to the students. It is clear that the teachers' understandings affected their practice through the types of problems they chose to give to their students and through the sequencing of those problems. The teachers also used their understandings to comprehend and carry out the objectives for each of their lessons. They were able to make more specific lesson objectives and have those objectives guide their interactions with the students throughout their teaching. They could decide whether the purpose of their lessons was to introduce a certain type of problem, develop interpretations for problems, discriminate among different types of problems, or develop strategies to solve problems.

The teachers' understandings enabled them to better address the needs of their students, not only through the building of their lesson plans but also in daily interactions with students. Prior to this study, the teachers were unhappy with how they taught the lessons, but were unable to make changes because they did not understand the mathematics well enough. By their own admissions, their knowledge only enabled them to change the numbers in the problem to increase or decrease difficulty. Because of their increased mathematical understandings they were able to construct tasks that would build on student understandings. The teachers gained insight into students' weaknesses and strengths, and could modify their practice accordingly. For example, Julie noticed that a group of two higher achieving students answered all of their problems correctly except for the Compare (Referent Unknown) problems. She was able to design more problems of this type and target her verbal questions to address the relationships involved.

All three teachers identified that the questions they asked their students were more targeted than in previous years on the mathematics they wanted their students to understand, because they understood the mathematics better. For example, the teachers' understandings of structure were evident in their teaching when they continuously asked students to explain the relationships within the word problems. Carmen made a practice of asking students to explain the story without using any numbers. These teachers did not simply expect students initially to be able to explain the nature of the problem; they used scaffolding to press students to interpret the numbers in the problem, the objects to which those numbers refer, and the relationships among the referents within the problem, instead of focusing on getting an answer or which operation to use to solve the problem.

The teachers used multiple representations in their teaching, such as blocks, pictures, equations, and diagrams. Teachers asked students to model high-level performance with explanations for how they wrote their equations, including interpreting why they decided to represent the quantities in the manner they chose and where to place the quantities in the equation. The teachers encouraged the use of direct modeling to understand the nature of difficult problems. The teachers spent time not only showing examples of ways to solve problems but also explaining how the strategies worked and how the strategies were contingent upon the nature of the problems they were attempting to solve. Asking the students to solve problems in multiple ways increased the flexibility of their understanding (Star & Rittle-Johnson, 2008). It gave the students further incentive and avenues to develop their strategies and develop their ability to explain and interpret the mathematics they used.

Finally, teachers were able to find validity in the arguments students made that were different from their own, and encouraged students to continue in their thinking. For example, Pam found herself wanting to ask students to solve Separate (Change Unknown) problems using the missing addend approach, because that is how she conceptualizes those problems. But she allowed her students to develop their own strategies, based on their interpretation. She identified that there are many appropriate ways to think about mathematical problems, and decided not to instruct her students specifically how to solve them. Teachers can allow students to develop their correct interpretations, merely by knowing when to refrain from imposing their own ideas that might hinder productive student thinking.

Discussion

After analyzing the data, I hypothesize that teachers' mathematical knowledge for teaching determines the teacher's ability to design and implement a task at a high level of cognitive demand as well as their ability to maintain that level throughout implementation. The following figure adapted from Henningsen and Stein (1997) illustrates the three phases through which tasks pass; the task in the curriculum, the task set up by the teacher, and the task as implemented in the classroom. The way the task is implemented in the classroom directly affects students' learning outcomes.

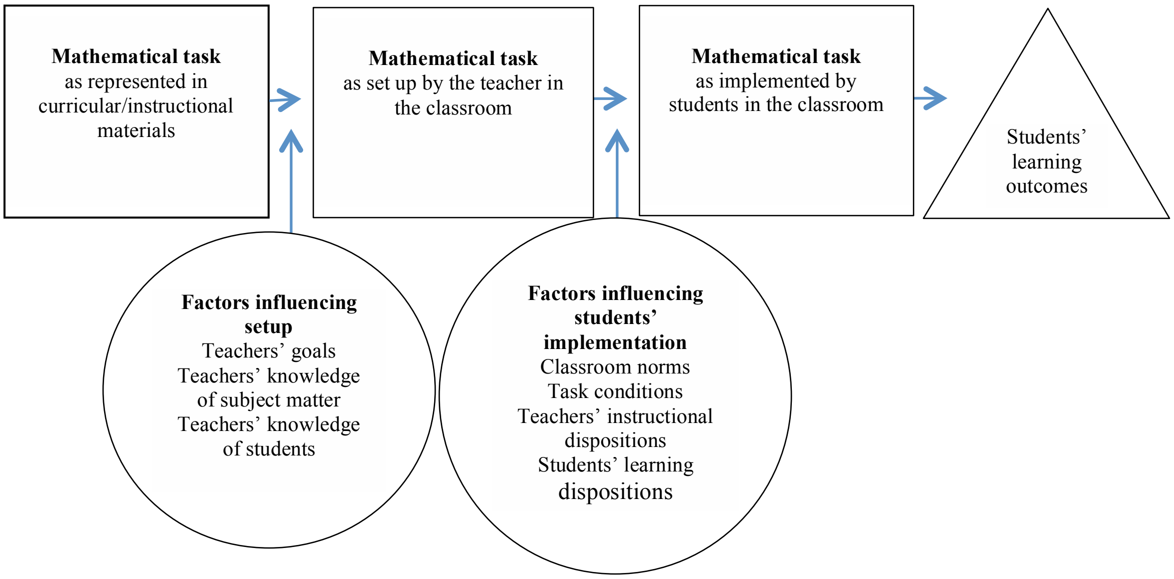

Figure 2. Framework for Mathematical Tasks (adapted from Henningsen & Stein, 1997, p. 528)

Research shows that the level of cognitive demand of a task tends to decrease as the task proceeds through the framework. In the framework, teacher knowledge of subject matter appears as a factor influencing the setup of the task, but it does not appear as a factor influencing the implementation of the task. I agree that teachers' instructional dispositions determine the way a teacher desires to implement a task, but, a cross-case analysis of the participants in my study showed that not only did they desire to implement tasks at a high level of cognitive demand – but they also used their understanding to create situations in which students were using tasks at a high level of cognitive demand and maintaining that demand throughout implementation.

The teachers used their understanding to create the two types of higher level cognitive demand tasks, "using procedures with connections" and "doing mathematics" indicated by Henningsen and Stein. Tasks that use procedures with connections were chosen to "focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas" (Stein, Smith, Henningsen, & Silver, 2000, p. 16). For example, Carmen asked her students to use the procedure of answering a list of questions she constructed to encourage them to think about the relationships between the quantities in the word problems, such as "Who has more?" "What clue tells you this?" etc. The students followed this procedure trying to make connections among the numbers in the problem, the quantities they represented, and the relationship between those quantities. The teachers also used problems that asked students to represent the quantities and actions in the problem in multiple ways, such as with diagrams, manipulatives, symbols, and problem situations (Stein et al., 2000, p. 16). Because the teachers were aware of the connections they were able to incorporate those connections with procedures. Teachers consistently asked students to make connections among the representations to develop their meaning, such as when Pam asked students to write mathematical equations to lead students to understand the types of knowns and unknowns in the addition and subtraction word problems.

The teachers also used their understanding to create higher-level cognitive demand tasks called “doing mathematics.” Instead of giving students the same type of problem repeatedly and telling them how to solve it, students were given problems that required nonalgorithmic thinking. Students were purposefully not given algorithms so they could develop more complex thinking about the operations. Teachers thought flexibly about the problems–not defining them by a single operation,–allowing different operations, depending on the strategies students chose. All three teachers chose tasks that required considerable cognitive effort so that some level of anxiety was experienced by the students at some time during their work, which was a change from previous years when the students were described as “bored.”

The teachers were also able to use their understanding to maintain that demand level over time. According to Stein and colleagues there are seven factors associated with maintenance of high-level cognitive demands (Stein et al., 2000). These factors coincide with the aspects of teacher practice that I claimed to be influenced by teacher understanding. I will discuss four of the factors I saw evidenced by the teachers in this study.

Eighty-two percent of tasks that stayed at a high cognitive demand were tasks that built on students’ prior knowledge (Henningsen & Stein, 1997). The participants in my study decided to write a pretest for their students. Through this test the teachers realized the students were capable of much more than the teachers were previously expecting. The teachers knew the demand of the tasks in the current curriculum was too low, but they did not have the understanding necessary to change them. During this study they completely changed their curriculum, based on the results of the pretest they gave their students. The teachers also encouraged the students to use strategies to solve problems based on their understanding of the students’ ability levels. Using their increased mathematical understandings, teachers were better able to determine the students’ prior understandings and design activities that could improve those understandings—without their increased mathematical understanding they could only increase or decrease the size of the numbers.

Scaffolding of student thinking and reasoning is another significant factor to maintaining tasks’ high level of cognitive demand. With each teacher I witnessed situations in which students were unable to complete a task and the teacher provided assistance through questioning and comments that asked students to think about the structure of the problem without reducing the demand of the task. For example, when one student was unable to understand a Separate (Change Unknown) problem, Carmen helped her develop a strategy using tally marks to solve the problem. The student was eventually able to build off this interaction and solve these problems without Carmen’s assistance.

Carmen also demonstrated a fourth factor, giving students a means of monitoring their own progress. She gave the students a checklist to fill out for each problem to enable students to guide their own progress in problem solving. 1) Read the problem. 2) Look for clues in the problem, find the important information. 3) Think of some strategies. 4) Solve the problem. 5) See if it makes sense. Carmen reported that this type of list was unnecessary before because the students “didn’t have to think” about the problems. Now that her curriculum did contain problem solving that required students’ cognitive effort, she addressed the development of students’ ability to monitor each step toward solving the problem.

These teachers modeled high-level solution strategies, another tool for effective maintenance of high cognitive demand. They explained how solution strategies were appropriate based on the nature of the problems. They acknowledged that they were unable to do this successfully before learning more about addition and subtraction word problems. They were aware of the important relationships in problems and could therefore model these relationships in a variety of ways. Part of these demonstrations involved drawing conceptual connections, another important factor in maintaining the task demand level. For example, teachers could draw connections between addition and subtraction when demonstrating the use of both of the operations within one word problem.

I suggest that student understanding was affected by the teachers' maintenance of the level of cognitive demand. The students demonstrated the type of understandings fostered by tasks with higher cognitive demand level that focused on "procedures with connections" (Stein et al., 2000). Students were able to develop deeper levels of understanding of mathematical concepts, such as their expanded understanding of the operations of addition and subtraction, through their use of procedures. They represented problems in multiple ways, such as modeling, equation writing, and creating their own word problems. Students were able to develop meaning by making connections among multiple representations. For example, the students in Pam's class were able to use equation writing, picture drawing, and creating of problem situations to better understand the relationships between types of knowns and unknowns in an addition and subtraction word problem. There were many examples of students showing cognitive effort, not following procedures mindlessly, and engaging with underlying conceptual ideas. There is also evidence that students were able to demonstrate the type of understandings that "doing mathematics" tasks are intended to engender. Students showed nonalgorithmic thinking while solving the addition and subtraction problems in multiple ways. There is evidence of students beginning to explore aspects of mathematical concepts such as invariance, generalizing, justifying, and focusing on the structure of the problem instead of just how to solve a problem. There were countless examples of students accessing relevant knowledge and using it appropriately to work through a task.

The teachers each expressed that they knew they were not effectively teaching this topic prior to the study. They did not realize they were lacking the mathematical understanding necessary to enhance their practice. Within days of increasing their SCK, KCS, and KCT they were able to change their practice. Their teaching practices, affected by their deeper understanding, influenced the quality of their students' understandings.

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CONNECTING PD TO PRACTICE: USING A TASK IN 7TH GRADE MATH[1]

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In this study, we expand the Cohen and Ball triangle of interactions to explore the relationship of professional development to classroom practice. We consider a case study of one teacher's implementation of a task from professional development in her 7th grade classroom. We were specifically interested in how the content and pedagogy of the professional development would be adopted by the teacher. Our findings suggest that this teacher treated pedagogy and mathematical content as separable, which led to problematic implementation of PD practices.

Research and theory around the design of high-quality professional development suggests that using materials like those a teacher would use in her classroom is one particularly effective strategy for influencing teaching (e.g., Banilower, Heck, & Weiss, 2007; Hill, 2004) as is modeling the kind of learning environment that is desired for the K-12 classroom (e.g., Desimone, et al., 2002; Elmore, 2002; Hill, 2004). Further, an emerging body of research highlights the need for high-quality teaching practices to be implemented in ways that are grounded in the mathematics. For example, Wood, Williams, and McNeal (2006) found that teachers who asked questions that pushed students to engage in inquiry and argument led to higher levels of success among mathematics students than teachers who asked questions that only required the students to report their thinking. While both instances involve *questioning*, the first is necessarily grounded in the mathematics and students' reasoning while the latter can be used generically (e.g., 'How did you get your answer?'). Similarly, Kazemi and Stipek (2001) identified the characteristics of classrooms in which there was more push for conceptual learning. Their findings suggest that these classrooms include norms such as: explanations need to include mathematical arguments rather than descriptions; students are expected to understand connections among multiple problem-solving strategies; errors are treated as a means for enhancing learning, thereby serving a generative role; and collaboration features consensus building through mathematical argumentation. Like the Wood et al. study, this study demonstrates that the precise ways in which pedagogical moves are implemented fundamentally shapes the classroom environment. Given what is known about professional development and what is known about teacher practice, we sought to understand what aspects of professional development (PD) a teacher might carry into her classroom and what the movement between the PD and classroom environment might look like.

Theoretical Framework

We frame our effort by extending the triangle of interactions metaphor introduced by Cohen and Ball (1999, 2001). In their model, the classroom learning environment is shaped by the interactions of three primary elements: teachers, students, and content as embodied in materials. We extend this metaphor to include a second triangle, which represents the same elements interacting in professional development (see Figure 1). The two triangles are joined at the vertex representing the teacher/participant. This vertex is noteworthy because the teacher is not only present in both environments, but also because her interpretation of the PD fundamentally shapes both environments as she brings experiences from her classroom to the PD and experiences from the PD into her classroom. While it is true that the materials from the PD can be taken into the classroom, the teacher is responsible for turning the intended (written) curriculum into the enacted curriculum (Remillard, 2005).

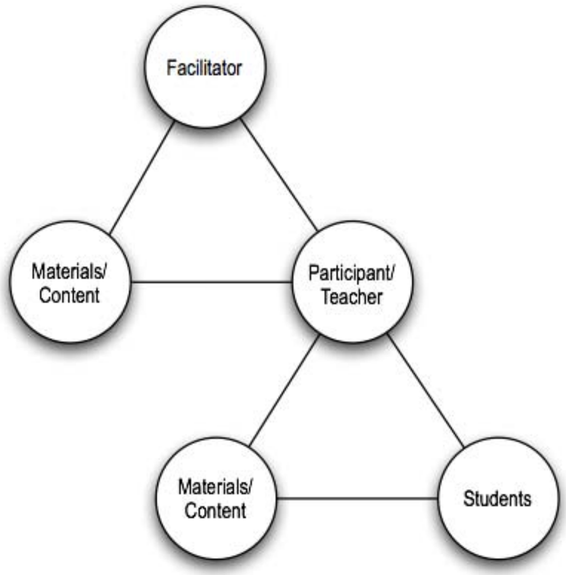

Figure 1. Double triangle of interactions representing the classroom (bottom) and PD environment (top).

The teacher is ultimately responsible for implementing ideas from PD in her classroom. This study, therefore, documents one teacher's experience in PD and the way this experience shaped her classroom. Since the teacher is the point joining the two triangles it is important to examine the teacher in the professional development context and her classroom context. To frame our analysis, we focused on a specific set of instructional moves as well as the teachers' content knowledge. Specifically, we were interested in the ways the teacher supported connection-making in her classroom as this had been a significant factor in her PD experience. We defined connection-making with a 4-facet framework for connection making. The four facets include:

- *Questioning & Communicating*: in this framework, these serve as tools for articulating, expanding, and refining mathematical ideas. Specific kinds of communication in which we were interested included that which supported exploring connections among representations, problematizing ideas, promoting conjecture-making and testing, and engaging with ideas in ways that move the learner to deeper levels of mathematical understanding.
- *Reasoning with Representations*: in the framework, reasoning with representations is seen as connection making. To reason with a representation, a person must draw on a set of mathematical understandings, embody them in the representation, and communicate to others about those ideas using language and the representation itself.
- *Embracing Multiple Approaches*: this framework builds from the perspective that different people have different solution paths. The path taken depends on knowledge invoked and the solver's knowledge. The value of highlighting these differences is in its effectiveness for introducing all of the learners to new perspectives and promoting their ability to create connections among ideas.
- *Scaffolding*: this framework is built from the perspective that facilitators make a number of moves that support learners in moving from their current levels of

> understanding to new understandings. Scaffolding moves include, but are not limited to, those consistent with the five practices for facilitating discourse around cognitively challenging tasks (Stein, Engle, Smith, & Hughes, 2008). These include anticipating responses, monitoring responses as learners work, select particular approaches to highlight during group discussion, sequence responses to highlight particular ideas, and support learners in making connections between different approaches.

These four facets of connection making formed the basis of our analysis of both the PD and classroom settings as we considered how the teacher translated the PD experience into her classroom practice.

Methods

Data were collected as part of a larger research project in which we used a mixed methods design to understand teacher learning in PD and the impact it has on classroom experiences and student performance. For this study, we considered one videotaped session of a 14-week mathematics PD experience and one videotaped session from one participant's 7th grade classroom. This teacher, Donna, was the only one in our study who invited us to videotape an implementation of a task taken directly from the PD course.

Professional Development

The PD workshop was a 14-week (total of 42 hours) experience for middle school teachers in which they specifically engaged in content knowledge development. The course engaged the 14 participants in exploring fraction multiplication and division as well as proportional reasoning. Each session, which was taught in the district office in a large, underachieving, urban school district, lasted three-hours and engaged the participants in hands-on, technology-supported engagement with the mathematics. The three stated goals of the PD were (1) understanding referent units (the *whole* to which a fraction refers) in a variety of situations; (2) using drawn representations; and (3) proportional reasoning. To meet these goals, participants were asked to engage with a number of open-middle tasks either as a group or in small groups. For the small group work, each teacher was responsible for preparing a *write-up* that documented the approach taken to solving the task, any dead-ends hit, and a fully discussed solution.

The facilitator, a member of the research team, had extensive experience as a professional developer and was a former high school teacher. Each session was videotaped using two cameras. One was focused on any written work being discussed, while the other focused on the people speaking. These two sources were combined into a single view using picture-in-picture technology to create a restored view (Hall, 2000).

7th Grade Class

As part of the larger research project, we asked several teachers representing a cross-section of abilities on our pre-course assessment to allow us into their classrooms so we could understand how the professional development impacted their teaching. We specifically asked to see examples of a typical lesson and a lesson the teacher felt was like the PD. For this analysis, we consider Donna's implementation of a task she thought was like the PD. The lesson was implemented approximately three months after the completion of the PD. As in the PD, the classroom was videotaped using two cameras and the sources were combined.

Data Analysis

Both the PD session and Donna's 7th grade lesson video were analyzed using memoing (Strauss & Corbin, 1998). Every instance of any of the four facets of connection making was noted. This included those instances that did not capture our full definition (e.g., the teacher may ask a closed question rather than a generative one). We chose this approach in place of grounded theory to help us focus on those aspects of classroom practice that have been demonstrated to be important for learning as well as those specifically built into the PD experience. Once we had memoed each lesson, we built a model of it to help us understand how the facets of connection making worked in that lesson.

Secondary data, including weekly telephone interviews with Donna completed during the 14-week PD, write-ups of her tasks, and reflections (e.g., specific questions about generalizing the mathematics) collected in the PD sessions were analyzed. We used the same facets of connection making and memoing technique as was used with the video data.

Results

In this section, we briefly present an overview of Donna's experience in the PD as well as some elaboration on her content knowledge. This is followed by a discussion of Donna's PD-inspired lesson. Data are summarized due to space limitations. Both the PD and the 7th grade lesson like the PD focused on a pair of mathematical tasks (shown in Figure 2).

Problem 2:

a) Joe the farmer drained his grain silo into a holding bin. The silo was full and held 300 cubic meters of grain. It drained at a rate of 5 cubic meters per hour. How fast would Joe have to drain a silo that held 420 cubic meters of grain so that it took the same amount of time? How fast would Joe have to drain a silo that held 210 cubic meters of grain so that it took the same amount of time? Is the relationship between volume and rate the same across these three examples? If so, describe the general relationship.

b) Grain elevators are large containers used to fill train cars with grain that farmers have harvested. If the elevator dispenses grain at 500 cubic feet per minute, it takes 12 minutes to fill a train car. If the elevator dispenses grain at 200 cubic feet per minute, how long will it take? If the grain elevator dispenses grain at 250 cubic feet per minute, how long will it take? Is the relationship between rate andtime the same across these three examples? If so, describe the general relationship.

Figure 2. Task used in both lessons. Discussion in this paper focuses on task (b).

Donna's PD Experience

A typical session of PD opened with the facilitator asking participants to work on a task for a short time that was then discussed in the whole group. Then, participants were introduced to the tasks from which they could select to complete a write-up. During the times in which the participants worked, they could choose to work alone or with a partner.

Donna, who was an 11-year veteran teacher, began the PD with a slightly above average mathematical knowledge in the areas of interest to us. On our pretest, her z-score was 0.2 but by the end of the PD, she scored a 1.4, showing much more than the 0.3 growth considered significant. In observations of the class from which these data are drawn, it was clear that she had some confusion about aspects of proportional reasoning and that her definitions for proportional relationships were superficial. For example, she spoke of direct proportions as being "up/up" relationships and inverse proportions as "up/down" relationships. In both whole class discussions and one-on-one discussions, the facilitator communicated the need for precision in defining the relationships by saying things like, "I think this speaks really nicely to what that is that we're talking about. I think just saying 'as one goes up the other goes up' and 'as one goes up the other goes down' — talking like that — additive and multiplicative — I don't think we're really being clear enough."

In the course of the PD class session, Donna developed a conjecture for herself about inverse variation. She asserted that if you multiply one value (e.g., *y*) by an amount, then you would need to multiply the other value (e.g., *y*) by the reciprocal of the amount. As shown in Figure 3, for example, to increase the 12 to 24 requires multiplying by 2, therefore, according to Donna's rule, the corresponding value to 24 would have to be ½ of 500. The facilitator worked one-on-one with Donna to explore this conjecture by asking questions about the situation in Figure 2B. The facilitator challenged Donna by asking questions about the t-chart Donna had made (Figure 3) such as, "So, if we take 500 to 200 and then we did 12 to 30, we should see the reciprocal?" Donna and the facilitator worked on this situation together for several minutes using the representations Donna had created and Donna's own conjectures. In the whole-class debriefing, she scaffolded the entire group's thinking by again emphasizing the need for more precision in defining proportional situations than simply describing them in terms of the idea of one value increasing or decreasing as the other increases. Through these conversations, we assert that Donna had an opportunity to begin thinking about proportional relationships as being multiplicative in nature.

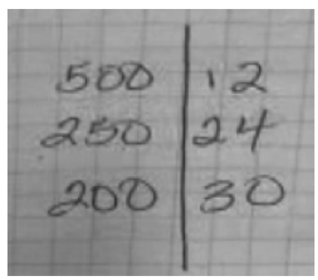

Figure 3: Donna's work for the situation in Figure 2, Question B.

At the end of the session, participants were asked in the reflection activity whether data shown in a table were directly proportional, inversely proportional, or neither. Donna showed evidence of both properly identifying the situations and providing mathematically grounded evidence for those. For example, her rationale for (appropriately) rejecting one table of data was that it "… is not a proportional relationship because there is not a constant pattern that fits either description (xy=k or y=kx). The multiplicative aspect is not there." She also provided a coherent definition for constant of proportionality that suggested she may be starting to think differently about direct and inverse relationships than she previously had: "A constant of proportionality is a number that can be multiplied by one term to get the other. Doesn't apply to inverse?"

Despite the promising evidence that Donna was beginning to sort out proportionality in November, by March when we recorded her classroom, she reported that she was still confused about how and why inverse and direct proportions were different. She said, "…I'm not even absolutely sure that's ever clearly defined for me…[in my own classroom] I recognized the difference… I'm not sure that it's defined. And I guess it's because the materials I work from don't define it clearly to me. I probably don't define it clearly to [my students]."

Donna's Classroom Lesson

Donna introduced the two tasks in Figure 2 (parts A and B) to her students by asking them to read aloud and using communication moves to ensure the students understood the problem. She encouraged them to work with "the mathematician next to you". She also used questioning to remind them of the representation they had been using with proportional situations (graphs) but also promoted other approaches saying, "… you can use the graph if you want to. But if you have an idea in your head of how you could visualize this in a different way, some other kind of model, then you can draw that." Donna circulated the room as the students worked asking them questions. Her questioning was largely confined to reporting questions (Wood et al, 2006) in which students explained *how* they did the problem but not *why* that had chosen an approach. For example, she asked one group, "What did you do when you did what you did?" We saw a few frequent questioning patterns in Donna's interactions with her students. One was to ask students to explain what the problem was asking if they were confused about what to do. Another pattern was her invitation to students to use their own strategies when solving a problem. However, this became problematic for Donna when she needed to evaluate students' approaches. Their approaches, at times, differed from what she expected and her only responses were either encouragement or to explain her own thinking—both without supporting students in connecting their understanding to her own. In one case, a student used a solution path Donna had not expected, so she explained her own to him. His response to her was to ask if he could have done it her way. When she said he could have, he commented on his own effort saying, "All that work for nothing." Communication was about evaluation and efficiency, not about engaging in mathematical discussions or pushing students thinking forward as Donna had experienced in the questioning she had in PD.

Donna had suggested to students that they should use representations in her launch of the problem, but as she circulated, she was clearly surprised by some of the representations she saw. In the spirit of supporting multiple approaches, Donna seemed hesitant as she tried to accept the ways students chose to approach the situation. For example, Donna encountered a group that created a bar graph for their inverse proportion. She asked the group, "Is this the kind of graph we've been doing with direct and inverse variation?" A student responded, "We've been doing line graphs, but I didn't feel like the line graph would be comfortable…" Donna responded, "If you think this is going to show you a better picture then go for it." Another student in the group commented, "I don't feel right about doing a bar graph." Donna articulated that students could use different approaches as long as they were comfortable with their answers, and she reminded them that the question asked them to determine which kind of variation (direct or inverse) was represented in the problem. Unlike the PD where mathematical discussions were had about the representations and their mathematical affordances, in Donna's class, the discussions focused on student comfort with no elaboration on that comfort or on aspects of the mathematics.

Analysis

In the PD, the framework of connection making was clearly present with all aspects of the class working together to scaffold learners. The facilitator used questioning to either understand participants' thinking or to push their understanding. She used precise communication in terms of not only incorporating mathematical terms but specifically discussing those terms and why she was focused on the precise use of them. She encouraged the use of representations and focused teachers on explaining them to her and to each other. The whole class debriefings of tasks always focused on connecting the various approaches together.

In contrast, when Donna implemented the task in her class questioning stopped at explaining how a problem was worked. Although multiple approaches were tolerated, they were not used to promote thinking and connection making among students. In fact, students saw Donna's approaches as being better than their own and, in the class we observed, they did not get to see any approaches other than their own or ones that Donna talked to them about in their partner work. Communication was not mathematically precise and, while not presented in the data due to space limitations, she pursued using the up/up and up/down explanations for the proportional relationships. She also emphasized the shape of the graph without pursuing any discussion about understanding the graphs in more detail. There was no sense of connecting ideas to each other and Donna did not seem to scaffold students' efforts so that the pieces fit together. In fact, she would give different pairs of students different directions without purpose—for example, one group was told to work the problem another way while another was asked about a different representation. But, there was no follow-up of any kind on these challenges.

Conclusions

Our goal in this study was to explore aspects of professional development a teacher might carry into her classroom and what the movement between the professional development and classroom environment might look like. In our case, we were offered the opportunity to observe a single teacher engage with a task taken from the PD. We assert that this analysis contributes to our understanding of how a teacher mediates the double triangle of interaction.

Donna was a good teacher in many respects. Her classroom was pleasant and her students clearly liked her. Her mathematics knowledge was clearly above average, despite some problematic understandings. According to her interviews, she enjoyed the PD and thought it was helpful to her teaching, though she could not provide examples of specific ways in which it was helpful.

From the perspective of the double triangle of interaction, however, we can see that the teacher mediation of the two environments plays an important role in the experience the students have. In PD, the task served as a conversation starter. It had been intentionally chosen by the facilitator to highlight particular relationships among the numbers (and the facilitator had discussed this intentionality in the PD). The focus of the task activities was on developing conceptual understanding that was mathematically precise. To do this, the facilitator used all four aspects of the framework for connection making in ways that were consistent with high press practices (Kazemi & Stipek, 2001).

Donna, having experienced this environment as a learner, chose to take the exact materials to her own students. She framed the task as a challenge for the students and used it to engage them in talking to each other and to problem solve. The focus of her interactions was often on correct calculations and the take-away was the shape of the graphs. There was no attention to precise use of mathematical terms nor was there any attention to the multiplicative relationship of the values in the task. Whereas the task had been used to engage participants with mathematical concepts to develop understanding in the PD, in the 7th grade classroom, the task was about doing hands-on mathematics with a goal of knowing that the graph shapes were different.

This suggests that while Donna was able to see the moves the facilitator made and take those into her classroom (e.g., she used questioning, supported multiple approaches and representations, etc.). Her lack of tying these to the mathematics in ways that supported a high press learning environment meant that the students missed out on opportunities for learning.

As suggested by the double triangle of interactions, in order for PD to impact classroom practice, it must be relevant and apparent to the teacher because she is the one who moves the opportunities from one environment to the other. While Donna valued her experiences as a learner, she did not take them, except in a superficial way, into her classroom practice. This suggests two things. First, the Donna viewed pedagogical moves and content knowledge as being separate from each other—thus her ability to use the same pedagogical moves as the facilitator but in less rigorous ways. Second, it suggests that the goals of PD may need to be reconceptualized from supporting the teacher in her personal development to providing support for developing the learning environment in which the teacher practices. This does not mean telling the teacher how to conduct her practice, rather it means focusing the PD so that the teacher's personal development is situated, for her, in the practice of teaching. For example, we wonder whether Donna's content knowledge development around the relationship of quantities in the inverse proportion would have been more salient to her teaching had we presented her with student reasoning similar to her own and let her think about how she would interpret and respond to that reasoning in her own practice. Pursuing opportunities to more fully integrate the two triangles of interaction may support teachers in better translating their own experience as learners into their practice as teachers.

Endnotes

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STUDYING ONLINE PARTICIPATION AND ITS IMPACT ON THE DEVELOPMENTOF MATHEMATICAL CONTENT KNOWLEDGE FOR TEACHING

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We report on our efforts to support teachers' development of mathematical knowledge for teaching through online professional development. In particular, we report on our investigation into the relationship between online interaction and teachers' development of mathematics content for teaching. Through the integration of content analysis and social network analysis, we identify underlying relationships between aspects of online interaction and teacher learning. Results indicate that while interaction, broadly speaking, was not correlated with teacher learning, particular combinations of content and the centrality of an individual in the interaction were. Implications of these significant correlations for mathematics teacher education are discussed.

For more then a decade, the importance of teachers' content knowledge has been a major focus in the literature (Ball, 1993; Ma, 1999; Shulman, 1986) and increasing teachers' mathematical knowledge continues to be a major focus in both education research and policy (Greenberg & Walsh, 2008; National Mathematics Advisory Panel, 2008). Despite these calls, a great number of elementary teachers continue to be underprepared and uncomfortable with the mathematics content they are expected to teach (Greenberg & Walsh, 2008). At the secondary level, studies have shown that students who have been successful in high school and university mathematics classes – even those with grades of A in calculus – often have weak or underdeveloped understandings of the concept of function, a core concept in the K-12 mathematics curriculum (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Monk, 1992; Thompson, 1994). With regards to the mathematics preparation of school teachers, Cuoco (2001) notes the solutions "will not come from rearranging the topics in a syllabus or by adding more topics to an already bloated undergraduate curriculum. Making lists of topics that teachers should know … won't do it either" (p. 170). He argues that teachers need experiences *doing* mathematics and to develop a "taste" and passion for it. Bass (2005) notes that regardless of their level mathematical preparation, evidence suggests that that teachers lack *Mathematical Knowledge for Teaching* (MKT): "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching" (p 429).

Throughout the past two years, we have engaged in an extended research project focusing on supporting teachers' mathematical content knowledge and mathematical knowledge for teaching through online professional development. The overarching goal has been to support teachers' mathematical development through authentic engagement in collaborative mathematical problem solving and connecting that engagement with the teaching of school mathematics. In this paper, we report on the results of a subset of this project that sought to support teachers' developing a coherent view of the school algebra curriculum and mathematics content knowledge for teaching school algebra. The primary research question we explore is *What is the relationship between online participation in teacher development activities and teachers' development of MKT?*

Theoretical Background

The majority of online instruction emphasizes the importance of interaction and the affordances of the internet in supporting interaction at a distance, but pays little attention to the key features and functions of those interactions. We have found Lotman's (1988) characterizations of text as *univocal* or *dialogic* particularly useful in understanding the function of the individual posts that make up an online interaction and the interaction as a whole. Univocal discourse functions "to convey meaning adequately" (Lotman, 1988, p. 34). In contrast, dialogic discourse can be characterized as a "thinking device." Lotman describes dialogic function as generating new meaning: "In this respect a text ceases to be a passive link in conveying some constant information between input (sender) and output (receiver). … [I]n its second function a text is not a passive receptacle, or bearer of some content placed in it from without, but a generator" (Lotman, 1988, pp. 36-37). This dualism has guided both the design of our teacher development activities and our analysis of the online interactions.

In our work with teachers, we seek to provide environments that support the dialogic function of text, where teachers view and use each others posts as "thinking devices." We, then, attempt to understand these interactions in terms of their potential and success for catalyzing the generation of new mathematical knowledge and knowledge for teaching. This perspective is consistent with a variety of research in the learning sciences that emphasizes the importance of discourse and interaction in learning (Mercer, 2000; Paloff & Pratt, 1999; Ravenscroft, 2001; Shale & Garrison, 1990; Su, Bonk, Magjuka, Liu, & Lee, 2005). Previous results documenting the relationship between interaction and learning include that learning is promoted through dialogue (Cunningham, 1992) and that learning occurs online through social activity in communities (Haythornthwaite, 2002; Rovai, 2002).

Through a design research methodology, we have developed and explicated a model for Online Asynchronous Collaboration (OAC) in mathematics teacher education, which has at its core cycles of individual, small group, and whole class interaction (see Silverman & Clay, 2010 for a detailed discussion of OAC). OAC begins with participants drafting a solution, initial approach, or questions on a set of purposefully selected, open-ended mathematics tasks in a private online workspace. In these initial posts, we do not expect that each teacher will be able to complete each of these activities, but we do expect each participant to attempt the assigned task and either pose a solution method and solution or ask relevant questions or wonderings that reflect their current state of thinking about the task. After the individual work phase, each participant's initial postings are made public for small-group and dyadic interactions where participants add comments, respond to classmates' questions, or ask for clarifications. Finally, the instructors orchestrate a whole-class discussion that lies at the confluence of the participants' collective sense making and the sessions' instructional objectives.

We have had success documenting participants' mathematical development (Clay & Silverman, 2008; Silverman & Clay, 2009) and the role of the teacher in supporting that development (Clay & Silverman, 2009). Despite these successes, we have struggled to identify correlations between the online teacher development activity and teacher development. In this paper, we focus our attention explicitly on this relationship.

Setting and Participants

In this article, we focus on an online graduate class in mathematics education that focused specifically on proportional and algebraic reasoning. The course was designed to support teachers as they deepen and extend their mathematical understandings and develop schemas within which a variety of mathematical ideas are conceptually connected (Silverman & Clay, 2009). First, we began by exploring mathematical ideas that the teachers teach regularly and ostensibly know well. We then sought to problematize the teachers' current mathematical understandings through posing "changes" to the mathematical idea (for example the exclusion, replacement or addition of properties, generalization, application to a new setting, etc.). These new conceptualizations of school math concepts were then used to support instructional conversations that bridged teachers' developing understandings and the school mathematics curriculum and focused on how various mathematical ideas fit together.

The wholly online class took place in the winter term of 2008 at an urban research university in the Northeastern United States. The class consisted of 17 (13 women, 4 men) who were geographically distributed throughout the United States. The 10-week course was hosted in the Blackboard Learning Management System. Within Blackboard, participants accessed course materials (readings and activities) from their homes or workplaces and participated in both threaded discussion boards and blogs. Each week, participants were presented with prompts to consider when creating their posts, but were consistently reminded that the prompts were meant as guides and not as a list of questions that must be answered. For each discussion board, participants were required to post a minimum of one initial post by Tuesday evening and two replies by Saturday evening.

Data Sources and Analytical Procedures

The two primary sources of data for this study include online interaction data, taken from the participants' interaction in the online discussion forums, and teacher learning data. Teacher learning was measured using instruments developed and validated by the Learning Mathematics for Teaching Project (http://sitemaker.umich.edu/lmt) for measuring teachers *Mathematical Knowledge for Teaching.* For this paper, the two versions of the LMT instrument for patterns, functions, and algebra were administered and are used as a validated measure of participant learning of the relevant content. Teachers' gain scores were converted to Item Response Theory (IRT) scaled scores using conversion tables provided by the developers.

Coding and Content Analysis

While raw interaction data (who talked to whom) is relatively easy to generate and can provide us with useful information about the interactions between teachers, this information does not provide us with any information about the content of and function of those interactions. In essence, it neglects the basic question posed by Schlager, Farooq, Fusco, Schank & Dwyer (2009): "What constitutes a meaningful relation or tie between individuals?" (p. 87). In order to better understand this question, we employed a variant of content analysis to compare, contrast, and categorize the posts. Coding of interaction data began with Lowes, Lin & Lang's (2007)'s coding scheme for coding interactivity in an online class, which was adapted to meet the needs of this project through a recursive process. Ultimately, the following codes became stable for our data set:

Cheerleading/Affirming (C/A)

- Offering praise and encouragement
- Focus on ideas not related to mathematics or teaching (ie. use of learning management system, course logistics, personal issues, etc.)
- Affirmation of work without any evidence of reflection

-

Students, Teaching, and Schools (STS)
- New information about students and teaching from teachers' own classroom or classroom experience

Direct Instruction (DI)
- Providing direct instruction to classmates; telling how to do or solve a problem

Doing Mathematics (DM)
- Making mathematical activity (thinking, representing, conjecturing, etc.) public

Questioning/ Challenging (Q/C)
- Raise questions that extend previous post, or express disagreement with it

All posts were coded on each of the two dimensions by two researchers using the above scheme. In order to ensure the reliability of the coding process, the researchers coded the first 30 posts together and came to agreement on the codes, definitions, and examples. The researchers then coded the entire data set independently and approximately 77% of the items were coded similarly by both researchers. Each item that was coded differently was discussed and deliberated until agreement was reached on how to code the item in question.

Social Network Analysis

Social network analysis (SNA), a mathematical approach for analyzing interactions and the structure of social network and the strength of the ties between actors in the network (Wasserman & Faust, 1994), was used to quantify the interactions being studied. One primary use of social network analysis is to identify important actors in the interaction and we use these measures of importance to further parse the online interaction data and to quantify teachers' role in the online interaction. Within social network analysis, there are two primary ways of quantifying the "importance" of actors in an interaction: centrality and prestige. An actor's centrality, or the number of posts generated by the actor is a measure of his or her influence on others within the social environment. In particular, it is the actor's potential influence on the interaction. When one has a high centrality, he or she is in direct contact with many others: "this actor should then be recognized by others as a major channel of relational information … a crucial cog in the network, occupying a central location" (Wasserman & Faust, 1994, p. 179). An actor's degree prestige, or the number of posts received by the actor (the in-degree), is a measure of how he or she is potentially influenced by others in the social environment. In terms of online learning, an actor's prestige can be interpreted as the amount to which colleagues (other actors) seek out and support an individual (Russo & Koesten, 2005).

Analysis of Relationship Between Online Participation and Increases in Mathematical Knowledge for Teaching

In order to quantify participants' participation, each participant's centrality and prestige was calculated (both their overall centrality and prestige as well as centrality and prestige broken down by each of the codes discussed above). Finally, Pearson's correlation was used to identify statistically significant linear relationships between participants' centrality and prestige and gains in mathematical knowledge for teaching, as measured by the LMT instruments.

Results

Analysis of the relationship between individual participation and LMT gain scores was conducted on the twelve independent variables shown below in Table 1.

Table 1: Twelve Independent Variables

	Centrality	Prestige
Overall	[1]	[2]
Cheerleading/Affirming	[3]	[4]
Students, Teaching and Schools	[5]	[6]
Direct Instruction	[7]	[8]
Doing Mathematics	[9]*	[10]
Questioning/Challenging	[11]*	[12]

An analysis of the linear relationship between each of the independent variables for participants' content prestige and centrality and participant learning was conducted using Pearson's correlation coefficient. This analysis indicated the following positive relationships (indicated by * in Table 1 above):

- A positive relationship between participants' Doing Mathematics (DM) centrality and their LMT gain score, $r(15) = 0.624$, $p < 0.05$.
- A positive relationship between participants' Questioning and Challenging (Q/C) centrality and their LMT gain score, $r(15) = 0.631$, $p < 0.05$.

Conclusion and Discussion

In this study, we bring into question the common belief and research results that emphasize the importance of interaction in online learning (Paloff & Pratt, 1999; Shale & Garrison, 1990; Su et al., 2005). In particular, we present evidence for the claim that it is not *simply* interaction that is important for learning in an online context. We extend Schlager, Farooq, Fusco, Schank & Dwyer's (2009) question of "What constitutes a meaningful relation or tie between individuals?" (p. 87) in two key directions: the content of the contribution and the individuals' patterns of activity. Results indicate that while overall levels of activity in online interactions were not correlated with gains in mathematical knowledge for teaching, the centrality of individuals Doing Mathematics and Questioning/Challenging were. Put another way, teachers with high activity (ie. teachers who generated a large number of posts) coded as Doing Mathematics and Questioning/Challenging tended to have higher LMT gain scores.

We recognize that this result might seem counterintuitive: one of the benefits of social networks (like online classes – or schools, for that matter) is that individuals can learn and grow through individual interactions with group members. For example, research has shown that *who* a teacher "gets help from" and not *the amount* of help a teacher gets was related to changes in teaching practices (Penuel & Riel, 2007). In other words, it seems that the posts directed towards an individual should be an important predictor of teacher learning. This was not the case in this study: participants' learning was correlated with the centrality of an individual (the number of posts an individual directed towards others), provided the posts involved Doing Mathematics or Questioning/Challenging.

We reconcile this apparent contradiction by focusing on the nature of MKT. We have argued previously that the development of MKT starts with coherent mathematical understandings and involves personal reconstruction of one's mathematical understandings to include both (1) the variety of ways that individuals may understand the idea and (2) how particular ways of understanding can empower individuals to learn other, related mathematical ideas (Silverman & Thompson, 2008). We also argued that this personal reconstruction is necessarily the result of a *reflecting abstraction* (Steffe, 1991) and not easily developed through explanation or demonstration: “the transition requires a building up of the understanding through students’ activity and reflection and usually comes about over multiple experiences” (Simon, 2006, p. 362). Framed this way, it appears to make sense that our participants’ learning was not correlated with explanations and demonstrations directed towards the participant (others’ posts directed to the participant). Doing Mathematics and Questioning/Challenging centrality can be interpreted as a metric for the one’s level of personal engagement and reflection in interactions designed to support the development of MKT. We speculate that participants with a high Cheerleading or Direct Instruction are experiencing the same discussion board quite differently than those with high Doing Mathematics and Questioning/Challenging centrality: rather than “telling” they are “doing” and “wondering” and placing themselves in a state of cognitive disequilibrium, which has the possibility of stimulating equilibration (learning).

Implications for Mathematics Teacher Education

The primary implication for mathematics teacher education is that, while teachers need to be challenged and encouraged to engage with their colleagues as part of their professional development activities, efforts need to be made to increase the amount of Doing Mathematics and Questioning/Challenging that they engage in. One obvious method, which we have implemented with little success, is being more specific about the posting requirements (for example, requiring at least one Doing Mathematics or Questioning/ Challenging post). We believe that there are two reasons for our lack of significant results from more specific requirements: our past participants (1) want to talk about what they believe their current needs to be and doing or learning math is often not a current perceived need and (2) often believe that direct instruction *is* the essence of doing mathematics. With these two factors in mind, it is clear that the challenge is not one of imposing requirements but rather one of a cultural shift: teachers need to come to see the value of MKT for their teaching and that the more teachers engage in *doing* mathematics, the more effective they will be at supporting their students as they do mathematics. Thus, it seems that one way to increase the effectiveness of online implementations designed to foster MKT would be focused teacher development activities designed to support this cultural shift. Examples of such activities include the Math Forum’s Online Mentoring Project (Shumar, 2006) and dialogue games (Ravenscroft, 2007).

Future Work

We fully recognize that all of the emphasis cannot be placed on increasing participants’ Doing Mathematics and Questioning and Challenging centrality – these posts need not and do not happen in a vacuum. A great many Doing Mathematics posts were in response to others’ posts and it is necessary to have something (a post) to question or challenge. Further, we acknowledge that other types of posts (cheerleading, for example) may be useful in supporting generative discourse. We have begun to explore sequential analysis (Bakeman & Gottman, 1997; England, 1985; Jeong, 2003) to study group interactions. Sequential analysis involves using probability to study relationships between individual posts and predict the characteristics of interactions that support individual contributions with particular characteristics (for example, “Doing Mathematics” posts) and the conditions under which those posts are more likely. In addition to increasing the effectiveness of our online teacher development, we believe that this research can help us bridge our work and the current emphasis on online communities in teacher professional development.

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EXAMINING EXPRESSIVE DISCOURSE IN MULTI-MODAL TECHNOLOGY ENVIRONMENTS

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We investigate the impact of integrating dynamic geometry environments with haptic devices, which allow users to not only see and manipulate geometric figures on a screen but also feel, through force-feedback, the result of interacting with such objects. The feedback loop can be assigned to varying attributes of objects, for example, the changing area of a deformable triangle, as well as invariant ones. We report preliminary findings from our work in informal settings. Our findings indicate that such experiences allow young children as well as undergraduates to yield expressive discourse that is intimately connected to the mathematical concepts being presented.

Introduction

Our project combines various technological ingredients to develop a new mathematical and scientific learning environment for allowing more students to access conceptually demanding ideas through various senses in K-5th grade classrooms. We are using a modified version of *The Geometer's Sketchpad®* software, the H3D Application Programming Interface and specific haptic devices.

Haptic literally means "ability to touch" or "ability to lay hold of" (Revesz, 1950) and has evolved in a technological era to be an interface for users to virtually touch, push, or manipulate objects created and/or displayed in a digital visual environment (McLaughlin, Hespanha, & Sukhatme, 2002). Haptic devices can provide force-feedback or tactile feedback. Haptic interfaces allow users simultaneous information regarding their input and reaction to that input as programmed within a computational environment.

The activities we are creating allow users to construct, interact and explore geometric figures and shapes, and so engage in model-eliciting activities in various mathematical topics. Our study is assessing and evaluating what new or enhanced learning experiences can be created by the synergistic integration of dynamic geometry with new haptic hardware, why this is necessary and how it can improve present practice.

We are designing new mathematical investigations that build on established curricular standards and frameworks, and assess the potential scalability of such an approach in the future given these intersecting research traditions. We can finally do this because of the availability of new affordable devices, but such technological advances are irrelevant without an aim to transform the mathematical and scientific activities of the classroom. The first stage of our project, which we report on here, investigates the educational potential for introducing such a multi-modal environment in different settings by investigating how it can engage different aged students to investigate mathematical ideas focused on change and variation. We particularly focused on features of *expressive discourse* that are elicited when students interact with our dynamic haptic geometry environment. We have analyzed whether the added modality of force-feedback can be engaging and rich for students in terms of how they can potentially discover new concepts, which has led to the emergence of design principles for future implementation in formal mathematics settings.

Theory

Haptic technology has evolved over the past 10 years, particularly out of a focus on virtual reality in the 90s, and has become more available in a variety of commercial and educational applications, including 3D design and modeling, medical, dental and industrial applications. However, research specifically examining haptics in relation to learning and education is scarce. A large proportion of the existing studies focus on "haptic perception" a major field in psychology focused on haptic sense, and the second main set was focused on multimodality. Multimodality reaches into education in various ways intersecting deeply with a multi-media approach. Recently, multimodal approaches have also focused on the role of gesture with increasing interest (Wagner Cook, Mitchell, Goldin-Meadow, 2008) with particular focus on this mode as a form of mathematical expressivity (Hegedus & Moreno-Armella, 2008).

Historically, this has been translated as a way to create multiple learning pathways for students to work within auditory and visual modalities. In fact, the audio/visual modality is still the predominant "multi-" media form. But, students can interpret visual, auditory and haptic displays to gather information, while using their proprioceptive system (making sense of the relative positions of one's own body parts) to navigate and control objects in their synthetic environment (Dede, Salzman, Loftin & Sprague, 1999). In this work, multiple sensory representations can offer mutually reinforcing information that a user can collect to develop an understanding of a mathematical or scientific model. In addition, haptics have been said to be superior to vision in the perception of properties of texture and microspatial properties of pattern (Zangaladze, Epstein, Grafton, & Sathian, 1999), while vision is more useful in the perception of macrogeometry particularly shape and color (Sathian, Zangaladze, Hoffman, & Grafton, 1997). Leveraging these relative strengths, our proposed activities will incorporate the need to investigate both shape and properties of construction (i.e., the mathematical structure of a figure or surface). We propose that it is relevant to integrate haptic technology with dynamic geometry software to offer multiple sources of information-feedback for students; it is not enough to offer a way for students to just see a mathematical object or a scientific model in a static way they must also engage with it dynamically, tactically and naturally.

Setup of Multi-Modal Technology Environments

Description of the haptic activity setup

The SenseAble PHANTOM Omni haptic device (See Figure 1) is capable of reading position and orientation data of a stylus in 3D space, which serves as the physical interface to the device. The device also provides force feedback to the user through the stylus. Users can grip the stylus and move it freely in 3D space with their choice of grip suitable to their intentions at any given time. We paired PHANTOM Omni with a laptop so that users are able to view a visual simulation as they simultaneously manipulate the device. Interviewers used the mouse and a separate keyboard to manage switching among activities and manipulating the parameters of activity variations.

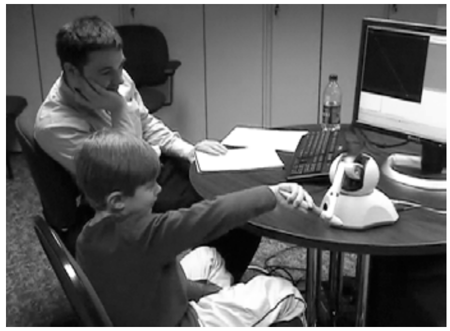

Figure 1. The SenseAble Phantom Omni haptic device

Description of the Types of Simulations

We aimed to elicit student expressivity associated with two types of simulations: *breaking point* and *impact*. A *breaking point* is defined as a point of resistance against the haptic stylus, where the resistance is removed once a large enough force is exerted on the stylus. An *impact* is defined as bumping into, or being bumped by objects within the haptic environment. These simulations were embedded into activities in which the students were not initially presented with any visuals. This protocol gave us the opportunity to hear students draw heavily on their haptic experiences without the additional influence of the visual simulation. Later, visuals were revealed in order to examine ongoing expressive discourse in the presence of additional stimulus.

One activity we embedded the simulation of *breaking point* is called *Falling Off*. Haptic experience in this activity consists of a resistant force occurring from a user's contact with a visual surface using the haptic device. As the user moves off the edge of the surface, the resistance is removed, which results in a feeling of falling off. Another activity involving the simulation of breaking point is called *Break Through*, wherein users encounter an unavoidable barrier in the center of the haptic environment. When contacted by the user, the barrier provides a resistant force, which can be removed once the user exerts sufficient force on the barrier. Additional haptic experiences can be added to either side of the barrier through the exertion of forces that oppose a user's movements within the haptic environment.

In, *Impact*, our second simulation type, students encountered two types of obstacles as they explored the haptic environment. The *Spheres* activity presented a stationary 8x8x8 lattice of spheres. Spaces among the spheres allow users to probe around and through the lattice, bumping into the spheres in the process. The second activity involving the simulation of impact is called *Bumped*. In this activity, the obstacle moves perpetually and repeatedly from left to right with a speed controlled by the researcher. The users feel being bumped by the obstacle when they explore the center of the haptic environment.

Methods

During the spring of 2010, we collected preliminary observational data in the form of video and field notes from 32 students (27 elementary and five undergraduate) interacting with the haptic activities. The students involved were chosen from three different populations within the Southcoast region of Massachusetts including local elementary schools, university freshmen and young children from a local urban city *Boys and Girls Club of America*. The observations occurred during informal interview sessions, where researchers asked questions to the students about what they thought and *felt* and how they related these experiences to the dynamic visuals on the screen. For undergraduate students, the interview sessions lasted about an hour; for elementary students, they lasted between 25-40 minutes. In this paper, we do not make a distinction among these students with respect to their developmental differences. Instead, we only provide a descriptive approach towards their haptic experiences to explore the features of their expressive discourse. Pseudonyms for student names are used.

We used the qualitative software product NVivo® to examine the use of expressive discourse. For example, we addressed the following: How did the student dialogue cohere with the expectations of the activity? What can we learn about the types of interview question and response patterns that inform future interface and activity design? We have used NVivo® to code for these types of expression as attributes for our particular cases (both students and activities). In order to answer the questions surrounding the activities' expectations the coded student discourse was cross-referenced with the set of expectations for each particular activity.

Our analysis of students' expressive discourse focused on how they could differentiate haptic experiences (e.g., "this felt like the ball was moving faster"). We initially identified three categories of expressive discourse. The first category is *metaphorical word use*. We define metaphorical word use as the set of utterances about objects or actions that are not present in the haptic-visual environment, but result from students' previous experiences. We further classified the use of metaphorical word use as scholastic (mathematical/scientific) or non-scholastic (non-mathematical/scientific). Scholastic use of metaphors relates objects or actions to mathematical or scientific objects students have previously interacted with. Non-scholastic use of metaphors is not related to mathematical/scientific objects but to students' everyday experiences.

The second category of expressive discourse is *reaction to sensual experiences.* We define a sensual experience as any form of expressivity that is a result of the students' sensations and reactions to the feedback (e.g., force, sound, or visual) they receive from the haptic device. We subdivided this category into descriptive aesthetics, evaluative aesthetics, and gesturing. Descriptive aesthetics pertains to utterances about the color, size, or feel of what students are seeing, feeling or hearing (Sinclair, 2004). Evaluative aesthetics differs from the descriptive aesthetics in that, instead of just describing the form (e.g. it's green), students are assigning some affective statement to their experiences (e.g. it's scary or it's beautiful). In addition to the verbal reactions pertaining to an activity's aesthetics, we also focused on students' sensual reactions through gesturing. We consider gesturing as a set of bodily movement for communicating in context, or collaboration with speech; either to oneself, a fellow student, or one of the interviewers.

The final category of expressive discourse is *developing a way of measuring and calibrating visual and physical spaces*. This corresponds to the students' routines as they attempt to measure and calibrate the physical and visual space during their haptic experience.

Results

Metaphorical Word Use

Examples of scholastic metaphorical word use were seen during an activity where students were presented a flat surface, which they could drop onto a small, white ball. When asked what would happen when the surface was dropped, one elementary school student (aged 8), Annie, predicted that it would fall to one side. This prompted another similarly aged student, Beatrice, to relate it to a "see-saw", and the white ball as the fulcrum. When asked to give a definition of fulcrum, Beatrice said "the center that keeps it together, like scissors". More metaphorical word use was seen with elementary school students during the *Break Through* activity. Students were presented with a barrier that provided resistance to passing through it, until enough force was exerted using the haptic stylus, thus releasing the resistance. As an explanation for this barrier, which some students described as a wall or "pressure", two groups of students reasoned that it had magnetic properties that "[didn't let] you go where you want". Undergraduate participants also used expressions containing mathematical metaphors. In one activity three undergraduates were shown a shallow, concave parabolic curve. The haptic stylus controlled a point displayed on the x-axis and, as they moved the point in the positive direction, students felt an increasing force, proportional to the parabola's slope, in the negative x-direction. In one student's explanation of this force-feedback, he referred to the feeling as "riding across sine waves".

The use of non-scholastic metaphorical word use, however, was more prevalent in our interviews. In one *Break Through* activity, when the force opposing user movement was activated, students described their movements as "heavier", "like rubber" or an "eraser on paper". In contrast, movements without the force were described as "lighter" or "freer". One student even compared their movements with, and without the force to being chased in a scary movie. The student related the opposing force to when they are walking and "you feel all that pressure, like someone's behind you, and you're scared". Movement without the force present represented a "relief" for the student.

Another metaphor discussed in a *Break Through* activity, related the force opposing movement to walking on a windy day. In the metaphor, the force was described by an elementary school student as "wind actually blowing you" towards the barrier. In a separate session, an undergraduate used the same metaphor, saying the force was like wind when you're walking, "there would be resistance to your movements." The *Bumped* activity also provided similarities between the elementary and undergraduate students. Two groups of students, one elementary and one undergraduate, described the periodic impacts as the beating or pumping of a heart, or a pulse. Along with the theme of periodicity, these two age groups also likened the period impact to a clock.

Reaction to Sensual Experiences

This type of expressive discourse classifies the communications of student reactions to sensual experiences during our activities. As an example, we again reference the force opposing movement in the *Break Through* activity. One student explained that when the force is not active the haptic environment "feels like it's empty", but with the force the environment has "something inside it." This explanation makes a description, in terms of a container, about the haptic sensation associated with moving through the environment.

The *Spheres* activity intends for students to describe how a lattice of spheres hindered or modified their movement within the haptic environment. During the interviews with both elementary and undergraduate students, movements across, and within the lattice were described as bumpy 20 times. Again, these "bumpy" descriptions pertain to student reactions to sensations caused by their movement through the environment.

Evaluative aesthetics differs from the descriptive brand, because students make affective statements rather than just talking about the forms of their sensual experiences. For example, in the *Break Through* activity, when one student pushed against the barrier he described it as "weird". When further evaluating the experience he said, "it feels like rubbing on rubber…it's weird." Some elementary school students exploring the spheres in the *Spheres* activity began to associate the colors of the spheres with how "hard" they felt. Interestingly, however, the stiffness of the spheres in the activity was identical. In the *Bumped* activity, one group of students indicated that the impact from the moving obstacle was emotionally affective. The students said that the unexpected impacts became "scary", because the sphere "just jumps out of nowhere".

In addition to the verbal reactions pertaining to an activity's aesthetics, students in our interviews also expressed their sensual reactions through gesturing. The *Bumped* activity provided a number of gestural reactions, mainly because it was the only activity offering objects that were out of the students' control. In each of our other activities, students were the initiators and producers of sensation on objects in the haptic environment. However, the students could not control the moving obstacle in *Bumped*. Therefore, students would make gestures to either avoid coming into contact with the obstacle, or seek out contact with it. This type of gesturing occurred when the obstacle was both invisible and visible.

In one interview, a student named Catie discovers that there is a different haptic sensation that results from pulling the device close to her body, rather than towards the base of the device. When the next student, David, begins using the device, Catie gets the attention of David and makes a gesture, pulling her arm towards her body, in an effort to show David the difference. While Catie makes the gesture, David mimics it, presumably in an attempt to achieve the same sensation as Catie.

One student, while moving through the *Bumped* activity used a gesture to test if they were causing the impacts or if the impacts were independent of their movement. While the student, Ethan, was moving the haptic device he noticed the device providing force-feedback. Ethan claims that he is not the source of the feedback, saying the device is "doing it on its own." Another student, Fred, questions Ethan by saying, "how do we know *you're* not touching it." In response, Ethan rests the haptic stylus in the palm of his hand, looks at Fred, and says, "see, I'm not doing it!" Fred appears to be satisfied with the evidence. This episode is quite significant because it illustrates a gesture powerful enough to convince another student that the sensation felt by Ethan was not caused by Ethan's movements.

Developing a way of measuring and calibrating visual and physical spaces

In some cases, fifth-grade students compared the regular structure of the lattice of spheres within the *Spheres* activity to familiar objects with regular structure. Students most often described the spheres as discrete objects, such as "bumps" or "rocks." As they explored, students also characterized the arrangement of the objects in space, starting with horizontal structure and then adding in a vertical component. One group offered, "it feels like you're going over bump after bump," noting repetition left to right. They later decided "it feels like stairs because it's going higher and higher," adding in the vertical. In this group, "stairs" became an accepted description adopted among the students and repeated: "like going down stairs," "like rocks piled up as stairs." Another group, that followed a similar pattern of describing horizontal and vertical structure, was more systematic in their investigations and specific in describing how the horizontal and vertical structure interrelated. Moving from left to right, a student sought to describe the repetition additively: "If I move one over, there's one more, and one more over there. Once you go over here, there's no more." This group also explored the vertical pattern in conjunction with the horizontal, using stairs as a way to describe change in two dimensions: "Every time I move down there's something stopping, and up, and sideways, kinda. […] As you go up, it stops, up, stop. [...] It's kinda like stairs, you stop it goes down, stop it goes down." This group also adopted "stairs" as one shared way to describe one aspect of the structure of the matrix.

The use of stairs to relate vertical and horizontal structure appeared during the non-visual portion of the *Spheres* activity. Once the students were shown the sphere lattice, one face of this three dimensional lattice was immediately revealed to them. The nature of flat computer monitors meant that the depth of the 3D model was obscured by the front of the lattice. But, students still noticed the depth in their explorations with the haptic stylus and in the visuals. In one third-grade group, Doug, Ellen and Fiona wondered aloud about the structure of this simulation in the depth dimension. By counting the visible spheres and using multiplication, Fiona decided that a face had 35 spheres (the proximity of the virtual camera to the lattice meant that some of the spheres were out of the field of view, and students counted a 5 x 7 lattice in the front face). As Ellen began to count the spheres by tapping them with the haptic stylus, Doug and Ellen agreed that the front face was not the entirety of the lattice. “35 is just the top.” They sought a way to complete the count in the depth dimension, which was more difficult to see on the screen. Ellen said, “You can't tell how many there are down, so you can't tell how many there are.” She went on to elaborate as she explored in the depth dimension: “If you, like, move it farther away and closer to the machine you can kind of feel the ones on the bottom, but not really.” The laptop screen was tilted away from her, and her descriptions appear to associate “down” with depth. As she continued to probe, they discussed how one might arrive at a full count of the spheres. “You would do 37, I mean 35 times how many there is down.” While Doug attempted to tell how many there were based on the shadows of the spheres in the display, Ellen began counting aloud. As she moved the stylus closer, and then farther away from herself she counted: “One, two, three, four...” When asked whether she was counting the spheres on the laptop screen, she said, “No. Feeling them.” It appeared that the desire to get a full count of the lattice and the difficulty of getting accurate visual confirmation led Ellen to count by feeling variation along one specific dimension. Communication among the students and the researchers during this activity involved both expressive word use and gesture, as Ellen used deliberate and exaggerated poking motions to illustrate her count while we all attended to her efforts.

Discussion

From the findings that relate the expectations of each activity and students’ expressive discourse, we formulated a set of haptic design principles. For example, activities that provide resistance in the form of a breakable wall (i.e. resistance against moving then, after a certain level of force is exerted, no resistance) are described by students as containing stoppages, “getting stuck” and preventing them from “going where you want”. We also discovered that students were able to discriminate varying levels of viscosity, when the viscosity change is preceded by a breaking point.

No visuals in an activity often led to students exploring positions within the haptic environment where they posited haptic sensations would occur. Student discourse also indicated that haptic sensations could incite affective statements. For example, when students encountered an object moving along a track, the frequency of impact led to unsettled emotions (e.g. “creepy”, “scared”, “losing control”).

The analysis also led to the identification of distracters in the activities, defined as any element of an activity (i.e. visual or haptic) that distracts the user from the design intentions of the activity. For example, in the *Falling Off* activity, students were more focused on the visible, squiggly lines on the surface instead of exploring the feeling of moving off the surface. Further, we found that students who worked within haptic activities more than once were engaged (i.e. attentive to the interview questions and activity tasks) throughout the 25-30 minute interview. In fact, some students were more active as their interaction with the haptic environment increased.

In creating haptic principles and identifying distracters, we have now discovered a set of haptic experiences that have been shown to be successful in eliciting certain student discourse. Using this knowledge, we are designing more mathematical haptic activities, in preparation for a main study during the spring of 2011.

Students' expressive discourse was particularly rich in their reaction to sensual experiences and use of metaphors. Students' genuine participation in the experience was reflected in their gestures and utterances of surprise, excitement, shock, etc. which provided a natural context in which they developed and tested self-generated hypotheses. The instances in which students shifted their everyday metaphors to scholastic (mathematical/scientific) ones made us wonder the characteristics of the contexts in which they moved between the informal and formal discourse during their experiences. Since our future goal is to use the haptics environment to develop and implement mathematical activities, adding a fourth category to our initial classification of expressive discourse seems necessary. The fourth category would mainly focus on how, and under what conditions, students move back and forth between everyday words to the more technical/formal, mathematical words. We are also interested in how students cope with the notions of variance and invariance as well as continuity and discreteness as they conjecture about mathematics through their haptic experiences in physical environments.

In summary, a new set of design principles emerged from our analysis of expressive discourse as we take steps towards making the haptics experience mathematically relevant and significant. We believe that students' self-motivated hypothesizing/theorizing through interaction with peers, as reflected by their discourse in our preliminary study, is crucial for mathematics learning. The haptics environment presents many opportunities to contribute to the teaching and learning of mathematics.

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DECIDING WHAT TO TRUST: CONFLICT RESOLUTION WHEN CHECKING WITH A GRAPHING CALCULATOR

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This paper reports on a mixed methods study of 111 AP calculus students' self-reports of their graphing calculator use, comfort, and rationale when choosing between mathematical solutions produced with and without a graphing calculator.

Graphing calculators have taken on an important role in learning mathematics, especially at the secondary level (Dion et al., 2001). Graphing calculators have been reported to have been used by over 80% of U.S. high school mathematics teachers and are currently allowed on more than 70% of the U.S. state's mandated standardized tests (Texas Instruments, 2010; Weiss, Banilower, & Smith, 2001). In addition, they are allowed on 100% of college entrance exams and AP calculus, statistics, physics, and chemistry exams (College Board, 2010). One way in which a graphing calculator is often promoted in the classroom is as a tool for checking or verifying work done by hand (e.g., Doerr & Zangor, 2000; Harskamp, Suhre, & van Streun, 2000; Hennesey, Fung, & Scanlon, 2001; McCulloch, 2005; 2009; Quesada & Maxwell, 1994). The fact that students use the tool to check their work is not surprising. However, the literature has not considered the benefits and constraints for students when using the tool in this way. In this paper we report on a mixed methods study of 111 high school Advanced Placement (AP) calculus students' uses of graphing calculators to check their work and their methods for reconciling any differences between solutions found by hand and with the technology.

Background

Graphing Calculator as a Checking Tool

Doerr and Zangor (2000) investigated the ways in which students use graphing calculators to support mathematical learning. They found that the graphing calculator was often used as a checking tool "when it was used to check conjectures made by students as they engaged with the problem investigations" (p. 156). Quesada and Maxwell (1994) used open-ended questions to analyze students' thoughts on the positive and negative aspects of the graphing calculator. They found that a majority of students felt that the ability of the graphing calculator to check answers was a positive aspect of the tool. From additional interviews with students, the researchers found,

> The statement 'the ability to check their answers' was interpreted by some to mean not only the capability of confirming graphically or numerically the answers obtained algebraically, but also the ability (new for many of them) of thinking graphically about problems before trying to solve them algebraically" (Quesada & Maxwell, 1994, p. 213).

Other researchers have observed the promotion and use of the graphing calculator as a checking or verification tool in the classroom (e.g. Berry, Graham, & Smith, 2006; Doerr & Zangor, 2000; McCulloch, 2009), and researchers have seen that the use of the tool in this way can be important for creating an understanding of graphing (Hennesey, Fung, and Scanlon, 2001), and for allowing students to support their analytical work with graphs and tables (e.g. Waits & Demana, 1994). Researchers have not yet, however, clearly determined why a student chooses to use the tool to check (Berry et al., 2006).

Confidence and the Graphing Calculator

Many students recognize the effectiveness of the calculator for confirming or checking the reasonableness of answers, but they struggle when answers obtained on the graphing calculator do not match their own expectations or work they have done on paper (Kenney, 2008; McCulloch, 2005; 2009). This relates to a level of confidence that students have both for their own work and for the graphing calculator as an effective tool in problem solving. Many researchers have been able to find improvements in assessment scores when students use graphing calculators (e.g., Ellington, 2003), and some have suggested that a possible reason for such improvements is that students are more comfortable or confident when they have a graphing calculator to use (Dunham, 2000). However, it seems that even when students feel that graphing calculators are useful, they may often lack confidence in the calculator's ability to help them in problem solving (Graham, Headlam, Honey, Sharp, & Smith, 2003) and may put more trust in their own work than in the calculator. In other cases, students' confidence in their mathematical ability contributes to a lack of perceived need for verification with the graphing calculator (Mesa, n.d.)

On the other hand, there is concern that students can become over-confident in the graphing calculator, using the device as a "black box" and blindly accepting calculator output (Doerr & Zangor, 2000; Forster & Taylor, 2000). Doerr and Zangor (2000) explain that this occurs when learners depend on calculators to produce answers without attending to the meaning, purpose, or interpretations of the problem situation. They found that this "black box" use results when neither the student nor the teacher provides meaningful strategies for calculator use. Similarly, Goos, Galbraith, Renshaw, and Geiger (2003) suggest the calculator can take on the role of "master" for the user and that students can become overly dependent on the tool when "lack of mathematical understanding prevents them from evaluating the accuracy of the output generated by the calculator" (p. 78). When used in these ways, the graphing calculator can become a source of mathematical authority for the user (Williams, 1993; Wilson & Krapfl, 1994) and be over-used to the point that students rely on the calculator with little critical analysis of the results (Burrill et al., 2002).

Research has not yet looked in detail at the level of confidence that students have for their own work or for the graphing calculator itself when using the tool in problem solving. In this study, we focus on this issue by examining where students place their trust when conflicts arise in problem solving with a graphing calculator. In this study, we address the research questions: What do students say they do when a solution produced without technology does not match one produced with technology? How do they say they reconcile this situation?

Methods

Data Sources

Advanced Placement (AP) Calculus classes were chosen as the focus for this study because the curriculum and expectation of calculator use is relatively consistent nationwide as it is set by The College Board. To ensure that the population of students was as diverse as possible, this study was set in four high schools in the northeastern United States. High School A is located in a low-income urban community, High School B serves students from both suburban and rural communities, High School C serves students in an affluent suburban community, and High School D serves students in a middle class suburban community. All four of these schools provide their AP Calculus students with a graphing calculator to use at home and at school. High Schools A, B, and D provided their students with a TI-83+, while High School C provides the TI-89 (which has Computer Algebra System (CAS) capabilities). For the purposes of this study, the term graphing calculator refers to both calculators.

All of the students at these four schools (n = 111; 49 female; 62 male) completed a survey instrument designed (based on pilot studies) to identify the ways that they typically use a graphing calculator, both in and out of the classroom, and their comfort in doing so. In particular, the survey provided data on student demographics, mathematical achievement as determined by their math teacher, frequency of graphing calculator use and comfort with the tool. In addition, an open ended item was included that read:

> *Imagine the following situation: You solved a problem on your own and then used your graphing calculator to check your solution. The calculator gave you a different solution than the one you got when you worked the problem on your own. Which answer do you trust? Why?*

All 111 respondents provided answers to this question on the survey, in many cases surprising the researchers with the amount of detail included in their responses. The responses to this open-ended item are the focus of this paper.

Data Analysis

All categorical data from the survey instrument were entered into an Excel file and each written response to the open-ended item above was first coded for solution chosen in the reconciliation process (i.e. graphing calculator (GC), non-graphing calculator (non-GC), or neither). We then examined all possible associations between descriptive student characteristics and solution chosen. Since the descriptive data were categorical in nature, a chi-squared test for association was used. Associations were examined between solution choice and each of gender, teacher, teacher-rated mathematics ability, and student-reported comfort using a graphing calculator. Next, we analyzed the written responses to the open-ended item using a thematic content analysis process (Coffey & Atkinson, 1996). After coding each response based on the solution ultimately chosen in the reconciliation process, we examined students' written responses for emerging themes within each solution group. This resulted in the development of a codebook with 12 data-driven codes. The codes that emerged within each group of solution choices (GC, non-GC and neither) directly corresponded with each other, regardless of the ultimate solution choice, and fell into four larger categories. Our resulting list of these four categories appears in Table 1 below. It is important to note that the assignment of codes to student responses were not discrete. For example, the response "If I'm not sure, the calculator. If I'm sure, my answer. I probably plugged something wrong into the calc," was coded as both confidence in math ability and careless errors. In the next section, we provide further details and interpretations for the coding results.

Table 1

Categories for rationale of solution choice

Category	Definition
Careless errors	Student notes that "careless errors" (either arithmetic or syntactical) are possibly the cause of any discrepancies between GC and non-GC solutions.
Check work	Student notes that either the GC or non-GC (or both) solution(s) must be checked for small errors and, barring any small errors, ultimately accepted.
Recognition of GC affordances and limitations	Student notes either affordances or limitations of the GC in their reasons for accepting or rejecting a GC solution.
Confidence in math ability	Student notes acceptance or rejection of a GC solution is based on confidence (or lack there of) in own math ability.

Given the situation posed in the item, that of having to reconcile a GC produced solution with a different non-GC produced solution, 60 out of 111 students (54%) wrote that they would ultimately choose a GC produced solution, 39 (35%) said they would choose their own work (non-GC produced solution), and 12 (11%) did not make a definitive choice between the two (Figure 1). No associations were found between the solution choice and gender $(X^2(2, N=111)=2.649, p=.266)$, teacher $(X^2(6, N=111)=8.231, p=.222)$, teacher rated mathematics ability $(X^2(4, N=11)=2.603, p=.626)$, or student reported comfort using a graphing calculator $(X^2(4, N=111)=4.051, p=.399)$. The most commonly provided explanations for deciding what to trust, regardless of solution choice, were students' concerns about making *careless errors* (n = 58, 52%) and students' suggestions that they would *check their work* before choosing which solution they trust (n = 43, 38%). A summary of the frequency of codes within each solution category appears in Table 2 below.

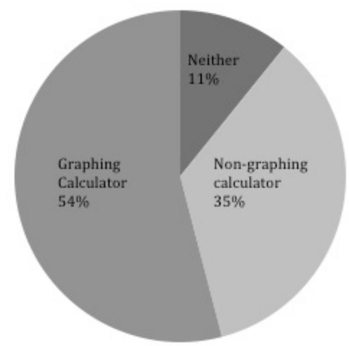

Figure 1: Student solution choices

Table 2

Code frequency in each solution preference category

	GC solution	Non-GC solution	Neither	Total
Careless errors	36	18	4	58
Check work	22	14	7	43
Recognition of GC affordances and limitations	13	6	1	20
Confidence in math ability	5	6	1	12

Note: Codes are not discrete

The following sections report a more detailed look at these findings. We have sorted the findings by the three choices made by students – GC solution, non-GC calculator solution, and neither – and include at least two examples of students' statements as they relate to the categories of rational identified.

Trust the Graphing Calculator Solution

Careless Errors

As noted above, 54% of the students responded that they would choose a GC produced solution over a non-GC produced one. More than half of these students' reasoned that they would choose the GC solution because it is easy to make careless errors when working by hand. The following statements by students in this category exemplify their responses:

- "I would trust the calculator because it is easy to make a careless mistake in computation when without a calculator and the numbers are large."
- "The calculator as long as I am entering it correctly because I have a greater margin of error than does the calculator."
- "I trust the calculator because of human error."

Students not only recognized that careless errors by hand were likely, but some (n = 6) also noted that they would use the situation to help identify their errors (e.g. "I would trust the calculator because everyone makes mistakes, so I would use that proposed answer and work back and see my mistake and fix it"). On the other extreme, a few students noted that they trust their calculator solutions so much that they would change their written work to match it. For example, one student wrote, "Calculator. I am a very confident calculator user. I'd try to change my work to match the calculator answer."

Check Work

Not all of the students automatically identified the GC solution as the one they would trust. Sixteen students noted that they would first check their work, both written and GC produced, before choosing the GC solution. The following responses are examples of this:

- "I would double check my work and also how I entered the problem into the calculator. If they still don't match, I would trust the calculator's answer."
- "Well I would actually make sure I plugged in everything correctly into the calculator. If that was right, then I would doubt my own solution. So I'd trust the calculator answer."

Although these responses might imply a similar feeling to those in the careless error category, we see these students' willingness to check all work first as demonstrating more than just blind trust in the technology.

Recognition of GC Affordances

Justification for choosing a GC produced solution was also attributed to a belief in the infallibility of the GC by 13 students. This reasoning was evident in responses such as:

- "Calculator – we can make math errors but the calculator doesn't."
- "Unlike humans, calculators don't make computational mistakes for no apparent reason."
- "Calculator. The only possible error made by a calculator occurs when a wrong number, equation, etc. is entered. Room for error on the calculator is restricted."

We have identified this reasoning under the broader category of limitations and affordances of the graphing calculator due to students' beliefs that one of the affordances of the calculator is that it does not make errors.

Confidence in Math Ability

Five of the 60 students reported that they trusted the GC due to a lack of confidence in their own mathematics abilities. For example, students answered:

- “Calculator, I have no confidence in my math abilities.
- “I would trust the calculator because I am usually wrong and I have had success in the past with trusting the calculator.”
- “The calculator. It is better at algebra than me.”

Interestingly, only one of the students in this category was rated as a relatively lower ability student by the teacher, one other was rated average, and the other three were considered to be among the strong students according to the teacher. Thus, student responses show a disconnect between their confidence in their ability to do mathematics and their actual performance in class.

Trust the Non-Graphing Calculator Solution

Careless Errors

Thirty-five percent of the calculus students reported that they would choose a non-GC produced solution in the situation that it did not match a GC produced one. Like those that chose GC produced solutions, many in the non-GC category also considered the possibility of careless errors when making their decisions (n=17). The difference, however, was that these students were not as concerned about errors in their written work as they were about errors they may make pressing the calculator buttons. For example, students suggested:

- “I probably entered it incorrectly on the calculator. I’d try to punch it in again.”
- “My own answer is probably right; I have finger-calculator problems. Usually the reason for the disparity between my value and the calculator value is the lack of parentheses or wrong decimal place.”

Unlike the GC choosers, none of these non-GC students who used the careless error reasoning suggested using the situation to help identify their errors. They were not as concerned with determining how to get the correct solution on the calculator if they trusted the work they had done by hand.

Check Work

Thirteen of the non-GC choosers responded that they would trust their non-GC produced solution after checking their work for mistakes. For example, students stated:

- “I’ll check my work again and if I didn’t do anything wrong then I’ll trust my work.”
- “I would double check my work, and then use my answer because I could have easily input something wrong in the calculator.”

As with the GC group, these students were not willing to blindly accept one answer over the other, but did demonstrate more confidence in their own work. Many of the reasons in both this and the careless error category were related primarily to students’ personal experiences, as evidenced by their discussions of what they “often,” “easily,” or “usually” do.

Recognition of GC Affordances

Like the GC choosers, non-GC solution choosers noted confidence in their mathematical abilities or their beliefs about the affordances or constraints of the GC in their justifications. However, unlike the GC produced solution choosers, these students placed the authority in the situation with themselves. For example, six responses for the non-GC choosers focused on the limitations of the GC as a reason for not trusting its solution. Students shared:

- “I trust the answer I would have gotten on my own. The calculator does not show all the steps and it is easy to make mistakes when putting information into the calculator.”
- “I trust my own, sometimes the graphing calculator comes up with weird answers using trig functions or does not find the right answer.”

- "I would trust myself because the calculator tends to make mistakes sometimes in graphing when the equation isn't entered properly."

It is interesting to note in the last two examples that students place the blame for mistakes on the calculator, when in fact mistakes may be a result of their own careless errors. We placed them in this category, however, based on the students perceptions of the tool, not our own.

Confidence in Math Ability

Six students noted their *confidence in their mathematical abilities* (rather than their lack of confidence) as their reasons for trusting the non-GC solution. Responses included:

- "As long as I am confident with the answer I got, and was not very unsure with it in the first place, then I would trust my own answer not the calculator's."
- "Myself, because I am able to do the problem on my own, I wouldn't even need to use the calculator."

It is important to note that none of the students who noted confidence in their mathematical abilities as justification for choosing a non-GC produced solution were considered by their teacher to be among the highest ability students as compared to their peers.

Trust neither solution

Twelve of the 111 students that participated in the study did not make a definitive choice between either the GC or non-GC produced solutions. The responses of these students indicate that they do not necessarily value or rely on one solution over the other. Instead, they all noted the importance of rechecking their work, on both the GC and paper, to identify errors and to understand why the solutions differed. For example,

- "I recheck the calculator first and then my own answer. I check both and trust neither."
- "I don't trust either until I figure out where the mistake is. I either made a mistake with the math or I typed something into the calculator wrong. When the answers match, I trust the answer."

Again, these students appear to be considering the same things as the GC and non-GC choosers (i.e. checking their work and the possibility of careless errors on paper and in button pushing), however they place equal value with each method. Most importantly, they do not exhibit blind trust in either solution method. However, the responses do suggest that, as long as the two solutions do match, students would feel confident with the answer.

Discussion

Our findings in this study show that, whether they would choose the GC or non-GC solution when a conflict in answers arises, students are considering similar factors:

- The possibility of careless errors
- The importance of checking their work
- The constraints and/or affordances of the graphing calculator
- Confidence in their mathematical abilities

More than half of the students chose the GC solution over the non-GC produced solution. Many of the students' reasons for doing so were attributed to lack of confidence in their mathematical ability or to an over confidence in the infallibility of the GC. This finding is consistent with Goos et al. (2003) who found that students sometimes develop relationships with graphing calculators in which the graphing calculator is viewed as the "master". This suggests that it is truly important for teachers to be aware of the issue of mathematical authority. The students surveyed here are high school calculus students – often the best and brightest at their schools – and we find that more than half of them are handing the authority in a mathematical situation to the tool over themselves. This raises concerns for the results we might find with a group of less strong students.

On the other hand, the large number of students that noted they would choose the GC because they are concerned about making careless errors in their written work gives credence to the literature that has suggested that students attitudes towards mathematics increases when GC's are available because having the tool increases their levels of confidence (e.g. Dunham, 2000). If having a GC available can decrease students' concerns about the occurrence of careless errors, it is possible that students may spend less time worrying about small mistakes and more time focusing on thinking deeply about the mathematics.

It is promising to see that more than half of the students, regardless of solution choice, noted the need to check their work. This suggests that these students are thinking critically about what might have caused the difference in answers, rather than just accepting one as true. On the other hand, it is possible that those students who did not mention checking might have assumed that the situation was placed in a testing situation, one in which they may have assumed they would not have time to go back and check their work and would thus be forced to choose a solution and move on. Such interpretations should be examined more carefully in future studies.

In conclusion, given the prevalence of graphing calculator use in U.S. high school classrooms and standardized tests, these tools are likely going to be a mainstay in school mathematics for quite some time (Weiss, Banilower, & Smith, 2001; Texas Instruments, 2010; College Board, 2010). The results of this study suggest that while such promotion may be beneficial, it needs to be handled carefully so that students do not blindly place mathematical authority with the tool when reconciling differing solutions. To build a better understanding of these phenomena, further research is needed on how graphing calculator use and decision making is being promoted by teachers as well as how students perceive this promotion. In addition, it is important to investigate if these results hold true for other technology tools such as computer algebra systems, spreadsheets, and dynamic geometry systems.

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A COMPARISON OF MIDDLE SCHOOL STUDENTS' ARGUMENTS CREATED WHILE WORKING IN TECHNOLOGICAL AND NON-TECHNOLOGICAL ENVIRONMENTS

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This study characterized and compared the arguments eighth grade mathematics students created while working in technological and non-technological environments. Toulmin's (1958/2003) argumentation model was used to analyze the content and structure of the arguments, including the ways in which the students used the tools (technological and non-technological). Students in both classes were less likely to make their reasoning explicit when using tools, technological or non-technological, which may be related to the task on which the students were working and the use of the tools as a visual referent.

Background

The development of mathematical reasoning and understanding are culturally mediated, through language and through the use of tools (Lave & Wenger, 1991; Vygotsky, 1978). From this socio-cultural perspective, there is a strong connection between students' uses of tools and their learning because tools are the instruments of access to the knowledge, activities, and practices of a classroom community (Lave & Wenger, 1991). Productive activity with the tools and the understanding of them are not separate. Verillon and Rabardel (1995) indicate that tools do not exist as tools until the user is able "to appropriate it for himself and has integrated it with his activity" (p. 84). The use of tools and understanding their significance interact to become a single learning process.

In mathematics classrooms, tools are frequently used. Traditional tools such as compass, straight-edge, ruler, and scale have been used for thousands of years to explore, conjecture, and make meaning of mathematics, especially geometry. With the advent of the microprocessor, technology tools have become more prevalently used in the mathematics classroom. One type of technology tool that has become popular in the teaching and learning of geometry are dynamic geometry environments (DGEs), such as The Geometer's Sketchpad (Jackiw, 2001). Many teachers, researchers, and professional organizations have suggested the use of dynamic geometry environments to teach geometry (e.g. NCTM, 2000).

The use of tools is an important aspect of participating in a community of practice. Similarly, the use of language plays an important role for an individual to participate in the community. Although it is important for individuals to learn from the discussions of other members of the community, it is even more important to learn how to speak in the community of practice (Lave & Wenger, 1991). The affordances of the tools dictate the understandings that one can construct within the community and, in turn, the understandings that can be developed by the classroom community. Thus, the discourse of the classroom, which includes argumentation, is based, in part, on the use of tools. As technological tools become more prevalently used in the classroom, in what ways does the content and structure of the arguments students create differ when using technological and non-technological tools? The purpose of this study is to describe the arguments created by students enrolled in two eighth grade mathematics classes, one in which the students regularly use a DGE and one in which the students use non-technological tools, and compare the content and structure of the arguments created by pairs of students in each class.

One model that has been used by several researchers (e.g., Hollebrands, Conner, & Smith, 2010; Lavy, 2006; Stephan & Rasmussen, 2002) in mathematics education to examine students' mathematical arguments is Toulmin's model of argumentation (Toulmin, 1958/2003). Toulmin decomposed an argument into six components: claim (the conclusion whose validity is being established), data (the facts being appealed to as the foundation of the claim), warrant (the link between the data and the claim), backing (circumstances in which the warrant would otherwise be invalid), qualifier (confers the strength of the warrant), and rebuttal (circumstances in which the warrant does not hold). Figure 1 shows how these six components (data, claim, warrant, backing, qualifier, and rebuttal) fit together. Data is provided or constructed and "so" a claim is made based on this data. This claim can be made based on this data "since" the warrant. The warrant is relevant "on account" of the backing. The claim is valid "unless" the rebuttal occurs.

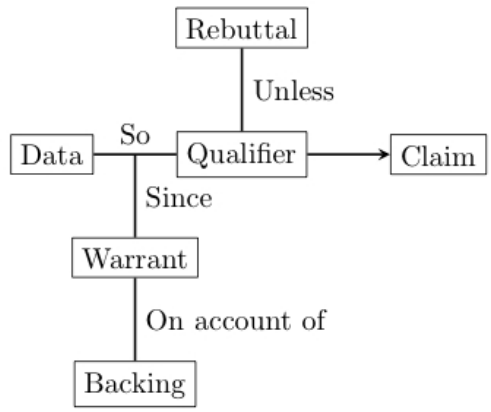

Figure 1. Toulmin's (1958/2003) Model of Argumentation

Methods

For this study, two classes of eighth grade students were selected to participate in a two-week classroom teaching-experiment on triangles taught by the author. I conducted the study with students of varying ethnicities and socio-economic status at an urban public middle school in the southeast United States. For one of the classes, technology played an integral role. Students used a dynamic geometry environment, The Geometer's Sketchpad (Jackiw, 2001), to explore and investigate geometric concepts. The majority of the tasks developed for this unit utilized this technology. For the other class, the students used traditional mathematical tools such as scissors, rulers, compasses, protractors, and snap-cubes. For both classes, the tools provided students with a means to reason and develop new understandings about the geometry concepts. In order to minimize the amount of variation between the two classes, the selection and sequence of tasks for each class were similar. Furthermore, the design and implementation of the majority of the tasks and activities for both classes were similar in nature with the major difference being the tools available to the students. In some instances, the tasks differed to capitalize on the affordances of the tools.

During the study, the students in both classes were placed in pairs. By having the students work in pairs, the students were given the opportunity to have discussions with their partners while working on the tasks. These discussions were the primary focus of the study's analysis. I chose to use pairs rather than larger groups to maximize the opportunities for students to interact with the mathematical task while still having peer-to-peer discourse. Four pairs from each class were purposefully selected to be the focus of the data collection and three pairs were selected for analysis based on the students' attendance. To select the pairings, I asked the regular classroom teacher to identify students that would be willing to verbalize their thinking and work well together. The pairs of students used in the analysis from the technology class were Amy and Judy; David and Erica; and, Heather and Mary[1]. The pairs of students from the non-technology class were Andy and Frank, Bob and Ellen; and, Clair and Jim[1].

Data collection consisted of: video and audio recordings of the two classes both small group and whole class discussions; video-recordings of the computer screen to capture the students' uses of technology; and, artifacts which include copies of students' written work including in-class work, homework, quizzes, and exams. Of each class's eight class meetings, the researcher only analyzed the small group and whole-class discussion for three tasks, although results from the analysis of one task, the triangle inequality, will be presented in this paper. I transcribed the video recordings of the whole class discussions and small group discussions. From these transcriptions, reasoning episodes were established by identifying claims. I then created a description of the argumentation episodes for that claim which included the participants' words (from the transcripts) and actions including the students' uses of the mathematical tools. Then, I diagrammed the argument according to the model developed by Toulmin (1958/2003) including each of the six constructs, data, claim, warrant, backing, qualifier, and rebuttal. For a given argument, some of these components were not specified and I had to make some inferences. In these cases, I noted these inferences and attributed them to a student, the teacher, or a combination of students and/or teacher. In the diagrams, I used a box to outline each of the spoken or known constructs. If I made an inference, I used a "cloud" to note the inference in the diagram.

Results

The results reported in this paper are based on data taken from tasks in which the instructional goal was for the students to develop an understanding of the triangle inequality theorem. The students in the technology class used a pre-constructed sketch to explore whether given sets of segments would form a triangle. The sketch allowed the students to adjust the lengths of the segments using sliders and to drag the segments without changing their lengths. The students in the non-technology class completed the same task using snap-cubes and rulers. While working on the triangle inequality tasks, the three pairs of students from both classes created arguments of various structures and contents. Three themes emerge in the comparison of the content and structure of arguments: the difference in the number of arguments between the two classes and the frequency in the use of the technology/tools; the explicitness of the warrants and the use of technology/tools; and, the content of the additional data.

The Number of Arguments and the Frequency in the Use of Technology/Tools.

The first theme to emerge in the analysis of the arguments is the difference in the number of arguments created by the pairs of students in each class. The pairs of students working in the technological environment created 75 arguments. In comparison, the pairs of students working in the non-technological environment created 56 arguments. This difference is even greater considering the students in the non-technology class had twenty additional minutes of class time to create arguments when they were asked to compare their answers to exercises on the homework with their partners. Removing these arguments, the students in the non-technological class only created 44 arguments. In addition to the discrepancy in the number of arguments, there was also a difference in the proportion of arguments in which the students used the technology/tools.

While working on the triangle inequality task, the pairs of students in the technology class employed the technology in 59 (79%) of their arguments. The pairs of students in the non-technology class created 19 (34%) arguments in which they used the tools. The higher proportion of arguments in which the pairs of students employed technology is most likely due to its use during the opening activity. During this activity, the teacher did not intend for the students to use the technology. However, the students requested to use it to assist them in solving the problems. The students in the non-technology class did not request to use the snap-cubes during a similar opening activity. At times, the pairs of students in the non-technology class used a ruler to make a sketch, but this was infrequent. Perhaps, the students in the technology class viewed the technology as a tool that can assist them in solving problems. The students in the non-technology class may not have viewed the snap-cubes in this manner. Rather, the students may have seen the snap-cubes as a means to create data to make a generalization, but not to solve problems.

The Explicitness of the Warrants and the Use of Technology/Tools

Although the pairs of students in the technology class created more arguments, the content and structure of the arguments in relation to the use of tools, both technological and non-technological, were similar for both classes. When working on the triangle inequality tasks, the pairs of students in both classes were more likely to provide an explicit warrant when not actively using the technology/tools than when the pairs of students created an argument using the technology/tool.

Arguments created by the pairs of students working in the technological environment. Of the 59 arguments in which the pairs of students in the technology class employed technology, they only provided an explicit warrant for 6 (10%) of these arguments. Of the 16 arguments in which technology was not actively employed, the pairs of students provided an explicit warrant for 12 (75%) of these arguments. The explicitness of the warrants may be related to the type of tasks on which the students were working. In general, when the students were using the technology, they were merely attempting to determine whether a triangle could be formed with a set of segments. For example, Judy and Amy attempted to determine whether a triangle could be formed with segments of lengths 3, 4, and 4. Amy adjusted the sliders accordingly and dragged the endpoints of the diagram to form a triangle. Judy claimed that a triangle can be formed by stating, "Possible" and Amy agreed by stating, "Yep." This argument is illustrated in Figure 2a.

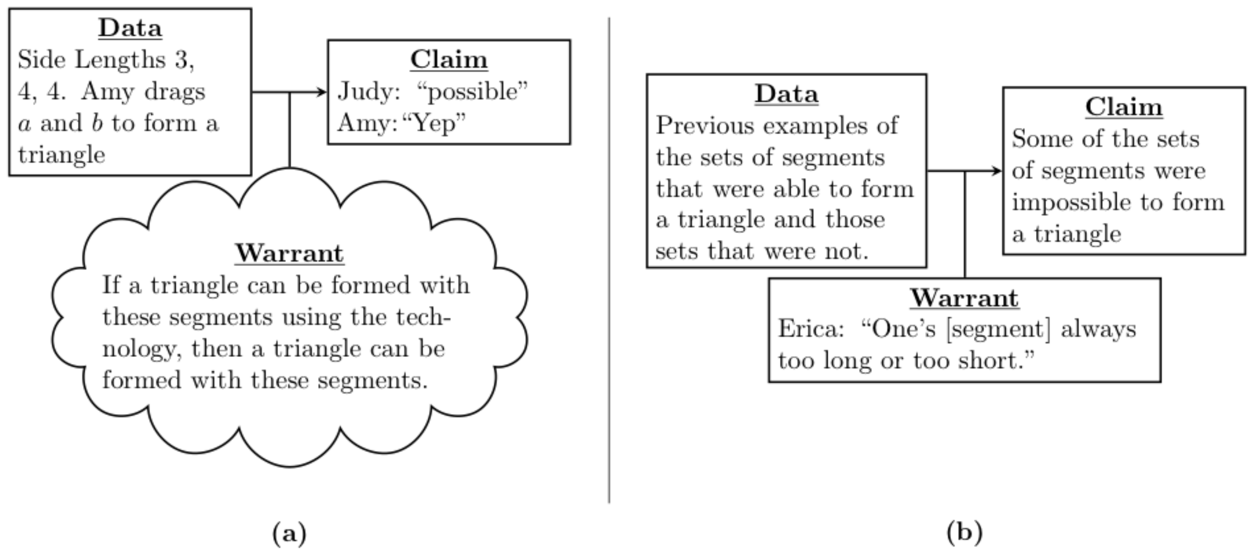

Figure 2. Arguments in which (a) students use technology and provide an implicit warrant and (b) students do not use technology and provide an explicit warrant

However, when the pairs of students were not actively using technology, the students were mainly working on generalization type tasks. For example, one of the arguments common to all pairs of students in structure was in response to the question on the task sheet, "Why was it impossible to construct a triangle with some of the given lengths?" One student, Erica, stated, "One's [segment] too long or too short." This argument is illustrated in Figure 2b. The question asked the students to generalize across the examples. The data used by the pairs of students to support their claims were their answers to the examples sets of segments on their task sheet. To gather this data, the students used technology. However, when responding to this question, the data had been previously collected and their reasoning was not based on their active use of technology, but on the product of their previous uses.

Arguments created by the pairs of students working in the non-technological environment. The arguments created by the pairs of students in the non-technology class were similar in content and structure with regards to the use of the non-technological tools and the explicitness of the warrant. Of the 19 arguments in which the pairs of students used the tools, they only provided an explicit warrant for 1 (5%) of these arguments. Of the 37 arguments in which tools were not actively employed, the pairs of students provided an explicit warrant for 30 (81%) of these arguments.

Similar to the pairs of students in the technology class, when the pairs of students were using the tools, they were attempting to determine whether a triangle could be formed. In these arguments, the data were the lengths of the segments; the arranging of the snap-cube segments; and, the appearance of the figures the students formed using the snap-cube segments. In one argumentation episode, Clair and Jim were determining whether a triangle could be formed with segments of length 3, 4, and 4. Clair arranged the snap-cube segments and was able to form a triangle. Jim said, "Possible." Clair agreed saying, "Yep." This argument is illustrated in Figure 3a.

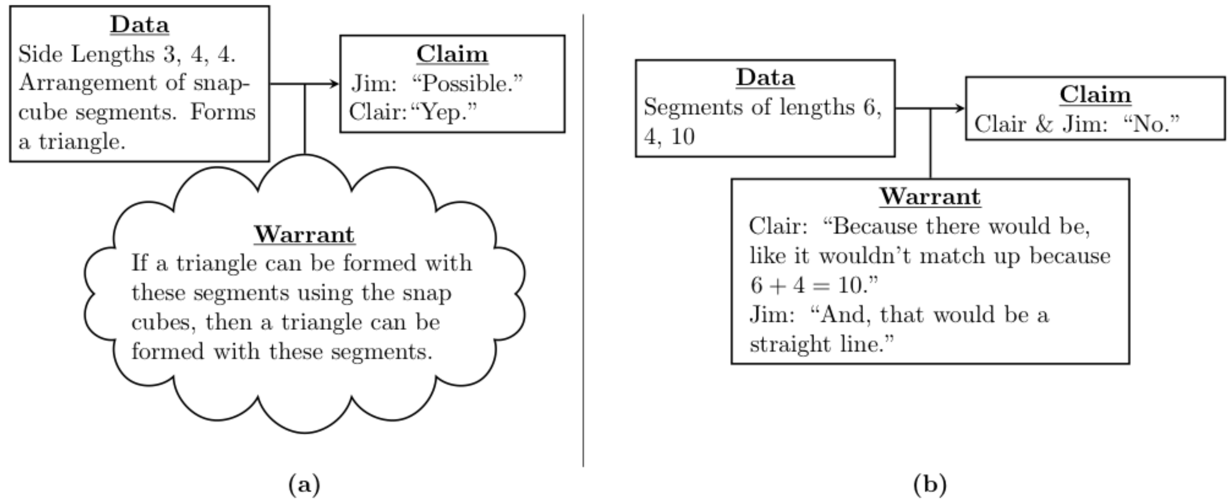

Figure 3. Arguments in which (a) students use non-technological tools and provide an implicit warrant and (b) students do not use tools and provide an explicit warrant

When not actively using the tools in the creation of their arguments, the pairs of students were generally working on generalization tasks and the majority of their warrants were explicit. However, the pairs also created arguments with an explicit warrant that was based on their previous uses of the non-technological tools. For example, Clair and Jim created an argument of this structure as they discuss their solutions to a homework problem. The problem asked the students to determine whether a triangle can be formed with segments of lengths 6, 4, and 10. Clair said, "The first one you said no and I agreed. I said no because there would be like it wouldn't match up because 6+4=10." Jim added, "And that would be a straight line." This argument is illustrated in Figure 3b. Although, in this example, the claim created by pair of students was not a generalization, they justified their claim by using a known theorem and stated what would occur if they attempted to form a triangle with segments of the given lengths.

Additional Data Collection

Many times the pairs of students in both classes collected additional data to verify or refute a previous claim. All three pairs of students in both classes created arguments of this structure. The students' decision to seek additional data may be due to a number of factors including an explicit challenge to a claim, the uncertainty of a claim, the uncertainty of a claim due to the lack of precision in the use of the technological tool, and the ways in which the students used the technology to collect the initial data. Even though the pairs of students collected additional data for a variety of reasons, the students in the technology class always used technology to collect this additional data, usually using the drag feature of the technology. For example, Amy and Judy were determining whether segments of lengths 2, 7, and 4 would form a triangle. Amy dragged the sliders accordingly and Judy claimed, "That looks totally impossible, that is huge." Amy immediately responded, "You don't know that though; that's the only thing." Judy agreed with the sentiment and stated, "I know, the weirdest looking stuff may be possible." Amy dragged the endpoints and is unable to form a triangle. She stated, "It's impossible." This argument is illustrated in Figure 4. In this argument, Amy challenged Judy's claim indicating appearances can be deceiving. Judy agreed and the pair of students sought additional data to verify or refute that claim.

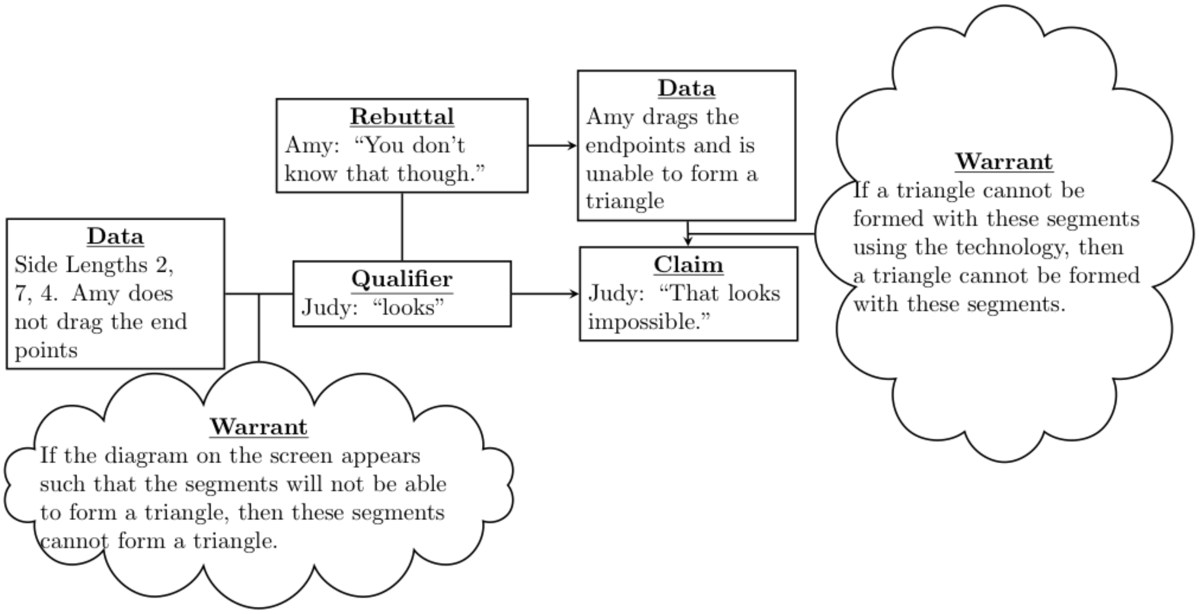

Figure 4. An argument in which additional data is collected using technology.

For the majority of the arguments in which the pairs of students in the non-technology class collected additional data, the students used tools by rearranging the snap-cube segments to determine whether a triangle could be formed, using a ruler to create a sketch to determine if a triangle could be formed, or focusing on different features of the tools. Although the pairs of students mainly used the tools to collect additional data, this was not always the case. At times, the students used known facts such as definitions and theorem as additional data to verify or refute a claim. For example, Ellen had written on her paper that a triangle could be formed with segments of lengths 10, 7, 24. The teacher asked her, "So, you think 24?" She responded, "As long as it's more than 17, it don't [sic] matter." Then, the teacher simply stated, "So 10, 7, 24…okay, okay." Ellen asked, "Hold on is it more or less?" The teacher pointed to the board and asked, "What does it say up there?" Written on the board was the triangle inequality theorem. Ellen began erasing her paper and changed her answer. Ellen collected additional data by reading the theorem on the board. Realizing that her initial claim was incorrect she made a new claim.

In the first example of this section, Amy challenged Judy's claim regarding the appearance of the diagram, which prompted the pair to collect additional data by dragging the figure to determine if the initial claim was correct. Perhaps, when these students' initial claims were challenged, the students felt the need to collect additional data using tools to verify their claims. In the second example, Ellen's uncertainty of her claim prompted her to question the basis for her reasoning and sought additional data to determine whether her reasoning was correct and, hence, whether her claim was valid. In other words, when a claim is challenged, the students used tools to collect additional data in order to resolve the challenge. When the students were uncertain about their claim, they collected additional data, either using tools or definitions and theorems, to determine whether they were using sound reasoning.

Discussion

One major finding of this study is when students actively used the technological or non-technological tools, they were more likely to create arguments with non-explicit warrants. In contrast, when the students did not actively use the tools, they were more likely to create arguments with an explicit warrant. Hollebrands, Conner, and Smith (2010) had similar findings in their study of the arguments college geometry students created when working with technology. When the college geometry students provided explicit warrants for their claims, the students were generally not using technology. The authors attributed this finding to the students' prior experiences in learning mathematics at the collegiate level where the students were expected to provide formal proofs, which require explicit warrants. This same attribution cannot be made to the middle school students in this study because it is unlikely they had been exposed to formal proofs. Rather, the lack of explicit warrants when the pairs of students were using the technological and non-technological tools may be attributed to the visual nature of the tools and the task on which the students were working.

The students' lack of explicit warrants in their arguments created while actively using the technological and non-technological tools may be related to the visual nature of the tools. Each student did not have his or her own personal set of tools. Rather, the pairs of students shared the tools and would alternate between using them. Thus, the students shared the same visual display presented by the students' uses of the tools (the screen for the students in the technology class and the arrangement of the snap-cube segments for the students in the non-technology class). Because the pairs of students had the same visual referent, they may not have been compelled to provide an explicit warrant. Lavy (2006) investigated the types of arguments middle school students created while using an interactive computerized environment. The author found that the students not only used images on the screen and commands as data, but also in their reasoning. Lavy concluded, "In a visualized environment, it is obvious that visual evidences can serve as reasoning in an argument, since all the work in this setting has the same character" (p. 168). Thus, the pairs of student may not have been compelled to provide an explicit warrant because they felt their reasoning was obvious to each other. In fact, when the students' challenged each other's claims the students generally used the tools to collect additional data rather than make their reasoning explicit (see Amy and Judy's argument illustrated in Figure 4).

In general, when the students were using the technological or non-technological tools, they were determining whether given sets of segment lengths would form a triangle. However, when the pairs of students were not actively using the tools, the students were mainly working on generalization type tasks. Even though the students in the current study did not actively use the tools when working on generalization type tasks, other researchers (e.g. Healy & Hoyles, 2001) found that students will create generalizations while using technology. Some pairs of students in Healy and Hoyle's (2001) study used a DGE to investigate relationships among the angle bisectors of a quadrilateral. The pairs of students did not create the same constructions and did not arrive at the same conclusions. However, those students that were successful were able to construct and measure aspects of their diagrams and developed generalizations while using the DGE. The students in the current study did not have the option of creating their own diagram. Instead, the students in the current study used a teacher-generated pre-constructed sketch that limited the students in how they could modify and/or measure aspects of the diagram. Perhaps, the students in the current study would have been more likely to use the technology while working on generalization type tasks if they had been given the opportunity to create their own diagrams.

Endnotes

1. All names are pseudonyms

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DESIGNED AND EMERGENT PEDAGOGICAL SUPPORTS FOR COORDINATING QUANTITATIVE AND AGENT-BASED DESCRIPTIONS OF COMPLEX DYNAMIC SYSTEMS

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Often, quantitative and mathematical models are used to study complex dynamic systems–where collections of individual components interact to produce an emergent pattern of behavior. To interpret such models involves 'unpacking' the collective behaviors and interactions that underlie a given pattern. We investigate what representational supports in a technology-mediated agent-based modeling environment led secondary students to question and investigate the relationship between individual behaviors and collective patterns in a population dynamics unit. Our goals are to inform 1) our own design of educational tools and 2) the theoretical elaboration of general "conceptual leverage points" for mathematics and complex systems education.

Objectives / Purpose

Over the past several decades, there have been a number of technological and theoretical developments that have enabled scientists to explore the interconnected and complex nature of natural and social phenomena in ways that were not previously possible (Bar-Yam, 1997; Mitchell, 2009). This increasing capacity to document and study the world's complexity has led calls for complex systems topics and principles to be integrated into the K-16 curriculum (Forrester, 1994/2009; Jacobson & Wilensky, 2006; Kaput et al., 2005; Sabelli, 2006). Indeed, in her plenary session at PME-NA last year, English noted that technology and complexity "have led to significant changes in the forms of mathematical thinking that are required beyond the classroom" – and that technology has increased the demand for "the interpretation of data and communication of results" (2010, p. 33).

Our objective is to better understand how we can support students as they engage in one such "new" form of mathematical thinking – to interpret mathematical and quantitative models in terms of the complex, interactional events that they encapsulate and measure. This report is part of a larger design-based research project (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) that seeks to (1) explore student reasoning about complex dynamic systems and associated mathematical ideas including the measurement and encapsulation of complex behavior, notions of rate of change and accumulation in multivariate systems, and nonlinearity; and (2) develop educational tools and activities that support such reasoning.

Background and Theoretical Framework

One important form of mathematical thinking related to complex systems involves understanding the relationship the behavior of a system at different *levels* (Wilensky & Reisman, 2006). Often, the relationships between the *individual* and *collective* levels of a system can seem inconsistent, since many individual behaviors and interactions underlie a given collective result – for example, individual cars in a traffic jam each move forward even as the jam itself propagates backward; and chemical reactions occur at the atomic level even as chemical concentrations reach equilibrium (Wilensky & Resnick, 1999). Similarly, it is common for information about complex systems to be represented in dynamic quantitative and mathematical forms at the aggregate level, even while their underlying individual behaviors are not described mathematically (Blikstein & Wilensky, 2009; Stieff & Wilensky, 2003; Wilensky, 2003). Hence a primary issue in the interpretation and communication of data and mathematical models of complex systems is to infer and reason about what behaviors they may encapsulate (Author, 2010).

One way to encourage students to interpret mathematical behavior is by actually enabling them to simulate and model the phenomena that underlie it. Two decades of work has explored how this can be done with dynamic phenomena such as motion (Kaput, 1994; Nemirovsky, Tierney, & Wright, 1998; Stroup, 2002), banking (Wilhelm & Confrey, 2003), and other physical scenarios (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Yerushalmy, 1997). This is not limited to systems where patterns are generated by and reflect a single causal mechanism, but also collections of mechanisms – Confrey and Smith have shown that multiplicative reasoning can be supported by considering individual elements in a collection each "splitting" to create exponentially more entities (1995). Doerr has shown that even the relationship between probabilistic collective behavior and theoretical idealized models can be explored by students by juxtaposing theoretical and empirical models (2000).

We are exploring the potential for *agent-based modeling* (ABM) environments to provide a flexible context for students to learn about quantitative / mathematical patterns that represent complex dynamic systems. In an agent-based model, individuals that comprise a given complex system are simulated by computational *agents* whose behavior and interactions are governed by simple rules. Each agent executes these rules iteratively to simulate a collective emergent outcome, which can be expressed both visually and quantitatively with numbers and plots. While ABM has been shown to help students make sense of the relationship between individual and aggregate behaviors in many *scientific* domains (Jacobson & Wilensky, 2006), we argue that it can also be used to support students as they explore trends represented via data and mathematical models (Authors, 2009).

Of course, any multirepresentational environment is successful only insofar as its representations relate to educationally relevant tasks and students' existing knowledge (Goldman, 2003; Kaput, 1998). To the extent that complex systems are described at multiple levels using multiple representational systems, we argue that it is very much an educationally relevant task to students to navigate and coordinate those descriptions and representations. In this paper, we concern ourselves with supporting students in accomplishing this task by exploring how well the representations that students interact with while engaging in agent-based modeling resonate with and build upon their existing knowledge.

Research Question

We are interested in how students' interactions with agent-based modeling and associated representations can help them make sense of the relationship between a collective, quantitative pattern exhibited by a complex system, and the individuals, behaviors, and interactions that pattern encapsulates. Toward this end, we utilize design-based research to answer the question: <u>What representational supports made available through our agent-based modeling activities led students to *attend to* the particularities of the encapsulated individual-to-collective relationship in complex systems, and what supports do they use to *resolve* the relationship that leads to those particularities in population dynamic systems?</u>

Methods and Design

Participants

We draw our data from two iterations of a 200-minute classroom-based educational intervention implemented across six classrooms at two public metropolitan high schools in the Midwestern United States. "Preparatory High" (PH) is an academically selective senior high school with a student population of over 1,000. At the time of our study, roughly one-third of PH students were identified as low-income, about 40%, 30%, 25%, and 5% as White, Asian, Hispanic, and Black/Native American/Other, respectively, and over 98% of students met or exceeded state standards. We conducted our intervention in two AP Biology classes, each which included between 20 and 25 students from grades 10 through 12 who were simultaneously enrolled in Precalculus, Calculus, and/or Statistics courses. "Local High" (LH) is an open enrollment senior high school with student population of less than 200. At the time of our study, 99% of LH students were identified as low-income, 100% as Hispanic, and about 25% met or exceeded state standards. We conducted our intervention in four Precalculus classes at LH, each having between 15 and 25 students from grades 10 through 12. Both schools had high numbers of students that met or exceeded expected levels of growth as predicted through state standardized exams.

Design and Activities

Our implementation included the use of a *modeling toolkit* and *associated activities*, each which reflect our design hypothesis regarding what aspects of the environment may attune students to the complex connection between individual and collective behavior. Though it is beyond the scope of this paper to review our design in detail, it was heavily informed by projects including SimCalc Mathworlds (Roschelle & Kaput, 1996), Algebra Sketchbook , Modeling4All (Kahn, 2007), Agentsheets (Repenning & Ambach, 1997), Trips (Clements, Nemirovsky, & Sarama, 1995), Function Probe (Confrey & Maloney, 1996), My Graph Rules (Wilensky & Abrahamson, 2006), and various population dynamics activities (Blanton, Hollar, & Coulombe, 1996). The three supports emphasized by these tools and activities that we examine in this paper include the *graphical noise* produced by agent-based models, the *behavioral descriptions* used to create the models, and *spatiotemporal visualization* of agent behavior in models synchronized with plots of the quantitative patterns that emerge from their execution.

The Modeling Toolkit

The modeling toolkit utilized by students in our study consisted of new construction and analysis modules for the NetLogo agent-based modeling environment (Wilensky, 1999). The first module, DeltaTick (Figure 1), enables users to rapidly construct computational agent-based models from a library of visual programming blocks that each represent behaviors, constraints, or quantities that may be of interest when modeling quantitative trends within a specific domain of inquiry. For the population dynamics activities described in this paper, we used a library that provided students with agent instructions such as "reproduce" and "die" with certain probabilities, "wander" around a spatial world, and constrain their reproductive and death behavior based on environmental or individual "if" statements (concerning the availability of space or partners or an agent's age) factors. The full set of "behavior blocks" available for students to construct models from is featured below. The second module, HotLink Replay (Figure 2), enables users to replay a simulation, and interact with enhanced plots that enable the user to click on a given graphical feature and observe that point of time in the visualization of the simulation, to overlay and compare graphs across simulations, and to zoom in and annotate specific regions.

The Classroom Activities

Over the course of each 200-minute implementation, students completed a set of activities designed to help them recognize and build connections across individual and quantitative aggregate behavior in population dynamic models. For the purposes of this paper we focus our analysis on two activities that were completed at both sites: the "Probabilistic to Exponential" activity and the "Make Graph Fit" activity. In "Probabilistic to Exponential", activity students were guided through the construction of a simple simulation wherein 100 computational agents each have a .01 chance of reproducing

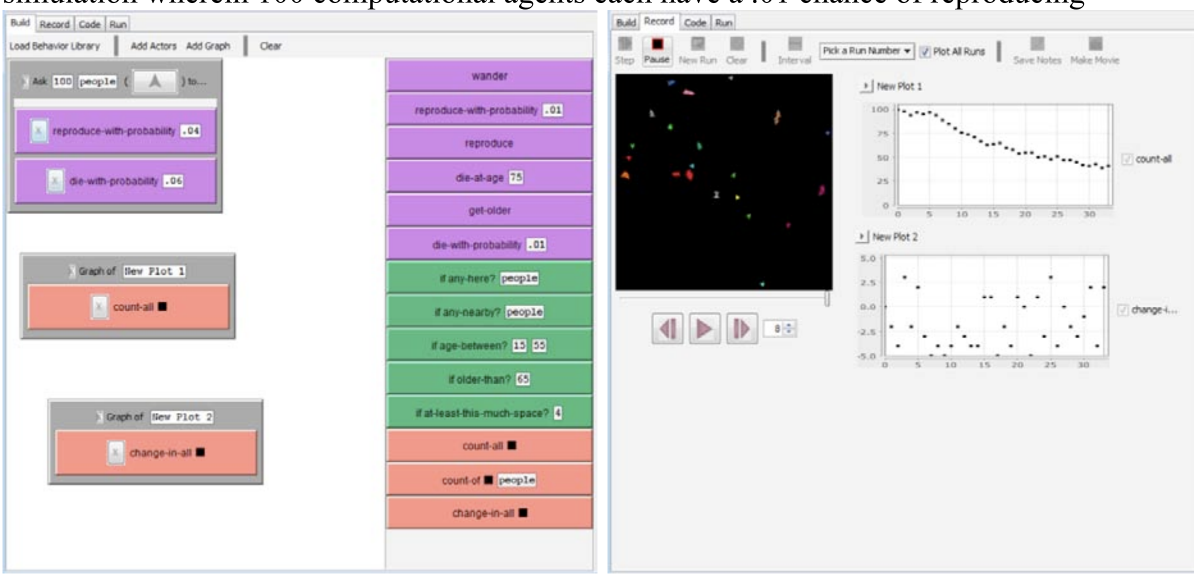

Figure 1. The *DeltaTick* (left) and *HotLink Replay* (right) modules.

per unit of time, and plots of both the number of births and the total number of people per iteration of time were tracked. They were then asked questions designed to emphasize the connection between the "noisy" data and the standard exponential population growth model. In the "Make Graph Fit" activity, students were provided a collection of population patterns including exponential decay, relative stability, logistic-like growth, linear-like growth, and density-induced growth (whereby individuals reproduce only when spatially near others), and asked to find combinations of behaviors that recreated each pattern.

Data Collection

During each implementation, groups of 2 to 3 students worked on laptops equipped with Camtasia Screen Recorder Software, which enabled us to capture and synchronize students' on-screen activity with their group discussions as captured by video cameras mounted within each laptop computer. These captured videos were then segmented by activity sequence and transcribed using Inqscribe Transcription Software. Since for the purpose of this paper we are interested in students' interaction with the designed tool and the role that activity prompts played in guiding students to attend to the connections between individual behavioral and aggregate quantitative representations of population dynamics, we focus our current analysis on the collection of video data segments that capture the Probabilistic to Exponential and Make Graph Fit activities.

Analytic Approach

To explore our primary research question, we coded each activity segment video to for the students in a given group *questioned* a representational support or *attended to* the fact that a support illustrated a particularity between individual and collective system behavior, and *resolved* to actually explore and better understand that relationship. By "explore and better understand the relationship", we mean attempting to resolve the particularity. We attended both to supports that we hypothesized would emphasize individual-to-collective connections (graphical noise, behavioral enactment, and temporal visualization) as well as unexpected supports that emerged from our data in order to pursue the dual goals of testing our existing hypothesis regarding how agent-based modeling can engage and support students in reasoning about the "individual-to-collective" relationship, while also identifying other potential supports that represent additional pathways for learning (Cobb et al., 2003).

Results

Since we are interested in exploring our hypothesized supports as well as identifying those that emerged from students' interactions with our designed environment, we present our results in two sections. First, we describe students' interactions with our three hypothesized supports for bridging quantitative to individual differences: graphical noise, behavioral enactment, and spatiotemporal visualization. Second, we briefly review other supports that students leveraged to question and begin to interpret the relationship between individual probabilistic and collective quantitative behavior.

Designed Supports for Bridging Quantitative to Individual Descriptions Graphical Noise

We expected graphical noise to alert students to the complex relationship between individual behavior and collective quantitative trends by emphasizing the probabilistic nature of the underlying behavior that graphs measured. Specifically, during the "Probabilistic to Exponential" activity, we guided students to construct a graph measuring the *change in population* to emphasize that the probabilistic nature of individual agents' reproduction manifested quantitatively. We found that while nearly half (7 out of 16) of the groups explicitly questioned the noise produced by the *change in population* graph even before they were prompted to do so by the accompanying activities, only one group found this to be enough motivation to explore the issue in more depth. When explicitly prompted to explain the noise in the accompanying activities, many groups did in terms of the relationship between change and accumulation in the available graphs, without connecting to probabilistic agent behavior.

Initial Questioning 7/16 groups	**Change Explanation 1/16 groups no prompting 7/16 groups prompting**	**Probabilistic Explanation 3/16 groups prompting**
"Like, it's going up and down..." "Wow, why is it doing that? … Is this derivative? I don't think so… Why is it crazy?" "Like, it's...the graph is kind of like random"	Ana: I don't get the random dots. Jorge: They're not random. It's just...it's showing you the...let it get to 150 and I'll.. Alright let's go right here (zooms in to plot of change). 175 (zooms in to plot of population). It's just the change in one relation. So like from here...(adjusts graphs). Do you get what I'm saying, though?	"It says sometimes the graph of change-in-all goes down, even though the population is always growing. Because there's still probability that they might not have any...that they might not reproduce at all."
Other groups described the noise as the result of death, (1), an inherent feature of the program itself (1), or could not be classified (3).		

Individual Behavioral Enactment/Programming

A second factor that we expected to alert and support students in attending to to the connection between individual behavior and collective quantitative patterns was that during both activities, students engaged in the programming of individual rules as behavioral descriptions. We expected that when reasoning about the quantitative patterns generated by those models, students would recall either generic descriptions of individual behaviors that contribute to population growth (for example, notions of "having babies", "dying", or "counting people") or reference to the specific individual instructions they programmed ("one guy reproduces by 1%", "die with probability .05". Unlike the noise revealed in graphs, students did not express that the behaviors they programmed into the simulation and the resulting outcomes were inconsistent or unexpected. However, when they were asked to explain what individual behaviors contributed to the resultant trend in the "Probabilistic to Exponential" activity, they often did in only a generic form "change in population", "reproduction". It was not until the "Make Graph Fit" activities that students described trends in terms of the specific rules they programmed "reproduce by 1%", "die with probability .05". While it is not surprising that students would refer to the rules they used when working to build models in order to interpret their resultant behavior, this does illustrate the potential for construction activities to emphasize the multivariate, "encapsulative" nature of patterns in complex systems.

	Generic Explanations **8/16 groups Prob to Exp** **5/13 groups Make Graph Fit**	**Rule-Based Explanations** **3/16 groups Prob to Exp** **8/13 groups Make Graph Fit**
Probabilistic to Exponential	"I was thinking that maybe this relates to this, because it's actually telling you how many times they actually reproduce, which helps you understand what's going on..."	"Because individually they're still subject to reproduce by 1%. Over the long run, though, it increases..."
Other groups provided change-based descriptions (2) or could not be classified (3).		
Make Graph Fit	Herman: Give them their space. You can't reproduce that much. Eric: Alright...it's falling down. Janet: It's not growing fast enough.	John: It's not going to like replace though. Because if one guy reproduces, he'll make one new person. Whereas, if, if it's .99, then 99 people die. Franco: No. John : They wouldn't replace each other, though. Franco: Chance too high. John: They just die.

Spatiotemporal Visualization

Finally, we expected the side-by-side and synchronous nature of a spatiotemporal, agent-based visualization and a quantitative plot to emphasize that those quantitative patterns emerged from counting the addition and aggregation of individual agents. We found that few groups attended to this relationship during the "Probabilistic to Exponential" activity (only 2/16 groups mentioned any features of the visualization, and one group only did so to note that they hadn't paid attention to the visualization). In the "Make Graph Fit" activity, as students built models that featured spatial and interactive behaviors, they relied more on visualization to understand how those features led to patterns such as logistic growth (5/13 groups observed whether their behaviors played out in visualizations).

Emergent Supports for Bridging Quantitative to Individual Descriptions

Questioning Measurement of Behavior. The first, and perhaps the most impressive, support that led students to critically examine the relationship between simulated individual behaviors and the quantitative patterns those behaviors produced was to question what, exactly, each quantity was measuring. In our design, we expected that asking students what agent behaviors would *change* population would lead students to view population as encapsulating those behaviors. Instead, the results above imply that they may have seen the two factors as *related* without exploring whether they were explicitly causal.

Modeling Assumptions

A second support that led students to examine the relationship between individual and quantitative behavior was to question the assumptions about human behavior that underlied a given set of programmed rules. This manifested itself dramatically when one group of students questioned whether agents exhibited a 9 month gestation period or reflected any assumptions about a given demographic populations' behavior. The group then decided that in order to better understand the underlying assumptions of the model, they would conduct experiments with a single individual under different rules. This led them to consider how that single individual would influence a population.

The Reversibility Script

Finally, several of the students from our first research site, PH, used clues from the questions we asked in our activity set (including questions regarding the relationship between change and population graphs, and regarding the meaning of "change") to infer that it was a calculus-based reversibility relationship, rather than an individual-to-collective relationship, that was the main goal of their exploration. While these two concepts are not mutually exclusive, this "graphical/quantitative reversibility" appeared to supplant students' potential exploration of individual-to-collective behavior using the environment.

Discussion

We sought to identify what designed and emergent supports can help students to understand and navigate the relationship between quantitative patterns generated by complex systems, and the individual behaviors that underlie those patterns. We found that three supports – graphical noise, behavioral enactment, and spatiotemporal visualization, each played differential roles in drawing students' attention to and supporting their investigation of this relationship. We also identified emergent supports – specifically, the potential for framing graphical noise in the context of questioning what, exactly, is being measured; engaging students in discussions about the assumptions that underlie a given agent-based or mathematical model; and *downplaying* the reversibility of graphs until connections to the underlying behavior of a model is established, as potential paths forward in our development of computational modeling tools to support students as they engage in the new forms of mathematical thinking that are required in an age of complex systems.

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REIFICATION AND ABSTRACTION: DEVELOPING MATHEMATICAL CONTENT KNOWLEDGE THROUGH DISCOURSE AROUND ALGEBRAIC REASONING AND PROOF

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Moving from a reliance on physical representations of mathematical concepts or ideas like addition to a more abstract, mathematical object based understanding is a critical event in developing mathematical content knowledge. In a mixed methods study involving twelve in-service elementary and middle school teachers taking part in a two-week summer institute as part of an online Masters in Science program, the impact of using discourse-based teaching and learning strategies on developing their algebraic reasoning and proof understanding of their mathematical content knowledge was investigated. Using a combination of qualitative and quantitative data sources, a case-study analysis revealed that the teachers who increased their content knowledge made significant shifts in their investigations of mathematics concepts. These changes were evidenced by a movement from using discourse that relied on physical representations and examples to basing their discourse on properties, axioms, and generalized characteristics.

Introduction

Through leadership from NCTM and supported by the Common Core Standards, increased emphasis has been placed on the fostering and supporting of mathematical reasoning and justification as key educational elements in addressing mathematics educational reform needs (NCTM, 2000). Educators responding to these 21st century reforms for mathematics have identified continuing professional education supporting teachers in developing their own mathematical content and conceptual knowledge and understanding as critical for meeting this challenge (NCTM, 2000). The need for extending teachers' mathematics knowledge is in response to an increase in the level of mathematics proficiency required of primary and elementary teachers in order to prepare their students for success in making the all important transition from arithmetic, based on physically represented procedures, to the abstracted concepts of mathematics (Gray & Tall, 1994). This transition process begins with the introduction of algebraic topics and quite often continues for the remainder of a student's academic career (Sfard & Linchevski, 1994).

Research Question

As part of an online three-year Masters of Science program, a two-week long summer institute focusing on the development of algebraic reasoning and proof skills was conducted. In the context of that summer institute, the research question for this study is: what is the impact of using discourse-based instructional strategies focused on developing teacher's algebraic reasoning and proof understanding on their mathematics content knowledge?

Perspectives and Theoretical Framework

All too often, teacher preparation and licensure programs place more emphasis on methodology, pedagogy and practice than on content knowledge. This shift in emphasis often results in a focus on physically represented procedure with reduced attention to underlying abstract concepts. Additionally, the important role played by the dynamic relationship between procedure and concept in understanding and solving problems from world experience is seldom accessed or investigated.

In response to the importance of understanding how procedural and conceptual mathematical knowledge frame and inform the other, Gray and Tall (1994) presented their model of the role embodied objects play in learning mathematics. Their work, taking from Skemps' (1976) work on relational and instrumental understanding, described learning mathematics as "occurring in a biological brain" (E. Gray, Pinto, Pitta, & Tall, 1999, pg. 1) and could be investigated with support from a cognitive psychological perspective (Tall, 1999; Watson & Tall, 2002).

Continuing with research in mathematics learning using a cognitive psychology perspective, Sfard (1991) described abstract concepts as having two fundamentally different conceptualizations: "structural" and "operational" (Sfard, 1991, pg. 3). In Sfards' perspective, these conceptions of mathematics concepts, are instrumental in the process of learning and problem-solving. It is the interplay between a structural or object view and an operational or process view that frame and support the development of meaningful mathematics knowledge and understanding (Sfard, 1991; Sfard & Linchevski, 1994).

As these theories of mathematics learning developed, understanding the process of transitioning from a concrete, computational operations understanding to abstract objects understanding became a focus of research. Sfard & Linchevski, (1994) described this as a three part process, ending in "reification" (Sfard & Linchevski, 1994, pg. 20), or the ability to see something familiar in a new light. Moving through the transition process indicates a development of mathematical thinking that is difficult for learners to accomplish as well as for teachers to facilitate (Gray & Tall, 2001). In similar research, Edwards (2005) looked the transition from elementary to advanced mathematical thinking, defined as "thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses" (Edwards, Dubinsky, & McDonald, 2005, pg. 17). One characterization of advanced mathematical thinking is the development of the knowledge and reasoning skills to meaningfully think about problems where the concept of infinity plays a central role.

The importance of being able to move between these two perspectives of mathematics, (i.e. a concrete representation view versus a definition and axiomatic view), provided motivation for research into how this knowledge is developed. Gray and Tall (2007), extending their work in 1994, describe the role of language in providing pathways for the "compression" (Gray & Tall, 2007, pg. 2), where the brain is able to filter out irrelevant information and compact the important concepts, through a process where language plays a critical role. While important information is first just noticed in the thinking process, it is only when language is used to name the concept that the compaction process can result in the development of what Gray and Tall describe as a "thinkable object" (Gray & Tall, 2007, pg. 3).

The use of language as discourse in learning mathematics is the focus of the work of several researchers. Sfard (2003), taking a social constructivist approach and building on work by Vygotsky (1929) and Lave and Wenger (1991), describe a model of language, not just facilitating learning, but language *as* learning. In this model thinking is characterized as communication, whether internalized dialogue or externalized exchanges. It is this internalized dialogue, where concepts and ideas are named and objectified, that facilitates development of abstract, conceptualized understanding (Sfard, 2003). The use of language and discourse in helping students make the transition from a computational view of mathematics to a view based on abstract mathematical objects is centered on the need for using language to name abstract objects in order to think about them (Davis & Tall, 2002; Sfard, 2003; Tall, 2007; 2008; Umland & Hersh, 2006). It is precisely this perspective of the use of language that framed and supported the collection and analysis of the multiple data sources in investigating how the use of language in discussion and shifting usage patterns impact the development of the interplay between concrete, computational and abstract, objectified mathematical understanding.

Context and Methodology

This study was conducted in the context of the first year of a three year online Masters of Science degree program that focused on the integration of technology with science and mathematics instruction. Twelve K-8 teachers were selected from several school districts over a diverse geographical region covering approximately 500 square miles. The twelve teachers were part of a three year MSP grant project that included a face-to-face two-week summer institute consisting of a 3-credit mathematics content course and a 3-credit science content course.

The mathematics content course focused on the development of algebraic reasoning and proof skills and understanding. Each day, the students were presented with problems, ideas and topics for discussion. These problems used as problem solving prompts were sequenced to lead the students from concepts that could be explored through concrete examples, such as calculating the number of sodas consumed on the *n*th day if there had been 6 sodas consumed on the first day and 3 more each day after, to concepts and ideas where examples served as a starting place for more meaningful investigation and inquiry, for example determining the sum of the first *n* odd positive integers.

The mixed-methods structure used was a sequential explanatory design framework (Ivankova, 2010) where the quantitative data revealed changes in student understanding in both algebraic reasoning and proof knowledge. The qualitative data was collected in an effort to develop an understanding of the underlying elements of the classroom experience that influenced these changes.

The data collection and analysis methodology were structured to support a case-study analysis. This mode of inquiry was chosen to provide insight into both the changes in mathematical content knowledge as related to changes in algebraic reasoning and proof as well as if and how the students made the transition from a computational to an abstract perspective of mathematics.

The design of the case-study analysis methodology was framed by the work of Onwuegbuzie, Johnson and Collins (2009), where they presented philosophy of research that supported the use of mixed-methods designs and provide a typology of analysis strategies. This typology indicated that a case-study analysis was most appropriate where the goal is to "analyze and interpret the meanings, experiences, attitudes, opinions, or the like of one or more persons" (Onwuegbuzie, Johnson, & Collins, 2009, pg. 117).

Data Collected

Each day, as a result of their activities, the students produced written artifacts which were collected. These most often consisted of written scratch work, drawings, charts and illustrations used in problem solving and knowledge construction. In addition to these written artifacts, a video record of the verbal interactions during the classroom activities was made. This video captured both small group and whole group conversations where the students were engaged in sense-making interactions as well as expositions of ideas to the entire group.

In addition to this qualitative data, two different quantitative measures were utilized. The Diagnostic Teacher Assessment of Mathematics and Science (DTAMS) middle school algebra strand assessment was used in a pre/post-test context (Saderholm, Ronau, Todd Brown, & G. Collins, 2010). This assessment has a subscale focusing on mathematics content knowledge and a subscale that focuses on algebraic reasoning, allowing for gathering information about changes in both content knowledge and algebraic reasoning skills. This assessment was given on the first (pre-test) and last days (post-test) of the summer session. To assess changes in the students' understanding of proof, a diagnostic focusing on students' skills in constructing proofs from familiar and novel problems was used in a pre/post test context, also given on the first and last days of the summer session (Healy & Hoyles, 2000).

Data Analysis

Analysis of the quantitative data was accomplished through the use of an independent samples t-test with the teachers as the unit of analysis. The qualitative data was analyzed through an open coding process where the written and video data was coded following procedures developed by Corbin and Strauss (1990). The frame work for coding the qualitative data was based on Harel and Sowder's concepts of proof frames, where they describe a progressively increasing level of knowledge and sophistication in use of proof and algebraic reasoning to construct and communicate mathematics knowledge based on moving from a dependence on an inductive perspective based on random examples to using a deductive proof structure based on definitions, axioms, and logic (Harel & Sowder, 1998). This description process, using coding categories developed from characteristics of the different levels of proof abilities and knowledge framed by Knuth and Elliott (1998), identified changes in participant levels of proof understanding.

A case-study analysis was used to identify and describe any progress each student made in shifting from a computational perspective of mathematics to one based on abstract concepts. This identification process was framed by the work of Sfard and Linchevski (1994) in their description of the reification process[1] and by the work of Tall (2008) in his description of the transition process from a reliance on concrete examples to more formal, axiomatic mathematical thinking.

Results and Conclusions

Differences in Algebraic Reasoning and Proof Skills in Pre/Post Assessments

Assessment	Means: Pre-intervention	Means: Post-Intervention	t-value	p-value	Effect Size
DTAMS					
Content Knowledge[1]	.56	.76	-2.94	.032	.44
Reasoning and Problem Solving[2]	.51	.94	-3.02	.006	.54
Proof Construction Assessment[3]	2.08	3.58	-3.00	.012	.49

1. Mean score on a scale of 0-1. 2. Mean score on a scale of 0-2. 3. Mean score on a scale of 0-4.

Results of an independent samples t-test, comparing pre and post intervention mean scores, reveal a significant increase in the students' mathematical content knowledge, algebraic reasoning skills and their ability to construct algebraic proofs from familiar and novel contexts ($p<.05$ in all cases) with a large of substantial effect size (>.4 in all cases).

Through the coding process, it became apparent that there was a definitive shift in the content of both the written and the verbal sense-making interactions, where the participants' vocabulary and written word choices shifted from describing mathematics concepts in terms of concrete examples to describing mathematics concepts in terms of definitions and properties. The goals identified by the students for the problem-solving activities also evolved from finding a "solution" to developing their understanding of how the mathematics concepts or ideas in question could be extended to more generalized ideas, creating new (to them) connections and relationships.

The results of an analysis of the qualitative artifacts of the mathematical discourse engaged in by the students described how they developed their mathematical knowledge through mathematical discourse centered on making a transition from thinking based on computations based on examples to thinking based on definitions, properties and relationships.

The primary evidence for this transition came in the form of clearly presented changes in how the students talked about mathematics, both in their problem solving activities and when they were explaining their ideas. Analysis of video data revealed a shift in how the students talked about mathematics, both in their use of mathematics vocabulary and their use of mathematics notation in the context of displaying their ideas during sense making activities.

Day two:

Student A, in a discussion with another student: "I'm not sure how many examples we need to try. I checked a few numbers and it worked for all of them, so I'm OK with it".

Day seven:

Student B, in a small group discussion about the idea that adding two even numbers results in an even number: "I tried a bunch of examples and I think it has something to do with what makes an even number even. How do we need to define an even number to make sense of this?"

Student C, in same group discussion, "We need to write an expression that represents adding two even numbers where we use variables to represent the even numbers, not just examples of even numbers."

Their initial written work portrayed the use of primarily guess and check methodologies based a limited number of concrete examples, while artifacts from the end of the course presented multiple problem solving strategies used in concert to reveal patterns and relationships based on generalized properties and definitions.

These shifts from primarily computational methods based on concrete examples to more axiomatic, properties-based methods illustrated how the students' thinking shifted from a concrete, computational understanding of mathematics to an abstract, conceptual understanding. Initially, the students attacked each problem with a guess and check methodology, using examples in their sense making activities. At times, there were no reasons given for the choices of examples.

Day one:

Student D, displaying her work to the whole group and responding to a question about why she chose her examples, "I just used easy numbers, like 1, 2, and 3."

As the students spent time engaging in problem solving activities and developing their understanding of proof, they began to be unsatisfied with the limitations of understanding based on examples. Questions about generalization of ideas, properties and concepts became the focus of discourse, where extended discussions helped frame and develop shared knowledge.

Day 8:

Student A, in a discussion with another student about summing the first n positive odd integers and how to show that the summation equals n^2*, "I think we need to talk about odd numbers like we did with the problem about adding even numbers, with an expression instead of examples. That way we can add up the expressions and see how that might simplify to* n^2*."*

As the students became more adept at asking the kinds of questions that better represented their thinking processes, they became better at using questions to help craft their understanding and knowledge. The development of their questioning strategies implied a development in their thinking. Moreover, since the problems they were using as prompts for their investigations centered on aspects that were not accessible to them through examples, (i.e. infinite sums), their thinking necessarily shifted to a more advanced mathematical thinking structure (Edwards et al., 2005). This change in thinking, represented as changes in speaking and writing patterns (Hilbert, 2008), indicates a transition from an operational, through a structural, to a functional perception of algebra as described by Sfard & Linchevski (1994) or from a procedural, through a process to a procept (melding of process and concept) perception of mathematical thinking detailed by Gray, Pinto, Pitta, and Tall (1999).

Implications

Helping teachers extend their mathematical content knowledge is critical. Continuing education opportunities that support the development of conceptual mathematics understanding and the ability to smoothly shift between computational and abstract conceptual perceptions provide the teachers with a foundation of knowledge on which they can build, thus facilitating their mathematics teaching and the learning of their students. Structuring educational experiences where teachers have opportunities to engage in sense making of abstract mathematics concepts used in concert with computational mathematics to develop sophisticated problem solving understanding and strategies provides a framework where teachers are able to develop their own knowledge and skills of mathematical thinking. With well developed understanding, teachers are better equipped to help their students be successful on the same mathematical journey.

Continuing research towards a more complete understanding of how to help students (and teachers) better make the transition from a procedural understanding of mathematics to a more conceptual understanding and how to effectively use these different perspectives in problem solving and sense making activities would further efforts to make this difficult transition easier and more successful.

Endnotes

1. In this context, the term "reification" refers to converting processes applied to accepted abstract mathematical objects into self-contained static constructs in their own right (Sfard, 1991)

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[1]A TEACHING EXPERIMENT TO FACILITATE CONCEPT MODIFICATION: ASYMPTOTES, LIMITS, AND CONTINUITY

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After identifying students' concept images of rational functions , asymptotes, limits and continuity using a two-hour long problem solving interview, a four-week, eight sessions teaching experiment was conducted. While the teaching experiment helped the researcher understand how students developed their conceptual models of the concepts of asymptotes, limits, and continuity of rational functions, it also helped the researcher to investigate effective instructional practices that help facilitate mathematics learning with conceptual understanding. This paper summarizes the importance of student centered analysis, deconstruction, modification, and reconstruction of preexisting concept images in order to foster conceptual understanding of a newly introduced concept.

Purpose of Study

The purposes of this research were guiding students towards (1) summonsing a deeper understanding of the concept of asymptotes (2) helping them see the connections between the concepts of asymptotes and limits, and (3) affording them to use that connection to enhance their understanding of the concept of limits.

Asymptote, a concept that is closely related to the concepts of limits, continuity, and indeterminate forms is usually introduced to students in the precalculus course. Students often struggle with these concepts even in university analysis courses (Juter, 2004). Through a two-hour long problem-solving interview, the researcher investigated university calculus 2 students' conceptions of asymptotes and the connections between asymptotes, limits, and continuity. Nineteen students were interviewed using the problem solving instrument CORI that constituted problems requiring thinking *out of the box*.

The problem solving interview revealed that students generally possessed a process-oriented view towards mathematics problem solving. While holding numerous incomplete conceptions regarding rational functions, their asymptotes, limits and continuity, students also failed to establish connections between these closely related concepts. Students were unable to explain the behavior of functions around its asymptotes and they were unable to relate the asymptotic properties of functions in terms of limits. Without being able to relate to the ways in which function behavior was affected by its asymptotes, students were unable to find limit at infinity and infinite limit without a graphing calculator.

In light of the conceptual obstacles noted, a series of teaching episodes were carried out to help students attain essential modification in their conceptions of asymptotes, limits, and continuity. During the teaching episodes, a variety of activities that would encourage critical thinking and exploration of conflicting ideas were implemented.

Theoretical Framework

The prime goal of this research was to gain a deeper understanding of the complex cognitive processes involved in learning mathematics and the implications of those cognitive processes for mathematics instruction. Therefore, the theories that framed this research explored several crucial aspects of concept, and concept formation (Piaget, 1970; Tall & Vinner, 1981) in addition to knowledge and knowledge acquisition (Skemp, 1971). Theories of concept representation (Goldin & kaput, 1996) also influenced this research study.

Piaget asserted that students do not come to our classes as *blank slates*. The variety of notions students hold regarding a particular concept could be influenced by former formal or informal exposition to the concept. At times, prior experiences with a certain concept could negatively interfere with the newly introduced aspects of the concept. Piaget referred to these internal conceptual conglomerations as schema, while Tall and Vinner (1981) referred to these conceptual entities as concept images. Many associated images are consciously or unconsciously summoned while recalling and manipulating a concept. Concept images could seriously influence students' interpretations of the newly introduced mathematical concepts unless they are given the opportunity to consciously challenge and modify the pre-existing internal schema of the concept in accordance with the new conceptual input (Piaget, 1970; Tall & Vinner, 1981).

Goldin and Kaput (1996) elaborated on different ways of capturing experiences through their theory of representation. They argued that representation is a complex cognitive process that is signified by personal and idiosyncratic aspects or cultural and conventional aspects (Bruner, 1966, Goldin & Kaput, 1996). According to Goldin and Kaput there are internal (mental) and external (physical) representations. Internal mental representations involve the use of mental images. These mental images could be pictures, diagrams, or even written items such as words and definitions .These imageries do not include every detail of the object or every action performed on them but recall their important characteristics. These internal mental processes, though not directly observable, are often inferred by educators and researchers through external behaviors (words), or actions (problem solving).

Since there are different types of representations for human knowledge, different people have different mental images of mathematical contexts. For example, before formally learning the *concept of limits*, students are familiar with the *terminology limit* in daily life. The meaning of this word is interpreted differently by different people. For instance, when referred to an *off limits* situation, the meaning of *limit cannot be attained* is implied while in another situation such as *age limit* 5, the implication is that the *limit cannot be surpassed*. Due to this confusion, students acquire different meanings and interpretations while dealing with the concept of limits. One such dilemma causes uncertainty in students whether limit can be obtained, or can be surpassed.

In this instance, how is concept modification possible? According to Piaget's (1970) and Skemp's (1971) theory of knowledge acquisition, knowledge or a concept is meaning-driven and is actively constructed by the learner. Accordingly, concept is formed through experience and through classification and abstraction. Gathering together experiences through similarities is termed classification while the process of identifying similarities among experiences is called abstraction. The efficacy of concept images depends up on the level of proficiency and flexibility with which the person acquired the conceptual knowledge. Since learners are active builders of knowledge, not passive receivers of information, I was curious to explore ways in which active construction of mathematical knowledge could be facilitated and how meaningful learning that is less vulnerable to misconceptions could accomplished. I posed that active and meaningful construction of mathematical knowledge could more likely be recognized by constructivist instructional model.

From a constructivist viewpoint, knowledge mediation and construction can best be achieved if students are active and interactive participants in the process of learning. According to Piaget (1970), however, social interaction helps build knowledge by simply aiding the restoration of equilibrium. Constructivism has different forms. While they all support the notion of individual construction of knowledge, there are also profound distinctions. Paul Ernest (1996) discussed various forms of constructivism: weak constructivism, social constructivism, and radical constructivism. *Weak constructivism,* as Ernest (1996) describes it, assumes that there is a realm of knowledge out there which could be received via senses. Individuals personally construct knowledge to match the body of knowledge out there. *Radical constructivism* is based on the epistemology that all knowledge is being constructed by individuals on the basis of cognitive processes in dialogue with the experienced world. This form of constructivism maintains that individual knowledge is in a state of change through reevaluation via the process of adaptation and accommodation. This form of constructivism assumes no position regarding the world of knowledge outside the experience of the knower. Finally, *social constructivism* acknowledges the existence of a world of knowledge that supports the appearance we have shared access to, but which we have no way of knowing about for sure. Individuals construct knowledge with the force of language and social interaction. Thus knowledge is a social construction, a cultural product consensually negotiated by communities. Learners construct knowledge within the cultural context that gives meaning to that knowledge.

As far as the application of constructivist model to classroom teaching goes, Glasersfeld (1996) offered several pointers to help teachers create a classroom that is more constructivist and meaningful in nature. Piaget's (1970) assertion not to consider students as mere blank slates holds a substantial role in Glasersfeld's view of constructivist's classrooms. Instead, teachers were encouraged to acknowledge students' prior conceptual knowledge and how that can serve as a foundation to construct knowledge that is more meaningful (Piaget, 1970). Accordingly, the teacher must try to find out where students stand as far as the knowledge they hold prior to instruction.

According to Glasersfeld, a teacher must then try to modify students' ways of thinking of mathematical knowledge. Students must realize that knowledge is context specific, and that knowledge that is well explored within the context of discussion is more sensible. Subsequently, instead of *giving out* correct answers, a teacher must ask students how they arrived at their answer and then help them explore the reasons why the answer may or may not work under another similar circumstance. Thus, students may become proficient in making sense of their own mathematical thought process through analyzing the problem situation and by discriminating between similar and diverse circumstances.

Research Question

How can students' pre-existing concept images regarding the concepts of rational functions, their asymptotes, limits, and continuity be modified? What effect does the teaching experiment have on students' concept modification and forming connections between the closely related concepts of asymptotes, limits, and continuity?

Methodology

I conducted a small scale Teaching Experiment (Steffe & Thompson, 2000) with a small group of participants to gain insights into, and build explanations of students' mental constructions of mathematical concepts. Seven students were recruited based on their performance on the problem solving interview and also based on their immediate availability. Through a teaching experiment a researcher will be able to gain first-hand information on students' mathematical reasoning (Steffe & Thompson, 2000). Teaching experiments are usually situated within a constructivist theory of learning in which learning is believed to be happening because of learners' active construction of knowledge.

Steffe (1980) identified three major aspects of a teaching experiment as individual interviews, teaching episodes, and modeling. Teaching experiments are concerned with conceptual structures and models of the kinds of change that are considered learning or development. Modeling is the formulation of models that would explain the results of the research. In this research, modeling would explain the researcher's perception of how students construct, interpret, and refine their notions of the mathematical concepts of asymptotes, limits, and continuity.

By following Steffe's model, once students' concept images were identified after the exploratory interview, I carefully outlined the lesson plans to be used during the teaching episodes. Each lesson contained a set of problems that students were expected to solve individually. In some cases, for example, with rational functions, a packet of notes was provided that contained definitions of some concepts with examples and non-examples. Lesson plans were designed bearing in mind that in order for students' conceptual schemes to change, they should be presented with situations in which they deem their existing schemes inadequate. In such cases, the new situation cannot be solved unless students make a major reorganization in their existing conceptual schemes.

The setting of the teaching episodes was somewhat different from that of a typical classroom. There was no urge to rush; there were no deadlines to meet, and no curriculum that must be covered during these teaching episodes. In addition, direct instruction was not used and students were encouraged to first work on their own and then discuss their ideas in groups while the teacher-researcher took the role of a challenger and a negotiator of concepts. The focus was on how students' conceptual schemes changed due to specialized mathematical interactions that took place between students, between the student and the researcher, and autonomously within each student. The researcher, however, asked probing questions, provided scaffold at times, and afforded students with additional resources as needed.

Before the discussion of any mathematical concepts, I started the teaching episodes with a brief presentation of the nature of mathematical truth. At this time, a list nicknamed as *the rules to live by* were also discussed. These rules were: (1) do not try to memorize every single rule or fact you may come across, (2) mathematics does not comprise simply of memorization, (3) mathematics is a problem-solving art, not just a collection of facts, (4) do not give up on a problem if you couldn't solve it right away, (5) re-read it thoughtfully and try to understand it more clearly, and (6) struggle with it until you solve it (Stewart et al, n.d.).The eight learning episodes, conducted over a period of four consecutive weeks were dedicated to mathematics concepts of *rational functions*, *usage of correct terminology, definition of asymptotes, vertical asymptotes, horizontal asymptotes, other asymptotes, limits, finite limits, infinite limits,* and *limits at infinity*.

Teaching episodes were based on 6 assumptions on student concept formation and conceptual change. The *first* assumption was that student's concept images were influenced by their previous exposure and experience with the concept. The *second* assumption was that students actively construct their own mathematical knowledge and active assimilation, accommodation and equilibration is imperative for the full development of these conceptions. The *third* assumption was that, equilibrium can only be maintained through the active encounter with situations that may pose disequilibria in the already existing conceptual framework. *Fourth*, I assume that social interaction and argumentation are essential features of maintaining conceptual equilibrium. *Fifth*, teachers could intervene and pose cognitive confusion by simply invoking contradicting problem situations by the usage of appropriate examples or by redirecting students' attention to formal definitions and probing a constant check between intuitions and formal definitions. The *sixth* assumption is that in a situation where the second, third, fourth and fifth assumptions are realized, a teacher-researcher might be able to foster concept refinement in learners.

I believe that concept refinement becomes complete only through the processes of constant re-assimilation, accommodation, and the maintaining of equilibrium invoked by different types of cognitive demands. It should also be noted that students' mathematical schemes change only slowly, and there will be extended periods when students operate at the same learning level. For example, during the learning episodes, there were instances where some students simply *could not* differentiate between the phenomenon of a hole and a vertical asymptote. Sometimes, students who seemed to have understood a concept suddenly could lose their steadiness when the concept was presented in a newer problem context.

In the teaching experiment, the sources of data were the students and the teacher-researcher. Students and the teacher-researcher provided data in the form of audio and video tapes of the teaching episodes that were transcribed. The researcher provided field notes for all teaching episodes. Video images helped the researcher to go back and review the teaching episodes for clarification and reconfirmation of documented findings during the research. Video segments were also selected to record possible interactions suggesting that students were engaged in personal constructions of strategies to solve a problem or finding explanations for their mathematical realities.

Results

As elaborated by Tripp & Doerr (1999), the investigation focused on how shifts in student thinking occurred and in what ways such shifts in thinking supported the development of viable models. What kinds of events in their interactions with each other, with the problem situation, and with associated representations led to shifts in their thinking? This was accomplished through closely observing changes in students' interpretations and representations during the teaching episodes as they faced challenging and inconsistent problem situations that conflicted with their existing concept images.

As revealed by the initial problem-solving interview, students' incomplete concept images of rational functions fell mainly into three categories: *the rational number image, the fraction image,* and *the discontinuity image.* Students who possessed the rational number image described that the graphs of rational functions are "nice," "symmetric" "one-piece," "continuous," and "without any complications." Some students specified that like rational numbers, such as $\sqrt{4}$ and $\sqrt{9}$, these functions were "whole." These students' concept image of rational numbers were restricted to that of numbers like $\sqrt{4}$ and $\sqrt{9}$. The rational number conception was the most prevalent conception that students held of rational functions.

Some students believed that rational functions assumed fraction forms with no variable in the denominator and accordingly no asymptotes or discontinuity existed in their graphs. However, those students who held the discontinuity image believed that all rational functions had variables in the denominator, and therefore their graphs were discontinuous, they came in several pieces, and had vertical asymptotes. During teaching episodes the concept of rational numbers were first explored. This was done by definition analysis: examining the definition of rational number, analyzing the meaning of the term *rational,* and examining polynomial functions. During this session students were curios to explore the basic structure of rational numbers and realize that rational numbers include a wider variety of other numbers in addition to numbers like $\sqrt{4}$.

Limited perspective on rational number concept, the connection between rational and fraction form as perceived by students *with no variable in the denominator* had clearly affected student conception of rational functions and their properties. These various prior notions needed to be re-configured in order to assign new meaning to the concept of rational functions.

Student notions about asymptotes in general fell in to the categories of the *three-piece graph image*, the *invisible line image,* and *the no-concurrency image*. The three-piece graph image reflects graphs with two vertical asymptotes and one horizontal asymptote that were symmetric with respect to the Y-axis. Three-piece graphs were also comprised of graphs with two vertical asymptotes (VA) and no horizontal asymptote (HA) and were symmetric about the origin. Students also believed that asymptotes were dotted lines the graph approached but never reached. The no-concurrency images held by students lead them to believe that a graph must never be concurrent with any of its asymptotes.

During teaching episodes, analysis of the definitions of the function concept, the independent and dependent variables, were instigated before the definition of asymptotes were discussed. These deliberations helped settle a big issue of terminology concerning *what is approaching what*; for example is it *as x approaches a, y approaches b or is it as y approaches b, x approaches a*? The various meanings of the word *approach* as students perceived them were also discussed in addition to each student elaborating on and debating over what their notions of asymptotes are and the reasons behind each of those notions.

Various textbook definitions of the concept of asymptotes were analyzed. During this time students unanimously pointed to the popular textbook definition "informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line." After examining this definition in regards to both vertical and horizontal asymptotes the dilemma was resolved and the unanimous decision was reached to amend the informal definition by specifying that *the curve was allowed to cut through the HA as long as it came back and approached the horizontal line or that the end behavior of the function should be such that it must approach the asymptote.* It was also noted that if a curve was to cut through the vertical asymptote, it could not come back and approach the vertical line, since this behavior would violate the function requirement. Students further elaborated that asymptotes of the graph of *non-functions* could very well intersect vertical asymptotes as well.

Regarding particularly on vertical asymptotes, students held a variety of notions. All students associated vertical asymptotes in connection with multiple views of undefinedness. Most students stated that *vertical always occurred at points where the function was undefined.* Though a few students who knew about the possibility of having either a hole or a vertical asymptote at the points were the rational function was undefined, they were unable to distinguish between the conditions under which a hole occurred for a rational function. Even though students had seen graphs with holes and have realized that the function would be undefined there, many of them were unable to differentiate between the function behavior around holes and around vertical asymptotes. The confusion between holes and vertical asymptotes seems to have stemmed from noticing no distinction between the $\frac{0}{0}$ and $\frac{b}{0}, b \neq 0$ forms.

I encouraged students to examine their own newly reconstructed definition of asymptotes to establish the condition/s that warranted the occurrence of vertical asymptotes. To challenge beliefs such as "a curve could not intersect its VA", "the function was undefined at VA" and

"VA occurred at the zeros of the denominator," I encouraged students to compare the functions $K(x) = \frac{x^2 - 1}{x + 1}$, and $a(x) = \frac{3x - 1}{x + 1}$. I also decided to invoke three types of representations to help students clarify the pitfalls that could arise from only using the algebraic methods or the graphing method using a graphing calculator without paying attention to the deeper structures of these functions. As such, students were asked to work first independently, and then as a group to explore the behavior of functions $K(x) = \frac{x^2 - 1}{x + 1}$ and $K(x) = x - 1$ at x = -1, first without a calculator. Then, they were allowed to use a graphing calculator, to graph and to 'zoom in' and then to explain what they saw. Students were further asked to compare and contrast between the function behaviors of K*(x)*, and $a(x) = \frac{3x - 1}{x + 1}$, at x = -1. This activity lead to a common consensus that a function $f(x) = \frac{p(x)}{q(x)}$ will have a VA at $x = a$, if $p(a) = 0$ and $q(a) \neq 0$. Later a variety of functions such as $m(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$, $f(x) = \frac{3x + 2}{9x^2 - 1}$, $t(x) = \frac{2x^3 - 5x^2 - 2x + 5}{2x^2 + 5x + 3}$ were examined for an in-depth analysis of different properties of different types of asymptotes and function behaviors around the asymptotes.

Instances of spontaneous dealings were invoked by student curiosity on what would happen if the degree of the numerator was larger than the degree of the denominator and what if the degree of the numerator was 2 or more degrees higher than the degree of denominator. These curiosities stimulated the discussions of oblique and non-linear asymptotes. To reveal another example, once students realized that factoring and canceling common factors were the main activities during the identification of holes and vertical asymptotes (VA), I asked students to solve a problem described by a table of values below. I was simply curious on what students might come up with as a consequence of this particular problem. The problem stated that for the function *V(x)*, the denominator was $11x - 4$. By observing the given table of values, what can you say about the behavior of *V(x)* at or around $x = \frac{4}{11}$? Note that $\frac{4}{11} \approx .364$.

This problem incited cognitive conflict in students that lead to some prime realizations. While they narrowed down the possibilities to be either a hole with finite limit or a vertical asymptote with infinite limit, the table of values clearly promoted some discussion. Even though the function appeared to approach infinities from either side of 4/11, as observed from the table of vales, the function value at this point appeared to exist. While students curiously perused how this could be possible, consensus were formed except for exact value x = 4/11, the function will remain defined. These discussions lead to the realization that it was important to always use exact value to discriminate between a hole and a vertical asymptote.

To help students understand the concept of infinite limit I encouraged them to observe the connections between infinite limit and the asymptotic property of the function. To help them recognize the occurrence of infinite limit, I encouraged them to inspect the function value at the point where limit was computed. Students realized that corresponding to the undefined form infinite limit existed. The next issue was to identify the direction in which the function was approaching infinity. One of the problems discussed to help students realize various possibilities was to ask them to discuss and respond to the reasoning behind an imaginary peer's – Mr.

Magoo's – argument while solving problems $\lim_{x\to 2^+}\frac{5x-9}{x-2}$, and $\lim_{x\to 2^-}\frac{5x-9}{x-2}$. Mr. Magoo found *f(3) = 6,* a positive number, and therefore concluded that $\lim_{x\to 2^+}\frac{5x-9}{x-2}=\infty$. Similarly, he concluded that *as* $x \to 2^-$,since *f(1) = -4/-1 = 4,* positive and therefore, $\lim_{x\to 2^-}\frac{5x-9}{x-2}=\infty$. Analyzing problems of this type helped students to see that while choosing to compute function values to tell the direction in which the function was progressing without bound, it is important to always check values "close enough" in addition to computing function values for at least two close enough points before making a conclusion.

While computing limit at infinity, by observing what would happen to each of the terms in the numerator and the denominator of the function as x approached infinities, students were able to infer its implications on the entire function value. They eventually made the observation that what was observed through the step by step examination of function terms connected consistently with the rules that they were familiar with as to comparing the degrees of the numerator and the denominator while computing limits at infinity. As to the discussion of the concept of continuity, within the realm of the time that was left, among other things students were able to make observations as to the different types of discontinuities that contributed to the occurrence of vertical asymptotes and holes.

Conclusion and Discussion

During the teaching experiment, I was mindful about students' previous concept images and how they could interfere with acquiring proper understanding of the concept definition. Drawing from the theory of concept images and concept definition I believed some level of balance between the concept image and the concept definition could be achieved through instructional activities that highlighted the importance active learning via social interaction and group work. This allows students the opportunity to re-examine and re-configure their incomplete conceptions by realizing how the concept applies and work in diverse problem situations.

I realize that student's previous concept images of rational numbers impeded their understanding of rational functions and their properties. This was caused by a limited perspective of rational number which led students believe that rational functions cannot have discontinuities or asymptotes. Thus, from the beginning of the discussion of rational functions and their asymptotes students seemed to have an unresolved disconnect of ideas. Students' conceptual understanding of asymptotes was modified by actively challenging their previous concept images of *invisible lines*, that were *just approached, but never reached* by the graph of the function This was accomplished mainly by reviewing, deconstructing, and amending the textbook definition of asymptotes and by trying to align it with a wide variety of functions.

As a result students were able to make connections between the concepts of limits and asymptotes in a more meaningful manner. They were also able to compute limits without the aid of a graphing calculator by visualizing the behavior of the graph at the point of calculation of limit. Thus, students seemed to have attained certain level of flexibility in the manner in which they think through limits and continuity in connection with function's asymptotic properties.

I believe that as a result of the transaction of the teaching experiment, students seemed to realized that mathematic is more than just by-hearting a bunch of rules and gaining understanding of calculus concepts is a lot more than simply being able to calculate the derivatives and just be proficient in performing different types of integration by simply applying rules.

Endnotes

1. This paper is a part of my dissertation, *College Students' Concept Images of Asymptotes, Limits, and Continuity of Rational Functions*, completed under the direction of Dr. Douglas T. Owens; Professor, College of Teaching and Learning, The Ohio State University, Columbus, Ohio.

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The Relationship Between Learner Characteristics and Learning Outcomes in a Revised First-Semester Calculus Course

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There is a long record of student difficulty in calculus. Previous research has suggested that calculus students may benefit from instruction that emphasizes technology, the graphical representation, and the notion of local linearity. For this study, the researcher revised the curriculum of first-semester calculus to incorporate these three elements. For 28 students taking the course, the relationship between several learner characteristics (representation translation proficiency, prior knowledge of rate, prior knowledge of function, spatial ability, and representational preference) and calculus proficiency was assessed. Results showed that the vast majority of students achieved high calculus proficiency, and that the first four characteristics had a relationship with calculus learning outcomes, but representational preference did not. Further, some students revised incorrect prior knowledge.

Introduction

This study was inspired by a history of student difficulties in calculus and innovation in response to those difficulties. A significant body of research has addressed student difficulties and common misconceptions in calculus (Orton, 1983d, 1983i). Researchers and educators have suggested, used and assessed approaches involving a revised syllabus (Heid 1988), increased visualization (Hughes-Hallet, 1991), and technology such as Computer Algebra Systems (CAS) (Kendal & Stacey, 2001). Little is known about the impact of learner characteristics on learning outcomes for calculus students. In this study, a revised curriculum for first semester calculus was constructed which emphasized technology, visualization and the notion of local linearity. The aim was to assess the impact of various learner characteristics on eventual calculus proficiency for students in such a course.

Background

Previous research into student difficulties in calculus has addressed student conceptions and misconceptions of limits (Oehrtman, 2002; Tall, 1992; Bezuidenhout, 2001a), of the derivative (Ubuz, 2007; Bezuidenhout 2001b), and of the integral (Bezuidenhout & Olivier, 2000; Orton, 1983i). The most intransigent of the difficulties occur with limits (Tall & Vinner 1981; Cottrill et al., 1996). Tall (1991) defined a cognitive root to be a unit of knowledge which is a meaningful part of prior knowledge and allows for further theoretical development, and he referred to local linearity as the cognitive root of the derivative (Tall, 1992). Several studies have observed that students can successfully learn the derivative using local linearity instead of limits (Tall, 1986; Maschietto, 2002).

Mathematical situations may be represented in symbolic, graphical, numerical, and verbal ways (Hughes-Hallet, 1991). It is important for students to be able to use each of these representations, and to translate between them (Tall, 1991). There are specific problem-solving benefits to the graphical representation (Hershkowitz & Kieran, 2001; Larkin & Simon, 1987). In particular, more frequent use of visual methods has been associated with higher performance in calculus (Haciomeroglu et al., 2010).

Some research has focused on preconditions for success learning calculus. The importance of prior knowledge of rate (Pustejovsky, 1999) and of function (Tall, 1997) has been noted.

Translation between representations has been adopted into a calculus assessment framework by Kendal & Stacey (2003), but not examined as prior knowledge. Results have been mixed regarding the role of learner characteristics such as mathematical representation preference or spatial ability (Presmeg, 1985; Bektasli, 2006).

A shift away from procedure-driven instruction and toward broader concept-driven instruction is supported by research in psychology (Chen, 1999) and mathematics education (Rittle-Johnson & Alibali, 1999). In calculus, traditional instruction focuses on execution of computations which can now be done with a CAS or other mathematical software (Tall, 1997). Researchers and educators have used, studied, and recommended calculus instruction emphasizing visualization (Hughes-Hallet, 1991), visualization with custom software (Tall, 1986), or CAS (Kendal & Stacey, 2001). For example, Hahkioniemi (2004) suggested that students use the visual representation to intuit differentiation rules.

Java applets and similar computer applications for mathematics instruction ("mathlets") have led to improved learning outcomes in mathematics education (Kidron et al., 2001; Heath, 2002). A mathlet is a small platform-independent application, typically interfaced through a web browser, offering interactive tools to explore a particular mathematics topic. Prior research has identified three main reasons for the potential positive impact of applets throughout mathematics education, and particularly in calculus. First, dynamic interaction is beneficial to the learner (Arcavi & Hadas, 2000); second, lack of dissemination and other logistical difficulties have, in the past, been obstacles to instructional change, but this is not a problem with applets because they can be accessed for free on the internet (Hohenwarter & Preiner, 2007); and third, mathlets are easy to use (Heath, 2002).

Research Question

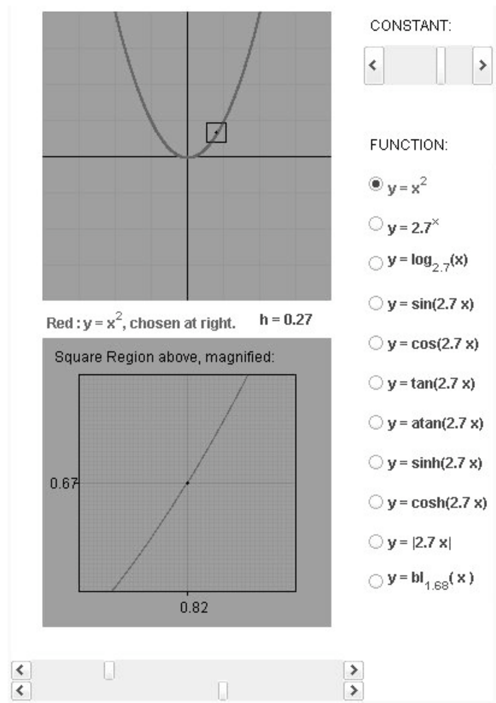
Research has shown that there are several advantages associated with using technology and visualization to teach first semester calculus. However, very few studies have considered the outcomes when a calculus course is reorganized around the notion of local linearity. Further, little is known about how students' different learner characteristics might impact their ability to benefit from this approach. Therefore, this study investigated whether there was a relationship between the learner characteristics of prior knowledge of rate, prior knowledge of function, spatial ability, representation preference, and proficiency translating between representations at the beginning of the semester and the achievement of proficiency in calculus by students by the end of a first semester calculus course which emphasized technology, visualization, and introduced the derivative via the notion of local linearity.

Procedures

The subjects were 28 students enrolled in a first-semester calculus course. Several sections were offered by the college, and students had no prior knowledge of the study, so a selection bias was unlikely. The site of the study was a large urban community college. The instructor was the researcher, who taught one calculus section.

The curriculum for the class was revised to incorporate technology, promote visualization, utilize the notion of local linearity, and delay the introduction of limits. Nine labs, each of which consisted of a mathlet and an activity sheet, were integrated into the course. Lab reports were collected as homework assignments. Students were introduced to local linearity in lab #3, which involved the local linearity mathlet designed by the researcher, displayed in Figure 1 below. The upper window displays the graph of the function selected, and the bottom window displays a small region of the graph. One slider allows the student to change the x-value at the center of the region, and the other slider allows the student to change the half-width *h* of that region. When the function is differentiable at that x-value, under sufficient magnification (sufficiently small *h*) a nearly straight line is graphed in the lower window. The last two choices for *f(x)* include points of non-differentiability. Therefore the introduction students received to the derivative at a point was graphical, as was the subsequent introduction to the derivative as a function. Students thereafter learned symbolic differentiation techniques. Formal limits were covered at the end of the semester.

Figure 1. A screenshot of the local linearity mathlet designed by the researcher



Data on spatial ability, representation preference, prior knowledge of rate and function, and proficiency translating between representations were collected at the beginning of the semester. Spatial ability and representation preference were assessed using instruments from Guay (1976) and Presmeg (1985), respectively, which have documented reliability and validity.

All other assessment followed frameworks constructed by the researcher. The instruments to assess prior knowledge of rate and function and proficiency translating between representations were constructed by the researcher and expert-validated for this study. Prior knowledge of rate and function were each assessed in the four mathematical representations (symbolic, graphical, numerical, verbal), and representation translation was assessed in 8 combinations (from symbolic to graphical or numerical; from graphical to verbal or numerical; from verbal to symbolic, graphical or numerical; and from numerical to graphical) using typical algebra and precalculus exercises assigned at the beginning of the semester.

The framework for calculus assessment was inspired by elements from Maschietto (2002), who used epistemological, cognitive and didactic considerations, Zandieh (2000), who divided comprehension of the derivative into 3 stages, and Kendal & Stacey (2003), who focused on multiple representations in their assessment. Calculus content was categorized according to topic and mathematical representation in the following way: find limits symbolically, graphically, and numerically; state the definition of the derivative symbolically (limit definition), graphically (slope of the graph, slope of the tangent line, slope indicated after magnifying the graph) and verbally (instantaneous rate); find the derivative at a point symbolically (calculate f '(a)), graphically (draw the tangent line, estimate from graph), numerically (estimate from a table), and verbally (calculate a rate in a word problem); find the derivative as a function symbolically (differentiation techniques) and graphically (graph the derivative given the graph of the function); determine non-differentiability symbolically and graphically; solve applications represented symbolically (find the slope or equation of the tangent line, find critical points or optimize a function), and verbally (solve contextual problems such as maximizing profit). Data on learning outcomes were collected during exams and other assignments throughout the semester using expert-validated items, as well as from three task-based semi-structured interviews during the semester. Each unit of student work was categorized and graded for proficiency on a scale from 0 to 4. A score of 3 indicated understanding or a solution that was nearly complete, or complete with several minor errors. A score of 3 or greater reflected high proficiency, and less than 3 reflected low proficiency. Calculus knowledge data were collected throughout the semester, and for each unit of the rubric, the highest level achieved by the student was the assigned score.

Results

Students generally developed a robust conception of the derivative by the end of the course. Overall, 21 students (75%) exhibited high proficiency in learning outcomes collectively. More than 70% of the students were highly proficient in 15 of the 18 rubric categories, including a robust conception of the derivative (in multiple representations and contexts), finding the tangent line (graphically and symbolically), finding the derivative (at a point or as a function, either symbolically or graphically), non-differentiability, symbolic differentiation techniques, and optimization. The other 3 rubric categories all involved limits.

The relationships between learner characteristics and derivative proficiency (as measured by an average of the rubric unit scores) are presented in Table 1.

Table 1. Derivative proficiency achieved by initial student learner characteristic (n=28)

Learner Characteristic	*Initial Level*	*Derivative Proficiency Achieved*	
		High (3.0-4.0)	Low (0.0-2.9)
Representation translation proficiency	High (3.0-4.0)	21	1
	Low (0.0-2.9)	0	6
Prior knowledge of rate	High (3.0-4.0)	20	1
	Low (0.0-2.9)	1	6
Prior knowledge of function	High (3.0-4.0)	12	0
	Low (0.0-2.9)	9	7
Spatial ability	High (15-24)	10	0
	Medium (10-14)	6	4
	Low (0-9)	3	3
Mathematics Representation Preference	Balanced or visual (-3 to 12)	8	1
	Slightly non-visual (-6 to -4)	7	2
	Very non-visual (-12 to -7)	5	2

There was a strong link between representation translation and calculus outcomes. Most students (22) exhibited high proficiency in translating between representations. Of those 22, 21

(95%) displayed high derivative proficiency. Of the 6 with low representation translation proficiency, 0 displayed high derivative proficiency.

There was a clear link between prior knowledge of rate and of function and calculus proficiency. Of the 21 students with high prior knowledge of rate, 20 (95%) achieved high derivative proficiency. Of 7 with low prior knowledge of rate, only 1 (14%) achieved high derivative proficiency. All 12 students with high prior knowledge of function demonstrated high derivative proficiency. Of 16 with low prior knowledge of function, 9 (56%) demonstrated high derivative proficiency.

All 10 (100%) students with high spatial ability demonstrated high calculus proficiency. Of the 10 students with medium spatial ability, 6 (60%) demonstrated high calculus proficiency, and of the 6 students with low spatial ability, 3 (50%) demonstrated high calculus proficiency.

Of the 9 students with a visual or balanced mathematical representation preference, 8 (89%) demonstrated high calculus proficiency. Of the 9 students with a slightly non-visual preference, 7 (78%) demonstrated high calculus proficiency. Of the 7 students with a very non-visual preference, 5 (71%) demonstrated high calculus proficiency.

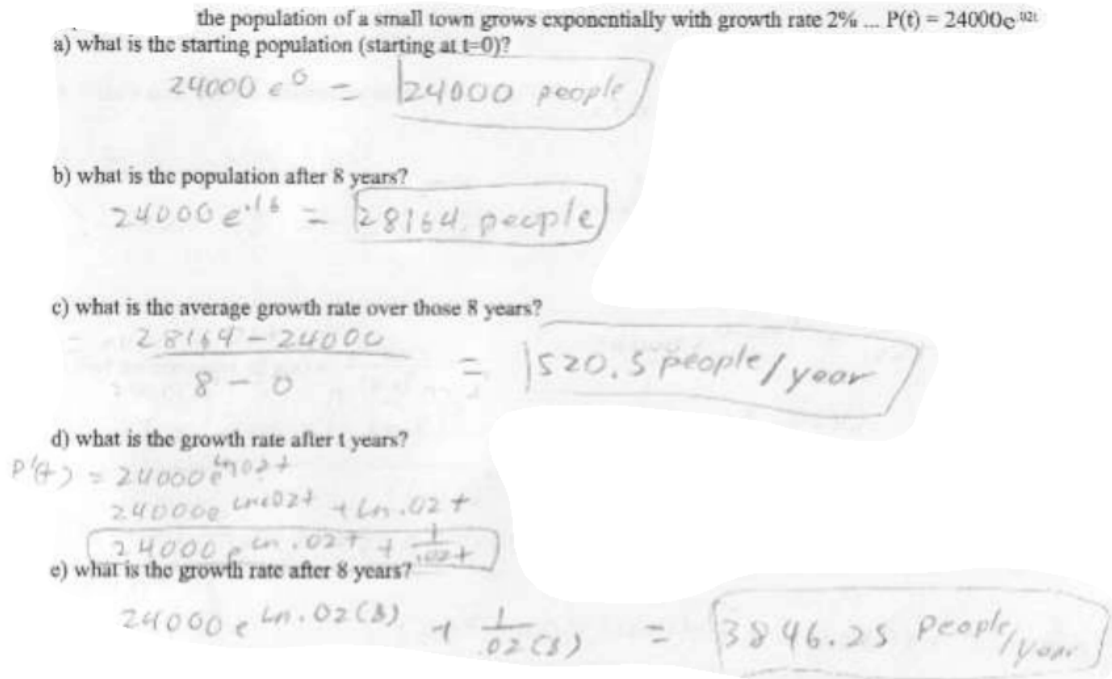
Although representation preference did not have a strong impact on student calculus achievement, some students with a high preference for visual representations demonstrated a very high level of enthusiasm for the course. One student called it his “favorite mathematics class ever.” Another used the mathlets to teach calculus to his friends. He also said “this calculus course has made me more interested in math” and as a result he decided to take more mathematics classes in the future.

Some students revised incorrect prior knowledge during the course. For example, one student demonstrated low proficiency in prior knowledge for distinguishing between average and instantaneous rates. He was asked to solve the following problem:

> The manager of a furniture factory finds that it costs $2200 to manufacture 100 chairs in one day and $4800 to produce 300 chairs in a day. Express the costs as a function of the number of chairs produced, assuming that it is linear. Then sketch the graph. What is the slope of the function and what does it represent?

He wrote, “f(c)=22c or f(c)=16c It first starts off high at 22 dollars a chair then at 300 drops to 16 dollars a chair.” He incorrectly calculated slope as ${}^{y}/_{x}$ determined at each point. After instruction on the derivative, however, the student demonstrated proficiency in making that distinction, as well as in symbolically calculating an instantaneous rate both at a point and as a function (with a minor error), as seen in Figure 2.

Figure 2. Work by one student on a problem about average and instantaneous rate



Another student misunderstood the concept of instantaneous rates given a graph or table of values at the beginning of the course. He also calculated $^{y}/_{x}$, and he asked about using $^{y}/_{x}$ to calculate the slope during a class session. On an early assignment, he wrote, "There are a few things I am not completely clear on: The slope of the point is rise over run, meaning y value over x value…?" Later in the course, he properly distinguished between them, as in the following interview excerpt:

Interviewer: Here's a graph, here are two points on the graph. I'm going to draw this line and take the slope. Is that a derivative?

Student: That is the average [slope].

Interviewer: Oh that's the average. If that's not the derivative then give me an example of a slope which is a derivative.

Student: Take this point for example. Zoom in on it and it becomes like a straight line. The slope would be like that.

Discussion

Three-quarters of the students in this first-semester calculus course, which emphasized technology, visualization, and local linearity, developed high calculus proficiency by the end of the semester. The high level of proficiency achieved by most of the students in this study deserves attention when contrasted with the sobering fact that nationwide half of students taking calculus in college fail (Tall, 1992). Although different initial learner characteristics were associated with varying outcomes, the prevailing result was one of overall high achievement. This finding alone makes the revised calculus curriculum at the heart of this study, and its elements of technology, visualization, and local linearity, worthy of further research.

Four of the five learner characteristics measured in this study were strongly associated with derivative proficiency. The strongest predictors were prior knowledge of rate and representation translation; for each indicator a high score almost guaranteed (over 90%) high calculus proficiency, whereas a low score made low calculus proficiency very likely (over 80%). The importance of prior knowledge of rate is consistent with the findings of Pustejovsky (1999), who used three case studies to conclude that prior knowledge of rate was essential for understanding the derivative. It follows from common sense that understanding rate in the linear case is a necessary precursor to understanding rate in the non-linear case. Further, if students cannot achieve understanding of rate in the simpler linear case, it is unlikely they will learn it in the more advanced non-linear case.

While spatial ability and prior knowledge of function were associated with achieving high derivative proficiency, low or medium spatial ability or low prior knowledge of function did not preclude achieving high derivative proficiency. Notably, representation preference did not have a strong impact on student calculus performance overall; there was almost no difference in rates of high calculus proficiency relative to mathematical representation preference. These are noteworthy outcomes in a calculus class in which graphs were the prime vehicle through which the derivative was introduced. The higher privileging of the visual representation, relative to traditional calculus instruction, did not seem to disadvantage students who were less facile with that representation.

Haciomeroglu et al. (2010) found that a preference for visual methods in solving calculus problems at the end of the semester was associated with higher achievement in the course. The current study found that most students in a calculus class that emphasized visual methods demonstrated high calculus proficiency by the end of the course. This was true even for students who entered without high spatial ability and without a preference for the visual representation. This result strongly supports the inclusion of visual representations in calculus instruction, in particular as part of the curriculum described in this study.

Students in this study revised incorrect prior notions on topics that were related to course content, such as average versus instantaneous rates. This counters the notion that knowledge, once inculcated, is immutable. Rather, acquired knowledge should not be considered static. Thus it is important to find an instructional path which simultaneously delivers new knowledge while enhancing prior knowledge. Some studies have looked at the possibilities for instruction in constant and non-constant rates of change before calculus. Five-year-olds have shown the ability to distinguish between linear and non-linear situations by attending to rate (Ebersbach et al., 2010), and middle school students successfully encountered early calculus concepts with an example of changing rate in kinetics (Barger & McCoy, 2010). The alignment of central mathematical themes and cognitive roots for advanced mathematical ideas, such as rate and local linearity in calculus, suggests a powerful way to organize curricula which encourages students to revisit a theme in courses of increasing sophistication, refine their conceptions, and build meaningful networks rich with connections between fundamental mathematical ideas.

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MATHEMATICAL AND NON-MATHEMATICAL UNIVERSITY STUDENTS' PROVING DIFFICULTIES

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This paper discusses university students' mathematical and non-mathematical proving difficulties. A total of forty-one difficulties have been observed and organized into nine categories. Of these difficulties, twenty-seven are briefly described below. These observations come from several years of teaching an experimental proving course to beginning graduate and advanced undergraduate mathematics students and from teaching an experimental voluntary proving supplement to an undergraduate real analysis course. We believe that discussing and categorizing these difficulties will lead to a greater understanding of students' thinking with regard to proof and to future research.

In this paper, we discuss and categorize students' difficulties that were observed and documented over several years in two whole class teaching experiments that emphasized active student participation in the construction of proofs.

Research Settings

The settings in which we observed students' proving difficulties were: (1) a course on proof construction designed to improve the proving skills of beginning graduate and advanced undergraduate mathematics students (hereafter referred to as the "proofs course"), and (2) a voluntary proving supplement to an undergraduate real analysis class (hereafter referred to as the "supplement").

In the proofs course, the students were given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students constructed their proofs at home and presented them in class. The proofs were then critiqued, sometimes extensively, and additionally suggestions for improvements in the notation used and the style of writing were given. There were no formal lectures, and all comments and conversations were based solely on students' work. Extended comments might occasionally turn into mini-lectures. The specific topics covered were of less importance than giving students opportunities to experience as many different types of proofs as possible and having them develop beneficial ways of reasoning. (For a detailed description of the course, see Selden, McKee, & Selden, 2010.)

The supplement was taught at the invitation of an undergraduate real analysis teacher because her students were having difficulty constructing proofs, because she did not have sufficient time to both cover the content and to work on students' proving as much as she would have liked, and because she was familiar with the above proofs course. Each week, one proof problem was selected or created to "resemble in construction" an assigned homework proof problem that the real analysis teacher intended to grade in detail, and that could be improved subsequently and resubmitted for additional credit. The supplement proof problem could be solved using actions similar to those useful in proving the corresponding assigned homework proof problem. However, the supplement proof problem was *not* a template problem, and "on the surface" would often not resemble the assigned proof problem.

In the 75-minute supplement period, the students co-constructed the proofs at the blackboard with the facilitation and encouragement of the supplement instructors. First the students would co-construct the *formal-rhetorical* part of the proof (Selden & Selden, 2009). This consisted of first writing the hypotheses at the beginning of their proof. Then, leaving a space for the body of the proof, the students would write the conclusion at the end of their nascent proof. Next students would unpack the conclusion and find and write the relevant definitions, such as that of sequence convergence, on the side board that had been set aside for "scratch work." Then the students would change the notation in the definition to "match" that of the theorem to be proved. They would then examine this definition to see where to start and end the body of the proof. For example, if the proof problem were one of showing a sequence $\{a_n\}_{n=1}^{\infty}$ converges to A, they would then write "Let $\varepsilon > 0$" immediately after the hypotheses, leave a space for the determination of N, write "Let $n \geq N$," leave another space, and finally write "Then $|a_n - A| < \varepsilon$" prior to the previously written conclusion at the bottom of their nascent proof. This brought them to the *problem-centered* part of the proof (Selden & Selden, 2009), where some "exploration" or "brainstorming" on the side board would ensue. The entire co-construction process, and accompanying discussions, was a slow one – so slow that only one theorem was proved and discussed in detail in each supplement class period. Finally, a pre-prepared written description of anticipated proving actions (which was usually three or four times as long as the final written proof), would be distributed. (For a detailed description of the supplement, see McKee, Savic, Selden, & Selden, 2010).

Theoretical Background for the Teaching

The teaching for both the proofs course and the supplement was informed by our theory of actions in the proving process and by our division of proofs into their formal-rhetorical and problem-centered parts (Selden & Selden, 2009). We see much of the proving process as a sequence of mental and physical actions, such as writing or thinking a line in a proof, drawing or visualizing a diagram, reflecting on the results of earlier actions, or trying to remember an example. As a person gains experience, much of proof construction appears to be separable into sequences of small parts, consisting of recognizing a situation and taking a mental or physical action. Actions which once may have required a conscious warrant can become automatically linked to triggering situations. We view such small, automated situation-action pairs as persistent mental structures that we have called *behavioral schemas* (Selden, McKee, & Selden, 2010). Such behavioral schemas are a form of procedural knowledge, that is, of knowing how to do something, as well as of "knowing *to* act in the moment" (Mason and Spence, 1999). We see behavioral schemas as very flexible in their implementation. They are not parts of fixed sequences that one might regard as procedures or as the implementation of algorithms. To acquire a beneficial schema, a person should actually carry out the appropriate action correctly a number of times (i.e., practice it) – not just understand its appropriateness. In our teaching, we provided students a variety of proving opportunities so they could acquire beneficial proving schemas. Changing a detrimental behavioral schema requires similar, perhaps longer, practice.

Because the students presented their work in the proofs course and because they co-constructed their proofs in the supplement, we had considerable opportunity to observe their actions. In addition, all classes were video-taped and subsequently analyzed to inform our subsequent teaching, as well as to make more general observations, such as those regarding students' proving difficulties.

Methods of Data Collection

The results reported here are part of a much larger study. Data were collected from 7 iterations of the proofs course beginning Fall 2007. Each semester between four and nine students were enrolled in the course. All class meetings were videotaped and field notes were taken by a graduate research assistant. These videos were viewed and analyzed by the authors and one or two graduate assistants the following day in order to determine what to do in the next class meeting, and interesting observations were noted so that similar instances could be watched for in future. These planning sessions were also videotaped and notes were taken by the first author. Individual tutoring sessions conducted by the second author were also videotaped. Take-home pretest, take-home and in-class final examinations, and student "scratch work" were collected in the proofs course. All this was done in the spirit of naturalistic inquiry (Lincoln & Guba, 1985) and to inform the future design of the course. The authors and the graduate assistants reanalyzed the data to confirm, elaborate, or adjust our earlier observations regarding students' actions and difficulties with proving and discussed these until we agreed on the accuracy of the observations.

In addition, data were collected from three iterations of the supplement beginning in Fall 2009 and these included: videotapes and field notes taken by a graduate research assistant, class homework and tests collected by the real analysis teacher, and observations on proving difficulties noted by the first author. The videotapes were viewed and discussed the following day to see how to design the next supplement class, and students' proving difficulties were observed and later discussed and categorized by the authors of this paper until they reached agreement. Some recurrent difficulties were observed in both the proofs course and the supplement.

Results: Categories of Students' Proving Difficulties

Although undergraduate students' proving difficulties have been observed by a number of researchers (e.g., Moore, 1994; Selden & Selden, 2003), we believe this is the first attempt to categorize and document as many of them as possible. Below we describe each category and give several examples of each. Some categories may overlap and this will be noted. When other researchers have reported similar difficulties in the literature, this will also be noted.

Non-mathematical Difficulties that Influence One's Ability to Prove

These came as a surprise to us and were noted in the supplement where we had ample opportunity to observe students in the process of proving theorems that were new to them. Examples include:

(1) There were students who were unable to copy a definition correctly onto the blackboard from the book or their notes while they were doing "scratchwork" or "brainstorming." This is important as missing even one word or phrase such as "for every" in a definition can lead to misunderstanding.

(2) There were students, who as spokespersons for their student groups, were unable to articulate notation or mathematical terms when reading or explaining a proof they had just constructed.

(3) There were students who had trouble adapting the letters (symbols) in a definition or previous theorem to the statement of a current theorem that they were attempting to

prove or use (in a line of their proof). This was merely a change in the symbolism; there was no change in the meaning. They merely needed to adjust the symbolism to fit the current context. Yet the students apparently found it difficult to go from one context to another. An example was using the definition of continuity given in terms of a function *f* to prove a theorem about a function $f \circ g$. Another is the real analysis student who used the letter *n* in his proof, and then found a relevant result in his textbook, and assumed (wrongly) that the letter *n* in the textbook theorem was the same as the one in the partially completed proof. We consider this, in part, as a difficulty with substitution – an area that has not been well researched.

Using Logic to Structure Proofs

It has been noted that university students have trouble using logic to structure their proofs (e.g., Moore, 1994; Selden & Selden, 2003). Indeed, a few transition-to-proof course textbooks pay special attention to this (e.g, Velleman, 1994, Chapter 3). By structuring proofs, we mean more than the usual discussion in such textbooks of how to begin and end direct proofs, contradiction proofs, contrapositve proofs, and mathematical induction proofs.

The extent and variety of these structuring difficulties seems not to have been fully considered and documented. Some situations involve making rather fine distinctions that probably do not need to be made in everyday conversation, and as such, may need special attention when teaching beginning university students. Examples include:

(1) Not knowing, when a situation with several possibilities occurs, that one can usefully consider cases, that the cases must cover all possibilities, and that the cases may overlap. A simple instance is considering cases when $x \in A \cup B$.

(2) Not understanding the distinction between *using* a definition to prove a statement (e.g., using the fact that a given sequence converges) versus proving a mathematical object *satisfies* a definition (e.g., showing that a particular sequence converges) is a difficult distinction to grasp and harder yet to implement. This was especially hard when our students had to make this distinction within a single proof, for example, in attempting to prove the product of two convergent sequences is convergent. Making this distinction involves knowing that a definition is an “if and only if” statement and keeping straight when and how to use the “if” part, in contrast to when and how to use the “only if” part.

(3) In a proof by mathematical induction, not knowing that one should start the induction step of what is to be proved with k+1 instead of k (especially when the statement of the theorem includes a “for all”), and only later using the induction hypothesis for k. This was observed more than once when a student in our proofs class attempted to prove that polynomials are continuous. Typically, the student began with an arbitrary polynomial of degree k, added to it a term of the form $a_{k+1}x^{k+1}$, and then proceeded to use the induction hypothesis. By doing this, the student had *not* proved that an arbitrary polynomial of degree k+1 is continuous, but rather only that a polynomial of this special form is continuous. The student might better have begun with an arbitrary polynomial of degree k+1; separated it into two parts, the term $a_{k+1}x^{k+1}$and the rest, which is a polynomial of degree k; applied the induction hypothesis to the latter; and finally applied the known (to the student) facts that sums and products of continuous functions are continuous, that a constant times a continuous function is continuous, and that polynomials of the form x^n are continuous.

"Metacognitive Knowledge/Actions"

Under this category, we have placed difficulties that seem to stem from the enactment of detrimental behavioral schemas or the lack of enactment of helpful behavioral schemas. The later might also be seen as a lack of strategic knowledge (Weber, 2001). Examples include:

(1) Knowing, or not knowing, that one can just "pick" an element that is known to exist (especially from an infinite set) instead of attempting to construct such an element. It would suffice to observe that such an element exists and give it a name, such as x_1.

(2) Focusing too soon on the hypotheses of a theorem to be proved, instead of first focusing on, and unpacking, the conclusion to see what is to be proved. We have documented in detail the case of Willy who did this for a theorem in topology (Selden, McKee, & Selden, 2010). We have observed this particular detrimental behavioral schema enacted many times by many students.

(3) Knowing, or not knowing, how to prove a compound element, or entity, is in a set. This was documented in detail and occurred when Sofia was attempting to prove the relative topology on a subset of a topological space is indeed a topology (Selden, McKee, & Selden, 2010).

(4) Knowing, or not knowing, when proving or disproving two groups (such as **N** and **Q**) are isomorphic, that one should first look for properties preserved by isomorphisms, rather than trying to find a bijection between the two groups and then attempting to show the operation is preserved (Weber & Alcock, 2004).

Going Outside of the "Proof Genre"

Students may have good ideas for a proof, but may not yet have mastered the art of writing proofs in the way their professors expect. A similar observation has been made by Mamona-Downs and Downs (2009). Examples include:

(1) Having a propensity to write definitions (that can be found outside the proof) into the proof. For example, some students wrote the entire definition for the continuity of a function into their proofs, instead of simply saying something like "By continuity of *g*, we have …".

(2) Writing directions to oneself, such as "I am now going to try to show that ..." or "We have to show that ..." in a (not exceptionally long) proof.

(3) Using informal language in a proof or using a word or concept that has not yet been formally defined. This occurred when students used function notation, such as $f^{-1}(x)=y$, instead of the ordered pair notation, before having proved that the inverse of the function f is also a function. Dreyfus (1999, p. 88) gave an example of a first-year linear algebra student who was asked to determine whether the following statement was true or false and explain. *If $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set, then $\{v_1, v_2, v_3\}$ is also a linearly independent set.* The student's answer was "True because taking down a vector does not help linear dependence." It is possible the student understood, but the use of "taking down" and "help" points to a lack of linguistic capability.

Conventions of the Mathematical Register

This category has a certain overlap with the above. Examples include:

(1) Not knowing the difference between "minimum" and "minimal."

(2) Knowing how to prove an "if and only if" statement by dividing the proof into two parts, indicating the beginning of one part by "($\rightarrow$)" and the beginning of the other

part by "(⟵)", thus demonstrating a knowledge of "if and only if" at an action level, but later asking about the meaning of "necessary and sufficient."

"Standard" Techniques of Proving

While these may have some overlap with the above mentioned "metacognitive knowledge/actions," one could also think of lacking these as having an "impoverished mathematical toolkit."

(1) Not knowing how to show "at most one" versus "at least one" or how to prove there is a unique object with some property by assuming, say x_1 and x_2, have the property and then showing $x_1 = x_2$.
(2) Not knowing how to use the Axiom of Choice, or which version to use.
(3) Not knowing how to prove "*p* or *q*" by assuming "not *p*" and proving *q*, or alternatively, by assuming "not *q*" and proving *p*.

Notational Difficulties

Using symbols carefully to "say what you mean" and "mean what you say" is not easy for students. Also, knowing when a symbol can be used again in a proof, and when it cannot, takes time to learn.

(1) Not knowing the *n* (or another variable) in a theorem one is trying to use is *not* the same as the *n* in one's proof.
(2) Problems dealing with three levels of abstraction: elements *u*, sets *U*, and families $\mathcal{U}$. For example, a set U can be open, but $\mathcal{U}$ (the topology) cannot be open.
(3) Being confused about when one can "reuse" a variable name. In a real analysis proof, a student used *Q* both for a neighborhood outside the proof (in the definition) and a neighborhood inside the proof.
(4) Not knowing how to avoid the use of indices, especially when the index set in not necessarily countable, or when it would be clearer to write without using indices. An example occurred when a student was trying to use mathematical induction to prove that any finite intersection of open sets is open, given that the intersection of any two open sets is open. The student could have considered an arbitrary family $\mathcal{B}$ of *k*+1 open sets, let $B \in \mathcal{B}$, and noted that $\bigcap \mathcal{B} = (\bigcap (\mathcal{B} \setminus \{B\})) \cap B$ is the intersection of two open sets and applied the induction hypothesis to the former intersection. However, the student used indices and got lost in the notation.
(5) Using the $\in$ and $\subseteq$ symbols interchangeably. This may be a confusion caused by everyday experiences with the use of "in". For example, if one has an aspirin in one's cosmetic bag which is in one's purse, it's OK to say "I have an aspirin in my purse" (Zazkis & Gunn, 1997). Part of students' confusion may also result from instructors using "in" for both "is an element of" and "is contained in."

Knowledge and Use of Quantifiers

That students have problems with quantifiers has been well-documented in the literature. This includes: not understanding that the order of the quantifiers influences the meaning of a statement (Dubinsky & Yiparaki, 2000); having difficulty interpreting implicit quantifers (Selden & Selden, 1995), and knowing the scope of a quantifer (Epp, 2003). We observed:

(1) In general, many students have difficulty with proving universally quantified statements, in which one customarily considers an arbitrary, but fixed, element, as in

□ real analysis proofs. They often wonder why the result has been proved for all elements, such as □, rather than only for the specific arbitrary, but fixed, element selected. Further, students often take a long time to feel comfortable that this is a legitimate method of proof; we have documented this with the case of Mary (Selden, McKee, & Selden, 2010).

The Problem-Solving Aspects of Proving

In an attempt to have students improve the problem-solving aspects of their proving and to have a "rough gauge" of (one dimension of) the difficulty level of proofs in our notes, we devised what we have called Type 1, 2, and 3 theorems (Selden, Selden, & McKee, 2008). How hard a theorem is to prove depends on many factors including a particular student's knowledge base and the method of proof. However, it is often possible, in the context of a given class, to make a judgment of difficulty for most students in the class. So in devising these three types of theorems, we were thinking of the kinds of proofs that our students had most often produced.

(1) A *Type 1 theorem* is one whose proof requires that students have to look for a useful theorem in the course notes. We put such theorems in our notes after it became clear that many students didn't look back over their notes to see if there were any useful theorems they could call on, rather than proceeding directly from the definitions of the concepts involved. For example, we have a theorem in our notes on the boundedness of a continuous function near a fixed number that would be useful in showing that the product of two continuous functions is continuous. However, our students rarely used that theorem and had difficulty proving the product theorem.

(2) A *Type 2 theorem* is one whose proof calls for stating and proving a lemma that is easily articulated and proved. We consider this to be cognitively more difficult than merely looking in the notes for a relevant theorem.

(3) A *Type 3 theorem* calls for stating and proving a lemma that is not easily articulated and proved. Doing this calls for considerable creativity. In our proofs course, *A commutative semigroup with no proper ideals is a group*, is such a theorem. There are two lemmas that would render the proof merely "a little tricky." However, these lemmas were purposefully not included in the notes so students could have the opportunity to discover them.

Discussion

While we do not claim the above nine categories are nonoverlapping or a complete list, we do believe they form a useful start. Also, although we have documented 41 difficulties, we have briefly described just 27 of them to illustrate the categories. We feel this is a beginning that could be helpful both to university and other teachers, as well as to researchers.

Several of the difficulties mentioned above have not been investigated in detail. For example, that simple non-mathematical difficulties can interfere with proving was a complete surprise to us and is open for investigation. Questions one might ask include: How do such difficulties as copying incorrectly directly from one's textbook or notes arise? Is this due to students' inadequate reading comprehension skills? How is it that students have not learned to articlulate (in words) various mathematical symbols? What does this imply about these students' mathematical thinking, if as Sfard (2009) asserted, thinking is just communicating with oneself? What do students think, or vocalize in inner speech when they encounter a mathematical symbol such as $\int_a^b f(x)dx$? What do students do with respect to articulating symbols when presenting their proof attempts for whole class discussion? Do students simply avoid vocalization by saying such things as, "We calculated, or we considered, this" and then point to the symbols? If so, why do they do this?

Another area that has not been investigated concerns structuring proofs. What does it imply about students' understanding of definitions if they do not know, in proving, the difference between *using* a definition of an object to assert a fact and proving an object, such as a function *g*, *satisfies* the definition of continuity. To date, much has been written about students' not knowing that mathematical definitions are analytic (or stipulated) rather than descriptive (e.g., Edwards & Ward, 2004); however, even after knowing this, students seem to have trouble making the much more subtle distinction between using a definition and satisfying a definition.

These are just some of the possible interesting questions that could be investigated.

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ASSESSING AFFECT AMONG UPPER ELEMENTARY STUDENTS WHO ARE GIFTED IN MATHEMATICS: VALIDATING THE CHAMBERLIN AFFECTIVE INSTRUMENT FOR MATHEMATICAL PROBLEM SOLVING

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The focus of the article is the validation of an instrument to assess gifted students' affect after mathematical problem solving tasks. Participants were 225 students identified by their district as gifted in grades 4-6. The Chamberlin Affective Instrument for Mathematical Problem Solving (CAIMPS) was used to assess feelings, emotions, and dispositions after students solved Model-eliciting Activities (MEAs) in groups of three. Through the use of principal component analysis, it was determined that three factors should be retained. The instrument holds promise because it may be used to assess affect which has implications for identification and curricular adjustments to optimize affect.

Literature Review

Conception of Affect in Mathematics Education

Myriad conceptions of affect appear to exist in mathematics education (Mann, Carmody, Chamberlin, 2011). Hart and Walker (1993), Hoffman (1986) and McLeod and Adams (1989), as examples, have referred to affect in mathematics using words such as motivation, dispositions, emotions, beliefs, and attitudes. Most definitions contain some variation of the aforementioned terms. It is therefore instrumental that one conception of affect be adopted. Anderson and Bourke's (2000) conception of affect is one commonly accepted given its degree of comprehensiveness. Consequently their conception of affect has been used for this research. They refer to affect as comprised of eight sub-components including: anxiety, aspiration, attitude, interest, locus of control, self-efficacy, self-esteem, and value.

Conception of Giftedness in Mathematics

Like the concept of affect, the concept of giftedness is a rather nebulous one. This is the case because countless definitions exist for giftedness and when applied to a discipline, e.g. mathematics, additional definitions seem to be propagated. Nevertheless, the definition of giftedness for this study is based on Krutetskii's (1976) premise that giftedness in mathematics is multifaceted. Through empirical research, Krutetskii found that nine areas of giftedness exist, but it could be argued that more exist. In specific, Krutetskii's nine areas of giftedness are: "the ability to formalize, the ability to generalize mathematical material, the ability to operate with numerals and mathematical symbols, an ability for sequential, properly segmented logical reasoning, an ability to shorten the reasoning process, an ability to reverse a mental process, flexibility of thought, mathematical memory, and an ability for spatial concepts" (pp. 87-88). Typically, students gifted in mathematics have at least one and perhaps more than one area of giftedness in mathematics.

History of Affect

The call to assess affect occurred as early as 1916, when Binet and Simon suggested that efficient cognition is impacted by feelings, emotions, and dispositions. They referred to affect as non-intellectual factors and they mentioned the ability to assess non-intellectual factors in school learning as of paramount importance. They viewed non-intellectual characteristics as equal importance to intellectual characteristics. Binet and Simon further claimed that the development of non-intellectual characteristics was antecedent and prerequisite to the development of intellectual characteristics. They suggested that by neglecting affect, learning conditions may be seriously compromised.

Non-intellectual characteristics then became known as non-cognitive characteristics (Chamberlin, 2010; Messick, 1979). The biggest challenge facing researchers, then and now, is the problem of quantifying a psychological construct. In short, researchers are faced with the question, “How do we measure something that does not readily lend itself to being measured?” Clearly defined concepts such as speed, volume, and mass are much easier to quantify since agreed upon metrics exist, but (psychological) constructs such as interest, attitude, and value do not have agreed upon metrics to quantify the degree of interest, attitude, and value. The question therefore remained, “how are these constructs quantified?”

Binet and Simon (1916) mentioned the significance of non-intellectual characteristics, but it was not until some 55 years later that a reputable instrument was created. In 1970, the National Longitudinal Study of Mathematical Abilities (NLSMA) completed a study conducted on a curriculum developed by the School Mathematics Study Group (Higgins, 1970). Researchers assessed student attitude with the use of 18 scales and the instrument had moderate to strong reliability coefficients (.59 to .85), but validity was not reported in the study. Interestingly, the impact of the instrument itself may not have been as significant as the creation of an instrument designed to assess a non-cognitive component. Two years later, Richardson and Suinn (1972) created the Math Anxiety Rating Scale (MARS) which was designed to assess student anxiety, and it was highly respected in the field of assessment of (mathematics) affect. Reliability coefficients were high for this instrument (.85 to .97) and the instrument’s validity was normed relative to the Differential Aptitude Test.

The aforementioned instruments were created to assess only one component of affect (e.g. attitude or anxiety). Hence, Aiken’s instrument (1974) not only promoted the idea that attitude may be comprised of enjoyment and value, but he actually followed up his theory with an instrument to assess each component. The true impact of the instrument came with the notion that Aiken promoted the theory that affect is multifaceted. Similar to the Richardson and Suinn (1972) instrument, the reliability was moderate to strong (.6 to .85), but the criterion validity of the instrument was questionable (.10 to .40).

The Fennema-Sherman Mathematics Attitude Scale (Fennema & Sherman, 1976), the first one widely used by researchers, was composed of nine scales that are ostensibly separate, but in reality are closely connected to one another. Four scales provide data on student affect and five additional scales provide data on teacher and parent affect. In addition, gender of participant and usefulness of mathematics is a domain. Two contributions arose from the development of this instrument. The first was that gender issues were brought to the fore of mathematics education and the second was that four components of affect were assessed at one time. The somewhat outdated scale was revised and re-validated in 1999 (Forgasz, Leder, & Gardner, 1999).

Only one study can be found that pertains to assessing gifted students’ affect in mathematics. In 1996, Pajares conducted a study on mathematical problem solving of gifted students and he used his self-efficacy instrument that had never been validated with gifted students so the results may be questionable. Despite all of this work, quite literally, no instruments have been created or validated for use with gifted students. Moreover, almost no instruments were created to assess affect during or after mathematical problem solving as most have been created generically to assess affect in mathematics. Consequently, the need for an instrument that has been specifically designed for use with gifted students that focuses on affect and mathematical problem solving would be an asset to the field of mathematics and gifted education.

Regarding affect and mathematics, one question perpetually surfaces which is, 'why does an instrument to assess affect among GT students in mathematics need to be created and subsequently validated?' This is a legitimate question. The responses to this question are threefold. First, affect is one of the most crucial impacts and determinants regarding students' achievement in mathematics (Dettmers et al., 2011; Shores, Shannon, & Smith, 2010). Second, it has been hypothesized that this age level, i.e., grades 4-6, are the most important for the development of long-term attitudes, emotions, and dispositions regarding mathematics (Hart & Walker, 1993). The ramifications therefore of neglecting affect among GT students is that if students develop negative affect towards mathematics in grades 4-6, the most promising mathematics students will be lost to other disciplines. Third, it is the moral imperative of the American educational system to substantially challenge all students, including the most capable students (Sheffield, Chamberlin, Tassle, Mann, & Carmody, 2011). Affect and challenge are intricately intertwined and neglecting such students does not bode well for international ratings of American students in mathematics and it may further have negative ramifications for maximizing mathematical learning in the classroom.

Methods

Mathematical problem solving is the focus of doing mathematics (National Council of Supervisors of Mathematics, 1978; National Council of Teachers of Mathematics, 1980; 1989; 2000). Given the high priority of mathematical problem solving (Chamberlin, 2008), it was determined that an instrument was necessary to monitor student affect after the process.

Participants

Participants in the study came from four elementary and middle schools throughout the inter-mountain region of the United States. There were 225 participants from a potential sample of 250 or a 90% response rate. The respective number of boys and girls in the actual sample is not known because all data were collected anonymously and gender was not recorded as per the schools' requests. However, there is no indication that a gender bias existed among participants because the actual sample appeared similar to the target sample which contained 51% girls and 49% boys.

Identification Procedures

Prior to soliciting schools, the first author investigated the identification procedures to see that similarity existed from district to district. Each school had minimally different identification procedures and they were all aligned with identification procedures consistent with those outlined by the National Association for Gifted Children.

Implementation of Problems

All of the problems implemented were open-ended tasks so they had more than one solution (process) and more than one product as potential responses. Further, students were given no direction regarding what mathematical procedure to use in an attempt to solve the problem. For each task, students were asked to read the newspaper article that accompanied the handout at home the night before the task was implemented and to complete a series of questions designed to create conversation about the topic. On the day of the activity, students met as an entire class, discussed responses to the comprehension questions with the facilitator (i.e., the first author), and read the problem statement. Once the facilitator was assured that all students understood what was asked in the problem statement, students worked with peers in groups of three to solve the problem. Solving the problems and writing up the solutions typically required 50 to 70 minutes so some class periods were extended to accommodate the research protocol. Upon immediate completion of the problem solving task, students were asked to complete the instrument designed to assess affect. This was done in an attempt to garner accurate data regarding student affect during the mathematical problem solving process. Solution discussion followed data collection.

Tasks Used

The tasks used were a special type of mathematical problem solving activity called Model-eliciting Activities (MEAs). To date, the most comprehensive review of MEAs, i.e., their purpose and general characteristics, who and how they were created, some examples, characteristics of student interpretations, four disclaimers, the six principles for design, common misconceptions, and questions about them, is detailed by Lesh, Hoover, Hole, Kelly, & Post (2000). The design principles, explicated in Chamberlin and Moon (2005) and Lesh et al., provide a basis for understanding what comprises all MEAs. At the heart of how MEAs are created are the six principles used to design MEAs. These principles are: the Model-construction principle, the Reality principle, the Self-assessment principle, the Construct Documentation principle, the Construct Shareability and Re-usability principle, and the Effective Prototype principle. The objective in using the six design principles is to increase the likelihood of uniformity in the structure of tasks. The Model-construction principle suggests that all MEAs demand that a mathematical model is constructed to explain the mathematical phenomenon or phenomena. The developed model should have immediate utility and future utility (reusability). The Reality principle does not insure that all tasks are immediately applicable to all students' lives. Instead the Reality principle demands that all tasks demand that students try to make sense of a mathematical situation. The Self-assessment principle suggests that each MEA should have some component that enables students to determine whether or not their response is logical. One question often posed to assess this principle is, "Will students know when they're finished with the problem?" The Construct Documentation principle suggests that each MEA must demand that students document their work. This documentation should closely detail their thinking (especially their mathematizing of the situation). The Construct Shareability and Re-usability principle demands that each MEA must have the construction of a model for it to be an MEA. Moreover, the model should meet the demands of the immediate problem, and as models go, should be able to be used in subsequent, similar mathematical situations. Therefore each model should satisfy an immediate as well as a long-range need. The Effective Prototype principle suggests that each MEA should leave some residue (Skemp, 1982) for future mathematical situations. The Construct Shareability and Re-usability principle demanded that the model is transferrable to future situations. The Effective Prototype principle demands that mathematical concepts learned may be applied to future, non-identical, situations.

Examples of MEAs may be found online at http://crlt.indiana.edu/research/csk.html and at https://engineering.purdue.edu/ENE/Research/SGMM/CASESTUDIESKIDSWEB/index.htm.

The activities used for this research were ones that have traditionally been perceived by teachers as free of or with significantly reduced gender bias. Activities used were: On-time arrival, Track programs for sale, Summer reading, Amusement park, and Track and field teams.

Instrumentation

The Chamberlin Affective Instrument for Mathematical Problem Solving (CAIMPS) was designed for grade 4-6 mathematics students of advanced academic ability. It is important to note that students in the convenience sample were not all identified as gifted in mathematics. Pending identification procedures, some students were simply identified as gifted in general. The 40-item instrument was initially designed to assess Anderson and Bourke's (2000) eight factors of affect with five items comprising each scale. These factors are: anxiety, aspiration, attitude, interest, locus of control, self-efficacy, self-esteem, and value. Respondents were provided with five options regarding how they would respond to each item: strongly agree, agree, undecided, disagree, and strongly disagree.

Results

Validation of the CAIMPS

The statistical procedure employed to investigate the actual number of factors that the CAIMPS would yield was exploratory factor analysis (EFA), a kind of principal component analysis. EFA is a process whereby items are analyzed repeatedly in an attempt to "identify the fewest possible constructs needed to reproduce the original data" (p. 533, Gorsuch, 1997). The number of factors was determined by the Cattell Scree-plot Test (Cattell, 1966). Based on the component factors of the scree, the analysis proceeded to extract three principal components (Stellefson & Hanik, 2008; Tanguma, 2000).

Statistical Procedures Used

A varimax roation method was employed to extract the three principal components (Bauer, Jackson, Skwarchuk, & Zelefsky, 2006). The Keiser-Meyer-Olkin test determined acceptable sampling adequacy (.916), indicating that few common factors were shared in the three factors. Bartlett's test of sphericity confirmed that the data were non-collinear. Based on the items that loaded onto each factor, the researchers identified the three factors as:

- Attitude/value/interest
- Self-esteem/self-efficacy
- Aspiration

Additionally, three items did not load onto these factors. Although each item was intended originally to investigate anxiety, their internal reliability of .519 was not satisfactory enough to identify them as a stand-alone construct and the items were consequently discarded from the analysis. Each factor is described below.

Factor 1: Attitude, value, interest

The first factor is comprised of the items associated with attitude, value, and interest. The construct is comprised of 14 items: 1, 2, 6, 9, 13, 20, 23, 24, 27, 30, 31, 32, 34, and 37. The measure of internal reliability, Cronbach's alpha was very high (α= .918).

Factor 2: Self-esteem, self-efficacy

The second factor is comprised of the items associated with self-esteem and self-efficacy. The construct is comprised of 14 items: 3, 4, 5, 10, 12, 16, 17, 19, 26, 29, 33, 36, 39, and 40. The measure of internal reliability was also high (α=.908).

Factor 3: Aspiration

The third factor is comprised of the items associated with aspiration. The construct is comprised of 8 items: 8, 14, 18, 21, 25, 28, 35, and 38. The Cronbach's alpha is moderately high (α=.805), and certainly within acceptable levels.

Discussion

Implication 1: Use of Instrument

This instrument was designed for specific use with upper elementary and lower middle grade gifted students at the immediate conclusion of an open-ended mathematical problem solving task. Given the diversity of characteristics of the sample, the instrument may be used with a broad range of students as it was validated with students identified as gifted in general. The application of this instrument to classroom settings with gifted students may provide immediate, reliable data regarding students' affect at the conclusion of mathematical problem solving activities.

Implication 2: Use of Instrument for Identification Procedures

This instrument may have particularly strong implications for identifying advanced mathematicians and it may point to prospective success in mathematics. Identification of students who have the greatest likelihood of success in a program is a necessary component in strong identification procedures and affective ratings should be considered in this equation. With initial identification procedures, seeking very high intellect was the only emphasis in GT programs. However, more contemporary beliefs are that multi-modal procedures can bolster and provide more comprehensive data than simply viewing IQ data.

Implication 3: Ease of Ese

Given the relative ease with which this instrument may be implemented, it is noted that teachers of the gifted can have the ability to make immediate changes in an effort to maximize learning. Moreover, this investigation of affect, through the use of an empirically proven instrument, may be used with certainty given data attained.

The implications contribute to GT mathematics students' learning insofar as they are all intended to provide immediate data for teachers to make adjustments to curricular approaches and selections. Moreover, the data secured was designed for assessment of affect with highly open-ended tasks. The positive component to this instrument is that students' affect links directly to their achievement. Negative affective ratings do not insure poor learning, but statistically, they correlate very closely. Similarly, positive affective ratings do not insure the maximization of learning in mathematics, but they set up the pre-conditions for the optimization of learning in mathematics. Fundamentally, affect cannot be intentionally impacted without a cognizance of the level of affect. Moreover, an instrument that reveals affective ratings on problem solving tasks enables teachers to make instructional decisions immediately to positively impact learning environments.

Limitations

The first limitation is that the sample that completed the problem solving activities and the instrument were predominately students identified as gifted in general (not gifted specifically in mathematics). Hence, some students were truly advanced in mathematics and some were simply average ability students in mathematics which is a fairly diverse sample. Second, the method for analyzing data was exploratory factor analysis. Confirmatory factor analysis is often regarded as a more powerful approach to solidifying results and would therefore be the more preferential of the two approaches.

Areas for Future Research

Two areas for future research exist. First, for subsequent data collection procedures, the more specific population of individuals gifted only or at least in mathematics should be explored although this potentially nebulous construct may prove problematic in assessing students. What one may consider gifted in mathematics may not be what another considers gifted in mathematics. Second, some limitations are evident with exploratory factor analysis (EFA). Chief among the concerns is that the process of EFA is just as the name implies…exploratory. Consequently, the revised instrument must be used with a new sample of students with the intent of using Confirmatory Factor Analysis (CFA) to investigate the statistical strength of the instrument.

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SELF-EFFICACY, PERFORMANCE, & CALIBRATION IN ADVANCED MATHEMATICS

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This study applied a social cognitive framework to investigate self-efficacy, calibration (accurate feelings of knowing), and exam performance among 195 students in the content courses of a secondary mathematics teaching program. Using structural equation modeling analysis of survey and final exam data, findings suggest high prior performance, strong self-efficacy, and slight overconfidence are most associated with increased exam performance.

A quote attributed to Henry Ford summarizes a key principle in the social cognitive view of learning: "Whether you think that you can or that you can't, you're usually right." This hypothetical relationship between performance and beliefs in one's capabilities hinges on effects of perceived *self-efficacy*, or self-evaluations of one's ability to accomplish given performances under specific constraints (Bandura, 1997). In the context of the undergraduate mathematics coursework of prospective secondary mathematics teachers, Ford's simple axiom about the influence of self-efficacy on success is just one part of multifaceted interactions among experiences, self-beliefs, academic motivation, and performance that can help to explain future teachers' trajectory toward meeting their goals in mathematics.

Research into self-efficacy has established that learners who express high self-efficacy in an academic domain tend to perform better on tasks in the domain than similarly knowledgeable peers who report low self-efficacy (Pajares & Schunk, 2001). More than 1,800 research studies in education have addressed self-efficacy, and results suggest moderate-to-strong positive effects of self-efficacy judgments on performance tasks in domains as diverse as reading comprehension, career choice, and problem-solving in mathematics (Lightsey, 1999). However, there are documented exceptions to this trend (e.g., Klassen, 2006), and some important aspects of mathematics self-efficacy, especially *calibration*, or the accuracy of students' self-efficacy judgments, have received little research attention.

Following a thorough review of literature on mathematics self-efficacy, the dissertation study reported here addresses three aspects of the existing research which suggest further study: (1) the self-efficacy of college students in advanced mathematics courses, (2) the calibration of students' beliefs in their mathematical abilities, and (3) the mathematics self-efficacy of prospective secondary mathematics teachers. The overarching purpose was to investigate the following question:

> *How do self-efficacy and calibration influence the exam performance of students enrolled in the mathematics courses of a secondary teacher preparation program?*

Theoretical Framework

When considered in the complicated context of mathematics content preparation for secondary teachers, a social cognitive conceptual framework can help to explain the level and accuracy of students' self-perceptions of their abilities in advanced mathematics courses.

Social Cognitive Theory and Self-Efficacy

Albert Bandura's social cognitive theory first began as a means for explaining observational learning mechanisms by positing that a *causal triadic reciprocality* exists between individuals' behavior, environmental stimuli, and internal cognitive factors. This approach has since developed into a robust theory increasingly focused on the cognitive and motivational processes supporting metacognition (Schraw, 1998), self-efficacy, and self-regulation among learners as they acquire knowledge and skills. Self-efficacy emerges as a prominent theme; Lightsey (1999) identified over 2500 hundred articles addressing relationships between self-efficacy and achievement.

Social cognitive research considers self-efficacy to be a mediating mechanism in human cognition because self-beliefs in ability act as a filter between prior experiences and subsequent development of abilities within a particular domain. In contrast to self-concept, which refers to more global self-beliefs and personal identity, Pajares and Schunk (2001) summarize the hypothesized direct role self-efficacy plays in the choices people make:

> Self-efficacy beliefs influence the choices people make and the courses of action they pursue. Individuals tend to engage in tasks about which they feel competent and confident and avoid those in which they do not. Efficacy beliefs also help determine how much effort people will expend on an activity, how long they will persevere when confronting obstacles, and how resilient they will be in the face of adverse situations. (p. 241)

Attributed in part to individuals' tendencies to rely heavily on self-efficacy beliefs during difficult tasks (Bandura, 1997), self-efficacy judgments are often better statistical predictors of performance in academic domains than standardized measures of ability or intelligence (Pajares & Kranzler, 1995). In fact, after controlling for instructional factors, path analyses of performance incorporating biographical (e.g., socio-economic status, gender), motivational, and instructional variables, suggest self-efficacy beliefs account for the largest portion of variation in academic performance (Madewell & Shaughnessy, 2003). Though measures of self-efficacy are often useful for predicting performance, strong self-efficacy beliefs may be less connected to success in difficult domains such as mathematics. In particular, developing both strong and accurate self-efficacy beliefs may be foundational to the benefits of self-efficacy in learning mathematics at the level of calculus and above.

Mathematics Self-Efficacy, and Calibration

Sometimes referred to as "feeling-of-knowing accuracy" (Schraw, 1995, p. 326), students' *calibration* (Pajares & Miller, 1994) in self-efficacy ratings is a relatively new area for mathematics education research (e.g., Chen & Zimmerman, 2007) with foundations in experimental psychology and reading education (Lin & Zabrucky, 1998). For example, the tendency of students across backgrounds and performance abilities toward *overconfidence*, or positively biased self-perceptions of abilities, was identified has been identified as a potential obstacle to learning: "overconfidence is a common phenomenon among young adult students that may result in inadequate learning due to premature termination of cognitive processing." (Lin & Zabrucky, p. 384)

In contrast, Bandura (1997) suggests slight overconfidence in one's abilities can be psychologically adaptive because overconfidence can have positive benefits on effort and persistence. In this view, poor calibration in the form of overconfidence can be reframed as a set of optimistic self-evaluations that may ultimately support taking-on challenges. Nonetheless, Bandura and other calibration researchers (e.g., Pajares & Kranzler, 1995) caution against grossly inflated overconfidence, suggesting that unrealistic overconfidence can lead students to engage in self-handicapping academic behaviors (Urdan, 2004) such as reduced studying and increased procrastination.

Since personal, social and cultural conditions are seen as important co-determinants of academic confidence, motivation, and behaviors, social cognitive theorists do not subscribe to global models of self-efficacy and performance (Bandura, 1997). Thus, it was important to develop a hypothesized model of self-efficacy, calibration, and performance which was based on the specific learning context at the research site.

Hypothesized Model and Research Questions

The hypothesized model for exam performance in advanced mathematics, shown in Figure 1, was based on an extensive review of related literature and was similar to models employed by Chen (2003) and Pajares and Kranzler (1995) in studies of mathematics self-efficacy among general student populations. The hypothesized model is a compact way of representing four quantitative research questions (Q1-Q4 below), which were addressed using structural equation modeling.

Q1 Does high school mathematics achievement have a significant effect on the amount of mathematics in participants' college major?

Q2 Do high school mathematics achievement and the amount of mathematics in participants' college major have significant effects on participants' calibration?

Q3 Do high school mathematics achievement, the amount of mathematics in participants' college major, and calibration have significant effects on participants' self-efficacy?

Q4 Do high school mathematics achievement, the amount of mathematics in participants' college major, calibration, and self-efficacy have significant effects on participants' performance on exams in advanced mathematics?

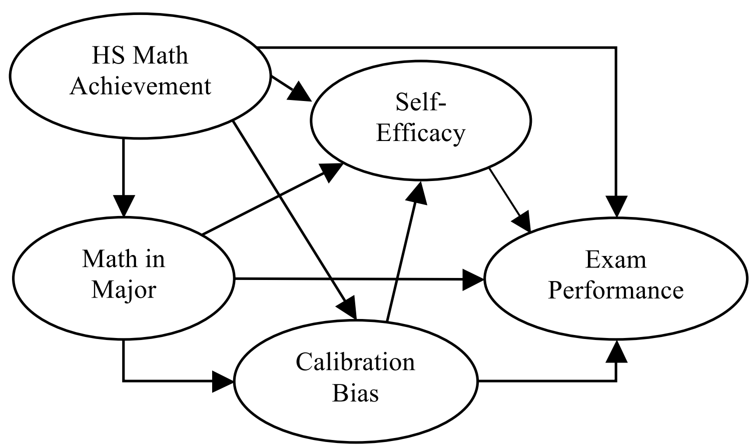

Figure 1. Hypothesized structural model for exam performance in advanced mathematics.

Methods

This study reports on findings from the quantitative portion of a mixed methods study which used a concurrent triangulation strategy (Creswell, 2003). Quantitative data sources included university enrollment data, background and self-efficacy survey responses, and copies of students' work on exams. Qualitative data sources included transcripts and artifacts from task-based interviews with 10 secondary teaching majors (see Champion, 2010).

Data Collection

Sample

There were 309 students enrolled in the 12 participating sections of advanced mathematics courses (M = 25.8, SD = 6.9) at the research site. Of the enrolled students, 17 (6%) did not take a final exam and 40 (17%) were enrolled in two or more of the classes, yielding a potential sample of 252 unique students who finished the classes. Of these, 210 (83%) consented to participate and complete data were available for N=195 students. Participating sections included three sections of Calculus I, two sections each of Calculus II, Calculus III, Discrete Math, and one section each of Linear Algebra, Abstract Algebra II, and Probability Theory. About half (49%) of the data comes from students' performance in Calculus I or II. When a student was enrolled in more than one participating section, only the data from the highest-numbered class was used for analysis.

Participants

Study participants ranged in age from 18 to 49 (M = 21.2, SD = 4.2). Most of the students (81%) were 18-22 years old, some of the students (11%) were 23-25 years old, or over 25 years old (7%). The percentages of study participants classified as Freshman, Sophomore, Junior, and Senior were 28%, 28%, 25%, and 18%, respectively. Study participants were almost exactly equally-distributed by gender (97 female, 98 male); the proportion of female students (50%) is substantially higher than national averages in advanced mathematics (Lutzer et al., 2007) but less than the overall proportion (60%) of female undergraduate students at the research site. Typical of the university, most of the participants self-identified as Caucasian (83%), followed in prevalence by Asian American (5%), Hispanic American (5%), and Other (7%). Though the study potentially included all students taking advanced mathematics at the research site, approximately half (49%) of study participants declared their primary major in mathematics or mathematics education. In particular, 12% indicated a major in Elementary Education – Mathematics Concentration and 37% indicated a major in Mathematics. About 79% (34/43) of the female mathematics majors chose the secondary teaching concentration, while just 37% (19/30) of the male mathematics majors chose the secondary teaching concentration. Other common student majors included Chemistry (10%), Earth Sciences (7%), Physics (6%), Biology (5%), Pre-Program (e.g., pre-medicine, pre-dentistry) (5%), and Undeclared (5%).

Data Analysis

The first four research questions (Q1- Q4) addressed effects posited by the structural path model of performance in advanced mathematics. Structural modeling was conducted using *R*, the open source implementation of S-Plus, and relied heavily upon structural model fitting routines in the package *sem* (Fox, 2009). The structural modeling procedures applied a strict interpretation of the underlying model, focusing on analysis of the correlation matrix and statistical evidence for inclusion or removal of the hypothesized directional effects of indicators and constructs (Suhr, 2008). Data collection instruments included a background survey, self-efficacy surveys in which students rated their confidence in being able to correctly complete exam tasks in the few minutes just before the actual exam, and graded copies of students' work on regularly administered final exams. Computation of self-efficacy and directional calibration scales followed procedures from several recent mathematics self-efficacy studies (e.g., Chen, 2003; Pajares & Miller, 1994).

Results

Descriptive Findings

Table 1 includes descriptive statistics of the indicator variables for the latent constructs High School Math, Math in Major, Self-Efficacy, and Calibration Bias. By definition, the means of the indicators for self-efficacy and calibration bias are ascending by “level.” For example, a student’s SE Level 1 rating indicates belief in being able to complete the mathematical tasks in which his or her classmates expressed the lowest collective rating. The means of self-efficacy indicators ranged from $M = 2.9$ to $M = 4.5$ on a scale of 0 to 5. The means of calibration indicators ranged from -0.4 for Level 1 (underconfidence) to 1.9 for Level 7 (overconfidence). Calibration means were significantly positive in 5 of 7 indicators at the $\alpha = .01$ criterion, suggesting general tendencies toward overconfidence in the calibration indicators.

Table 1.

Descriptive Summary of Indicators for High School Math Achievement, Math in Major, Self-Efficacy, and Calibration Bias

Construct	Indicator[a]	*n*	*M*	*SD*	Scale
HS Math Achievement	ACT Math	132	24.9	3.9	14 to 36
	HS GPA	133	3.4	0.6	0 to 4
	HS Self	195	4.6	1.1	0 to 7
Math in Major	Required Math	177[b]	23.0	15.2	3 to 45
Self-Efficacy	SE Level 1	195	2.9	1.3	0 to 5
	SE Level 2		3.2	1.2	
	SE Level 3		3.5	1.1	
	SE Level 4		3.7	1.1	
	SE Level 5		4.0	1.0	
	SE Level 6		4.2	1.0	
	SE Level 7		4.5	0.9	
Calibration Bias	Bias Level 1	195	-0.4	1.8	-5 to 5
	Bias Level 2		0.2	2.2	
	Bias Level 3		0.4	2.2	
	Bias Level 4		0.9	2.3	
	Bias Level 5		1.2	2.4	
	Bias Level 6		1.5	2.4	
	Bias Level 7		1.9	2.4	

Note. [a]“Level” indicator were formed by ascending within-class means. [b]Missing values for Required Math correspond to ambiguous majors (e.g., “undeclared”, “pre-program”). SE = self-efficacy rating, HS GPA = high school grade point average (capped at 4.0); HS Self = self-assessment of high school mathematics performance, Required Math = number of semester mathematics credits required by declared college major.

Table 2.

Distributions of Indicators for Final Exam Performance

Construct	Indicator	% Incorrect	% Correct
Final Exam Performance	Perf. Level 1	75	25

Perf. Level 2	63	37
Perf. Level 3	56	44
Perf. Level 4	42	58
Perf. Level 5	33	67
Perf. Level 6	27	73
Perf. Level 7	17	83

Note. Table entries indicate the proportion of students (N = 195) who correctly solved the corresponding final exam items. Items were randomly sampled from seven-level quantile groups based on item difficulty.

Indicators of final exam performance were based on dichotomous (correct-incorrect) scoring of seven final exam items by within-class means. As shown in Table 2, Performance Level 1, for example, represents students' performance on a difficult final exam item in their class – only about one in four students (25%) correctly solved the task corresponding to this first indicator. In contrast, 83% of participants correctly solved the final exam item corresponding to the Performance Level 7 indicator. Three indicators come from final exam tasks which were correctly solved by fewer than half of students; the remaining four indicators include greater than 50% correct responses.

The data suggested study participants experienced relatively strong performance in high school mathematics. The average ACT Math score of participants was M = 24.9, which corresponds to the 79th percentile of U.S. college-bound students (ACT, 2007) and is about one standard deviation above the university average (M = 21.6, SD = 3.7, t(1,236) = 11.3, d = .9, p < .001). In response to "Which of the following best describes how well you did in your high school math courses?", most students (87%) chose one of the descriptors OK, Good, or Very Good. The percentage (17%) of participants' GPAs at 4.0 or greater was larger than the average percentage (7%) of 4.0s at the university.

The measure of required semester mathematics credits in participants' primary college majors (Required Math) was calculated based on the distributions of students' majors. Since all mathematics majors were required to complete 40 college credits in mathematics, the large proportion of Mathematics majors in the sample produced a large departure from normality. The non-normal distribution, coupled with a lack of correlation between Required Math and the other study variables, led to its removal from the modeling estimation procedure.

Structural Equation Modeling Results

Model Estimation

After removing Required Math from the structural model specification, the *sem* estimation procedure for hypothesis-based model converged in 210 iterations. All directional effects in the model were significant at the α = .05 criterion with the exception of the posited direct effect of the latent variable High School Math Achievement on the latent variable Final Exam Performance (β = -.19, p = .40). Model fit indices included an overall chi-square of χ^2 (246) = 608.0, CFI = .70, NNFI = .67, SRMR = .08, and RMSEA = .09. The comparative fit indices (CFI and NNFI) were both below the .9 threshold for good fit, and the SRMR and RMSEA indices suggested marginal model fit. A likelihood-ratio test confirmed the structural model provided a significantly better fit than the measurement model ($\Delta\chi^2$ (6) = 146.7, p < .001).

Model FindingsFigure 2 shows the final structural equation model with the estimated standardized directional effects; the estimates of measurement errors are omitted from the diagram for readability. Standardized parameter estimates for the structural model were all significant at the α = .05 criterion, and values ranged from β = .21 (the loading of Bias Level 1 on Calibration Bias) to β = .78 (the loading of HS Self on HS Math Achievement). Effects *between* latent constructs (represented as ovals in the diagram) can be interpreted as estimates of the sign and relative magnitude of effects in the hypothesized model. For example, the review of literature supported direct effects of calibration bias on both final exam performance and self-efficacy, and the estimated coefficients suggested the negative effect of calibration bias (β = -.75) on final exam performance was comparatively larger than the positive effect of calibration bias on self-efficacy (β = .39).

As shown in Table 3, the large direct negative effect of calibration bias on final exam performance was mediated somewhat by an indirect positive effect of bias on self-efficacy. Similarly, high school math achievement had indirect effects on final exam performance through the separate effects of high school mathematics achievement on self-efficacy and calibration bias. The largest total effect on final exam performance was that of self-efficacy.

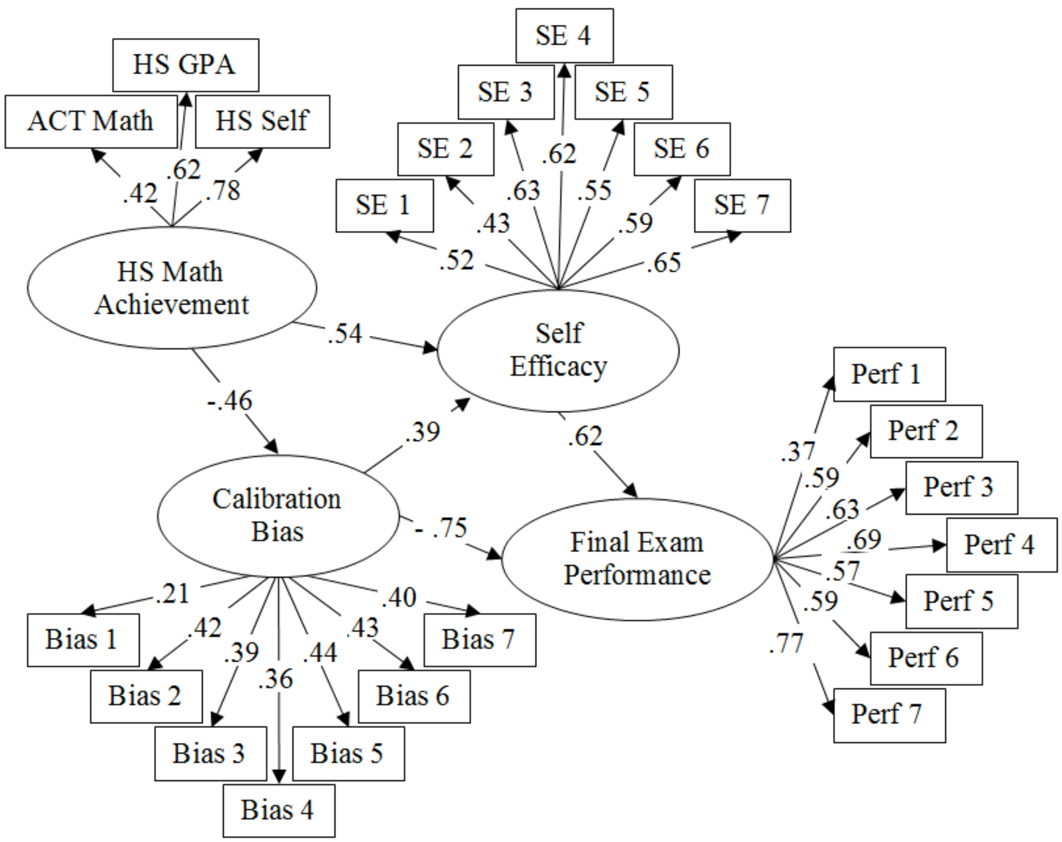

Figure 2. Standardized coefficients of directional effects in the final estimated structural equation model.

Table 3.
Standardized Direct, Indirect, and Total Effects between Latent Constructs

Effect of…	on…	Direct	Indirect	Total
HS Math Achievement				
	Calibration Bias	-.46**		-.46
	Self-Efficacy	.54**	-.18	.36
	Final Exam Performance		.57	.57

Calibration Bias				
	Self-Efficacy	.39*		.39
	Final Exam Performance	-.75**	.24	-.51
Self-Efficacy				
	Final Exam Performance	.62**		.62

Note. $^{*}p < .05$, $^{**}p < .01$.

Discussion

This study adds to existing literature on mathematics self-efficacy and calibration in mathematics. In particular, the study extends path analysis studies of calibration among middle school (Chen, 2003) and high school students (e.g., Pajares & Kranzler, 1995) to the important population of college students. The incorporation of structural equation modeling, with its affordances for multiple measures of underlying factors, also addresses implicit assumption in the prior path analysis studies that attitudinal variables can be perfectly measured by a single scale. The study findings suggest self-efficacy and calibration exhibit approximately equal and opposite effects on mathematics performance, providing support for Chen's (2003) findings that calibration has a weak effect on self-efficacy and that both self-efficacy and calibration have moderate to strong effects on mathematics performance. The similarities between estimates of directional effects in this and Chen, Pajares, and others' studies, taken in context of differing settings and measures of mathematics performance, suggest robustness for findings that self-efficacy and calibration have independent mediating effects within the long-understood influence of prior achievement on future performance in mathematics. Future research goals include (1) developing techniques for improving the calibration of students in advanced mathematics courses, (2) identifying potential differences in self-efficacy between teaching and non-teaching mathematics majors, and (3) documenting self-efficacy trajectories of secondary mathematics majors.

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EMPOWERING PARENTS THROUGH KNOWLEDGE OF HOW CHILDREN LEARN MATHEMATICS

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Parents want to help their children succeed in school, especially in mathematics however; they are often inadequately prepared to do so. The primary hypothesis of our research is that children's understanding of mathematics can be improved by empowering parents with resources focused on how children learn mathematics. Emergent themes that evolved as the result of these beginning research efforts are: 1) Parents desire and need more mathematical content knowledge 2) Parents think differently about how to help their child as a result of learning about children's thinking about math 3) Parents are frustrated by their child's struggle with mathematics.

Children are not the only ones doing homework these days. As more and more schoolwork is sent home at earlier and earlier ages, parents also feel the pressure to help their students succeed (Kohn, 2006). Many parents, however, are ill-prepared to work with children, especially in mathematics. Therefore, it is not surprising that parents expressed a desire to be more effective in helping their children with schoolwork (Mapp, 2003).

Despite altruistic intentions, most parents do not have a grasp of cognitive development or basic knowledge of how their children think about mathematics. Without ill-intent, they often rely on ineffective questioning, explaining and examples when working with children (Eloff, Maree, & Miller, 2006). Unfortunately, well-meaning parents generally do not know the importance conceptual understanding plays in assisting their child in becoming a life-long learner of mathematics.

It is not that all parents and primary caregivers have been unsuccessful in learning mathematics themselves, but more often than not those who have been successful have learned mathematics as rules and procedures. They might be able to tell children the rules but unless they understand the concepts behind the rules; they will not be effective in helping children understand mathematics. By knowing how children think mathematically, parents can make appropriate interventions that provide powerful learning opportunities.

Our initial research with parents (Feikes, Swchwingendorf & Stevenson, 2010) offered four emergent themes, three of which will be the focus for this paper. The purpose of this study is to explore how previously published materials on Connecting Mathematics for Elementary Teachers (CMET), can be adapted for use with parents to increase their knowledge of children's thinking about mathematics and efficacy to assist their children. Ultimately, the goal is to positively impact student learning of mathematics.

Review of Literature

The Connecting Mathematics for Elementary Teachers, (CMET), an NSF supported project (DUE 0126882 and DUE 0341217) funded the development of the commercially published textbook CMET (Feikes, Schwingendorf, & Gregg, 2009) and supplemental materials for prospective elementary teachers and their instructors (http://www.cmetonline.com). Literature on mathematical content knowledge (e.g. Hill & Ball, 2004) and the importance of understanding children's thinking (Empson & Junk, 2004; Steinberg et. al, 2004), contributed to the creation of these materials.

Results from CMET indicated that preservice teachers who used the CMET materials increased their self-efficacy (Pratt, Feikes, & Hough, 2007, 2007) developed beliefs that were more aligned with conceptual understanding of mathematics (Hough, Feikes, & Pratt, 2006), and developed greater pedagogical content knowledge—the mathematical knowledge necessary for teaching (Feikes, Pratt, & Hough, 2006).

There are currently two other nationally funded projects focus on parents and mathematics education. Math and Parent Partnerships, (MAPPS), attempts to engage parents in mathematics with their children. The intent is that parents would be to support their children's mathematical learning. After participating in the MAPPS project students had better attitudes about mathematics and liked math more. They also indicated that they were more likely to try harder after participating in this project with their parents (MAPPS, 2007). Another project, *The Role of Gender in Language Used by Children and Parents Working on Mathematical Tasks* at the University of Hawaii, focuses on gender-related differences in language and actions used by children and parents when working on mathematical tasks.

Methodology

Due to the limited literature on parents' work with children on mathematics outside the classroom and the exploratory nature of this study, a variety of data collection techniques were used to gather information from a wide range of participants. To develop an understanding of the effectiveness of the CMET materials to assist parents in helping their children with learning mathematics and to identify revisions that are needed to meet the needs of parents and their children, it was deemed appropriate to focus on a limited number of parents who could be easily accessed and were identified as wanting to assist their children with homework. It is left to later studies to contrast the behavior of these parents with others who are not as readily interested in assisting their children with mathematics homework.

Data Collection

The primary methodology is a design experiment (Brown, 1992, Collins, 1992; Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). This methodology has been modified to fit with the research conducted in this study. Features of the research that lend itself to a design study are the application of innovative instructional material, the testing of theories of how to help parents help children, and the cyclical nature of design.

A design experiment provides the tools to test theories of how parents work with their children on mathematics and how to improve these interactions. Theories on parent learning are 'humble' theories in that they target domain-specific learning processes (Cobb, et. al, 2003).

Further, this study entails five key features described by Cobb et al., (2003) it: 1) develops theories, specifically theories parent learning and parent enablement, 2) is interventionist—new activities will be employed, 3) tests these theories and materials with parents in real learning situations, 4) is iterative in that both materials and theories will be revised in a cyclical process, and 5) is pragmatic—the materials will be realized with real parents working with children.

The present study's varies in two ways from Cobb et. al. This study focuses on parents interacting with their children as they help them with mathematics outside of the classroom. A second key difference is that we have collected anecdotal evidence and interview data about the learning process without actually observing and intervening in the parent/child episodes. Despite these differences we consider the design experiment as the best fit to our research purposes.

Data Sources

Data for this study consisted of a) written reviews of the materials, b) course work in a graduate mathematics education course, c) transcripts from parent focus groups, and d) transcripts from parent interviews. For clarity, these are all independent data sources and there was no overlap of participants. The purpose of collecting these data was to examine how parents might use knowledge of how children learn mathematics in helping children learn mathematics.

Written Reviews of CMET Materials

There were two groups of parent participants who provided written reviews of using CMET with their children, one of which was a group of graduate students. Two parents were asked to review the CMET materials in regards to its possible contribution in helping them with their children. They completed a questionnaire in which they were asked to assess each chapter, what they learned, and how they used any of this information with their children. These participants were paid a small stipend. Several graduate students who were also parents reviewed the CMET materials as part of a mathematics education graduate course for elementary teachers. They were asked to evaluate the CMET materials in regards to the degree to which it contributed to their ability to assist their children with mathematics homework.

Focus Group Interviews

Two focus groups were conducted with parents at local elementary schools, one school had a high rate of free and reduced lunches (83%) and the second was in a middle class community and was Four Star School. The focus group sessions were one hour in length, audio-recorded and transcribed. This corpus of data will be used to determine how knowledge of children's mathematical thinking might be used by parents working with children, how parents interpret the resources, and more importantly how they use CMET resources working with children.

Individual Interviews

Two parents whose children were having difficulty in mathematics were presented with selected sections of CMET that related to the difficulties that their children were having. These parents were interviewed and the interviews transcribed. The questions were brief, asking them what did they learn by reading the resource materials and how did they use them with their child.

Data Analysis

The data sources listed above were submitted to Constant Comparative Analysis to produce emergent themes (Glaser and Strauss, 1967). These themes were substantiated, modified, or rejected through review by two researchers. Every effort was made to triangulate the themes with multiple data sources (see Results section).

Results

The overall goal was to discern if the CMET materials adapted for parents on how children learn mathematics are useful, how these types of resources might be used by parents, if using these resources impact children's learning of mathematics and how they can be improved. The results presented here are only include the initial emergent themes.

1. Parents see the need to learn mathematics to help their children.

One theme that emerged was that parents want to know more mathematics in order to help their children. A parent in the focus group was talking about her son's homework that night. "My son understands it much better than I do, but in order for me to help him, I have to understand it." She realized that she could not help her son on his homework unless she understood the mathematics. Later in the focus group she indicated that she could use the Internet to help her learn more mathematics, but she currently does not have the Internet. She wants to help her son and is even aware of resources but does not have the resources at her disposal.

Parents from both the focus groups had difficulty understanding lattice multiplication and partial products. In regards to lattice a parent said, "I was like man, how did they get this number? Because it didn't add up right at first. Then you have to start going an angle with it, and I thought okay, I got it now." Again, the parent may attribute his not understanding lattice multiplication to his lack of understanding of mathematics.

2. After using the adapted CMET materials, parents began using a different approach to help their children.

One parent who reviewed CMET indicated that using these materials changed how she approached working with her daughter. She found herself more willing to let her daughter experience mathematics rather than just telling her how to do it. This was most evident with problem solving activities where she found herself "restraining myself to not 'jump in' and allow her to experience problem solving for herself." She began to realize that her way of helping her daughter may not be the best approach. In a broader sense she took a more reform-oriented view to teaching. She moved from 'teaching as telling' and acted in a way that supports the notion that children must construct their own knowledge of mathematics.

The following self-reported example from a parent who reviewed CMET illustrates evidence of both learning and change in how this parent interacted with her son.

> My son Nathan had 'created' his own way to multiply two, 2-digit numbers. 23 x 45 = (20 x 40) + (3x40) + (3x5) + (20x5). I kept trying to convince him that the traditional algorithm was more efficient because his method became cumbersome for 3-digit numbers - I then found out his method was called partial sums and some children are taught this way to multiply.

After reading the CMET materials, rather than fight her son, this parent saw his work as a precursor to the traditional multiplication algorithm. She explained, "So we used his partial sums as a jumping off point to get to the traditional algorithm - trying to look where his pieces matched with "mine" as he calls it.

Another parent, who reviewed CMET in the graduate course, began to engage her child in statistical activities described in the materials. At first she thought the chapter in the CMET on statistics and data analysis was for children in the upper grades. Reading examples about how young children can collect and represent data inspired her to encourage her child to do the same.

> I found myself actually having my 4-year-old collect data with a mini clipboard. I had him go around our house when we had company lately and see who was wearing jeans, pants, or skirts. He drew pictures and reported his "data" to us at dinnertime.

When asked if the CMET materials were helpful a parent responded, "Yes and no. It helped me to understand what she was struggling with, but she still struggles with [that]." For example, if her daughter solved 4 x 5 = 20 and then is asked 5 x 5, her daughter had difficulties solving the problem. The mother explained, "So when I ask her what 5 x 5 is, she doesn't understand that you add another 5. She isn't seeing it as four groups of five. I don't know how to help her." This parent seems to have internalized some information from the materials. She sees that multiplication facts are related and that she wants her daughter to understand that multiplication is repeated groups.

These three examples above illustrate that parents were reading the CMET materials, learning from them and, interacting differently with their children from what they had prior to reading the materials.

3. Some parents are frustrated helping their children with mathematics.

Another theme that emerged was the frustration of parents whose children were struggling with mathematics. A parent, of a third grader in the focus group, expressed his frustration when his daughter solved a problem like 6 + 2 on her homework and then later in her homework she did not know how to immediately solve: 2 + 6. "And then their fingers go up. And I'm like, why would you, you know, why would you have to do [that]. Shouldn't have to do that we've been doing that since Kindergarten."

Regarding this frustration the same parent said, "She gets confused and she gets frustrated and she usually just lets it go- don't even want to mess with it no more… then I get upset, and I may start getting a little loud."

Conclusions

Parents want to be a valuable resource to their children but they frequently lack the knowledge necessary to help with math homework (Eloff, Maree, & Miller, 2006). One way to support parents is to provide resources to discover how their children actually learn mathematics. With this knowledge, parents will be empowered to contribute to their children's mathematical development, hopefully resulting in student learning.

The findings of this exploratory provide an impetus to continue research on adapting CMET materials for parents and continue our search for funding. The fact that parents see the importance of learning about how children learn mathematics reinforces the need for CMET adapted materials. The finding that parents are actually using the information to work with their students with a greater understanding of student thinking, seems to make the creation of materials for parents a necessity.

From the third emergent theme, it appears that not only must these materials be provided in an understandable format to parents, and suggestions should be given on how to use this knowledge; but that parents' ways of thinking about how their children learn must also be addressed to alleviate frustrations. Perhaps, addressing the beliefs and values of parents is beyond the scope of our intended resources, but it is an area of that must be considered.

Another significant question, which we have not addressed is, "Does parents' use of CMET resources help children gain a deeper mathematical understanding that results in measurable student learning?” This question is difficult to address outside of the school setting but is the ultimate aim of this and other intervention efforts to improve mathematics education.

With more homework and need for parents to help with mathematics, it seems necessary to provide assistance to not only students but also to parents and caregivers. Knowing that parents’ involvement with their children’s schoolwork improves students’ performance and their attitudes about math, parent assistance about how children learn mathematics become paramount. The adaption of CMET materials for parents appears to be a rich source of information to increase parent’s ability to improve their child’s learning of mathematics and further research is warranted.

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IDENTITY AND POWER ISSUES IN MATHEMATICS FOR ADOLESCENT FEMALES

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This paper presents the results of a study involving surveys and interviews conducted with 76 Northern Nevada Girls. We investigated how the girls perceived themselves and others in relation to mathematics, what factors influenced those perceptions, and how those perceptions influenced the girls in mathematics. Findings are organized into three main themes: identification with mathematics, outside influences on mathematics identity, and power relations

Females continue to be underrepresented in the mathematical sciences, and they perform more poorly in and have less positive dispositions toward mathematics than males (Wiest, 2011). Because females' dispositions toward mathematics tend to develop relationally within social contexts (Zeldin et al., 2008), issues of identity and power may be important to examine in efforts to support and encourage females in mathematics. Although identity has not occupied a central position in mathematics education research and thus lacks a common, accepted definition (Cobb, Gresalfi, & Hodge, 2009), the construct is emerging as a topic of interest in gender and science-oriented fields (Brotman & Moore, 2008). The purpose of this research was to investigate issues of power and identity in relation to mathematics for a group of adolescent girls. We investigated how the girls perceived themselves and others in relation to mathematics, what factors influenced those perceptions, and how those perceptions influenced the girls in mathematics.

Related Literature

The General Construct of Mathematics Identity and Its Impact

Wigfield and Wagner (2005) describe identity in general as "an overall sense of who one is" (p. 228). Personal identity encompasses perspectives individuals construct about the world around them and themselves in relation to that world, for example, how others perceive them and what role they occupy within a community (Hodge, 2006; McCarthey & Moje, 2002). It is a value-laden construct in which individuals determine the degree to which they adopt or resist others' expectations (Cobb et al., 2009).

Although personal identity tends to be stable at any moment (Esmonde, 2009), it is a dynamic construct that evolves over time (Gaganakis, 2006; Hodge, 2006; Owens, 2007/2008; Solomon, 2007). Identity can vary in different contexts, situations, and relationships and is thus complex and domain-specific (Esmonde, 2009; Gaganakis, 2006; Hodge, 2006; Leatham & Hill, 2010; McCarthey & Moje, 2002; Wigfield & Wagner, 2005). Therefore, student identities must be examined within the context of a specific subject area. Martin (2009) describes mathematics identity as

> the dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the conditions of their lives. A mathematics identity encompasses a person's self-understanding and how others see him or her in the context of doing mathematics. (pp. 136-137)

Mathematics Identity Formation

Identity is a negotiated construct that is influenced by a number of social, cultural, cognitive, and affective factors (Gaganakis, 2006; Martin, 2009; Owens, 2007/2008). Because identities are located within relationships (e.g., McCarthey & Moje, 2002), peers, teachers, school counselors, families, the community, and others influence identity formation (Axelsson, 2009; Esmonde, 2009; Gaganakis, 2006; Hodge, 2006; Martin, 2009; Mireles-Rios & Romo, 2010; West-Olantunji et al., 2010). For example, the way in which an individual is positioned within particular social and academic settings influences personal identity in those situations (Owens, 2007/2008; Solomon, 2007). Nevertheless, individuals exercise varying degrees of agency in determining how to process and respond to external assessments of themselves (Hodge, 2006). Peers have a particularly strong influence on young people's development, perspectives, and behaviors, such as their self-confidence and academic achievement motivation (Bissell-Havran & Loken, 2009; Hoogeveen, van Hell, & Verhoeven, 2009; Keck, 2008; Wigfield & Wagner, 2005).

Another factor that can influence mathematics identity is past and present mathematics performance, such as grades and test scores (Axelsson, 2009; Keck, 2008). One's sense of academic competence and affective characteristics such as confidence, motivation, and anxiety influence identity formation and stem from such factors as possession of academic knowhow and social support (Axelsson, 2009; Owens, 2007/2008; Wigfield & Wagner, 2005). Students' self-concept, including competency-related beliefs, has been shown to decline from early adolescence through secondary school (Nagy, Watt, & Eccles, 2010; Wigfield & Wagner, 2005).

Teachers' beliefs and classroom practices, such as those that foster or inhibit competence and motivation or promote peer comparisons, can shape mathematics identities (Martin, 2009; Wigfield & Wagner, 2005). Selected other instructional practices that can influence identity development are use of group work, which can proceed equitably or inequitably, ability grouping, and technological learning supports (Esmonde, 2009; Owens, 2007/2008; Solomon, 2007). Research shows that students develop more positive identities, including a stronger sense of agency, in classrooms that emphasize student thinking, responsibility, and active, collaborative participation versus authoritative, teacher-centered classrooms where students mainly follow directions and perform prescribed procedures (Cobb et al., 2009; Keck, 2008; Solomon, 2007).

Students' social identities, such as gender, race/ethnicity, national origin, and social class, are important factors in identity formation (Brotman & Moore, 2008; Gaganakis, 2006; Solomon, 2007; Wigfield & Wagner, 2005).

Identity and Power in Mathematics

Identities are constructed within social settings in which individuals are positioned in ways that give them differential status and power (Esmonde, 2009; McCarthey & Moje, 2002; Solomon, 2007). Hodge (2006) describes power relations as "situations in which ideas and practices of particular groups or individuals are valued over those of others" (p. 377), which can include ways of reasoning, communicating, and demonstrating knowledge. Like identity, power relations can change over time as interpersonal dynamics shift based on various influential factors from within or outside the group (e.g., Martin, 2009).

Research Purpose and Method

The purpose of this research was to investigate issues of power and identity in relation to mathematics for a group of adolescent girls. We investigated the following questions: How do adolescent girls perceive themselves and others in relation to mathematics? What factors influence these perceptions? How do these perceptions influence the girls in mathematics?

Participants in this study were 76 Northern Nevada girls who voluntarily attended a six-day, residential summer mathematics and technology camp during the summer of 2009 or 2010 (36 and 40 girls, respectively). They entered grade seven (50 girls) or grade eight (26 girls) the following fall. Of the 74 girls who disclosed their race/ethnicity, 50 were White/non-Hispanic, 13 were Hispanic, 8 were Asian/Pacific Islander, and 3 were a mix of two or more categories. About one-third of the girls participated in the program on a financial-need scholarship as determined by eligibility for free or reduced-cost lunch. Research participants completed a written five-item survey upon entering the summer girls' mathematics and technology camp. The open-ended questions included questions related to how the girls viewed themselves as "mathematicians," who influences those views, and what perceptions the girls held of others in mathematics. Four girls also participated in individual interviews at three points in time: the camp beginning, the camp end, and four months later when they were back in school. The girls were asked questions related to how they perceive and want to perceive themselves in relation to mathematics, what factors influence these perceptions, and potential future career aspirations in a mathematics-related field. We analyzed our survey and interview data by reviewing it carefully for themes related to our research. We sorted the data into three *a priori* themes that appeared in the professional literature on identity and power: identification with mathematics, outside influences on mathematics identity, and power relations.

Results

Identification with Mathematics

Ability. In reference to how "good" they are in mathematics, respondents described positive and negative associations related to their mathematics abilities or capabilities. Many indicated positive associations with mathematics by describing that they see themselves as "mathematicians." Participants used such terms as "kinda good," "smart," and "advanced" to describe themselves in mathematics. Several positioned themselves as average or in the "middle" in terms of mathematics ability. Participants who reported negative associations with mathematics expressed difficulties with learning mathematics or described mathematics as an area of needed academic improvement.

Performance. Some participants based their perspectives toward mathematics on performance in general or performance on specific topics and skills. During Interview 1, a respondent said, "I do think that I'm good at math, but sometimes I don't like if I don't get a problem." This participant later stated in Interview 2 that she felt less confident about herself during the camp in reference to "two hard math [word] problems in the mornings." Another interviewee reported in her first interview that she is good at mathematics because of her multiplication and integer computation skills. However, she did not believe she was good at long division. Other mathematics topics that participants commented on in describing themselves in relation to mathematics based on performance included speed in solving a problem, understanding a concept or recalling a mathematics fact, and frequency of errors or mistakes made when doing mathematics.

Dispositions. Over one-half described themselves as having a desire to learn mathematics and being serious about mathematics. They said they like mathematics and enjoy facing challenging problems. A few other respondents described disinterest in or apathy toward mathematics, often due to their struggles with the subject matter.

A common belief among participants was that “everyone has potential,” in other words, that personal agency plays a role in mathematics ability/performance. One-fourth of the survey respondents wrote that those who are good at mathematics typically demonstrate determination to do well in mathematics. They explained that people must “try” in class, pay attention during lessons, ask questions when they need help, and study. Whereas 16 of 47 girls of higher-SES expressed the belief that personal agency thus plays a role in mathematics, only 3 of 24 girls of low SES did so. Conversely, three participants claimed that some people are naturally better at mathematics than others. One stated that being better at mathematics is a “gift from god.”.

Almost all participants described the usefulness of mathematics in everyday life. Most participants also said mathematics was very important to their future career. Five participants, however, were more equivocal about their valuing and perceived usefulness of mathematics. For example, three weren’t sure how important it would be for their career (e.g., singer or author).

Participation. Some participants described their willingness and effort to actively participate in mathematics class at school. They reported exerting effort to learn new concepts and skills, such as paying attention in class, taking notes, doing homework, and studying for tests. Ten participants described providing mathematics help to peers, mainly friends, during mathematics class and at lunch or after school. Five expressed a lack of interest in pursuing additional mathematics participation by stating that they only did required assignments and activities to achieve a good grade in mathematics class. One participant explained that she was not doing well in mathematics, so she came to the summer camp to try to improve her skills. In reference to the math and technology camp participants had just entered while completing the surveys for this research, only 3 girls reported that they had ever attended a similar, supplementary, out-of-school time program in a mathematics-related field.

Outside Influences on Mathematics Identity

Family members, including parents, grandparents, aunts, uncles, and siblings were reported as influences on mathematics identity. Family members provided support, assistance, feedback, and encouragement. Participants said parents and siblings helped them with homework. Students also described parents’ attitudes towards mathematics as influences. One student wrote: “My parents are very supportive of my math skills. They always joke that they don’t know where it came from. My mom says it’s a gift that she can even add and subtract!” One-third of the girls described their parents’ love for mathematics as a means of support and encouragement.

Teachers were also identified as influences within the instructional environment. For positive influences, several participants described how their teachers expressed or modeled their enjoyment of mathematics or made it fun, and one wrote, “My teacher calls me a math wizard.” In contrast, teachers also exerted negative influence on participants’ mathematics identity. Three students described feeling “put down” by mathematics teachers, and one interviewee noted, “My last math teacher caused me to think a little less of my ability in math.”

About one-third of the participants described the level of support provided by teachers as influences on mathematics identity. Most reported that they received enough mathematics help from teachers. One participant explained that her teacher would slow down and work out the problem with her. Several students reported that they did not receive an adequate amount of help. Only 4 of the 24 low-SES participants compared with about half of higher-SES participants reported teachers as an influence in how they see themselves in mathematics. The role of the teacher and type of instruction is evident in this quote from a rising seventh grader:

> Math camp helped me a lot because we got to use pictures and stuff. At my school, they use pictures too, but they just do answers. My math teacher, I didn't really get to know math or do math or anything, because he tells us the answers…and he won't even let us work out the problem or anything…. So, I like being here because I get to know math and I get to work it out for like 10 minutes or so with a group and at my school, we had to work it out individually unless our desks were put into groups…. I was scared to come to math camp because I was scared that all of the other girls would know how to do the problems and I wouldn't because of my math teacher giving me all the answers.

Other influences indicated were peers and television/movies. About half of the respondents explained that friends and classmates supported or influenced their association with mathematics in various ways. Some respondents described friends and classmates passing negative judgment on their mathematics abilities. Students described peers calling them names, such as "nerds" or "brainiacs" for reasons such as enjoying mathematics, doing well in mathematics, and attending the math camp. Four participants mentioned television/movies as outside influences. Two participants commented on seeing characters on television shows with mathematics-related careers and who enjoy doing mathematics. Another participant wrote that movies have an impact on her because filmmakers cast people who are nerdy (seemingly a negative connotation).

Participating in "extra" math experiences is likely another influence on mathematics identity. All four girls who participated in interviews said at both follow-up interviews that the camp influenced how they perceived themselves in mathematics. They mostly discussed having gained new knowledge and skills, new methods for doing mathematics, and increased confidence as a result of these improvements and expansions.

Power Relations

One-fifth of the participants identified differences in positioning and opportunities within the classroom and society as a whole in relation to mathematics. These participants wrote that students are given different opportunities in class based on their level of understanding of mathematics. Three participants perceived a difference in the amount of respect given to certain types of students. One wrote, "Teachers take smarter kids more seriously and give them more respect than average or below average students." Another stated: "My 7th grade teacher wouldn't actually see me much because she would only be with the smart ones and leave the ones that really need help behind." In contrast, two of four interviewees who self-identified as Hispanic did not perceive differences in positioning within the math camp context. In response to what things at camp made her feel good about herself in mathematics, one stated: "I think that during math camp, I knew that the other girls around me were all on about the same page and that I didn't feel like I was too far ahead or too far behind." The other described fear about attending the math camp because she "was scared that all of the other girls would know how to do the problems…they would be mean to me or tease me." She explained that the setting turned out to differ from what she had anticipated. She found her peers helpful and supportive.

Within a societal context, six participants wrote survey comments indicating perceptions that certain groups or types of people have better ability or are treated differently than others in mathematics. One wrote, "Math is known as a hard subject with only men capable for succeeding," and another simply stated, "Men are respected more." Other girls addressed additional biological characteristics, including hair color (blond) and race/ethnicity, that they perceived to influence the treatment some people receive. In contrast, eight participants reported that everyone has similar mathematics ability.

Discussion and Closing

Most research participants expressed some type of opinion about themselves in relation to mathematics or vice versa. This indicates that most girls had some type of conscious opinion on this matter in one direction or another, which may indicate the salience of the mathematics-self relationship. Although most comments were positive, some were negative. The girls were also asked to comment on the value of mathematics and whether or not various groups of individuals are equally capable in mathematics. These perspectives also tended to be more positive than not. Of the majority positive responses to these questions, we suspect that some were rehearsed rhetoric the girls had been "fed" at some point from teachers, parents, or others. For example, the mathematics camp that all of these girls attended opened with a parent-camper session in which the first author (Wiest) emphasized that females and males have equal mathematics capabilities and that effort and experience are highly important factors. This occurred one to two hours before the girls completed the surveys prior to the main camp activities beginning. It is likely that this and similar experiences influenced some girls to state what they have heard and adopted, likely unexamined in many cases. Nevertheless, the girls did offer a number of examples of the utility of mathematics that did not arise in the camp's opening address. Again, it is uncertain whether these "tidy" examples were memorized from some source or were authentic. Similarly, a number of girls articulated the importance of mathematics to their future career by perceiving this importance in general or, especially, by naming a specific career. These may also be regurgitated comments heard from well-meaning adults.

We suspect that deeper probing would yield more fragile and less positive self-images in relation to mathematics and that the positive notions the girls repeated may not be fully integrated into some or many of these girls' self-perceptions. Mixed perspectives indicating confidence or lack thereof were evident across but also within girls. The variability demonstrated within girls, such as girls who described greater ability and confidence with some mathematics topics compared with others as well as awareness of societal perspectives compared with their own, can mean that these girls' have good self-awareness into their own strengths and weaknesses, which can be useful to guide future learning. However, it may also signify easily malleable and thus fragile self-conceptions in relation to mathematics that depend on mathematics topic, mathematics performance, and others' opinions, in other words, external factors beyond their control.

The role of personal agency in determining one's relationship with mathematics is important. A belief that one can strengthen one's positive connection to mathematics with appropriate experience and effort is important to help girls feel some sense of personal responsibility for and control over this relationship. A belief that predetermined or external factors alone shape this relationship, such as biology, the mathematics content itself, or others' perspectives, can lead to fatalistic notions that could cause some girls to "accept" their position in relation to mathematics and other individuals. In the present study, about one-fourth of the participants expressed the belief that personal agency plays a role in mathematics. However, one-third of higher-SES girls but only one-eighth of low-SES girls offered this perspective. This is an area for concern.

Higher-SES girls likely have more opportunity for furthering their academic skills through attendance at out-of-school-time programs, travel, and possession of educational materials at home. Perhaps their parents vocalize the importance of working hard and an expectation for success and may provide more role models in mathematics-oriented endeavors. Low-income students may have less access to opportunities due to financial constraints, which limit opportunities, resources, and time (e.g., Wiest, 2010). Further, they may be disenfranchised from school, finding it harder than higher-income students to believe they are taken seriously in school or that academics can play an important role in their future (see also Martin, 2009).

The opinion of others is important to girls. Peers clearly play an important role. However, adults also have an important impact on girls' self-perceptions, performance, and participation. This appears to be more complex than meets the eye. For example, one girl noted in a positive sense, "My parents are very supportive of my math skills. They always joke that they don't know where it came from. My mom says it's a gift that she can even add and subtract!" Whereas this comment appears to be positive on the surface (and is to some degree), the subtext is that the girl's mathematics ability came from some unknown place rather than being a result of the girls' own intelligence and effort. Further, it provides a weak female role model in the mother's comments about herself in relation to mathematics. This type of comment can plant deep seeds that are likely digested and may surface in the form of negative dispositions at some point. One interviewee clearly links her concern about her teacher only giving the students answers and not letting students figure them out for themselves to her lack of confidence doing mathematics because she believes she is not properly prepared with the needed skills.

Mathematics participation is affected by financial resources and opportunities for out-of-school-time (OST) experiences, as well as awareness and encouragement. The girls in this study had little opportunity or encouragement to pursue OST mathematics. The important role of the teacher is evident where one girl noted that she had not previously participated in programs similar to the mathematics camp because her teacher had not suggested it. Because greater participation in additional mathematics experiences appear to be linked to girls' dispositions (e.g., confidence) and self-assessed ability, it follows that greater mathematics participation can likely improve girls' mathematics identity. It is thus imperative to offer *all* girls more mathematics opportunities and to encourage and support their participation.

It is clear from this study and others that there is much room for hope in relation to girls' present and future relationship with mathematics. However, proactive work in this area is still needed. The fact that some girls, although small in number, perceive societal perspectives and power relations that academically and socially position people in particular ways that are not always positive and productive is another source of evidence that our work is not done.

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DEVELOPING COMPETENCY AND MATHEMATICAL KNOWLEDGE USING PROBLEM SOLVING

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This article reports on the type of mathematical competencies that students exhibit at the superior level of education to solve problems which involves concepts such as the unknown quantity, the variable, and the solution of a linear equation. The role of the professor is emphasized, as well as his/her interventions to help students develop comprehension. The discussion of competencies in which this paper is based is established by De Corte (2007). The problem reported forms part of a series of problems solved in an environment of Problem Solving. The results reflect the mathematical knowledge exhibited in the students' procedures.

Introduction

In this era of rapid advances both in science and in technology, there is a demand to have people who are not only more informed, but also better educated. We require an education that will form competent human beings able to make contributions for the further development of society. Various questions emerge upon this theme: What does it really mean to develop competence? How has mathematics formed part of this competence? What does it mean to develop mathematical competence? How should the learning process that we use for our students enhance their acquisition of mathematical skills? What is the role of the professor in the development of this competence?

This article will discuss questions such as: What mathematical competencies are exhibited at the superior educational level to solve problems that require linear equations? Another question is: What type of interventions can the professor use to help his students develop mathematical knowledge and competencies? Based on a final report given in by students in this study, it is possible to identify the procedures commonly used by students who solved Problem 1, here reported, in a Problem Solving environment (Schoenfeld, 1985). Thus it is possible to identify students' competencies and mathematical knowledge. The professor's role in helping students to develop their mathematical knowledge and skills will also be noted. The discussion about mathematical competence found in this paper is established in De Corte (2007).

Review of Literature

Obtaining mathematical competence implies acquiring mathematical disposition, of which is characterized by five aspects (De Corte, 2007, p. 20, 21): an accessible base of knowledge, well-organized and flexible, the use of heuristic methods and meta-knowledge, positive mathematics-related beliefs, and self regulatory skills. Furthermore, this competence implies that the student has the capacity to transfer these skills and knowledge to new tasks.

The acquisition of the mathematical competence according to De Corte (2007) requires an active-constructive, self-regulated, situated, and collaborative process of learning. An environment of collaboration, where it is permitted that the student establishes meaningful mathematical knowledge is important. One theoretical perspective in the teaching and learning of mathematics that considers and uses discussions such as the one hereby presented is Problem Solving (NCTM, 2000|2003; Schoenfeld, 1985; Santos, 1997).

This theory proposes that the teaching of mathematics should be approached as a problem-solving domain where students construct a deep understanding of mathematical ideas and processes by engaging them in doing mathematics (Schoenfeld, 1985; Santos, 1997). It implies discarding the idea of introducing the problems in the curriculum with the only purpose of memorizing a set of definitions, algorithms, and techniques to solve routine activities (Schoenfeld, 1985).

In Problem Solving theory, the role of the professor is fundamental and definitely influences the formation and creation of the classroom as a place of learning (Grouws, 2004) where resources, heuristics, control and beliefs are important. The professor can create favorable learning conditions in the classroom where the following points are taken into consideration: the development of a significant answer by the student himself /herself, student justification and explanation of answers, ability on the students' part to understand the meaning of explanations and justifications offered by other students. Classroom environment should be provided in which the students can feel confident to communicate their ideas, ask questions, use multiple representations, construct ideas, and formulate exceptions to the rule (Santos, 1997). On the other hand, the students who are in disagreement or those who do not understand something in the classroom feel sufficient self assurance to question and challenge their classmates. The social interaction that takes place in the classroom is essential for the development of reasoning and the process of acquiring meaning (Rasmussen, Yackel & King, 2004). The professor can help with configurations/teaching techniques in the classroom that encourages interaction in the group(s).

The design and use of the problems in the classroom should have the objective to promote the learning of mathematics concepts, to develop strategies, procedures and heuristics (NCTM 2000|2003). This is to say, it should have the goal that the student achieve the following: a reflexive attitude to form the structure of inference, the ability to use constructive arguing, the practice of critical reading, the use of effective writing and communication skills, and the faculty of decision making, as well as other aspects (NCTM, 2000|2003; Schoenfeld, 1985; Santos, 1997). These characteristics are necessary to form competence in students who study mathematics. Creating an environment in the classroom for the solution of problems, therefore, makes it possible to develop mathematical proficiency in students, or in other words, permit the students to acquire an aptitude for mathematics (De Corte, 2007).

Methodology

The problem described in this article forms part of a series of algebraic problems designed with the objective to understand and develop the following mathematical competencies (De Corte, 2007): an accessible base of knowledge, well-organized and flexible, the use of heuristic methods, meta-knowledge, and self regulatory skills. The development of concepts such as the unknown quantity, the variable and the solution of a linear equation were the core mathematical knowledge to be developed. The role of the professor was important in the classroom to get this objective.

The problems were implemented in a Problem Solving environment where students worked in groups to solve them. Each class session had duration of two hours. During this time, the students were given general instructions to solve a problem in separate work groups, each containing three participants. Advances were discussed at various times by the presentation of groups of students to their classmates and the class session ended with a global analysis of the problem. Finally, the homework assigned was to write an individual report demonstrating how the solution of the problem was. One of the best reports is included in this paper (figures 1-3) as an example of procedures to solve Problem 1.

The problem used for this paper was given to freshman students enrolled in a course of general mathematics, at a public university. The objective of the course was the learning of basic mathematical knowledge that students should acquire in their first year of university studies (topics of algebra, progressions, applications, probability, and statistics). The study whole group consisted of 20 students, most of whom were studying the major of commercial systems. The students level of knowledge for this group was based on their studies in high school.

The following problem was given to the students and is the one used in this paper:

Problem 1. The cost of a barrel of Maya Petroleum is \$395.00, and the cost of a barrel of Brent Petroleum is \$545.00.

a) If you mix one barrel of each brand of petroleum, how much does a barrel of this resulting mixture cost?
b) If you mix two barrels of Maya Petroleum, and one barrel of Brent Petroleum, how much does one barrel of this resulting mixture cost?
c) If you mix one barrel of Maya Petroleum and two barrels of Brent Petroleum, how much does one barrel of this resulting mixture cost?
d) If you mix 15 barrels of Maya Petroleum, and 20 barrels of Brent Petroleum, how much does one barrel of this resulting mixture cost?
e) How much should you mix from each brand of petroleum if you want the mixed barrel that you produce to cost \$450.00?
f) How much should you mix from each brand of petroleum if you want the mixed barrel that you produce to cost \$500.00?
g) How much should you mix from each brand of petroleum if you want the mixed barrel that you produce to cost \$600.00?

The questions lettered a-d can be solved in an arithmetic form. The aim is to offer students the opportunity to observe the relationship among data and identify a pattern in the structure of operations. The questions lettered e, f, and g require the structure of the following linear equation:

$$\frac{395x+545y}{x+y}=b \qquad \text{equation (1)}$$

In the above equation, the letter x corresponds to the number of barrels of Maya Petroleum, and y corresponds to the number of barrels of Brent Petroleum. The letter b corresponds to the cost of one barrel of the mixed petroleum.

Results and Discussion

It is important to emphasize some of the difficulties that were identified in students' mathematical skills at the beginning of the course when they solved word algebraic problems. Students were not aware that an equation could have an infinite number of solutions. Furthermore, they had not had previous experience in using both graphic and algebraic procedures for solving a word problem.

Students had a lot of difficulty in interpreting verbal language to algebraic language. They had difficulties in identifying the variable and writing equations. When they resolved a problem, they did not specify the solution in their report (as observed in Figure 2). In other words, they did not consult the word problem in order to analyze the viability of the solution. They only displayed the solution or solutions identified, without specifying the relationship between the solution and the question stated in the word problem. Students did not demonstrate the habit of validating their steps in the process of problem solving.

Taking the previous observations into consideration, the following aspects were developed throughout the course. Regarding accessibility, well-organization, and flexibility, mathematical concepts such as the variable, an equation, the solution of a linear equation and others were studied in depth. Concerning heuristic methods, the use and integration of different forms of representation was promoted. As for self-regulation skills, students expressed their ideas through discussions, argumentations, and presentation activities. Additionally, students were constantly reminded to validate their own procedures. They were also reminded to continuously go back and review the initial question stated in the word problem.

Procedures for solving Problem 1

Arithmetical Procedures: Looking for Patterns

Students read in groups the Problem 1. They did not have difficulties to answer the four questions *a*, *b*, *c* and *d* because of their arithmetic nature (Figure 1). However their procedures were not organized in a systematic way when they solved it in the classroom. It is possible to observe that there is a specific order and explanation of ideas in Figure 1 since this figure is part of a final report. Students could systematize the information as a homework because they had already had discussions in the classroom.

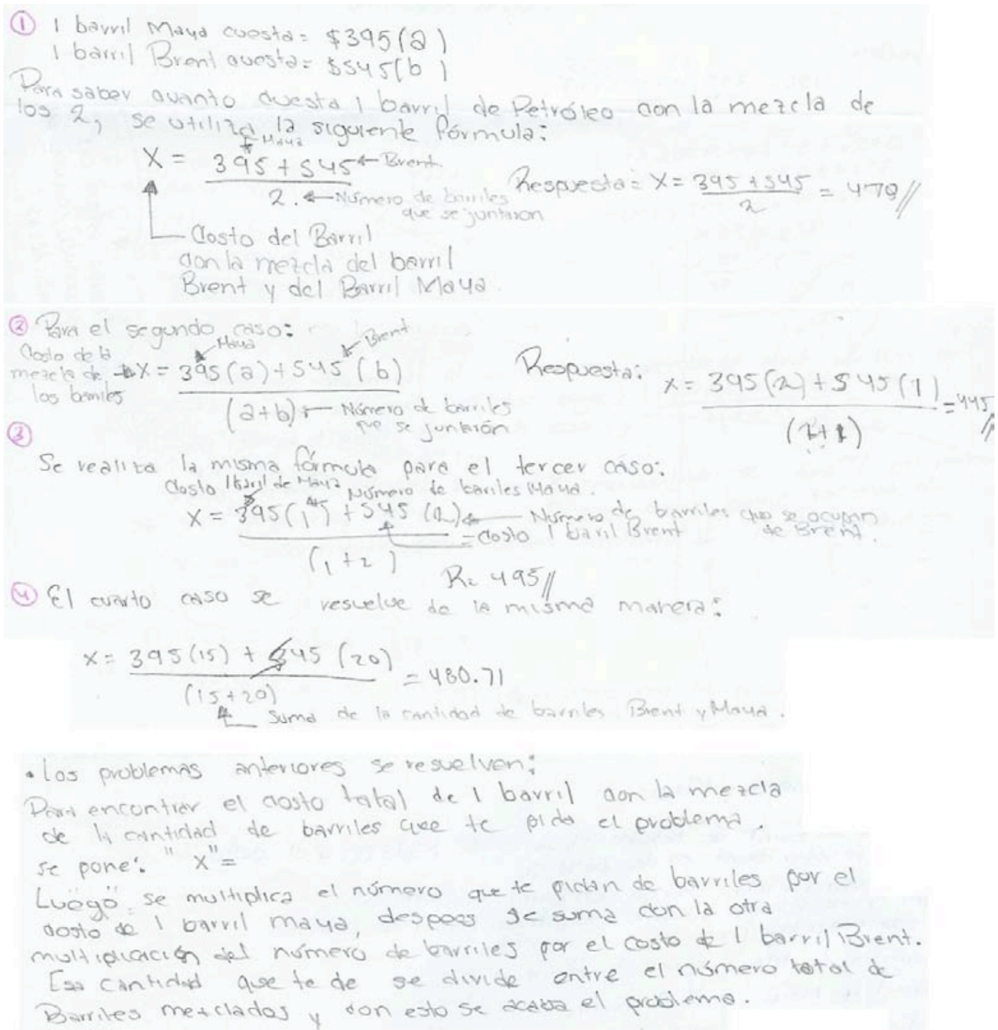

Figure 1. Process of finding the solution to *a-d* in Problem 1.

When answering the question *e* in Problem 1, groups of students tried to solve the problem using trial and error procedures. The students chose possible amounts of barrels (x and y in the equation 1) as solution, and carried out the corresponding operations. They then saw how to get closer to the answer they needed (450), and then did the calculation again.

The students demonstrated understanding of the relationship between the data in Problem 1, the structure of the relations included, and the dependency between the amounts they needed to calculate. Nevertheless, they did not consider writing this functional relation in algebraic language to solve the Problem 1(Letter *e*). This procedure was an arithmetic one.

Algebraic Procedures: Writing Equations and Using Different Mathematical Representations to Solve Letters e-g, Problem 1, and Understanding the Solution

The role of the professor in the classroom was important in this stage. The questions that the professor asked the groups to discuss the procedures were: Was there any other method that you could have used that would have helped you find the solution? Which operations were used that allowed you to answer the previous questions? What were the amounts that did not change in every calculation, and which were the amounts that varied? Can you write equations? These questions and the fact that three groups presented their organized arithmetic procedures used helped the class to utilize algebraic language and write the equation $\frac{395x+545y}{x+y}=450$ (equation 2). However, to find the answer of the problem using this equation became a challenge again. The students had only one equation with two variables so they were confused about the solution.

Professor intervened again in the classroom to promote discussion. She asked some questions as follows: What is the solution or solutions for this equation? How can we find them? How can we know what the values are for x and y? What happens if we suppose that x has a value of one, and we substitute this value in the equation. Can we find the value of y? What happens if I give different values to x? What happens if I graph them? The students were encouraged to use different representations to solve Problem 1, Letter e (Figure 2).

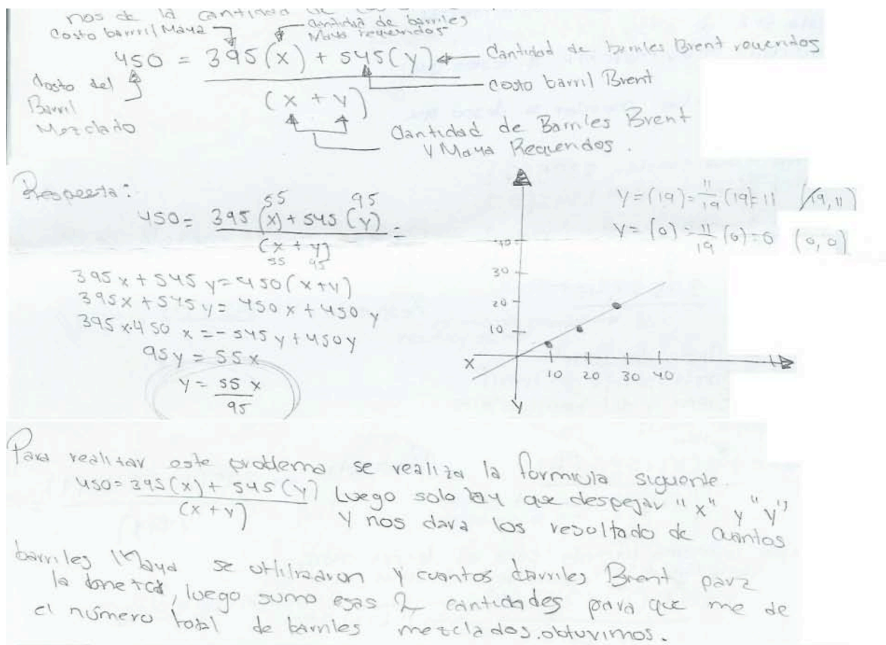

Figure 2. Process used to answer Letter e, Problem 1.

Nevertheless, it is possible to note that students were confused in the following entry (it is a translation from the text included in Figure 2): “To complete this problem, use the following formula. Later, you only have to find the value of x y y to get the results of how many barrels of Maya Petroleum and how many barrels of Brent Petroleum you will use for the mixture. Later, I can add these two quantities to give me the total number of barrels that are mixed [that] we obtained.” It is not clear if the students understood the equation used to find the solution, either the solution. The student (in this entry) was trying to explain the formula, but the student did not write something about the solution of the Problem 1. Student did not write the conclusion emerged in the classroom. The whole group presented the same difficulties in the reports.

Finding the solution to Letter f was somewhat easier for the class during the session. The whole group was able to discuss it with sufficient interpretation of the graph. During the class discussion, students found that there were an infinite number of solutions, and they obtained some of them from the equation but in the reports students did not write something about the solution as in the report of Figure 3. Students only registered the quantities used to draw the graph. Some of them explained in their reports that they followed the same process as before (in

Letter *e*), only that they changed the quantity of 450 in the equation. They explained that this modified the problem, but did not describe how this happened.

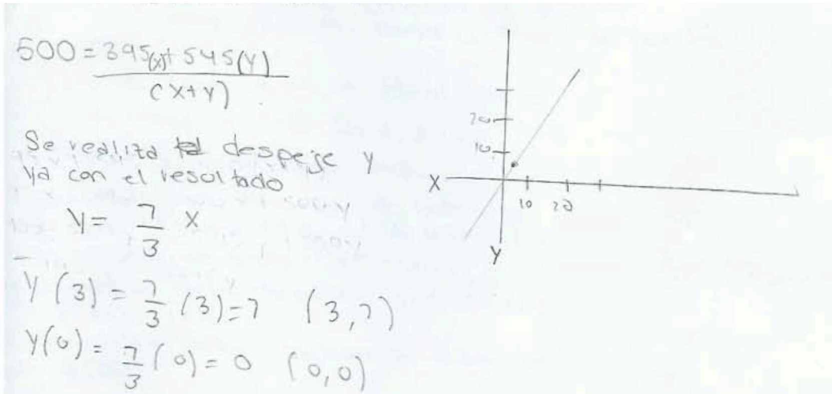

Figure 3. Process of the solution in Problem 1, Letter *f.*

When solving Letter *g* (Figure 4), the groups of students expressed surprise and asked: What is the significance of a straight line such as this? Can we talk about the negative quantities for barrels of petroleum? Is there a solution to the problem? These questions provided an advantage to discuss if there were limits in how many prices were possible for a barrel of petroleum of whatever mix we could produce. This in turn raised the question, "When can the Problem 1 have a solution?"

Students used graphs again to analyze the question of Letter *g*, Problem 1. In Figure 4 we can observe the procedure to solve it. The student in this report explained in it that in order to get the answer, he only needed to change the quantity of 600 in the equation (2). Then, the student said about the solution: "Since it gave me a negative answer, the equation doesn't have a solution."

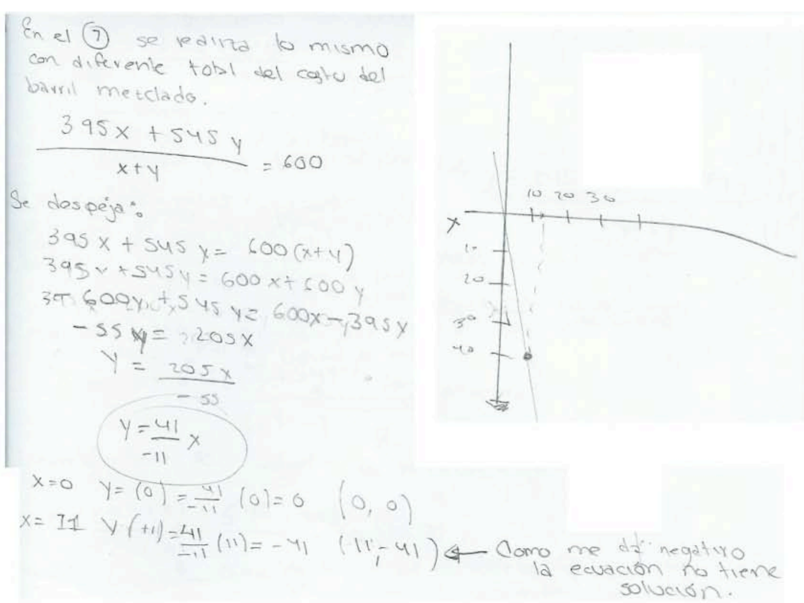

Figure 4. Process of obtaining the solution for Problem 1, Letter *g*.

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Discussion of Results and Conclusion

In an attempt to discuss the process used to answer the problem stated previously, it is necessary to analyze the following questions, which underlie this paper: What mathematical competencies are exhibited at the superior educational level to solve problems that require linear equations? Another question is: What type of interventions can the professor use to help his students develop mathematical knowledge and competencies?

Because the procedure used in this report represented the reports of the whole group of students, it could be said that it reflects the following: the students have a base of arithmetic knowledge that permits them to undertake algebraic problems, identify information in the problem, and recognize patterns and relationships. However, they have difficulty representing these relationships in equations. They also have difficulties with the concept of infinite number of solutions, the variable, and finding the solution to an equation. They do not have different heuristic methods formed to help them solve problems. Students have impediments with argumentation and validating their own procedures and solutions. Their concept of a linear equation is one that is limited to identifying only one answer, and does not include the existence of infinite number of solutions, or non existence of any solution.

On the other hand, in the report that has been included in this paper (figures 1-4), one can note great achievements in how many mathematical competencies have been developed by the students which is demonstrated by their knowledge of the unknown quantity, the variable, and in the solving of an equation. The results are presented in an organized form; there is evidence of relationships and patterns in Figure 1, as well as in figures 2-3, even though there is no evidence of a written argumentation for the solution of the problem. That is to say, there is not a conclusion in terms of what was discussed at the end of the class: When will the Problem 1 have a solution?

The questions asked by the teacher helped students reflect on the concept of solution and its relationship with other concepts, such as the unknown quantity and variable. Integrating the process of graphing in the process of solving the equation helped students to learn these concepts in depth. The role of the professor was essential in this sense, as well as proving help to the students to reformulate, contextualize, discuss, and analyze Problem 1. Constantly monitoring the procedures used by the groups was important to identify the mathematical knowledge of the students, the comprehension level of the problem, and the formation of pertinent questions.

This problem was implemented in other classroom settings, beyond the one discussed in this article. The class discussed the type of solutions they could use in order to solve the problem in terms of whole numbers and real numbers, as well as their significance in terms of barrels, and the possibility of having the a container to form the mixture. Students identified that the total amount of the mixture obtained would be limited to only one barrel, and therefore, as it is expected, other types of graphics and discussions surfaced.

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NAÏVE MENTAL MODELS: THE PERSISTENCE OF PROSPECTIVE ELEMENTARY TEACHERS' CONCEPTION OF PROPORTIONALITY

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Using dynamic learning technologies, we created dynamic conceptual models to address prospective elementary teachers' misconception of proportionality involving problems in the form of "a/b of X". Data collected at multiple points revealed the persistence of prospective teachers' naïve mental models and their interference with meaningful mathematical reasoning.

Learning is fundamentally about change, and change has been emphatically called for during the past two decades in mathematics education, where rule-based instruction masks the true nature of mathematical reasoning and hinders young students' academic progress and career opportunities (National Council of Teachers of Mathematics, 2000). There is growing evidence that, for a variety of reasons, many elementary school teachers do not have the kind of mathematical knowledge that is pedagogically conducive to meaningful classroom instruction (e.g., National Council on Teacher Quality, 2008). Teacher preparation programs and professional development projects have thus resorted to inquiry-based and theory-guided instruction in order to bring about substantive changes in K-12 teaching practices. Following the work of Milrad, Spector, and Davidsen (2003) and Seel (2003), we incorporated model-centered instruction into a mathematics methods course in our teacher preparation programs, taking advantage of emerging open-source dynamic mathematics technologies. We embedded significant mathematical topics in realistic situations and made connections with other focal ideas in a coherent manner, integrating multiple representations and dynamic modeling tools. In this paper, we document our research findings about prospective teachers' misconceptions as they interacted with a dynamic conceptual model specifically designed to explore their understanding of ratio and proportion. We analyze their responses under the perspective of mental models and discuss the implications for mathematics teacher preparation.

Theoretical Framework

We conducted our instructional intervention and data analysis under the theoretical perspective of Model-Centered Learning and Instruction (MCLI, Seel, 2003). Mental models are dynamic mental structures that learners create internally in support of inference and decision-making when they are faced with a problem situation. Cognitively, mental models build a bridge between propositional rules and perceptual images as a means of comprehension (Johnson-Laird, 1983). The interpretation of a problem or a discourse is a result of both the model and the processes that have led to the construction of a model. As Johnson-Laird contends, there is no need for mental logic in the theory of mental models and "people do not ordinarily think in a truthful-functional way" (p. 51). In our effort to make sense of prospective mathematics teachers' naïve mental models for fundamental ideas in school mathematics, we find the theory of mental models well suited for our purpose and MCLI an informative instructional design framework for reconsidering the teaching and learning of school mathematics in teacher preparation programs. Furthermore, conceptual modeling is one of the instructional strategies that can potentially support prospective teachers' creation of valid mental models as conventionally accepted in the field of the subject matter (Norman, 1983; Seel, 2005).

Methods

Context and Participants

This study was part of a mathematics content and methods course in the teacher preparation program at a state university in the Midwestern United States. The class met once a week in a computer lab and three times a week in a regular classroom for 50 minutes. Throughout the semester, the instructor adhered to mathematical modeling as an umbrella perspective on mathematical problem solving involving key topics in middle grades mathematics. Key theoretical constructs were shared with participants using specific examples from school mathematics, including schemata, mental models, modeling, and simulations. Multiple representations were introduced using the open-source mathematical environment GeoGebra (Hohenwarter & Hohenwarter, 2009).

There were 16 students who stayed through the semester; 14 of them gave consent to the study. They were freshman college students with an intention to major in the elementary education program, which requires them to be enrolled in the course. There were only four male students in the class. Most students had a neutral or negative attitude toward mathematics as shown in the institutional evaluation form.

Data Collection

During the first half of the semester, the instructor realized through quizzes and daily interactions with the participants that the majority had a rather rule-based knowledge of elementary mathematics, which was very consistent with the literature (Stigler & Hiebert, 1999). One area that most students were struggling was proportional reasoning, which required the coordination of several variables (Lesh, Post, & Behr, 1988). Therefore, on the mid-term exam, the following problem, among others, was given: *Please explain why "15% of 200" can be written as 15% × 200?* Surprisingly, no one was able to give a mathematically valid and reasonable explanation. The common response was "Because in math terms, '*of*' means *to multiply*. And the multiplication symbol in mathematics is (×)". The rest were more or less the same, with a few students adding more irrelevant comments such as "one cannot divide because that would make the number bigger."

It thus became evident that the majority of the participants held a naïve mental model of the proportional situation that led them to the superficial inferences, namely, symbolic word matching. There is no mathematical or pedagogical meaning in this kind of mental model, albeit it could produce correct numbers and seems meaningful to the participants.

To address this common conception held by virtually all the participants, a web-based dynamic conceptual model (see Figure 1) was designed using GeoGebra to demonstrate the structural relationship underlying the problem: *A% of Y* essentially means finding a number *X* such that *A* is to *100* as *X* is to *Y*. The conceptual model allows the exploration of similar problems and was discussed in class and uploaded to the course site. As observed in class, all participants seemed to accept the mathematical structure of the conceptual model as a reasonable way to explain "*why 15% of 200 can be found by 15% × 200.*"

After a month, a similar problem was given on a quiz, involving a fraction instead of a percent. A brief summary of the quiz was subsequently given to the whole class. After another month, two versions of the problem were given on the final exam, one involving a fraction at the beginning of the test and the other a percent at the end. With the second one, the conceptual model was printed on the test form and participants were asked to respond in reference to the model.

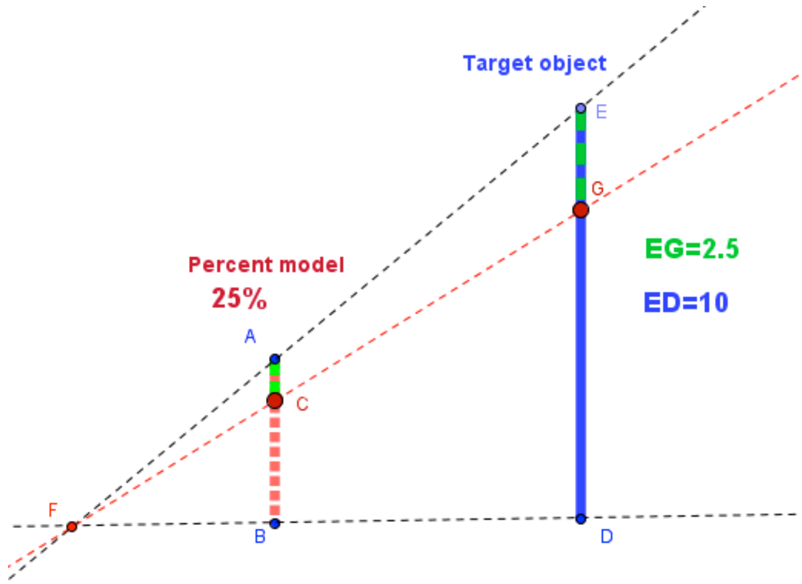

Figure 1. A dynamic conceptual models for problems in the form of "a/b of X"

Results and Discussions

Data collected at three points converged uniformly to the persistence of participants' preconceived mental models of the problem situation. The dynamic conceptual model failed to bring about significant changes to the way the participants would deal pedagogically with such problems. Only one-third of them started to refer to the conceptual model when it was presented along with the problem. For example, in response to " why *23%* of *57* can be found by *23%* × *57* ?", two participants wrote as follows:

Student A: You are able to do this because the following model set up a scale where your slider configures how much you view when taking a percent (or partial part/fraction) of a whole number (ex. *57*). In other words, percent is a fraction or part of the whole number you are using. This kind of set-up is called a proportion. It would be the same as the equation: *x/57= 23/100*.

Student B: Say *ED=57*. Follow the projected path of *23%* to *57*. Find where the path intersects at two points on *57*. That is *23%* of 57. Multiplying *23%* × *57* would get the same answer.

Findings of this study confirm Seel's (2003) observation that students do not always accept the conceptual models presented in model-centered instruction; instead, they create mental models on the fly in response to a specific situation, which is not necessarily consistent with the instruction they received. This, however, does not mean that students did not create, at least, a transitory valid mental model while interacting with the conceptual model. It does imply that on the one hand, mental models are not fixed mental entities and, on the other hand, certain naïve mental models are very resistant to changes (cf. Perkins, 1986; Schoenfeld, 1985) and thus may call for the use of mental censors, especially in teaching practice. It is worth noting that when conceptual models are given, some participants tend to respond in more meaningful ways. Further studies are needed to explore if such conceptual models can bring about long-term changes in participants' mental models with respect to this type of problem or similar problems in school mathematics. As web-based interactive and dynamic learning environments enter all levels of mathematics education, this study contributes to the literature of model-based learning and instruction (Doerr & Lesh, 2003; Zbiek & Conner, 2006) and identifies needs for future design and development, especially in teacher education.

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METHODS AND CONCEPTUAL DIFFICULTIES FOR SOLVING AND GENERALIZING LINEAR PATTERNING TASKS

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This study sought to understand the similarities and differences between middle and high school students' methods for generalizing linear patterning tasks. While high school students in Algebra 2 performed much better than middle school students, the success rate remained low. Whereas middle school students frequently applied proportional reasoning on such tasks, high school students favored a linear strategy. Individual interviews with 24 students in grades 6-8, Algebra 1, and Algebra 2 uncovered many conceptual difficulties for generalizing the linear pattern.

Objectives

Pattern generalization has been acknowledged as helping students transition from early algebra to formal algebra (Bezuszka & Kenney, 2008; Department for Education and Skills, 2001; English &Warren, 1998; National Council of Teachers of Mathematics [NCTM], 2000; Usiskin, 1988). Some studies (Cooper & Sakane, 1986; Lannin, 2005; Steele, 2008; Stacey, 1989) have focused their attention on investigating students' abilities and strategies to generalize and justify patterns presented in a geometric context and have suggested that the visual aspect of the geometric structure helped students identify the patterns of change and facilitate the generalizing process.

However, the participants of all these existing studies were middle school students prior to taking formal lessons in algebra. It is yet unknown how high school students, after having taken Algebra 1 or Algebra 2 classes, would perform on geometric generalizing activities and whether their strategies would be similar to or different from those of middle school students. Would they be more likely to approach this type of task by generating a rule, creating a table, or applying the concept of slope and *y*-intercept? This proposal reports partial findings of a study that investigated the rate of change concept with both middle and high school students. In particular, this report discusses middle and high school students' performance and their methods used in solving a linear geometrical patterning task (Figure 1) and the conceptual challenges they faced when attempting to generalize this pattern during the follow-up interviews.

Growing Dots Task

How many dots will be in the 100th picture? Show your work.

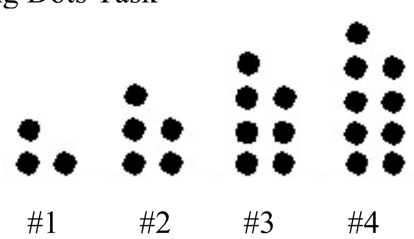

Figure 1. A linear geometric patterning task

Specifically, we seek to answer the following two research questions:

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

1. What methods do middle and high school students use to solve a linear patterning task, and what are the similarities and differences of methods between the middle school and high school students?
2. What conceptual challenges do middle and high school students face when generalizing a linear patterning task, and what are the similarities and differences of these challenges between the middle school and high school students?

Theoretical Framework

Blanton and Kaput (2005) describe "reasoning algebraically" as a "process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and appropriate ways" (p. 413). Students are expected to be proficient in developing two types of generalizing rules by the end of their algebra learning: recursive form or the close form (NCTM, 2000). For example, the rule of the recursive form for the Growing Dots task could be expressed as Next = Now + 2, starting at 3, where Now and Next represent the number of dots in the current and next stages. The rule of the close form can be expressed as $y = 1 + 2x$, where x represents the stage of the figure, and y represents the number of dots in that figure. There is a growing recognition that the development of algebraic reasoning is a long-term process that needs to be started at the elementary school level.

Linear numerical and geometrical (visual) patterns are among the ones that have been used most frequently at the elementary and middle school levels to lay the foundation for algebraic reasoning**.** Prior studies have identified many conceptual challenges students face when asked to represent a general rule in symbolic forms, as well as the difficulties they experience in connecting the recursive and close form (Lannin, 2005; Rivera & Becker, 2008). Stacey (1989) conducted a paper-and-pencil test to investigate the strategies used to solve three linear generalizing problems by students aged 9 to 13 years old. She found that the majority of students found generalizing to be challenging. The rate of correct responses varied from 8% to 24% for various age groups. Her analysis identified four main methods of solving this type of task (Table 1). This work was used as the framework for our initial coding, with new ones being added as needed.

Method	**Stacey's Explanations**	**Examples from the Growing Dots Task**
Counting	Counting from a drawing	"I used a calculator, started with the last number, added 2, until I got to 100."
Difference	Multiplying the figure number by the common difference 2, i.e., implicitly assuming that repeated addition of 2 implies that $M(n) = 2n$	"It went up by 2, $2 \times 100 = 200$"
Whole object	Taking a multiple of the number of squares in an earlier figure, i.e., implicitly assuming that $M(nm) = m \times M(n)$	"There were 3 dots in number one picture. $3 \times 100 = 300$"
Linear	Using a pattern that recognizes	"$1 + 100 \times 2 = 201$" or "$3 + 100 \times 2 =$

	both multiplication and addition are involved and that the order of operation matters; i.e., implicitly using a linear model $M(n) = an + b$ with $b \neq 0$.	203"
Unclassified		$3 \times 5 \times 7 \times 9 \times 11 = 10{,}395$

Table 1. Explanations of common student methods with Stacey and Growing Dots Task

Methods

Participants

The study was conducted in a rural school district in a Midwestern state of the United States. The district had only one middle school (grades 6-8) and one high school (grades 9-12) with about 650 and 840 students, respectively, in 2010. The student population was 80% non-Hispanic White and 11% Black. Fifty-one percent of the students received free or reduced lunch. Both middle and high schools had student test scores on state mandatory tests close to the state means. Both schools used mathematics curriculum from McDougal Littell. Students from grade 7 and above had studied units in terms of either linear equation or linear function in their mathematics classes prior to the start of the study.

Procedures

A paper-and-pencil test was administrated at the end of the school year in the spring of 2010 during the regular math class. One hundred pairs of parents/students gave consent to have their student data included in the study. The numerical break-down of these 100 students by class can be seen in Table 2 below.

The test had three main sections that contained tasks embedded in various patterns of change, such as linear, piece-wise linear, exponential, and quadratic. Based on the written work, 24 students (5 sixth graders, 4 seventh graders, 4 eighth graders, 5 Algebra 1 students, and 6 Algebra 2 students) were interviewed individually for 45-60 minutes to gain more in-depth knowledge of their mathematical reasoning on how they solved the tasks on the written test. When there was an error on the written test, probing questions based on the nature of the errors followed to help students clarifying their reasoning. Then they were asked to find another method to solve the given problems. The interviewer encouraged students to think aloud while developing their alterative methods.

In this paper, we will focus on only the results of the task in Figure 1 and the portion of the follow-up interviews that aimed at identifying the various conceptual difficulties students had in generalizing the given linear pattern. Generally speaking, the interviewer first asked students to describe the pattern they saw in words or in numerical expressions. Then the interviewer asked the students to justify their pattern by connecting it to the pictures. In addition, the high school students and a few advanced middle school students were encouraged to find a "formula" that would help them solve the question in general. If the students seemed to be lost, the interviewer encouraged them to generate a table or graph to help them identify the correct formula.

Data Processing and Analysis

The students' tests were first graded for the correctness of the final answers, then coded with the types of strategies developed by Stacey (1989). First, two individuals coded the data independently. Then they compared the codes and conducted further discussion to resolve the discrepancies. All interviews were transcribed with the help of Trasana© and analyzed to deepen our understanding of the reasoning behind students' methods and the sources of the misconception or partial understanding that prevented them from identifying the correct general rule.

Results and Discussions

In this section, we will first discuss the results and findings regarding students' performance on the task, methods they used to solve this task on the written test, and a comparison of the variance between middle and high school students. We will then report the findings based on the analysis of the follow-up interviews.

Methods for Solving the Task

Table 2 shows how the participating students performed on this study, organized by their grade/course level. Overall, the performance level was low across the grades, and significantly lower than those being reported in Stacey's study. There is a general trend of improvement over the grade/course levels, with the Algebra 1 students being the exception.

	Grade 6 (n = 19)	Grade 7 (n = 20)	Grade 8 (n = 22)	Algebra 1 (n = 20)	Algebra 2 (n = 19)
% of correct responses	5%	0%	13.6%	0%	26.3%

Table 2. Student performance on the pictorial linear problem

Similar to Stacey (1989), we also identified four methods in this study presented in the students' written work. However, our analysis found more complexity of student reasoning within the three most commonly used methods (Whole Object, Linear, and Counting/Estimating, referred to as Counting by Stacey), than was presented by Stacey. Whole Object was one of four methods (shown in Table 3) used frequently to find the number of dots in the 100th picture. Four different types of reasoning were identified in this category. The student multiplied the number of dots in the existing diagrams by 100, which is the term in the sequence for the 100th picture, such as 3 × 100 and 9 × 100, where 3 and 9 represent the number of dots in the 1st picture and 4th picture, respectively. Another group of students used the number of dots from the existing diagrams multiplied by the quotient of 100 and the figure number of that diagram. For example, a student's solution to "draw pictures up to #10, and then 21 × 10" showed that the students found there were 21 dots in the 10th picture, so the 100th picture would contain 10 times as many dots. In addition, some students thought they could obtain the answer by multiplying the number of dots in a particular picture by the difference between 100 and the position number of that picture. For instance, a student reasoned that since the 4th picture has 9 dots and 100 – 4 = 96, then the 100th picture must have 96 × 9 = 864 dots.

We expanded the original Counting Method identified by Stacey (1989) to be the Counting and Estimating method, where the student had the explicit awareness of the common difference of 2, but didn't provide further explanations that would indicate that the student had used another method. Some examples of this include: "I would say 100 dots, you are adding 2 every time," "Almost 101 dots, the dots go up by 2 and if you add 2 all the way up to 50 you get 101," or one student simply put down "plus 2 each column, 103." The answers given were quite varied in this method. While they were able to recognize the constant change of 2 in the pattern, students were not able to coordinate that piece of information with the overall number of dots.

While the Linear method was used by many students, not all of them reached the correct answer. The most common false linear method assumed the initial value to be 3, the number of dots in the 1st picture, and thus students gave 203 as the answer. Three high school students gave 199 or 197 as the answer, assuming 9 (the number of dots in 4th picture) + 2*x* (95 or 96). These students recognized the need to make an adjustment for the number of 2s being added, but failed to come up with the correct adjustment.

Table 3 illustrates the variance in methods used by the different grade levels. The gray cell indicates the most frequently used method for that particular grade. Nearly 50% of the sixth graders used the Whole Object method that was based on a faulty assumption that a proportional relationship existed in the given pattern. While many of them also recognized the constant difference of 2 in the pattern, the pull for proportional relationship was very strong. The dominating strategy used by seventh graders was the Difference method, based on the constant increase pattern. The more evenly distributed methods in eighth grade and Algebra 1 students reflected the transitional stage. Finally, more than half of the students from the Algebra 2 course used the Linear method.

	Grade 6 (n = 19)	Grade 7 (n = 20)	Grade 8 (n = 22)	Algebra 1 (n = 20)	Algebra 2 (n = 19)
Whole Object	9 (47%)	4 (20%)	4 (18%)	4 (20%)	2 (11%)
Difference	2 (11%)	10 (50%)	3 (9%)	4 (20%)	1 (5%)
Counting/Estimating	3 (16%)	3 (15%)	6 (32%)	2 (10%)	2 (11%)
Linear	1 (5%)	1 (5%)	6 (27%)	5 (25%)	11 (58%)
Unclassified	4 (21%)	2 (10%)	3 (14%)	5 (25%)	3 (15%)

Table 3. Distribution of student strategies by grades

As indicated in Table 3, the nature of the conceptual challenges for middle and high school students in this study was quite different. The application of the Whole Object method by close to 50% of the sixth and seventh grade students could be seen as part of a widespread phenomenon of overgeneralizing proportionality (Stacey, 1989; Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003). This could be the result of heavy emphasis on proportional reasoning in mathematics curriculum at the sixth and seventh grades. Both eighth graders and Algebra 1 students, while showing some tendency to move away from assuming the proportionality in this type of task, were not certain about the alternative methods. Fifty-eight percent of the students from the Algebra 2 course used the Linear method, which involved the coordination of two quantities—the initial value and the constant change. However, while there is an increasing awareness of these two quantities and the need to coordinate them, few were able to carry out the coordination successfully to get the correct answer of 201.

Strategies and Conceptual Challenges for Generalizing the Linear Pattern

The follow-up interviews with the 24 students selected to reflect a wide range of methods used for the paper-and-pencil test attempted to identify the roots of these conceptual difficulties.

All interviewees were encouraged to first verbally describe the pattern they saw in the picture. All recognized the "plus 2" pattern among the consecutive pictures. When encouraged to describe a way to describe the overall pattern, three different numerical patterns with four different geometrical configurations emerged (see Figure 2): (a) 3 dots plus 2 dots more each time (Pattern #1 and Pattern #2); (b) 1 dot and a pair of 2, adding a pair each time afterwards (Pattern #3); and (c) the left column has the same number of dots as the picture number, and the right column is just one more dot than the left column (Pattern #4). When reasoned correctly, both Pattern #1 and Pattern #2 led to the equation of $y = 3 + 2(100–1)$, with y being the total number of dots in that sequence. However, Pattern #3 would lead to the equation $y = 1 + 2(100)$, and Pattern #4 to $y = 100 + (100+1)$. All will lead to the answer of 201. Five out of the 24 interviewees were able to come up with the correct answers on their own. Three of them used Pattern #4 as the basis for their solutions, and one used Pattern #3. The last student saw Pattern #1, but did not use the physical configuration to generate his answer. Instead, he shifted his attention to the numerical pattern that the number of dots was about double the corresponding figure number, and he adjusted this estimation with "plus 1" to get his answer. All were able to describe the general rules in words or in symbol form, $y = 2x + 1$.

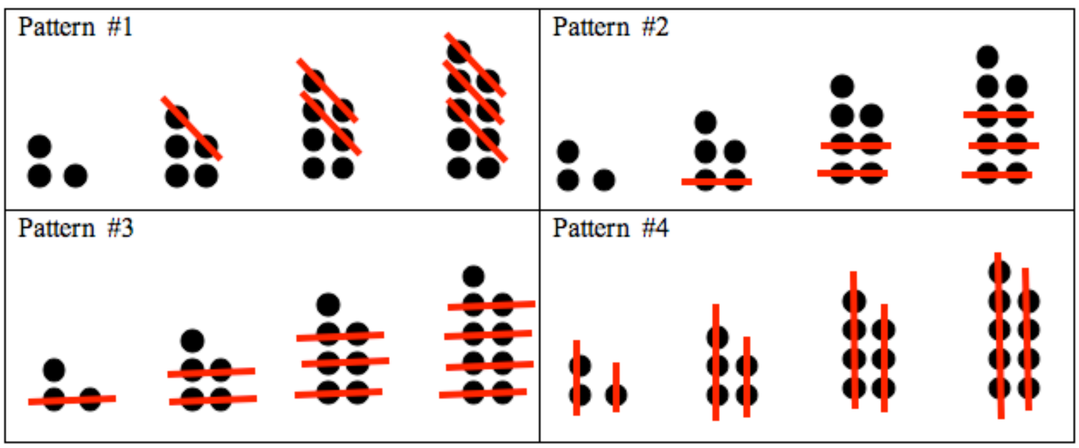

Figure 2. Students' strategies to generalize the linear pattern task

With probing questions that helped students to use available tools such as numerical expressions, graphs, or tables to make sense of the pattern they saw, 11 of the remaining 19 students were able to come up with the generalized rule in either words or symbols. In the remaining part of this section, we will discuss the conceptual difficulties students encountered when attempting to generalize with numerical expressions, tables, and graphs.

When the interviewees were unable to develop a generalized formula, they were encouraged to generate a table or a list to make the relationship between the "figure number" and the "number of dots in the corresponding figure" more explicit. To develop a generalized formula for the given pattern, it is necessarily to recognize both the "plus 2" pattern and the initial amount, which is either 1 or 3, depending on how students saw the pattern. When students failed to do so on their own, the interviewer suggested the student rewrite the 5 dots in the second set of dots either as 3 + 2 or 1 + 2 + 2, the 7 dots in the third set of dots as 3 + 2 + 2 or 1 + 2 + 2 + 2. The question then becomes, how many 2s are in each expression? There were four main conceptual challenges that the interviewees faced when attempting to find the generalized formula from this path. First, those who saw either Pattern #1 or Pattern #2 had a hard time articulating that the idea of the number of 2s being added in the figure is 1 less than the corresponding figure number. Second, some students were unable to connect repeated addition to multiplication. Third, some rejected the expression 3 + 99(2) because they read it as (3 + 99)2. Finally, some high school students were uncomfortable with the expression $3 + (x - 1)2$ because it did not fit their experience with $y = mx + b$ with the constant on the back and the variable in front of the coefficient.

Those who used a table or graph to identify the generalized formula faced different type of conceptual challenges. Out of nine students who attempted to use a table to make sense of the generalized pattern, only four succeeded. Three of these students did so after converting the table data to a graph. One made an estimate of $y = 2x$ and adjusted for it by comparing the y values. None connected the concept of constant rate of increase with the concept of slope and used it to find the correct formula. The analysis of those that were not successful suggested three main types of conceptual challenges. The first one was rooted in the limited conception of the patterning task. For example, one student explained that the 100th picture would have a different number of dots depending on which given figure you used. So the 100th picture for the 1st given picture would be 300, and the 100th picture for the 4th picture would be 900. The second challenge was caused by the difficulty shifting from the recursive pattern which existed in the number sequences formed by the number of dots in the picture, and the close form which asked for the relationship between the position number in the sequence and the number of dots in the corresponding picture. They believed that $y = x + 2$ was the correct answer. The last conceptual challenge was the false assumption of proportionality. In addition to the Whole Object method, some students believed that there was a constant multiplicative factor between the variables x, the figure number, and y the number of dots in the corresponding figure. For example, one student thought the equation would be $y = 2.5x$ because there were 5 dots in the 2nd picture, but then noticed right away that this formula would not work for the 3rd picture. When urged to check if the proportional pattern holds for the first four given figures, all students were able to see that the y=mx equation they proposed did not work.

Implications and Significance of the Study

The findings of this study indicated that geometrical patterning tasks are accessible and challenging to both middle and high school students. The availability of geometrical figures in the pattern helped to ground students' explanations of their methods and reasoning. Many research studies have reported the strong tendency for students to apply the cross-multiply algorithm indiscriminately which, unfortunately, is a side effect of the increased emphasis on proportional reasoning at the middle school level. Comparing and contrasting geometrical patterns that have zero and no-zero value in stage one can highlight the essential characteristics of linear versus proportional relationships. This type of task provides a natural setting to make distinctions between proportional and nonproportional relationships. The experience of using and interpreting complex numerical expressions is crucial to students' work with algebraic expressions. The findings of this study affirmed the potential of geometrical patterning tasks to help students develop algebraic reasoning as suggested by Lannin (2005) and Steele (2008).

The overall low performance of the participating students was a surprise to both the researchers and the mathematics teachers of these students. The follow-up interviews conducted with the students have led to the identification of the several conceptual challenges faced by students when attempting to generalize the patterns. Such information is useful in developing hypothetical instruction paths that would be effective to support students with different initial methods for this type of task.

The findings also suggested that while geometrical patterning tasks might be beneficial in helping students develop algebraic reasoning, such development is a complex process. Even though all the Algebra 1 and Algebra 2 students interviewed in this study have learned about linear functions, none of them used the concepts of slope and intercept to help them identify the general rule of this patterning task. The tendency to focus only on the change in y-values rather than on the rate of change that relates x and y values suggests the need of additional instruction on this important concept. Future studies are still needed to explore how geometrical patterning tasks might be used to assist students in developing a more robust understanding of the rate of change as a main conceptual tool to differentiate linear and nonlinear relationships.

Acknowledgement

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DEVELOPING TEACHERS' REPRESENTATIONAL FLUENCY AND ALGEBRAIC CONNECTIONS

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This study examined the development of thirty-seven elementary and middle grades teachers' algebraic connections and representational fluency during a six-month professional learning project. To evaluate the collaborative nature of designing professional development, the team of professional developers/researchers used the collective self-study method (Samaras & Freese, 2006) to examine how purposively designed experiences such as the content-focused institute in the summer with school-based follow-up Lesson Study cycles (Lewis, 2002) in the fall encouraged vertical articulation of algebraic connections. The analysis of teachers' reflections from problem solving tasks and Lesson Study and researchers' memos from observed research lessons revealed more flexibility in teachers' representational fluency with problem solving strategies in the classrooms.

Introduction

Research and initiatives in elementary and middle school mathematics emphasize the importance of both fostering algebraic reasoning through problem solving and laying the critical foundations before students encounter formal algebra instruction (Blanton, 2008; Driscoll, 1999; NCTM, 2000; NMAP, 2008). This leads to some important practical questions: what experiences are required for elementary and middle grades teachers to effectively teach algebraic reasoning through problem solving, especially if they never learned mathematics in a reform oriented manner? What opportunities can professional development providers design for teachers to develop teachers' representational fluency? Reform oriented mathematics teaching presents challenges as teachers learn to change their practices (Remillard, 2000; Boaler, 1999). Such changes that advocate a reform-based vision of teaching algebra along with the great need for effective algebraic instruction in the K-12 curriculum has motivated mathematics educators, mathematicians and school leaders to emphasize more on studies about teachers' understanding of algebraic thinking and reasoning. Teaching algebraic reasoning in the earlier grades through problem solving requires depth of mathematical content knowledge for teaching. Hill, Sleep, Lewis and Ball (2007) suggest that specialized content knowledge is partially domain-specific and that the specified content knowledge for mathematics is not obtained based on an intelligence level, or ability to understand mathematics. Mathematics knowledge for teaching (Hill et. al, 2007) includes understanding of general content but also having domain specific knowledge of students and knowledge of content in algebra, patterns, and functions.

The purpose of the study was to provide a coherent picture of how teachers' algebraic connections and representational fluency develop through a content focused and practice-based professional development project. Our study set out to identify the key practice-based skills necessary for teachers to acquire the specialized knowledge for early algebra that would help them make algebraic connections to enhance their teaching and promote algebraic thinking in students.

Theoretic Framework

"Early algebra is not the same as algebra early. To move algebra-as-most-of-us-were-taught-it to elementary school is a recipe for disaster" (Carraher, Schelimann & Schwarts, 2008, p. 235). The notion of teaching early algebra then is both a paradigm shift and a novel way of teaching that requires extensive professional development. According to Carraher et al. (2008), early algebra builds on background contexts of rich problems where early algebra tightly interweaves existing topics of early mathematics and formal notation is introduced gradually and "judiciously". Many researchers have explored teaching early algebra for the elementary grades and found that elementary students are capable of reasoning algebraically (Kaput, Carraher, & Blanton, 2008; Kaput, 2008; Bastable & Schifler, 2007; Kieren, 2004; Carpenter & Levi, 2000; Greenes & Findell, 1999; Moses, 1993). This integrative approach of algebraic reasoning and arithmetic connections is common in many international elementary and middle school mathematics curriculum (e.g., China, Russia, Singapore, and South Korea), where students begin the formal study of algebra much earlier (Cai, 2004). Such integrative approaches have helped these students in building mathematical understanding with skill development, in embedding the arithmetic in interesting and challenging problem situations, in enhancing their number sense through purposeful calculations, and in relating problems to one another and to other essential mathematics content knowledge they are building constantly.

According to the NCTM (2000) the underlying concepts of algebra are patterns, relations, and functions and the way in which they are utilized to: 1) represent and analyze mathematical situations and structures using algebraic symbols; 2) represent and understand quantitative relationships; and 3) analyze change in various contexts (NCTM, 2000, p. 39). Blanton and Kaput's (2008) Generalizing to Extend Arithmetic to Algebraic Reasoning (GEAAR), project goal was to embed algebraic thinking into instruction and to build teacher capacity to "algebrafy their classrooms" (p. 384). The evidence to support their findings was gathered from teachers' reflections and written work, students' reflections and written work, observations, and interviews. This professional development addressed both teacher content knowledge as well as classroom practices. Blanton and Kaput (2008) reported that teachers become better at teaching algebraic reasoning when the teachers' own mathematical knowledge and understanding are increased. According to Blanton and Kaput (2008), teachers develop their algebra "eyes and ears" which allow them to bring out the algebraic reasoning while looking at student work and carefully listening to their discussions and questions. This requires teachers having an integrated depth of algebra content knowledge to know what to look and listen for in the classroom. Developing algebraic habits of minds (Driscoll,1999) include the idea of seeing algebra as: 1) doing and undoing; 2) building rules to represent a function; and 3) abstracting from computation. Developing algebraic habits of mind is critically important to the success of our elementary, middle and high school teachers. It takes a teacher who has a deep and profound understanding of fundamental algebra to provide opportunities for elementary and middle grades students to explore the foundation concepts for algebraic reasoning through patterning, relations, functions, and representations using algebraic symbols and utilizing mathematical models to represent relationships (NCTM, 2000).

For our study, we defined 'algebraic connection' as the integrated knowledge developed by applying algebraic concepts to interdisciplinary problem-solving opportunities through multiple representations and flexible strategies. Such connections help to discover and comprehend the applications of mathematical concepts to real-world problems in a way that is beyond what is taught in a traditional classroom. Thus, 'algebraic connection for teachers' included some very specific practiced-based skills (such as, making: 1) cross-curricular (horizontal) connections- algebraic connections to real world situations and interdisciplinary problem-solving; 2) 'algebra-arithmetic' (vertical) connections- connecting arithmetic structures such as addition, subtraction, multiplication, division, ratios, proportional thinking, rational numbers and geometry to algebra; making generalizations about a concept; 3) algebraic connections through representations orchestrating mathematics discussions through the use of students' physical, tabular graphical models, verbal and symbolic notations to build representational fluency.

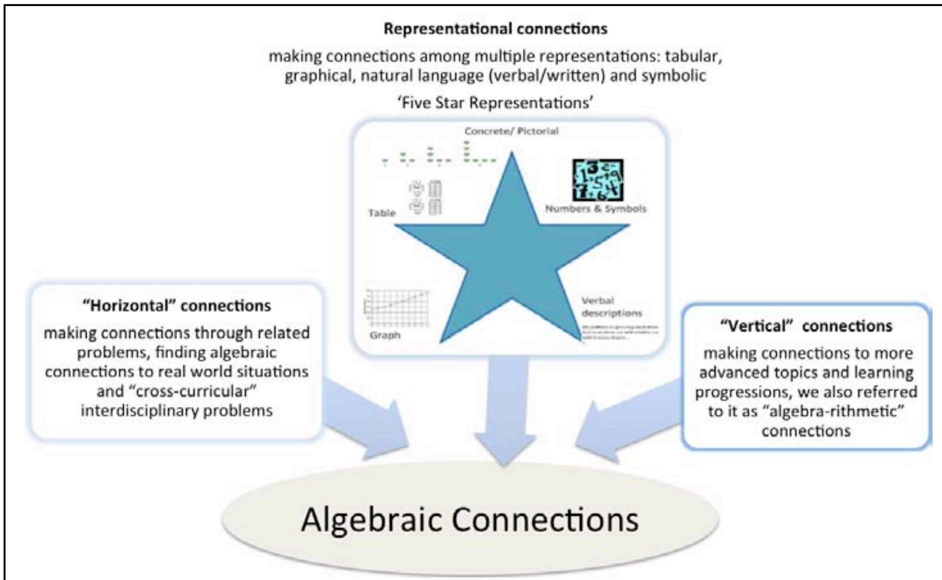

Figure 1

One of the key performance skills was for teachers to teach algebraic lessons using the five star representations for all algebra lessons. The five star representation connected representational systems such as using pictures, number lines, graphs, tables, arithmetic notations and verbal description. (Figure 1) Representational fluency, the ability to use multiple representations and translate among these models, has been shown to be critical in building students' mathematical understanding (Goldin & Shteingold, 2001).

Methods

Research Questions

The goal of the study was to understand the processes of change in participants' teaching practices and examining what professional development activities and events were pivotal in influencing that change. This study explored the following two research questions: 1) what changes in teachers' development of the algebraic connections were evident through the activities and professional learning opportunities and 2) what are some essential professional development opportunities and support necessary for teachers as they transition into teaching through problem solving and making algebraic connections?

Data Sources

This study examined teachers' reflections and field notes from two phases of the professional development project: the summer institute where teachers were immersed in algebra problem solving activities as learners and the second phase with the follow-up Lesson Study where teachers implemented teaching practices that promoted algebraic connections through rich problem solving.

Participants

Thirty-seven elementary and middle grades teachers from grades 3 - 8 met for a one week summer institute and continued as school teams in Lesson Study (Lewis, 2002) during the academic year. A majority of the teachers (78%) taught in Title One schools that served underrepresented and underserved populations. On average, teachers had 12 years of experience, where 18 held baccalaureate degrees and 19 held Master degrees. Six teachers were mathematics specialists, and three teachers were special educators.

Procedures

To evaluate the collaborative nature of designing professional development, the team of professional developers/researchers used the collective self-study method (Samaras & Freese, 2006) to examine how purposively designed experiences such as the content-focused institute in the summer with school-based follow-up Lesson Study cycles in the fall encouraged vertical articulation of algebraic connections. The method for analysis involved a mixed method approach of survey analysis and qualitative analysis. Using the Grounded Theory approach (Strauss & Corbin, 1994), we used a constant comparative method. This method allowed for us to systematically gather and analyze data through the summer institute and Lesson Study cycle and to generate a theory that was grounded in these data. The researchers used Self-Study to determine what designed activities promoted the development of representational fluency. During the summer and follow-up Lesson Study session, we debriefed on the project activities and analyzed the teachers' reflections from the activities. Upon analysis, we searched for recurring themes and categories that linked various data. The analysis of the data from teacher reflection began with the reading of the reflections from the summer institute during which patterns in the participants responses were identified. The data analysis was aimed at answering the research questions and identifying themes, categories, or types (Miles & Huberman, 1994). Successive readings of the data necessitated modification and further development of the coding categories.

Results

Opportunities to Deepen Teacher Mathematical Knowledge and Algebraic Connections

To address research questions, "what changes in teachers' development of the algebraic connections were evident through the activities and professional learning opportunities", and "what essential professional development opportunities and support are necessary for teachers as they transition into teaching through problem solving and making algebraic connections?" we analyzed the reflections from problem solving tasks during the content-focused summer institute and from the Lesson Study.

The analysis of teachers' reflections revealed four main themes: teachers developed a) an awareness of their own metacognitive process during problem solving tasks; b) an appreciation of the need for specialized mathematics knowledge; c) connections to teachers' classroom practices based on one's own reflection from tasks; d) awareness of the affective nature revealed during the learning processes. Below are representative excerpts from the qualitative analysis.

Awareness of Teacher's Own Metacognitive Process During Problem Solving Tasks

The developing awareness of one's own metacognitive process encouraged reflection upon personal learning and problem solving style preferences, identification of common pitfalls and misconceptions that accompanied the problem solving process, and the ability to internally perform task analysis in order to generate multiple problem solving ideas.

> After Dr. S. shared how Gauss' thinking connected to this problem, it seems to keep coming up in other problems we have solved. It all started with the handshake problem and then it appeared again in the toothpick problem, the fruit stack problem, and the penny jar problem.

> While I could identify and extend a triangular number pattern, the importance of the pattern was not something I thought was significant. I don't ever recall noticing this triangular number pattern, and it seems to keep coming up in numerous problems. 1, 3, 6, 10…now, I keep seeing this pattern everywhere. – Elementary math specialist

Appreciation of the Need for Specialized Mathematics Knowledge

Participants noted that using multiple strategies and representations to solve rich questions that required deeper mathematical thinking increased their understanding of algebra concepts. Teacher learners immersed in a content-focused institute and asked to explore novel approaches and unfamiliar tools and technology experienced an expanded appreciation of algebraic connections and representational fluency.

> I noticed that when he explained his strategy to our other partners, his visual really seemed to be clearer than my abstract expression. That was very powerful for me because I need to always keep in mind as a teacher that all of my students learn in different ways and are all at very different levels. While some of my students may be ready for the algebraic expression, there are so many important benefits that can be gathered from the pictorial/graphical representations as well. -7th grade teacher

Making Connections to Teachers' Classroom Practices Based on One's Own Reflection From Tasks

Throughout the institute, participants grappled with problems as students themselves and saw the value of rich problems that encourage multiple approaches and rich discussion. They personally experienced the connections that exist algebraically within and between the mathematical strands of number, geometry, measurement, and statistics. Scaffolding opportunities, differentiation potential and readiness assessment were all named as benefits of this type of learning environment. However, many did voice concern over barriers within their schools that could impede implementing these newly learned and appreciated reform practices.

> Obviously I have thought about how I might use this problem in my own classroom this year. I am excited to see what algebra I can bring out for my students and how I can modify a lesson focused on inequalities to a new lesson focused on algebraic expressions.
>
> As I think through how I will adjust and focus this lesson for my sixth grade students I am oriented on two main factors. First, I need to consider multiple representations and access points in my classroom. I need students to feel comfortable seeing problems in multiple representations and then get them comfortable trying to use these approaches (concrete, pictorial, graph, symbolic, verbal) to solve problems. I also must consider how I present a problem—how can I make sure all of my students have an access point to solving a problem?- 6th grade teacher

Awareness of the Affective Nature Revealed During the Learning Processes

The teacher learners voiced both struggles and satisfaction during the problem solving processes. They appreciated the collaboration and support from colleagues and emphasized the importance of mathematics communication to building mathematical ideas.

> The next two classes were painful to say the least, by Thursday I was ready to tell my partner I couldn't do this. She talked to me all the way to class but I was not really convinced. Then like a flash, all start falling into place. As I pass the pages, I look at the pizza problem. That was the one that gave me confidence. I saw the connection with the handshake problem. I thought, what was different this time. As I was thinking about that, Habits of Mind came to my thoughts. I realized that I was beginning to use them in my problem solving. That is something I want to impart in my students! – 4th grade teacher

Catalysts for Change: Essential Opportunities for Teachers to Develop Algebraic Connections

As we coded teachers' reflections from the designed project activities, key events stood out as being pivotal points in eliciting teachers to rethink their practices. We identified these key activities as the Catalysts for Change in our teachers because these activities seemed to be the events in which teachers realized they needed to change their way of thinking about their instructional practices for mathematics teaching and learning.

Key Activity 1: Teachers Immerse in Rich Problems with Algebraic Connections

It was important for teachers to grapple with the problems and experience disequilibrium. This opportunity allowed for teachers to make connections to the fundamental algebra and increase their understanding of the topic. It was a "relearning" process for many who were making sense of the algebra that they had learned procedurally; void of context or real life applications. These connections were made by both experienced and novice teachers. We coded this as '*making connections and see algebra in a new way'.* An experienced eighth grade teacher who currently taught Algebra was able to see the content in a new way. One excerpt from her reflection said,

> Just as I needed to make connections between the context, my rule, and why it worked, students of all ages need these same opportunities. Previously when teaching students about triangular numbers, I focused on the number pattern and the geometric pattern. It was solely identifying what is happening by doing and undoing and extending the pattern, perhaps even getting to build a rule. What was totally missing from my instruction? Problems in which triangular number patterns actually appeared! Pretty big I think. Just as my understanding of triangular numbers and algebra deepened, students will receive the same benefit by having opportunities to take their understanding of patterns and apply it to more complex situations. *-8th grade teacher*

We also noticed that these activities elicited teachers developing representational fluency and valuing the use of multiple representations. A novice 5th grade teacher excerpt was coded *'developing representational fluency'* because she expressed using multiple representations as she solved and made sense of the problem and also made connections to how her students might gain access to the problem through a visual approach.

> Reflecting back on how I solved the problem I notice that I used many different representations to help me solve the problem. First I drew a picture, then I used a table, and last I used symbols. This idea of multiple representations was accented in class and made me process how students might need different entry points to be able to engage in this problem. It made me think about how to set up a problem and get students started on a problem. The visual used in class was extremely helpful for me, even though I could access the problem in other ways. It reinforced what I saw was going on in the problem, but had I been a child that could not see what was going on, it would have given me an opportunity to engage in the problem (*5th grade math and science teacher).*

Key Activity 2: Opportunities for 'Teacher Math Talk' on Algebra and Vertical Articulation

The importance of collaboration and vertical articulation came to the forefront as teachers discussed problems with their colleagues and talked about the grade level mathematics expectations. One teachers commented how her team members helped her build on her existing ideas about algebra.

> As we have worked through problems this week, I have been very fortunate to sit with colleagues who teach at a range of grade levels. Our table actually represented grades 3-6. As

> we shared our thinking and problem solving strategies, I learned to see the big, intricate picture from them. There were times when it seemed that they gained a lens by hearing about my analysis of a table, my exploring with manipulatives, or through my illustrations. *-3rd grade teacher*

This was a great connection to what research has noted about how students learn from class discussion (Hufferd-Ackles, Fuson, & Sherin, 2004). We observed teachers learning from the mathematical discussions, questioning, explaining mathematical thinking, and debating mathematical ideas. This critical activity also elicited teachers' responses to recognizing the importance of breaking down the essential mathematical learning, identifying the common student misconceptions and scaffolding and differentiating for diverse learners.

> I feel most prepared to teach my students to use multiple representations, to notice patterns, and build rules. I think it was most helpful to solve problems collaboratively in the summer class and to share our solutions. It was interesting to see the various ways my classmates solved the problems. *-Special educator who teaches in an inclusive classroom*

In many ways, this opportunity allowed teachers to simulate the math talk that they would conduct in their own classroom. Having a chance to discuss potential misconception, analyze sophistication of solution strategies, and ways to create parallel tasks for differentiation allowed teachers to have a natural conversation of progression of mathematical ideas and vertical articulation.

Discussion

At the end of the course, participants reflected on how their participation in the yearlong professional development session had impacted their ideas of algebra in the early grades. The most common themes were that they "relearned" the math by being in the "shoes of the student" and having to solve challenging problems while breaking down the important mathematics helped them see the early building blocks for creating algebraic connections. These experiences not only increased their confidence in the teaching and learning of mathematics but also helped the participants to develop a productive disposition towards mathematics.

This study revealed that engaging teachers in challenging activities that created some sense of disequilibrium led teachers to reflect on their knowledge, their problem solving process, and their instructional practice. Participants' positive experiences during the learning process helped develop a more productive disposition towards teaching algebraic reasoning through problem solving and through multiple representations, and value collaborating with other mathematics educators in translating their learning into practice. In addition, the study demonstrated the effectiveness of the choice of rich problems that helped us to study teacher's developing knowledge and that also provided the teachers with useful examples for algebraic instruction and student learning assessment. It is the authors' hope that the results of this study contribute to the growing body of knowledge documenting how professional development serves as a catalyst for change in teachers as they take time to reflect and relearn specialized knowledge for teaching algebra in the early to middle grades.

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DIFFERENTIATING ALGEBRAIC EQUIVALENCES IN CLASSROOM NETWORKS

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Equivalence in algebra has different meanings in different contexts and a sophisticated understanding of equivalence is required to solve increasingly complex problems. This study builds upon previous design research studies that illuminate how mathematical constructs can be mapped to social relationships to support mathematical meaning making. We follow how one student's conceptions of equivalence changed as he worked with three different partners on four algebra learning activities in two novel classroom network based learning. We illustrate how differing affordances in the learning environment lead to different forms of mediation when disagreements accompany differing mathematical conceptions in a classroom network environment.

Introduction

Algebra gains much of its power from the flexible use of equivalence (Gattegno, 1974), and students often find it difficult to negotiate different kinds of equivalence in different contexts (Linchevski & Herscovics, 1996). Peer discussions mediated by technology and instruction support convergence of mathematical meaning (Moschkovich, 1996) and the development of more sophisticated conceptions of equivalence. Our own ongoing design based research project aims to foster mathematical conversations and interactions in pairs and small groups using handheld network technology (White, 2006; White, Sutherland, & Lai, 2010). Broadly, and building on prior research involving classroom network systems (Stroup, Ares, & Hurford, 2005), our design approach involves mapping social relationships between students to mathematical constructs embedded in technology to create opportunities for students and the teacher to interact in ways that make mathematical relationships explicit.

The present study focused on investigating students' developing ideas about equivalence in the context of two different learning activity designs, each of which featured an equivalence engine intended to support students' joint engagement with algebraic expressions as they constructed polynomials and transformed equations. We follow one student as he worked with three different partners over four different sessions using these two different activity designs, and examine instances in which disagreement between partners occurred when students differed in their conceptions of equivalence in a particular context. Disagreements in discourse signal a dialogical conflict of meaning (Bakhtin & Holquist, 1981) that both emerges from and is mediated by agents and tools in a particular context (Kozulin, 2003). The episodes presented here are the result of an ongoing exploration of how student understanding of equivalence in different contexts emerges from and is mediated by resources, tools, teachers and other students. In particular we highlight the way disagreements between students accompany differing mathematical conceptions of equivalence and how different affordances of two activity designs influence mediation of these disagreements in a classroom network learning environment (Ares, Stroup, & Schademan, 2009; Stroup, et al., 2005).

Equivalence in Algebra

Students initially conceive of the equal sign as a signal that a solution will follow. This operational understanding of equivalence gradually gives way to an understanding equivalence as a comparison of equal values (Kieran, 1981; Rittle-Johnson, Mathews, Taylor, & McEldoon, 2010; Sfard & Linchevski, 1994). A sophisticated understanding of equivalence becomes increasingly important as problems become more complex (Knuth, Stephens, McNeil, & Alibali, 2006). Operational conceptions of equivalence in certain contexts can persist into college even for students who do well on standardized tests (McNeil & Alibali, 2005). Below, we introduce two different classroom network-based collaborative learning activity designs intended to address these challenges by supporting an instructional sequence in which students jointly explore and negotiate meaning of algebraic equivalence relations in different settings.

Two Learning Environment Designs for Exploring Equivalence

The learning activities in this study featured two different classroom network designs: *Terms and Operations*, focused on students' collaborative efforts to construct polynomial expressions, and *Two-Sides*, centered on pairs' joint transformations of shared equations. Both learning environments were created with the NetLogo modeling platform (Wilensky, 1999) and HubNet architecture (Wilensky & Stroup, 1999) connected to a class set of TI-83 graphing calculators that function as handheld devices on the network.

Equivalence of Expressions in Terms and Operations

The *Terms and Operations* activity design allows the teacher to populate a field with floating monomial terms (Figure 1). Pairs of students each construct a collective expression by alternately adding, subtracting, multiplying or dividing a term captured from the field. To complete each transaction the student must enter the result of the operation as an equivalent expression.

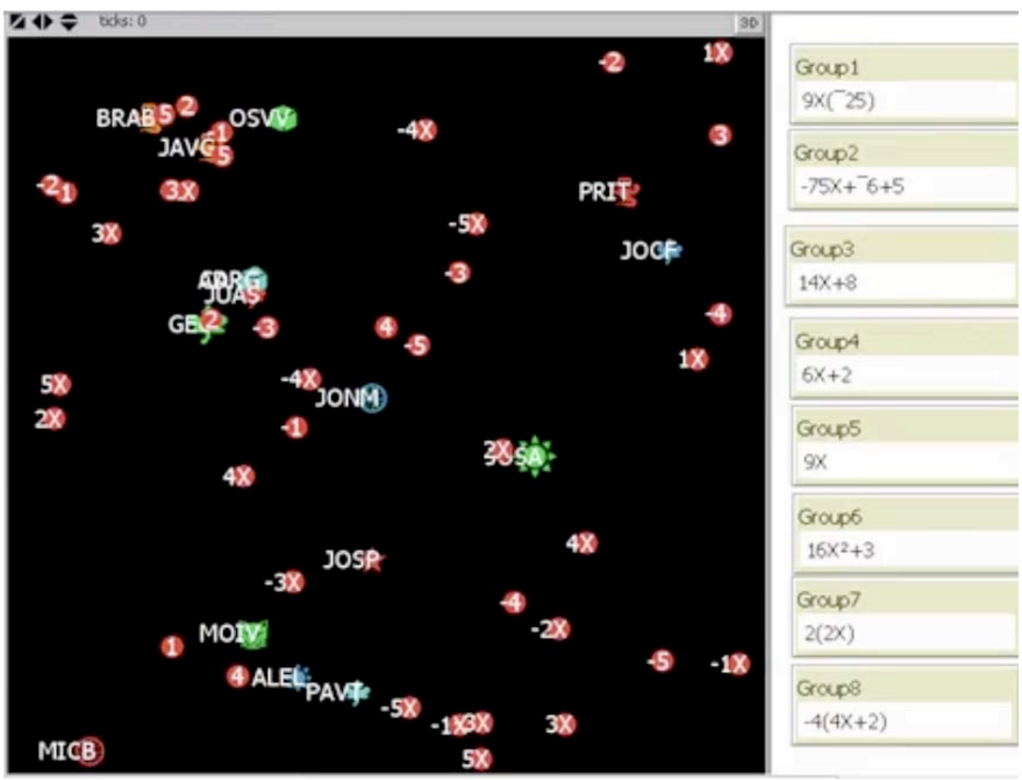
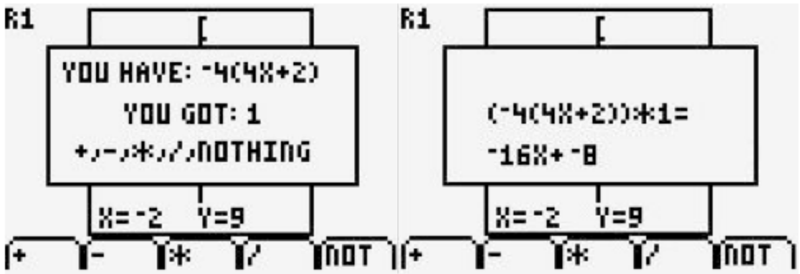


Figure 1. ***Terms and Operations*** **shared display with floating terms and collective expressions.**



Figures 2a and b. Student calculator screens featuring a) captured term and operation choices and b) a collective expression under the chosen term and operation and an equivalent student entry.

The system updates the collective expression only if the result of the operation and the expression entered by the student are equivalent. When the expression entered by the student is not equivalent the collective expression remains unchanged. The *Terms and Operations* design verifies equivalence of expressions by providing feedback in the form of updates to the collective object.

Equivalence of Equations in Two-Sides

The *Two-Sides* activity design builds on previous designs by verifying both equivalence in the context of expressions and equivalence in the context of equations. Each member of the student pair is assigned to either the left or right side of an algebraic equation. To create a new equation each student in the pair must enter a valid algebraic expression. The networked relationship between the two student' calculators in the *Two-Sides* environment acts as an equal sign for the equation they share. Each member of the pair can transform her own side of the equation by entering any equivalent expression. The system updates the equation to reflect the transformation only if the new expression is equivalent to the original expression. When the pair wants to transform the entire equation they must coordinate the transaction by pressing the operate button and entering equivalent operations on both sides.

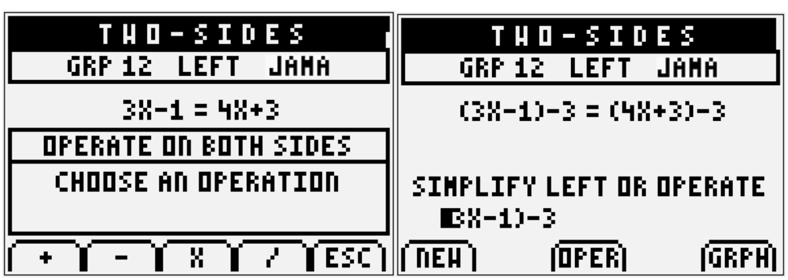


Figure 3a and b. ***Two-Sides*** **left student calculator screens showing the coordination of operating on both sides of an equation.**

The system updates the equation only if the operations by both students are equivalent. *Two-Sides* maps the algebraic equation to the partner relationship by acting as the equal sign between them, making it necessary for the pair to reach agreement when transforming the equation.

Method

The data considered in this study were drawn from a yearlong project in which ninth grade students at a non-traditional urban charter school organized around project-based learning participated in classroom network activities for one hour-long session each week as a supplement to regular pre-algebra and algebra 1 instruction. On the basis of informed consent two to three student pairs were selected as focus groups and videotaped for all activities. All screen states of the public computer display were recorded as a video file for each class session, and an additional camera with a wide zoom setting captured this projected display along with the whiteboard at the front of the room, as well as whole-class discussions and other teacher moves. Server logs recorded all activities on both the shared displays and the student calculators. Episodes involving disagreements between partners were flagged and coded by topic of the disagreement. Disagreements that did not involve a conflict of mathematical meaning were dropped. Conflicts over the meaning of equivalence in a particular context were selected, annotated and transcribed. The present study follows one student, Jamal, as he worked with three different partners over four different class sessions—two each in the *Terms and Operations* and *Two-Sides* activity designs. Disagreements that signaled dialogical conflict of the meaning of equivalence between Jamal and his partner during the four sessions were considered for analysis. The segments presented here were selected to illustrate how the activity design illuminated and mediated student conceptualization of algebraic equivalences in different contexts.

Analysis

Episode 1: Differentiation of X term and Constant Term Equivalence

Jamal and his partner (Sara) had been introduced to *Terms and Operations* in a previous session but had worked in different pairs. In this session the teacher instructed students to construct an expression using the four basic operations (add, subtract, multiply and divide), recording each step on paper and simplifying along the way. They had constructed a -20 using the sequence -1 * 5 * 4 when Jamal captured a -5X and subtracted it from the -20. The following disagreement occurred over the simplification of -20 - -5X:

1. Sara: That's going to be -25X
2. Jamal: Hold up. It's -20…
3. Sara: Yes it is…
4. Jamal: [Thinking aloud] -20 + 5X.
5. Sara: Hold up, hold up… -100X … -100X
6. Jamal: Really? What?
7. Sara: I'm subtracting…
8. Jamal: A hundred… How did you get a hundred?
9. Sara: No wait wait (he he)
10. Jamal: It's -20 + 5X
11. Sara It's a positive… wait it's a negative [as Jamal enters -20 + 5X into the calculator]
12. Jamal: Got it! It's -20 + 5X [both looking at the shared classroom display]
13. Sara So what's the answer?
14. Jamal: -20 + 5X
15. Sara: Huh?

The disagreement occurred when Sara attempted to construct a single equivalent term for an expression that included a linear term and a constant term. Note that Sara took into account all three negative signs when constructing a negative term (line 1). In contrast Jamal assigned the first negative symbol to the constant term and correctly interpreted the double negative on the X term (lines 2, 4, 10-12). Jamal's confidence in his version of the expression exposed Sara's uncertainty as she abandoned -25X in favor of -100X (line 5). Sara expected an answer in the form of a single term and expressed surprise (line 15) when the system verified the equivalence of Jamal's binomial expression. The *Terms and Operations* design exposed the differing conceptions of equivalence in the two students and provided important mediation by confirming the equivalence of Jamal's suggestion.

Episode 2: Ascribing the Rules for Equivalence in an Equation to an Expression

Jamal joined another partner (Irene) for the next *Terms and Operations* session. The teacher explained the distributive property and instructed the students to construct any expression equivalent to 2(X + 3) + 4. The following disagreement between the pair and the teacher occurred after Jamal and Irene had successfully constructed 2X + 10. They asked the teacher if they could divide by 2 to simplify the expression to get X = 5:

16.	Teacher:	X = 5. Okay. Where is the equal sign in here right now? [pointing to 2X + 10]
17.	Jamal:	Or, X + 5.
18.	Teacher:	X + 5. So you want to divide this thing by 2… [writes (2X + 10)/2] You're dividing this whole expression by 2. On the calculator you would write it out like this. Yes, Jamal you are right if I did divide this by two 2X/2 = X and 10/2 I = 5. So this is what (2X + 10)/2 would equal.
19.	Irene:	It's 5X.
20.	Jamal:	No 5.
21.	Teacher:	X + 5. No?
22.	Jamal:	[Looks stumped]
23.	Teacher:	Okay what were you saying Jamal.
24.	Jamal:	[Shakes head] Nothing.
25.	Teacher	What are you thinking Irene?
26.	Irene:	[giggles]
27.	Teacher:	Not sure? Here's the thing: is X + 5 = 2(X + 3) + 4? [Points to board]
28.	Jamal:	No.
29.	Teacher:	Why not?
30.	Jamal:	[Tentatively] Well, it is. It's confusing me.
31.	Teacher:	Here's one thing we could do. Let's say we knew what X was. Let's say X = 1. What would X + 5 be equal to if X = 1.
32.	Jamal:	[writing on paper] 6 and the other one would be 13… or no it would be 12.
33.	Teacher:	How do you make a 6 equal to a 12?

Implicit in the instructions for this task was the notion that 2(X + 3) + 4 would equal some equivalent expression. Jamal and Irene successfully constructed 2X + 10 which does equal 2(X + 3) + 4, but the *Terms and Operations* collective display showed only 2X + 10. *Terms and Operations* verifies equivalence of expressions but does not check for equivalence of equations, so Jamal attempted to verify the next move by asking the teacher if they could divide 2X + 10 by 2 to make X = 5. Clearly Jamal and Irene drew on prior experience of solving equations to make this suggestion, attempting to apply rules for equivalence in the context of equations to an expression. Irene combined the constant term and X term by suggesting that 5X might be a better candidate. The teacher mediated the disagreement by suggesting that a 1 be substituted for X. Jamal did the calculations and realized that 6 ≠ 12. The learning environment did illuminate the misconception of equivalence proposed by Jamal and Irene, but the teacher provided mediation because *Terms and Operations* does not address equations.

Episode 3: Equivalence in Equations versus Expressions Revisited

The teacher began the session by explaining that to solve an equation the X must be isolated on one side through the use of equivalent operations to both sides. Providing a list of equations to solve, the teacher introduced the *Two-Sides* activity design for the first time and challenged the students to solve as many of the equations as they could using the calculators. Jamal and his partner (Megan) had just restarted the program after a technical glitch and found that they had reversed roles. A disagreement occurred over what they should do to each side of 3X + 4 = X - 6 to move toward a solution.

34. Megan: Yes. Minus X. [Selects operate and chooses subtract X]
35. Jamal: I have to, like, subtract 4. Huh. [Selects operate, chooses subtract and types a 4]
36. Megan: No. Minus X. [Almost grabbing the calculator so Jamal will subtract X instead of 4.]
37. Jamal: Yeah.
38. Megan: No. We're supposed to do the same thing. You have to do the same thing so minus X.
39. Jamal: Minus…X. No… Dammit! [Types as he talks and presses enter]
40. Megan: You got it… There you go.
41. Jamal: Oh, dooney! [surprised and puzzled]

Both Jamal and Megan attempt to eliminate a term from their own side of the equation, but Megan understands the additional requirement that equivalent operations must be performed. Jamal expressed puzzled surprise (line 41) when the system validated subtracting X from both sides. The disagreement between Jamal and Megan corresponded to differing conceptions of equivalence in the context of equations. The *Two-Sides* system mediated the disagreement in a way that *Terms and Operations* did not by checking for equivalence of both equations and expressions.

Episode 4: Equivalence in Equations versus Expressions Resolved

Jamal rejoined Megan in a later session in which the students were challenged to enter any equation that contained at least one X term and a constant term on the same side. Megan entered 2X – 3 and Jamal entered 9 to form 2X – 3 = 9. The disagreement occurred over the first step in solving the equation.

42. Jamal: 2X – 3 = 9
43. Megan: …X – 3 = what?
44. Jamal: 9. I'm supposed to put the right 9. You're suppose to put 2X – 3… Then you subtract 3 and I add 3.
45. Megan: No. We both add it.
46. Jamal: Both add or both subtract?
47. Megan: No. We both add it.
48. Jamal: [Jamal watches the screen intently as the system updates] Oh, yah [unintelligible.]
49. Megan: Ok. Now simplify
50. Jamal: So 9 + 3 is 12. [They arrive at 2X = 12]

Jamal continued to struggle with the requirement that the same operation must be performed on both sides of an equation (line 44). Even after agreeing that the same operation must be performed on both sides Jamal struggled to anticipate the consequences on the equation as a whole (line 46). Jamal's intense concentration on the shared display indicated that the result meant something important to him. This conjecture was further supported by his affirmative response to the result (line 48), and his immediate simplification of his own expression (line 50).

The teacher began to explain to the class how the graph on the *Two-Sides* shared display related to the equation Megan and Jamal had made. Jamal decided to test his newfound understanding despite the fact that the teacher was using the graph of their equation as an example.

51.	Jamal:	[While Teacher talks Jamal initiates a divide by 2 and taps Megan to show her what he's doing.]
52.	Megan:	[Megan also divides by 2 and the shared display updates to X = 6. Changing the graph the teacher is using as an illustration]

Jamal recruited Megan to divide by 2 on both sides. The teacher was unaware of the update, which caused a few moments of levity in the class. Here the *Two-Sides* design was able to verify the equivalence of Jamal's proposal in the context of equations in a way that *Terms and Operations* could not, supporting Jamal's prediction that dividing both sides by 2 would produce the desired result. Unlike the prior instance during this episode (lines 45-47), Megan readily agrees because their conceptions of equivalence for this operation were the same. It is possible that Jamal used a guess and check strategy to arrive at the correct solution for the equation, but he was required to divide both sides by two to verify this. *Two-Sides* and Jamal's partner Megan provided mediation in the form of positive feedback.

Discussion

The *Terms and Operations* and *Two-Sides* activity designs enabled Jamal and his partners to jointly construct expressions and equations. In Episode 1 the *Terms and Operations* design mediated the disagreement between Jamal and Sara over the equivalence of expressions. However, when Irene and Jamal ascribed the rules of equivalence for equations to an expression in Episode 2, mediation shifted to the teacher because *Terms and Operations* did not verify equivalence for equations. The *Two-Sides* design tested equivalence for both expressions and equations providing mediation for Episodes 3 and 4. The different affordances of the two tools led to different forms of mediation of the disagreements between Jamal and his partners.

The design of the learning environment mapped the social relationships to the mathematical construct. Each move by the pair necessitated an agreement on a conception of equivalence in a particular context. Disagreement created the need to test one competing conception using the tools in the learning environment. When the *Terms and Operations* could not verify equivalence of equations the students turned to the teacher to resolve the disagreement.

The contrast in the mediation of equivalence between the two designs shifted mediation away from the teacher in certain contexts and toward the teacher in others. Collaboration supported by technology gave way to more teacher directed forms of instruction when the technology could not mediate disagreements. Student conceptions of equivalence were mediated, but the form of mediation differed depending upon the affordances of the learning environment.

Jamal shows increased sophistication in his understanding of equivalence over the course of the four sessions. He could negotiate equivalence of expressions (Episode 1), but ascribed rules for equivalence of equations to expressions (Episode 2). Working in the in the *Two-Sides* learning environment he began to differentiate between equations and expressions (Episodes 3 and 4). The technology both informed and supported different forms of instructional practice and promoted new kinds of classroom interactions that led to learning for Jamal.

This study took place in a non-traditional charter school with mix of both algebra 1 and pre-algebra students. Students volunteered to participate in the study on a once per week basis. The influence of mediation cannot be generalized beyond this study, and the influence of technology should be taken as an example of how pedagogical possibilities emerge from designs like these. Further research will be required to explore these possibilities in the context of classroom network learning environments.

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VALIDATING TWO PROBLEM-SOLVING INSTRUMENTS FOR USE WITH SIXTH-GRADE STUDENTS

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A validation study was conducted to explore properties of two problem-solving measures. One hundred sixty-nine students responded to two problem-solving instruments that were adapted from prior research (Verschaffel et al., 1999). Students completed the instruments during their mathematics class with one week between test administrations. There was sufficient evidence of good fit using a one-factor model for both measures. Item difficulty and discrimination were examined using Item Response Theory.

Introduction

Mathematically proficient students at any age make sense of word problems, evaluate their progress while problem solving, and interpret their solution (Chief of Council State School Officers (CCSSO), 2010; Kilpatrick, Swafford, & Findell, 2001). Unfortunately, data suggest that adolescents in the U.S. have difficulty solving complex word problems (Gonzales, Williams, Jocelyn, Roey, Kastberg, & Brenwald, 2009). With many states adopting standards that reflect the Standards for Mathematical Practice (CCSSO, 2010), assessments that capture students' problem-solving performance and align with these new standards are necessary. Instruments drawing on Item Response Theory (IRT) concepts and procedures are likely to produce more accurate estimates of problem-solvers' abilities than Classical Testing Theory (CTT) approaches (de Ayala, 2009; Embretson & Reise, 2000). Problem solving is briefly discussed followed by the benefits of IRT and a brief description of key terms typically used in IRT. The goal for this study was to revise and validate problem-solving tests from Verschaffel et al. (1999) using IRT and factor analysis.

Literature Review

Problem Solving

"A problem occurs when a problem solver has a goal but lacks an obvious way of achieving the goal" (Mayer & Wittrock, 2006, p. 288). Verschaffel et al. (1999) suggest that word problems should be open, realistic, and complex. Open problems provide learners multiple entry points and can be solved in different ways. A realistic task is developmentally appropriate and draws on a problem solver's prior knowledge. A complex problem requires an individual to employ sustained reasoning to solve it. More accurate and appropriate evaluations of students' problem-solving ability may be possible with problems drawing on these components. Problem solving goes beyond procedural thinking typically associated with solving exercises (Polya 1945/2004; Mayer & Wittrock, 2006). Problem solving

> usually involves several iterative cycles of expressing, testing and revising mathematical interpretations – and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics. (Lesh & Zawojewski, 2007, p. 782)

Exploring the ways students solve word problems in addition to their performance adds to the literature on students' problem solving and strategy use.

Prior research provides a foundation for creating new instruments with tasks that are open, realistic, and complex. Verschaffel and his research team (1999) explored the impact of teaching mathematics through problem-solving contexts on fifth-grade students' problem-solving performance. Four classes received 20 lessons meant to foster effective problem solving while seven classes continued with their normal instruction. A total of 232 participants completed both a pretest and posttest. Instruments were composed of ten open, realistic, and complex word problems. Each item on the pretest had a similar task on the posttest. Reliability for the tests was .56 and .75. Items were scored as correct, technical error, or incorrect. Technical errors were characterized as minor arithmetic errors, which were counted as correct responses in terms of students' performance. The tasks' open-ended nature would have supported explorations into students' strategy use; however, these analyses were not conducted. Results from the posttest indicate that on average, students from the intervention group had better problem-solving performance than those in the comparison group on the posttest, $p < .05$. Verschaffel et al. (1999) note a lack of problem-solving instruments using open, realistic, and complex tasks suitable for adolescents. Measures with well-documented and appropriate psychometric qualities could not be borrowed hence tests were created to suit the aim and scope of the study. Further research is necessary to make a version of the instruments for English-speaking students but also, the items' contexts may need updating. A second limitation Verschaffel et al. (1999) discuss was the lack of validity-related evidence associated with their instruments. The large number of participants would have afforded an investigation to gather statistical validity using IRT (de Ayala, 2009; Embretson & Reise, 2000).

Item Response Theory

IRT has four critical assumptions, (1) ability is a unidimensional trait, (2) the items are locally independent, (3) the probability of answering items increases as ability increases, and (4) item parameters are independent of respondents' abilities (de Ayala, 2009; Embretson & Reise, 2000). Results from IRT methods produce trait-level estimates of a person's abilities that depend on the individual's responses and the item properties (Embretson & Reise, 2000). Ability estimates are often more precise than CTT results and require fewer items (de Ayala, 2009). Each individual has a standard error estimate instead of assuming one for all respondents (Embretson & Reise, 2000). Respondents' scores can be linked when they take tests that measure the same trait; CTT requires strictly parallel tests (i.e., equal means, variances, and covariances). Finally, results from tests created with IRT procedures can be compared with results from different populations. Advancements in IRT methodologies facilitate updating and improving older problem-solving instruments that relied on CTT. By reexamining tests from earlier research, it may be possible to better estimate a learner's problem-solving ability and facilitate enhanced contexts that support problem solving.

The simplest IRT model is the Rasch model, which suggests that the odds of an examinee answering an item is modeled by the ratio of an examinee's ability and an item's difficulty (de Ayala, 2009; Embretson & Reise, 2000). Item difficulty characterizes the ability needed to have a 50% likelihood to give a correct response is 0.5 (de Ayala, 2009). Generally, an item difficulty below 0 indicates an easier problem whereas greater than 0 characterizes a harder item (de Ayala, 2009). Problem-solving tasks tend to be somewhat more difficult than standard mathematical exercises therefore larger item difficulties are expected. A useful trait of the Rasch model is the Sufficiency property, which indicates that respondents can be rank ordered by their total score. The second property of all IRT models is item discrimination, which characterizes the degree an item can distinguish respondents of differing abilities. The Rasch model constrains item discrimination to a specific value (e.g., 1) or it can vary for the measure. The 2-PL model allows item discrimination to vary for each task but participants cannot be rank ordered.

Method

Participants

All sixth-grade students in ten classes of two sixth-grade teachers from a school in Florida were approached for their consent, 169 students agreed to participate. Approximately 33% of students at the school are eligible for free-or-reduced-lunch (Florida Department of Education (FLDOE), 2010). Sixty-nine percent of students at the school are white, 12% are African-American, 11% characterize themselves as Hispanic, 5% are multiracial, and 3% are of an Asian ethnicity (FLDOE, 2010).

Instrumentation

The problem-solving instruments used by Verschaffel et al. (1999) were translated into English by an individual who previously taught Dutch at the university level. Items consisting of only one sentence were not revised because they were significantly shorter than most tasks. One item-pair relied on pictures and was not selected because the item format was unique from all other tasks. After examining the items, a total of eight item pairs were selected for translation and revision. The problems were revised to update contexts, to reflect U.S. students' experiences, and to clarify the language. For instance, an updated task described food options available in a U.S. school cafeteria (i.e., sandwich, pizza, and hamburger) rather than lunch choices typical of school cafeterias in the Netherlands. The first and second tests are located in Appendix A and B, respectively. When a near-exact translation of an instrument is used, a back-translation may be necessary; however, the contexts of the problems changed dramatically thus a reverse translation was unnecessary.

Two mathematics educators reviewed the tasks for open-ness, relevance to sixth-grade students in the U.S, and complexity. The tests were also evaluated by a fifth-grade mathematics teacher for these same characteristics as well as to verify that the items were realistic for children living in the U.S. and could be solved in multiple ways by students entering sixth-grade. The teacher indicated that the items were open, realistic, complex and further indicated that students entering sixth-grade could answer every question.

Data Collection

Measure administration occurred during students' mathematics class, which lasted 45 minutes. The measures were completed one week apart. Most students completed the first test over one-and-a-half mathematics periods whereas they finished the second test during one mathematics period. The measures were distributed to the students, the directions were read out loud, and students who asked for assistance were told to reread the problem and answer the question to the best of their abilities. Students did not have access to calculators during testing.

Analysis

Initially tests were scored using Verschaffel et al.'s (1999) scheme including incorrect (0 points), incorrect because of an arithmetic error (1 point), and correct (2 points). There were very few occasions of arithmetic errors (i.e., less than two percent of all responses) and a dichotomous

IRT model effectively suited the tests without sacrificing loss of information hence the scoring scheme was simplified to incorrect and correct (i.e., zero and one, respectively). Responses with arithmetic errors were collapsed into the correct category. A sample consisting of 20% of tests was randomly selected, two coders scored these tests independently and interrater agreement (IRA) using r_{wg} (James, Demaree, & Wolf, 1984) was calculated. IRA is more appropriate than interrater reliability because IRA provides evidence that ratings from the two coders would be identical for all other tests from that sample (LeBreton & Senter, 2008). IRA for both test administrations was within an acceptable range (r_{wg} = .99) (LeBreton & Senter, 2008).

Structural equation modeling procedures were employed to examine the fit of the tests with a single proposed construct: problem-solving ability. IRT analyses were conducted to determine item difficulty and item discrimination. Reliability of each measure was calculated using Cronbach's alpha and Pearson's *r* characterized alternate forms reliability. De Ayala (2009) stresses that a calibration sample with a few hundred respondents "should not be interpreted as a minimum, but rather as a desirable target" (p. 43) and a smaller sample may suffice to estimate item parameters. M-PLUS and R facilitated the model fit and IRT analyses, respectively.

Results

Test and Item Characteristics

The test of model fit for the one-factor solution for the first test was not significant $\chi^2(20) = 20.08$, p = .45 and Root Mean Square Error of Approximation (RMSEA) indicated excellent fit, *RMSEA* = .005. The Confirmatory Factor Index (CFI) and Tucker-Lewis Index (TLI) were both 0.99. Similarly, the test of model fit for the one-factor solution for the second test was not significant $\chi^2(20) = 21.42$, p = .37, *RMSEA* = .021. CFI and TLI for the second measure were 0.99. Root Mean Square Error of Approximation values less than or equal to .06 indicate close fit (Bryant & Yarnold, 1995/2005). CFI and TLI values greater than .90 provide evidence of good fit (Ullman & Bentler, 2009).

Three dichotomous IRT models were explored: Rasch with discrimination constrained to one, Rasch with unconstrained discrimination, and the 2-PL model. A 3-PL model was not considered because the sample was not large enough and guessing the answer on a constructed-response item is unlikely. Data were subjected to analyses and then each model was compared using ANOVA to support the decision for the most appropriate model. Statistical comparisons are helpful, but simplicity of models should take precedence when there appears to be little information gained from using a more complex model (de Ayala, 2009). The Rasch model with unconstrained discrimination showed a slight improvement over the Rasch model with constrained discrimination for the first test, $F(1)$=3.09, p = .08, and there was no statistical improvement with the 2-PL model, $F(7)$= 10, p = .19. For the second test, the Rasch model with unconstrained discrimination was an improvement over the Rasch model with constrained discrimination, $F(1)$=4.62, p = .03, and the 2-PL model was statistically better than the unconstrained Rasch model, $F(7)$=15.92, p =.03. The first test fit the constrained Rasch model fit best whereas the 2-PL model was selected for the second test. Tables 1 and 2 provide the item information for each instrument.

Table 1. Item information for first test

Test Parameters	Item							
	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
Difficulty	-0.49	2.82	2.47	1.74	0.99	1.65	1.69	1.53

Std. Error	0.17	0.39	0.34	0.25	0.19	0.24	0.24	0.23
Discrimination	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24
Std. Error	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14

Table 2. Item information for second test

Test Parameters	Item							
	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
Difficulty	-0.38	4.07	3.32	0.99	0.75	1.01	1.31	0.68
Std. Error	0.38	2.13	1.38	0.17	0.19	0.27	0.34	0.13
Discrimination	0.71	1.03	0.79	2.31	1.44	1.1	1.05	2.8
Std. Error	0.25	0.67	0.38	0.71	0.38	0.31	0.31	1.02

These results suggest that overall, the first test had fairly difficult items and it discriminated between participants well. The second test also had a number of difficult items and each item tended to discriminate fairly well. Cronbach's alpha for the first and second test was 0.60 and 0.62 respectively. Approximately 75% of the participants earned two or fewer total points on the first measure and 60% of the participants earned two or fewer points on the second measure. The minimum correlation coefficient to link scores across tests measuring the same latent trait was .60 (Ary et al., 2009). Pearson's correlation indicated the alternate forms reliability was 0.60, sufficient for exploratory studies with new instruments (Ary et al., 2009).

Discussion

These items were reasonably difficult and discriminated fairly well among participants. Most items' difficulty ranged from 0.69 to 1.74, which is within De Ayala's (2009) suggested range from negative one to positive two. This range for the problem-solving measures indicates that a student with above-average ability had a good probability of answering the item correctly. Items with a discrimination parameter value from 0.5 to 2.5 are satisfactory items (De Ayala, 2009); every item was within this range. The lack of negatively discriminating items suggests that students with higher abilities were able to answer more difficult tasks.

These results confirm that sixth-grade students experiencing typical mathematics instruction tend not to perform well on problem-solving tasks. Fifth-grade students' mean score, M=1.6 (Verschaffel et al., 1999) was similar to these sixth-grade students score on the first test. The first test is not necessarily harder than the second test, even though participants' average score was higher. For instance, items one, two, and three are easier items than items four through eight.

Results from this study indicate directions for future research using these instruments. First, items two and three on the measures should not be used in their present form because the item difficulties were exceedingly high. De Ayala (2009) suggests that item difficulties greater than two may not function well with a normal sample thus some items should be revised and reexamined. Second, students tended not to use pictorial strategies, which would have been efficient and effective. Participants might have fared better had they considered pictorial approaches, which would have elucidated the solution to questions two and three rather quickly. Research investigating the degree to which non-symbolic strategies are a part of instruction is necessary. Future studies using these instruments should continue to refine the tasks while maintaining their open, realistic, and complex nature.

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Appendix A

1) Ruth is planning to serve ice cream sundaes to guests at her birthday party. She purchased 3 flavors of ice cream: vanilla, chocolate, and strawberry, 2 different sauces: chocolate and caramel, and 4 different toppings: bananas, nuts, sprinkles, and whipped cream. How many different types of sundaes can be made if every guest selects only one ice cream flavor, one type of sauce, and one topping?

2) Jerome needs 1 gallon of paint in order to paint a bedroom ceiling that is shaped like a square and measures 12 feet on each side. How many gallons of paint would he need to paint a living room ceiling that is shaped like a square and each side measures 24 feet?

3) Bill is making a gate for a wooden fence to keep his dogs in his yard. He bought four boards of wood from the home improvement store. Each board measures 10 feet in length. He needs 3 foot 6 inch pieces of wood to build the gate. How many pieces can Bill make from his four boards?

4) A youth group and their adult chaperones want to visit a water park. The admission fees for this water park are listed below:

Children: $6.00 Adults: $10.50

The total cost for all 17 people in the group to enter the park is $129.00. How many children were in this group?

5) A group of 150 tourists were waiting for a shuttle to take them from a parking lot to a theme park's entrance. The only way they could reach the park's entrance was by taking this shuttle. The shuttle can carry 18 tourists at a time. After one hour, everyone in the group of 150 tourists reached the theme park's entrance. What is the fewest number of times that the shuttle picked tourists up from the parking lot?

6) Aunt Marie purchased 80 Silly Bandz for her two nephews Elijah and Jordan. She gave Elijah 10 more Silly Bandz than Jordan. How many Silly Bandz did Elijah and Jordan each receive?

7) A family is planning a camping trip to a national park and receives the following information about the costs per day:

Camping Fee	
Children 12 years and younger	$3.00 per day
All others	$7.00 per day
Parking for trailer	$9.00 per day
Use of common areas	$1.50 per person per day

The family will camp for 10 days and need to park their trailer each day. The family consists of 4 people including a father, mother, 8 year-old child, and a 14 year-old child. Each person will need to use the common areas on a daily basis. How much will they pay for their camping trip?

8) Maria wanted a bicycle so she started saving all of her money. For every $6.00 that Maria saved, her mother gave her $2.00. Maria had $56.00 after three months. How much money did Maria's mother give her?

Appendix B

1) Students at Sandhill Elementary School purchase their lunches from the cafeteria. There are 3 choices for a main dish: hamburger, slice of pizza, or a turkey sandwich, 4 different fruit options: apple, banana, orange, or peach, and 2 drink options: milk or juice. How many different types of lunches can be made if every student selects only one main dish, one piece of fruit, and one drink?

2) It takes Jeff 1 hour to mow a lawn that is shaped like a square and is 200 feet on each side. How many hours would it take him to mow a lawn that is shaped like a square if each side measures 400 feet?

3) Mr. Lee wants to make jump ropes for his students to use on the playground. He purchases four packages of rope from the home improvement store. Each package contains one piece of rope that measures 25 feet. Each jump rope needs to measure 8 feet 6 inches. He can cut the rope but cannot join pieces together. How many jump ropes can Mr. Lee make?

4) A youth group and their chaperones want to visit the Butterfly Rainforest exhibit at the museum. The admission fees for the Butterfly Rainforest exhibit are listed below:

Children: $6.00 Adults: $10.50

The total cost for all 17 people in the group to enter the museum is $129.00. How many adults were in this group?

5) A group of 150 people were waiting for a glass bottom boat to take them on a trip through a nature preserve. The boat can carry 18 people on each trip. After several hours, everyone in the group of 150 people had gone through the nature preserve. What is the fewest number of trips made by the boat?

6) Natasha and Marianne went to a theme park. Together, they spent $80.00. Natasha spent $10.00 more than Marianne. How much money did Natasha and Marianne each spend?

7) A family is planning to leave their pets at Animal Day Care while they are on vacation. They receive the following information about costs per day:

Kennel costs	
Dog	$11.00 per day
Cat	$9.00 per day
Food	$1.50 per animal per day
Walk	$3.00 per dog per day

The family will need to leave their pets at Animal Day Care for 10 days. They have 1 dog and 3 cats. All of the animals need to receive food on a daily basis and the dog must be walked each day. How much will the family pay for their pets' stay at Animal Day Care?

8) Janice went shopping at the grocery store and saw the following special offer. "If you buy 6 oranges, you get 2 free." She purchased her groceries and left the grocery store with 56 oranges. How many oranges did she get for free?

USING AN ASSESSMENT FOR LEARNING FRAMEWORK TO PERFORM A TEXTUAL ANALYSIS

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This report describes data from a pilot study conducting a textual analysis of a recently published Algebra 1 curriculum using a conceptual framework developed from Assessment for Learning (AfL) principles. AfL uses assessment results to change instruction in order to increase student learning. The study sought to explore how the first unit of the chosen curriculum conceived of AfL. Results of this analysis indicated that three categories of AfL were found in the curriculum (Goals, Teaching Planning, and Evidence of Learning) while a fourth category was nonexistent (Constructive Feedback).

Introduction

The literature on Assessment for Learning (AfL) and Formative Assessment (FA) is expansive. In the past twenty-five years, multiple reviews of literature have taken place (Black & Wiliam, 1998a; Crooks, 1988; Natriello, 1987; Sadler, 1989; Wiliam, 2007) as well as critiques of this research base (Dunn & Mulvenon, 2009). The field of mathematics education in particular has accepted the results of this AfL research, as can be seen in the Assessment Standards in the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000) as well as Wiliam's (2007) review in a recent mathematics education research handbook.

In his review of the AfL literature, Wiliam (2007) expands on the research behind five AfL strategies also found elsewhere (Leahy, Lyon, Thompson, & Wiliam, 2005; Wiliam & Thompson, 2007):

1) "Clarifying and sharing learning intentions and criteria for success"
2) "Engineering effective classroom discussions, questions, and learning tasks that elicit evidence of learning"
3) "Providing feedback that moves learning forward"
4) "Activating students as instructional resources for one another"
5) "Activating students as the owners of their own learning" (Wiliam, 2007, p. 1054).

Wiliam and Thompson (2007) connect these five research-based Afl strategies to three questions derived from Ramaprasad (1983): (1) Where is the learner right now? (2) Where is the learner going? and (3) How does the learner get there? This research has been used to help shape professional development (Black, Harrison, Lee, Marshall, & Wiliam, 2003) to elicit more evidence of the positive effects of AfL already reported (Black & Wiliam, 1998b). Research in AfL has not yet included studies examining written mathematics curriculum, nor the connections any AfL in the written curriculum might have with the enacted curriculum.

The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) incorporates AfL into one of its six principles. The assessment principle suggests that "assessment should enhance student learning" (p. 22) and "assessment is a valuable tool for making instructional decisions" (p. 23). The enhancement of learning espoused by this document closely follows what Black and Wiliam (1998a) reviewed. In terms of decision-making, this assessment principle emulates the three AfL questions mentioned already, especially the final one: how the learner reaches his or her learning goal. As the major standards document distributed nationally, the NCTM shows its support of AfL in the mathematics curriculum across school mathematics.

Harlen and Winter (2004) maintain that there is a need to assess problem solving, communication, and reasoning as part of the mathematics curriculum. The authors also contend that these strands are not readily assessed in normal testing situations. Harlen and Winter promote FA and AfL as being able to help assess aspects of the curriculum that regular testing are unable to assess. As one criterion for effective formative assessment, Shepard (2007) asserts that the formative assessments should be embedded within the written, enacted, and assessed curriculum. This pilot study sought to explore the depth of AfL in the written curriculum, specifically asking the following: What potential assessment for learning design features are contained in one unit of a standards-based curriculum?

Perspectives

The following analytic framework (Figure 1), based in part on the Assessment Reform Group's (2002) 10 Principles of AfL and in part on the previous research already mentioned, was developed by a colleague and I (Bentz & Engelman, 2010). Figure 1 represents half of the full framework. This framework acted as the tool for analyzing the design features of the curriculum studied in this project. In the framework, "curriculum resources" refer to all curriculum resources available to the teacher and the student. Evidence of Learning seeks to answer AfL question 1, Goals help answer AfL question 2, and Constructive Feedback attempts to answer AfL question 3. Teacher Planning is required for all three questions to be answered.

AfL Guiding Principle	Exemplary Practices					
Constructive Feedback	CF 1: Curriculum materials suggests teachers give students descriptive feedback regarding their learning	CF 2: Curriculum materials suggest that feedback should pinpoint student's strengths and how to develop them further	CF 3: Curriculum materials suggest that feedback recognize learner's efforts	CF 4: Curriculum materials suggest that feedback pinpoints student's weaknesses and is clear about how to improve on work	CF 5: Curriculum materials suggest that feedback takes into account what students can effectively manage at one time	CF 6: Curriculum materials suggest that teachers provide descriptive feedback when the students need/ request it
Goals	G 1: Curriculum materials ensure pupils understand the goals of learning	G 2: Curriculum materials suggest that students partner with teachers to set goals	G 3: Curriculum materials provide students with a clear and understandable vision of the learning goals			
Teacher Planning	TP 1: Curriculum materials provides opportunities to obtain and use information about students' progress towards learning goals	TP 2: Curriculum materials ensure teacher plans can be flexible to respond to students' initial and emerging ideas	TP 3: Curriculum materials enhance all learners' opportunities to learn	TP 4: Curriculum materials offer opportunities for all learners to achieve their best		
Evidence of Learning	EL 1: Curriculum materials provide opportunities for teachers formatively interpret and judge for evidence of student learning and improvement	EL 2: Assessments from the curriculum materials provide students with an opportunity to demonstrate acquisition of knowledge through tasks and question	EL 3: Assessments from the curriculum materials provide students with an opportunity to demonstrate acquisition of skills through tasks and questions	EL 4: Assessments from the curriculum materials provide students with an opportunity to demonstrate acquisition of understanding through tasks and questions		

Figure 1: AfL Analytic Framework

Methods

Research Design

To answer the research question, an exploratory study was conducted. Maxwell clarifies that the goal of an exploratory study is to "generate an understanding of the concepts and theories held by the people you are studying" (p. 79). The pilot study in this report seeks to understand a curriculum rather than people or behavior. The main goal is to describe phenomena and document the characteristics of the phenomena, in this case, phenomena in a mathematics curriculum (Johnson, 2001).

In order to answer the pilot study research question, the case study (Yin, 1998) is modified to investigate a single unit of one reform oriented mathematics curriculum. One unit is defined as the case itself. Since case studies require multiple sources of evidence in order to triangulate (Yin, 1998), three different sources of data were chosen from the same curriculum. My goal in applying the AfL analytic framework is to both discover how easily it can be applied to mathematics curricula while at the same time discover the extent to which that curriculum pre-packages assessment for learning strategies.

Sources of Data

The curriculum chosen was the National Science Foundation funded Center for Mathematics Education (CME) Algebra 1 curriculum (Education Development Center, 2009). The author team incorporated Cuoco, Goldenberg, and Mark's (1996) habits of mind for mathematics into a four-course high school mathematics curriculum. These habits of mind closely follow the ideas put forth by the 2000 NCTM Standards. Because these standards include language supporting AfL, this study endeavored to determine the extent to which the CME curriculum contained AfL strategies.

Within the CME Algebra 1 curriculum, three main data sources were used. The first data source analyzed was the Implementing and Teaching Guide (IG). The entire IG was coded first to get a sense of the intent of the authors – the principles of design the curriculum was built around – as well as certain structural components. By coding the IG first, specific aspects of the analytic framework began to be formulated for their use with certain elements of the student textbook and teacher resource materials. This helped code later sources more accurately.

The second data source was the Student Edition (SE) of the textbook. This source was analyzed second to determine what AfL the students would see independent from anything found in the annotations of the Teacher's Edition. The final data source analyzed was the Teacher's Edition (TE). When analyzing this source, information that exactly duplicated information from the student text was not coded. Only the annotations from the peripheral materials in the TE were analyzed.

The chosen unit analyzed in CME Algebra 1 was Chapter 1 (Arithmetic to Algebra). This curriculum has just eight chapters, each about fifteen lessons long. One chapter of this curriculum is considered a unit. This chapter was chosen to see how the first chapter of the first course in the CME high school curriculum began to use AfL strategies from the outset of the entire CME program.

Potential for Assessment for Learning

Throughout the coding process, any elements of the curriculum coded as AfL did not necessarily make those elements AfL in actual classrooms. Rather, these elements have the *potential* to become AfL depending on how they are enacted in the classroom. For instance, the *Check Your Understanding* element is coded as EL 1, EL 4, and TP 1. This means that any *Check Your Understanding* section in the curriculum offers the teacher three AfL strategies:

(1) an opportunity to formatively interpret and judge for evidence of student learning and improvement (EL 1)

(2) an opportunity for students to demonstrate acquisition of understanding (EL 4), and

(3) opportunities for teachers to obtain and use information about students' progress towards learning goals (TP 1).

In order for the *Check Your Understanding* section to be considered AfL, the results of student work in these sections must be used to make instructional decisions. Thus, any code applied to the CME curriculum in this project can be seen as an element of the curriculum with the *potential* to become enacted AfL

Data Analysis

Coding the three data sources took place in two stages. The first stage coded the sources in the order mentioned in a previous section. After the preliminary coding, an initial analysis of this data was completed, triangulating among the three data sources. As the coding progressed from one source to another, more was learned about various elements of the curriculum. Specific design elements found throughout the curriculum received new codes over time. Through triangulation, the codes for a given element were combined from the three data sources, thus developing a more thorough coding scheme. With this new coding scheme, all three data sources were coded a second time, obtaining coding not found during the first round of coding. This ensured that items coded early on in the coding process had the updated code for that element.

The *Check Your Understanding* sections occur throughout the lessons in Chapter 1, and each of these sections was considered a single element of the curriculum. The IG states that "students should do Check Your Understanding exercises in class. By giving students some class time to work on these exercises, you can assess their preparation for individual work on the On Your Own exercises that follow" (p. 84). The *Check Your Understanding* exercises are intended to discover where students are in their learning for the purpose of formatively interpreting whether the students are ready to move on to the *On Your Own* exercises. From the framework (Figure 1), this curriculum element receives the codes EL 1 and TP 1.

During this process of triangulation, the SE helps show a different aspect of the *Check Your Understanding* exercises. This exercises focuses on eliciting evidence of student understanding:

"1. Derman opens a checking account on June 14. He records his check withdrawals and deposits, but he does not calculate his balance for a week. On June 20, the bank tells Derman that he has overdrawn his account.

a. When did Derman first overdraw his account?
b. How much money does he need to deposit to reach a positive balance?
c. How could Derman have avoided overdrawing his account?" (EDC, Student Edition, p. 14).

These three questions delve into the understanding students have about the mathematics. Thus, this section received the code EL 4.

The TE adds to what the SE brings to *Check Your Understanding* by offering multiple ways the teacher can elicit student understanding of the material: "**Exercise 1** Some students may never have used a checkbook ledger before. Explain what each column represents [in a checkbook ledger]. Explain the difference between a deposit and a check (withdrawal). You may wish to provide students with copies of Blackline Master 1.2 to complete" (EDC, Teacher Edition, p. 14). This peripheral segment points the teacher to possible *misunderstandings* students may have. This confirms the code EL 4 given when looking at the SE.

Therefore, the *Check Your Understanding* sections have the *potential* to be used formatively to change the direction of the lesson should students not be ready to move on in their learning (EL 1, PL 1). As this item elicits evidence of understanding, it had to be coded with EL 4. This process of triangulating using all three data sources to determine which codes to apply was duplicated for each element in the curriculum that was repeated throughout the chapter.

Results

Teacher Planning

Teacher planning yielded more codes that any other piece of the analytic framework. The Assessment Reform Group's 10 Principles of AfL (2002) focus so much on the idea that AfL must be pre-planned – they claim that AfL cannot just happen in the course of a lesson. Thus, it is crucial for a curriculum to present a teacher with AfL strategies. The CME curriculum offers numerous instances where opportunities are given for the teacher to obtain and use information about students' progress towards learning goals (TP 1) – see Figure 2. This necessarily requires that G 1 and G 3 be appropriately accomplished. In the SE, TP 1 occurs nine times through *In Class Experiments*, thirteen times through *Check Your Understanding*, and sixteen times through *Additional Practice* web codes.

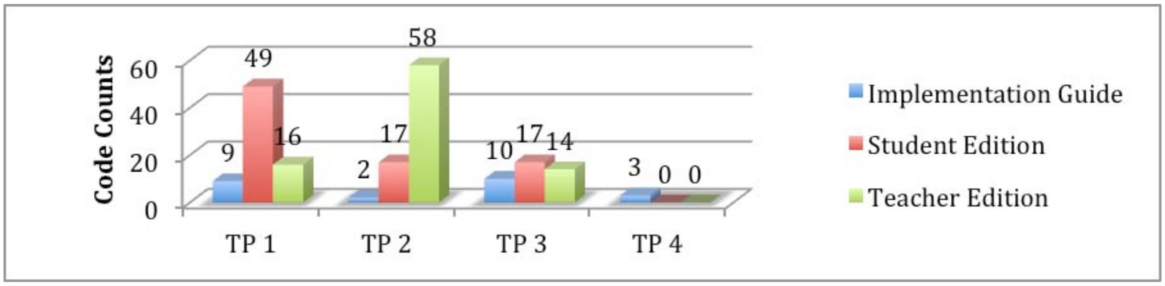

Figure 2: Teacher Planning Code Counts

Another aspect of AfL that is so important is the idea that instruction should change based on what students know and understand during a lesson (TP 2). The TE gives 58 such opportunities in the first chapter alone. Twelve occurrences of TP 2 came from *Additional Practice* sheets (the teacher could assign these based on the needs of their students), while thirty-one instances came from pacing suggestions. Pacing suggestions came in the form of exercises in the text being assigned as core problems (for all students to complete), extra practice worksheets (for those who need more practice), and extension problems (for those who understand the material thus far and can move on). As this chapter contains fifteen lessons total, a teacher averages two instances in class each day where they have the potential to make pacing and instructional changes.

One of the strengths in this curriculum is the intention that there is some flexibility with which the teacher can implement the curriculum. This flexibility enhances the learning for all students (TP 3) and encourages students to achieve their best (TP 4). Many of these instances come from the IG, which mentioned that it was the authors' goal to ensure that all students of CME have the opportunity to succeed at mathematics.

Goals

This category of the framework addresses the need for AfL to direct students toward the goals of learning. Of the three codes within this category, one of the three codes was never found (G 2). There were no instances where the curriculum suggested that students should partner with the teachers to set goals. The remaining two codes were used with the potential for AfL in mind – see Figure 3 for a summary of these code counts. Goals in the curriculum were similar to the following: "You will learn how to identify, describe, and justify patterns in arithmetic and in multiplication tables…You will develop these habits and skills: Gain a sense of how a mathematician works" (EDC, Student Edition, P. 8). These goal statements were in the curriculum using student friendly language (G3). They also have the potential to be G1 (students understand the goals of learning) if the teacher actually has the students read through them as well as keep the goals in mind throughout the lesson.

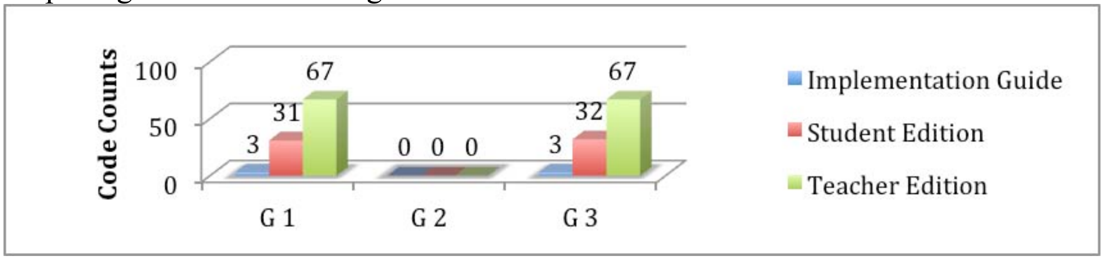

Figure 3: Goal Code Counts

Evidence of Learning

In order for the first AfL question to be answered (Wiliam & Thompson, 2007), evidence of learning must be elicited. A vast majority of these codes come from the SE – see Figure 4. The first code is another of the major codes that has the potential for being AfL. In the first chapter, teachers have the opportunity to formatively interpret and judge for evidence of student learning and improvement 72 times combined in the three data sources. The key word in this code is *formatively*, where the teacher has the chance to change the direction of student learning based on current levels of skills, knowledge, and understanding. This means that the interpretations must take place in order to change instruction. This includes any assessment item completed at home as well as those completed in class. In the first chapter, the curriculum materials include 51 opportunities to elicit knowledge, 48 opportunities to elicit skills, and 92 opportunities to elicit understanding.

Six categories of student exercises contain most of the codes for Evidence of Learning. *Check Your Understanding* occurred thirteen times in the chapter, and contained EL 1 and EL 4. *For You To Do* occurred twenty times, coded with EL 2 and EL 3. *Maintain Your Skills* occurred sixteen times, as did *On Your Own, Take It Further, and Additional Practice*. Codes used for these four sections were EL 3; EL 2 and EL 4; EL 4; and EL 1, respectively.

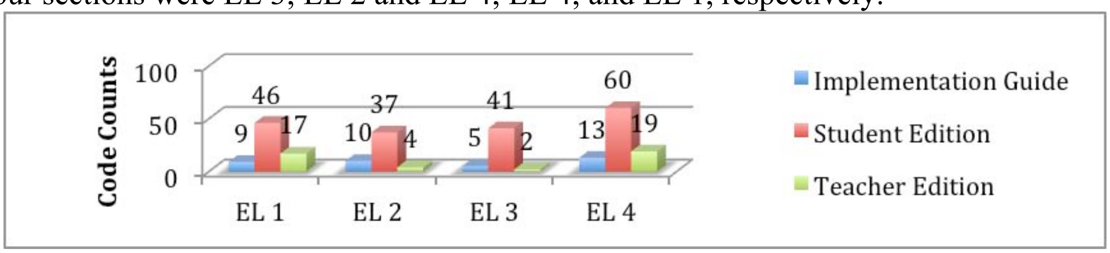

Figure 4: Evidence of Learning Code Counts

Constructive Feedback

Feedback is the backbone of the AfL literature (Black & Wiliam, 1998b; Wiliam & Thompson, 2007). The IG, SE (Chapter 1), and TE (Chapter 1) did not contain a single instance of constructive feedback. Constructive feedback is required for AfL to be completed (Sadler, 1989). There were numerous instances where the TE suggested ways for teachers to work with student answers. However, since these instances did not both point students to where they should go and how to get there, these could not be classified as constructive feedback.

Discussion

Overall, the first chapter of CME Algebra 1 presents learning goals in student-friendly language. The curriculum has great potential for ensuring students understand these goals. If a teacher endeavors to enact AfL, this curriculum will help begin that process by clearly outlining the learning goals prior to learning. In order for this potential to be realized, the teacher must do more than simply place these goals in front of the students. As the analytic framework suggests, teacher plans should ensure that information is elicited about students' progress toward these goals (TP 1). These goal statements also help to answer the second AfL question: Where are students going? (Wiliam & Thompson, 2007). CME Algebra 1 accomplishes this.

The numerous instances where students have the opportunity to show their knowledge, skills, and understanding help the teacher to have the chance to formatively interpret this evidence for further instruction. Moreover, the teacher is directed to use this information to make instructional decisions. This process helps answer the first AfL question: Where are students at in their learning? (Wiliam & Thompson, 2007). The CME curriculum offers the teacher opportunities each day of class to make instructional decisions based on where students are at in their learning. It is not enough that teachers simply find out where students are at in their learning – the teacher must change their instruction to help the students move on.

The final AfL question, directly related to feedback, is not addressed in the CME Algebra 1. This would mostly come through feedback. The fact that this major piece of AfL is missing could lead one to believe that the CME Algebra 1 curriculum does not accomplish this strategy of AfL. Bringing in the concept of potential for AfL, though, helps temper this major gap. In other words, the CME Algebra 1 curriculum has the potential to answer the first two AfL questions all by itself. The teacher still must enact these pieces of the curriculum. If these are enacted, then the teacher, outside of what the curriculum brings to the classroom, has the potential to put constructive feedback into the classroom on their own to complete the three AfL questions.

Because CME Algebra 1 helps answer the first two AfL questions, it does have the potential to provide the conditions under which AfL can be enacted. The key to making this so is how the teacher decides to use the curriculum. The teacher must decide whether or not they will take the time to not only go through the goals of learning with the students, but also point back to these learning goals throughout the learning process, especially as they elicit evidence of student learning. The lack of information the curriculum brings to help answer the final AfL question leads me to conclude that this curriculum is not a complete, pre-packaged, AfL curriculum but it does have great potential to enact AfL in the classroom.

Endnotes

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ASSESSING SECOND GRADERS' MATHEMATICAL COMMUNICATION ON A STANDARDIZED TEST

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The processes of communication and representation are essential mathematical ways of thinking, and assessments can provide opportunities to evaluate the evidence of their communication and representation of their problem solving. We examined the pretest and posttest responses to an open-ended story problem of 83 second-grade students from four classrooms. The implemented coding system differentiated the responses not only by the correctness of the response, but also by how explicitly the strategy or solution process had been communicated. We found that not only did students' responses show evidence of greater communication of strategy and process from pre- to posttest, but that answer accuracy was associated with well-communicated responses. Our results provide the grounds for emphasis on process communication in early elementary school and for future investigation into effective instruction and assessment.

Young Children's Communication of Mathematical Thinking

Variations of the idea that "the answer is not enough" pervade current mathematics teaching at all levels (e.g., Sterenberg, 2004). Intertwined with mathematics students' ability to use strategies to solve problems is their ability to communicate their mathematical thinking effectively – to show more than that final answer. According to the National Council of Teachers of Mathematics (2000), communication offers "a way for students to articulate, clarify, organize, and consolidate their thinking" (p. 128). For young mathematics students, in particular, to use strategies effectively, especially for more advanced arithmetic such as multi-digit subtraction, they must be able to "organize subgoals into a coherent process without getting lost within particular subgoals" (Fuson et al., 1997, p. 151).

To support learners' mathematical communication, teachers' instruction should emphasize documenting their solution processes. Whitenack, Knipping, Novinger, and Underwood (2001) studied a second-grade class during a unit on multi-digit addition and subtraction. Throughout the unit, the teacher emphasized students' organization of recording the regrouping process, applying strategies across multiple contexts, and sharing their processes. This instructional sequence not only allowed students to deepen their understanding of multi-digit operations, but also fostered organized problem solving and communication for their benefit and for the benefit of their peers and the teacher.

Assessment of Children's Mathematical Understanding and Communication

In spite of the emphases on problem solving and communication in instruction, assessments have traditionally focused largely on students' ability to reproduce facts and computational procedures (Pegg, 2002). Problem solving and communication are essential mathematical skills, and thus, assessments should focus not only on students' accuracy, but also on the *quality* of students' thinking (Pegg, 2002; Cai, Jakabcsin, & Lane, 1996; NCTM, 2003).

In the early elementary grades, when students are just learning to communicate in written form, assessments of students' problem solving and communication skills can be challenging to design. On the other hand, written assessments are often the most efficient to administer, compared to other methods that might yield problem-solving insights such as observation or interviews with students. Thus, well-designed written assessments hold the potential to assess students' problem solving and be efficient to administer, but these assessments should not only include ways to incorporate students' communication of their solution process, but also ways for evaluators to make sense of these responses to maximize the information they yield. Open-ended tasks, scored with a holistic rubric, may be effective to assess the quality of students' thinking (Cai, Lane, & Jakabcsin, 1996).

In the present study, we examined the quality of second-graders' responses on an open-ended mathematics task that dealt with multi-digit subtraction concepts. We were specifically interested in examining how students' communication and representation of their thinking changed over the course of an academic year. We were also interested in examining possible connections between quality of communication and accuracy.

Method

Data Source

In order to explore these questions, we examined pre- and post-test responses to an Ohio Department of Education second-grade assessment, designed to provide diagnostic information about students' ability to meet the second-grade state content standards. We chose to examine students' responses to one open-ended, constructed response item which read: *A girl has a dollar. She wants to buy a new pencil for 76 cents. How much money will she have left after she buys a pencil? Show how you found your answer.* This item had several key features: it addressed multi-digit subtraction concepts, important grade-level content, its story problem format required students to identify and apply an appropriate operation, and it involved money, which poses both conceptual and representational challenges for students. Finally, we felt that the open-ended format of the question would yield a rich and diverse range of responses.

Participants

We examined the responses of 83 second-grade students from four classes in a suburban primary school for whom both pretest (October) and posttest (May) scores were available. The students came from four classrooms. A mathematics coach worked with two of the four classrooms for several weeks during the year.

Coding

Coding Scheme. Our interests were centered on the quality of students' communication and representation of their mathematical thinking. The state-supplied three-point rubric for this item did not differentiate responses based on the quality of students' communication, so we decided to design a new rubric with this specific purpose (see Appendix). Our rubric categorized responses into five major levels of quality of communication, including responses: 1) that showed no work, 2) that showed some work but did not provide evidence of strategy use, 3) that provided evidence of an inappropriate strategy, 4) that provided evidence of an appropriate but insufficiently communicated strategy, and 5) that sufficiently communicated an appropriate strategy. Within each of these five levels, the score for a correct answer was given one point more than the score for an incorrect answer. We reserved the highest score (10) for responses that included an especially clear communication of a student's thinking either by demonstrating how their work was connected to the givens in the problem or by using multiple representations to communicate their thinking. Two coders achieved 90.1% simple agreement, periodically performing coding checks and ultimately reaching agreement on all coding.

Analysis. Upon coding completion, we calculated the frequency of responses that received each value on our rubric. Within each of the five main categories of communication quality, we examined scores for differences between correct and incorrect responses, with regard to communication quality and accuracy.

Results

From fall to spring, the mean scores increased not only in accuracy but also in the students' mathematical communication, from an overall mean score of 3.95 (of 10) in autumn to 5.77 in spring. Examining the responses more closely, we also observed trends in the way that students' mathematical communication and strategy use was correlated with their accuracy. We used the frequency of each response type to determine the percentage of correct and incorrect responses that were given within each level of communication. For example, of the 20 responses in which no work was shown, only 2 (10%) were correct. Similarly, we observed low rates of correct responses that used no identifiable strategy (35% of 26 responses) or an incorrect strategy (19% of 30 responses). Unsurprisingly, students who did not record appropriate strategies were less likely to provide correct answers than those students who did.

What was more notable, though, was the trend among responses that *did* indicate the use of an appropriate strategy. Our scoring rubric included two categories of responses that used an appropriate strategy but differentiated those that were sufficiently communicated (either an 8 or 9) from those that were not (either a 6 or 7). The difference between insufficient and sufficient communication, even when an appropriate strategy is used, does matter in terms of accuracy: while 92% (23 of 25) of the well-communicated responses yielded an accurate answer (see Figure 1), only 54% (23 of 50) of the insufficiently communicated responses yielded an accurate answer. This association between correctness and explication of mathematical communication provides a rationale for encouraging the development of communication in problem solving.

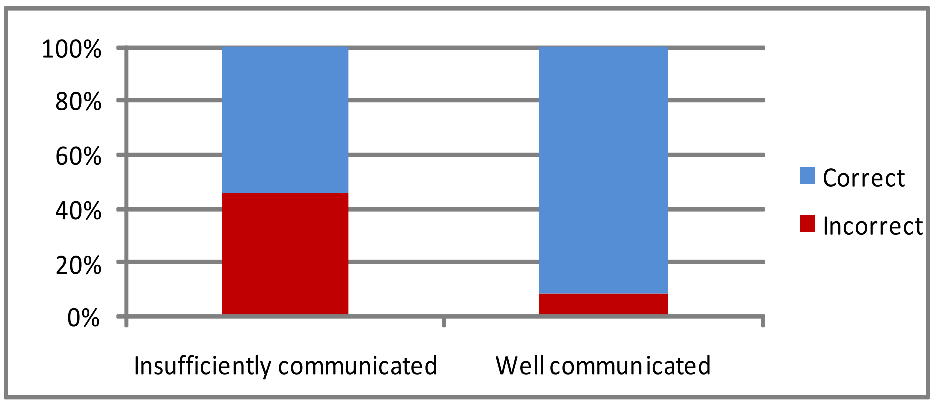

Figure 1: Proportion of Correct and Incorrect Responses for Two Types of Appropriate Responses

Although we were not able to conduct classroom observations, the student data came from four classrooms in the same school, and we examined the student responses by classroom for trends in responses. We found that the students in the coached classrooms, classrooms 1 (mean improvement of 2.76 points) and 4 (2.57 points) improved more on average than students in the uncoached classrooms, classrooms 2 (1.22 points) and 3 (0.63 points). When we looked at each class's spring data, we found that certain forms of communication and strategy use were favored by certain classes: in class 1, many responses (43%) included drawn coins; in class 2, a plurality of the responses (30%) used tally marks, used by none of the students in the other classes; Class 3 had the highest percentage of responses (74%) which included subtraction sentences. Within these, half had subtraction sentences shown both horizontally and vertically, and 43% showed the partial differences (e.g. 100 – 70 = 30; 30 – 6 = 24; 100 - 76 = 24), used by respondents in no other class. In class 4, a majority of responses (57%) involved drawing base ten blocks, while only 3 responses from other classes did.

The mathematics coaching experienced in two of the classrooms focused on engaging students in problem solving and developing their understanding of the five NCTM mathematical processes, including communication and reasoning and proof. We note that in Class 4, one of the coached classrooms, predominant responses involved multiple representations, including base-ten iconic representations, and in the other coached classroom, students favored another iconic representation, the coins. In the uncoached classrooms, students favored either basic iconic representations, tally marks to count (Class 2), or numeric equations. The relation between teachers' instructional and expectations for students' responses within class and students' methods of communication on standardized assessments merits further investigation, but these by-classroom trends support the idea that specific expectations for students' mathematical communication can be taught and that students respond to instruction about how to communicate their thinking.

Discussion

In our study, we sought to examine students' mathematical communication and strategy use on an open-ended story problem task on a standardized assessment, and we documented the relation between students' mathematical communication and answer accuracy on the mathematics assessment. Traditionally, mathematics educators and assessments may have emphasized accuracy more than communication, but our results suggest that good communication may support accuracy.

In examining the differences between insufficiently communicated responses and well-communicated responses, we found that students' organization and labeling of their work was associated with both the clarity of their communication and their accuracy. Teachers can help students learn to communicate their mathematical thinking clearly and support students' accuracy on mathematical tasks by encouraging them to record their thinking in an organized and well-labeled manner. In tasks like the one studied, teachers could emphasize the importance of labeling drawings clearly by encouraging students to cross out the coin or base-ten block that they are partitioning and draw an arrow to the group of coins or blocks that they have drawn in its place. More generally, teachers should expect students to "show their work" and should provide ongoing opportunities for students to engage in mathematical communication and by offering feedback on students' communication of their mathematical thinking.

Our study also has implications for those who design mathematics assessments for early grades children. With mathematical communication as an important goal for instruction, designers of standardized mathematics assessments should ensure that students' communication is encouraged, clearly stated as an expectation, and measured. With such provisions built into assessment tasks, test results could provide more specific, and thus, more formative, information about students' performance. While the rubric used for this study is likely more detailed than a rubric a school district would use, it is important that scoring rubrics give some attention to mathematical communication. Assessing students' communication on standardized mathematics assessments not only emphasizes that communication is an essential component of mathematics, but it also gives teachers and district staff evidence of strategies and different representations that students used. Future investigations will involve classroom observations and documentation of instruction and student work to identify the processes that lead to growth in students' mathematical communication.

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Appendix: Scoring Rubric

10 Response is very well articulated; as is evidence by:

- *A clear connection between student work and the information given in the problem. (e.g. indicates that the initial value is $1.00 rather than simply representing the number 100)*
- *The presence of multiple clear representations of the strategy used (e.g. student includes a clearly labeled drawing of base-ten blocks AND a subtraction sentence)*

9 Response is correct and well communicated. An appropriate strategy is used.

8 Response is incorrect, but well communicated. An appropriate strategy is used.

7 Response is correct and there is evidence that an appropriate strategy was used. However, the process of arriving at an answer is not well enough communicated. This may occur as the following:

- *A response including a subtraction sentence does not communicate the solution steps.*
- *A response including a counting strategy is disorganized, and thus requires some inference on the part o the evaluator.*
- *A response including drawings of coins or base-ten blocks does not label trades that were made, requiring some inference on the part of the evaluator.*

6 Response is incorrect, but there is evidence that an appropriate strategy was used. However, the process of arriving at an answer is not well communicated. This may occur in any of the following ways:

- *A response including a subtraction sentence does not communicate the solution steps.*
- *A response including a counting strategy is disorganized, and thus requires some inference on the part o the evaluator.*
- *A response including drawings of coins or base-ten blocks does not label trades that were made, requiring some inference on the part of the evaluator.*

5 Response is correct; there is evidence of a strategy, but it is not appropriate for the problem.

4 Response is incorrect; there is evidence that a strategy was used, but the strategy is not appropriate for the problem or does not seem to be connected to the problem.

3 Response is correct, but no identifiable strategy is used. Some work is shown, such as a representation of coins or tallies, but there is no evidence of the process used to get an answer.

2 Response is incorrect, and no identifiable strategy is used. Some work is shown, such as a representation of coins or tallies, but there is no evidence of the process used to arrive at an answer.

1 Response is correct, but no work is shown.

0 Response is incorrect, and no work is shown.

CALCULUS READINESS: COMPARING STUDENT OUTCOMES FROM TRADITIONAL PRECALCULUS AND AP CALCULUS AB WITH A NOVEL PRECALCULUS PROGRAM

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This comparative study examines three groups of high school students' understanding of the foundational concepts needed for calculus. Students completed one of three courses: (1) traditional precalculus, (2) novel precalculus, or (3) Advanced Placement (AP) Calculus AB. Student scores on the Precalculus Concept Assessment (PCA) and two open-ended tasks which focused on functions and rate of change provided the data. Student work was analyzed for strategies employed and efficiencies with their strategies on the two tasks. The results revealed that students who completed the novel precalculus curriculum gained a deeper understanding of the fundamental concepts for calculus by performing higher than the traditional precalculus students and comparably to the AP Calculus AB students.

Introduction

Precalculus is a mathematics course offered in high schools across the United States with approximately 18.5% of all high school students enrolling during their third or fourth year of high school. Yet a growing student population is required to enroll in precalculus when they enter college because they either did not complete it in high school, but more often students do not score high enough on the individual college mathematics placement exams and therefore are required to enroll in a course they may have completed during high school. Parents, students and educators are asking: (1) What are students not learning in high school precalculus that is causing them to have to retake the course when they enter college and (2) What mathematics content is missing from the high school precalculus curriculum that could help students better prepare to enter college calculus?

Over the past 20 years researchers have focused on college students understanding of function and the different reasoning patterns they need to be successful in higher level mathematics, such as calculus (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson & Oehrtman, 2005; Engelke, Oehrtman, & Carlson, 2005, October; Ferrini-Mundy & Gaudard, 1992; Ferrini-Mundy & Graham, 1991, 1994; Markovits, Eylon, & Bruckheimer, 1988; David Tall, 1992; David Tall, 1997; Thompson, 1994a, 1994b). However, less research has been conducted at the high school level to understand what secondary students who are enrolled in precalculus understand or do not understand about function and if the reasoning patterns and frameworks used for undergraduate students can or should be applied to high school precalculus students. Therefore, in this study we seek to answer the following two research questions: (1) How do high school precalculus students who used either a traditional precalculus curriculum or a research based conceptually–oriented curriculum compare on the foundational concepts needed for calculus and their efficiency and use of mathematical strategies? (2) How do high school precalculus students who completed a conceptually-oriented curriculum compare with AP Calculus AB students on the foundational concepts needed for calculus and their efficiency and use of mathematical strategies?

Theoretical Framework

The concept of function is an essential knowledge students need to be successful in higher level mathematics. For students to understand the concept of function they must first obtain a process view of function (Breidenbach et al., 1992), meaning they understand that they are mapping a set of input values to a set of output values and it is a continuum. Second, students must develop covariational reasoning (Carlson, 1998), described as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354).

Typically precalculus teachers and the curriculum they use focus student attention on learning the algebraic manipulations and procedures rather than the conceptual understanding of function that provides foundational knowledge needed to build on for studying higher level mathematics. Breidenback et al. (1992) and Dubinsky and Harel (1992) described two views of functions students have: *action* or *process*. An action view of function is described as "the ability to plug numbers into an algebraic expression and calculate" (Dubinsky & Harel, 1992, p. 85). Students tend to think about the function in a procedural way as they input a value into a specific function to calculate the correct output doing "plug and chug" without any conceptual understanding of function which may eventually hinder their understanding of calculus concepts such as limit, derivative, and integrals.

Carlson and Oehrtman (2005) extended Breidenbach et al. (1992) and Dubinsky and Harel's (1992) research on the action and process views students have of function arguing that student's ability to answer function focused tasks was related to two types of reasoning.

> First, … students must develop an understanding of functions as general processes that accept input and produce output. Second, they must be able to attend to both the changing value of output and rate of its change as the independent variable is varied through an interval in the domain. (Oehrtman, Carlson, & Thompson, 2008, p. 5)

For example, students who hold an action view of function have multiple misconceptions about piecewise functions being several functions and do not reason over an entire interval. Students tend to focus on specific points instead of interpreting the entire function. They also tend to not relate the domain and range to the inputs and outputs of the function. Without an understanding of functions accepting inputs and producing outputs students will struggle to reverse (inverse functions) the process. Most students are able to find the inverse of a function algebraically (switch the x and y and solve for the y variable) or geometrically (reflect the function over the line y = x); however, their answer typically has no meaning (Carlson & Oehrtman, 2005). Students who hold an action view are able to work with functions procedurally, but have little conceptual understanding.

On the other hand, students who hold a process view of function can imagine an entire function and how the input values affected the output values. This is described as covariational reasoning, the ability to vary inputs and outputs at the same time and interpret or understand their influence on the rate of change. Students who demonstrate a process view of inverse functions use a reversal process that defines a mapping of output values to input values. A process view of function is essential for understanding calculus concepts such as the limit, derivative, and integral.

Carlson, Oehrtman, and Engelke (2010) developed a taxonomy of foundational knowledge for beginning calculus representing the key concepts of precalculus students should know and understand prior to entering calculus. The *PCA* taxonomy was used to develop the *Precalculus Concept Assessment (PCA)*, a 25 multiple–choice item exam that assesses rate of change, function, and covariational reasoning that students will build upon in their study of calculus. The *PCA* and PCA taxonomy have gone through multiple refinements as research on student thinking is conducted (Carlson et al., 2010). Although the details of the PCA taxonomy are not part of this paper it is imperative to note that the existence of such a taxonomy is one of the reasons why the *PCA* is a valid tool for assessing students understanding of the foundational concepts of precalculus prior to entering calculus.

The *PCA* was developed over a 15 year time period based on numerous college students' interviews of what they thought and understood as they answered open–ended questions about functions. The distracters on the *PCA* are common misconceptions that college students have about functions (Carlson et al., 2010). More recently, the *PCA* was used by high school precalculus teachers to determine their students' understandings and misconceptions after completing the year long course and entering AP calculus (Teuscher, 2008). These data were influential in helping teachers discover holes in their mathematics curriculum and making changes to help students develop the key precalculus concepts prior to entering calculus.

Methods

In this study we examined high school students' understanding of the foundational concepts of calculus as described by the *PCA* and two open–ended tasks focused on rate of change. We coded students' strategies in solving these tasks and compared their efficiency in using the strategies. In this section we describe the instruments and participants.

Precalculus Concept Assessment

The *PCA* was developed by faculty from the Department of Mathematics at Arizona State University and was designed to reflect core content and common misconceptions students have about functions (Carlson et al., 2010). Carlson et al. (2010) administered the *PCA* to 902 college precalculus students and found the "Cronbach's alpha of 0.73 indicating a high degree of overall coherence" (p. 137). Carlson and colleagues also conducted clinical interviews with more than 150 college students so as to validate the *PCA* items. Each question was tested and validated to guarantee that students who selected a specific distracter (i.e., answer choice) consistently provided similar justifications during interviews (Engelke et al., 2005, October).

Open–ended Tasks

The *Piecewise Functions* (*PF*) task was adapted from a released 2003 AP Calculus free response item by an AP Calculus teacher for precalculus students; it gives students the graph of a piecewise function and has them find rates of change and then write equations for the function. The *Filling the Tank* (*FT*) task was taken from examples developed by Peter Taylor (1992) and it requires students to compare average flow (rate of change) with instantaneous flow as water is flowing in and out of a tank simultaneously. Both tasks build on the foundational concepts of functions and rate of change that students need before calculus and require students to use reasoning and demonstrate their understanding of the meaning of functions; however, one task is based in a contextual setting and the other is not.

Students worked on the two tasks individually during a scheduled class period during the first week of class and the student work was scored for correctness using scoring guides and coded for solution strategies on specific questions for both open–ended tasks. The items coded for solution strategies were *PF* questions 1, 2 and 4 and *FT* questions B and E. The five categories of strategies were: Formula, Graph, Table, Multiple, and None or can't tell. The scoring guides for the open–ended tasks required 22 scoring decisions on each of the *PF* and *FT* tasks and the average reliability of scoring was 95% and 96% for the *PF* and *FT* tasks respectively.

Participants

High school students from Rover High School located in the southwestern region of the United States, who completed either Precalculus or AP Calculus AB during the 2009-2010 school year and enrolled in AP Calculus BC for the 2010–2011 school year were asked to participate in the study. All mathematics teachers at Rover High School are part of a National Science Foundation Math and Science Partnership grant (No. EHR–0412537). The students in this study (N = 36) were taught by one of four different teachers in their previous mathematics course (Precalculus or AP Calculus AB). Four mathematics teachers taught students in precalculus during the 2009–2010 school year. Three of the four teachers used *Precalculus: Mathematics for Calculus* (Stewart, Redlin, & Watson, 2005) and the fourth teacher used a research based conceptually–oriented curriculum *Precalculus: A Pathway to Precalculus* (Carlson & Oehrtman, 2010). All precalculus students took the *PCA* at the end of the 2009–2010 school year. Students who enrolled in AP Calculus BC for the 2010–2011 school year completed the two open–ended tasks, *PF* and *FT* during the first week of class.

Please note that the three groups of students were not selected deliberately; they naturally emerged as a result of the mathematics classes they were enrolled and the choice of curriculum followed by their teachers. As a matter of fact, only one teacher chose to use *Precalculus: A Pathway to Precalculus* and this is how the PP students, who are of major interest in this paper, were selected.

Results

The goal of this study was to assess student understanding of the foundational concepts for calculus. Therefore, we compared precalculus students' performance based on the curriculum they used. We also compared the precalculus students (N = 25) who used the Pathways curriculum to the AP Calculus AB students (N = 11) who enrolled in AP Calculus BC the following year. To summarize and clarify, the three groups of students of interest in this study were: (1) students who were taught using P*recalculus: Mathematics for Calculus* (Stewart et al., 2005), which we refer to as Traditional Precalculus (TP; N = 11); (2) students who were taught using *Precalculus: A Pathway to Precalculus* (Carlson & Oehrtman, 2010), which we refer to as Pathways Precalculus (PP; N = 14); and (3) students who completed AP Calculus AB which we will refer to as AP Calculus AB (AB).

The analyses in this study are presented in four parts. First, we provide descriptive and comparative analyses to compare the PP students with both the TP students and the AB students. Second, we present a breakdown and distribution of student scores to the items on each task and compare the PP students with the other two groups (TP and AB). Third, we describe the solution strategies used by students based on the three groups. Finally, we compare the effectiveness of students' solution strategies for the PP students with the other two groups (TP and AB).

Statistical analyses and descriptives

The first analysis compares TP and PP students; in order to identify whether these groups were statistically different prior to entering AP Calculus and completing the open–ended tasks the *PCA* scores were compared. A t–test was conducted and PP students had a statistically significant higher mean on the *PCA* than TP students (t = –3.37, p = .003); therefore, further analysis to compare students' performance from these two groups on the open-ended tasks used *PCA* scores as a covariate to account for difference in learning prior to entering AP Calculus BC.

An analysis of covariance (ANCOVA) was conducted to determine the statistical difference between the mean total score on the *PF* task for the two groups (PP and TP) of students adjusted for *PCA* scores. No statistically significant difference was found between the two groups (PP and TP) for the mean total score on the *PF* task (F = 0.98, p = 0.333). A similar analysis was completed to determine the statistical difference between the mean total score on the *FT* task for the two groups (PP and TP) of students adjusted for PCA scores. Results indicated no statistically significant difference between the two groups (PP and TP) for the mean total score on the *FT* task (F = 0.20, p = 0.658).

An analysis of variance (ANOVA) was conducted to determine the statistical difference between the mean total score on the *PF* task for the two groups (PP and AB) of students. Results indicated no statistically significant difference between the two groups (PP and AB) for the mean total score on the *PF* task (F = 0.10, p = 0.753). A similar analysis was completed to determine the statistical difference between the mean total score on the *FT* task for the same two groups of students. No statistically significant difference was found between the two groups for the mean total scores on the *FT* task (F = 2.10, p = 0.161).

PCA scores for the AB students were not available because data collection on precalculus students for the MSP project began the year these students were in AP Calculus. However, the results of the two ANOVA's confirmed that there was no statistically significant difference between the PP and AB students for the mean total scores on both open-ended tasks meaning these students' prior knowledge was similar.

Breakdown and distribution of the scores

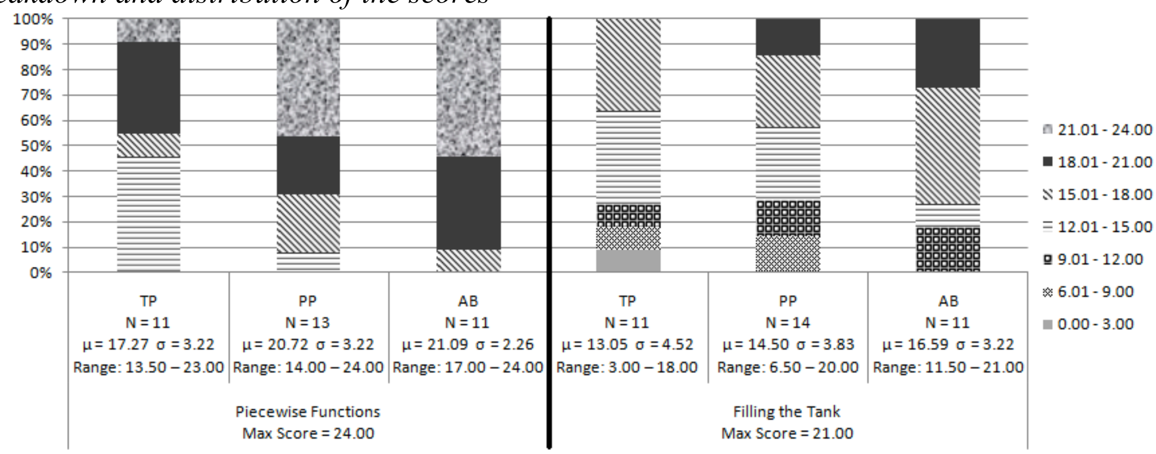

Figure 1. Score distributions on the total scores of the PF and FT tasks for the three groups of students

The distributions of the total scores for the three groups of students on the PF and FT tasks are displayed in Figure 1. For the PF task, respectively, 45% and 55% of the PP and AB students scored higher than 21 out of the total score of 24 while only 9% of the TP students scored in this range. Approximately 37% of the TP and AB students scored between 18 and 21 inclusively, out of 24. The TP group had the greatest percentage (37%) of students receive a score in the 15.01 – 18.00 range with 36% of AB students and 22% of PP students scoring in the same range. Finally, the standard deviation (3.22) was the same for the PP and TP students whereas it was less for the AP Calculus AB students (2.26).

For the *FT* task, respectively, 37%, 28%, 10% of the TP, PP and AB students scored in the 12.01 – 15 range. Meanwhile, 46% of the AB students, 37% of the TP students and 28% of the PP students scored in the 15.01 – 18 range. Although none of the TP students scored in the highest range of 18.01 – 21.00, 14% of PP students and 27% of AB students scored in this range.

Finally, the standard deviation was the greatest for TP students (4.52); whereas, the AB students had the smallest (3.22).

Distribution of the solution strategies

The distributions of the strategies for all three groups on items 1, 2, and 4 of the *PF* task and items b and e of the *FT* task are shown in Figure 2. For item 1 of the *PF* task, 77% of the PP students, 55% of the TP students and 37% of the AB students used graphing as their solution strategy. While none of the PP students used a formula, approximately 45% of AB students and 18% of TP students employed this strategy. Finally 18% of the PP students used multiple strategies.

For item 2 of the *PF* task, graphing again was the dominant strategy for a considerable number of students. PP students used graphing the most (77%); while the AB students used this method the least (37%). None of the PP students used the formula strategy; however, approximately 18% of AB students and 10% of TP students employed this strategy. Only PP students (7%) used multiple strategies when solving this question. The AB group had the highest percentage (46%) of students whose work for this item revealed no strategy and the TP group had a slightly less percentage (36%); however, the PP group only had 16% of students whose work did not reveal a strategy.

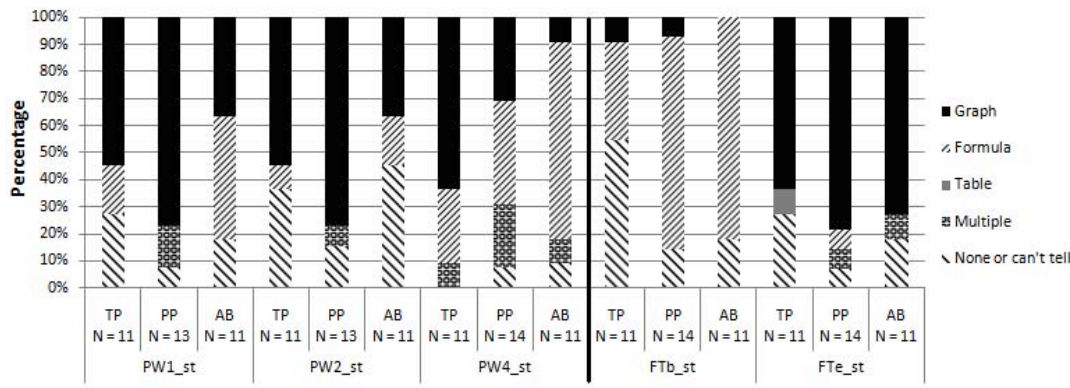

Figure 2. Distributions of strategies for the three groups of students on both of the open ended tasks.

For item 4 of the *PF* task, the TP group had the highest percentage (64%) of students who used graphing to solve this item. The PP group had the highest percentage (23%) of students who used multiple strategies, typically graphing and formula strategies, to solve this item. The AB group had the highest percentage (74%) of students who used a formula to solve *PF* item 4. Only 8% of PP and 9% of AB students work was not possible to code for a strategy.

For item b of the *FT* task, the dominant strategy was formulas for all three groups. Of the TP students, 36% used formulas, 9% used graphing and the remaining students it was not possible to code for a strategy. Of the PP students, 79% used formulas, 7% used graphing and the remaining 14% of the students it was not possible to code for a strategy. The majority (82%) of the AB students used formulas and the remaining 18% of students it was not possible to code for a strategy.

For item e of the FT task, the dominant strategy (graphing) was the same for all three groups; respectively 64%, 79% and 74% of the TP, PP and AB students used this strategy. Only TP students (9%) used tables and this was the only instance where tables were employed as a strategy. The PP group had 7% of students use formulas; whereas, 7% of PP and 8% of AB students used multiple strategies. The TP group had the largest percentage of students (27%)

whose work was not able to be coded for a strategy, while the PP group had the lowest percentage (7%). The percentage of students for whom it was impossible to tell which strategy was used was the smallest for the PP group on items 1 and 2 of the *PF* tasks and items b and e of the *FT* task.

Simultaneous Comparison of the Solution Strategies and Scores for Evaluating the Efficiency among the Groups

While it is interesting to indicate strategies students used in solving the open-ended task, it is also vital to know their respective efficiencies (i.e. what strategies do students use when they receive higher scores?). For this purpose we examined simultaneously the scores and strategies to assess for each group the strategies employed within each score range. Particular interest was given to the students who could score 19 or above out of 24 on the *PF* task and 14 or above out of 21 on the *FT* task; these students constituted the majority (approximately 70%) of the PP and AB students and a minority (35%) of the TP students. For this highest scoring group, the dominant strategies were graphs and formulas, although AB students preferred formulas to graphs whenever they could whereas this was the opposite for the PP students. The TP students in this group primarily used graphs as well.

Discussion

This study assessed students' understanding of foundational concepts of calculus. Two open-ended tasks were used to compare students' understanding after completing a precalculus or AP Calculus course, yet prior to entering AP Calculus BC. Thus, there was a gap of three months between the administration of the open-ended tasks and the last time students were in school learning mathematics.

The statistical tests that compared the TP and PP students showed no statistical significant difference between these two groups based on their performance on the tasks; however, the statistical tests accounted for prior achievement by using the *PCA* scores as a covariate. The *PCA* is a powerful assessment tool used for evaluating students' understanding of the key concepts in precalculus (Carlson et al., 2010). The fact that the PP students mean *PCA* scores are statistically higher than the TP students mean *PCA* scores indicates that the PP students have a significantly better understanding of the key concepts in precalculus when compared to the TP students. Moreover a second conclusion is that the mean total score for the PP students on the open-ended tasks were consistently higher than those of the TP students. The PCA scores for the AB students were not available.

The statistical tests that compared PP and AB students showed no statistically significant difference between their mean scores on the two open-ended tasks (*PF* and *FT*); specifically, on individual items and total scores. The existence of no statistically significant differences between these groups (PP and AB) is particularly remarkable since theoretically the AB students should have performed better than precalculus student on the foundational concepts of calculus as they were exposed to a whole year of calculus. Yet, the AB students did not perform better than the PP students, which could be interpreted as AB students did not learn the foundational concepts of calculus deeply enough to retain them for three months over the summer break; however, the PP students did. This result also suggests that the precalculus curriculum, *Precalculus: A Pathway to Calculus* (Carlson & Oehrtman, 2010) is different than the traditional precalculus program.

In fact, *Precalculus: A Pathway to Calculus* was specifically designed to deepen the understanding of mathematical concepts by improving the learners' exploratory skills through analysis and synthesis; the learners in this curriculum continually study mathematical and scientific contexts and use mathematics as a means to perform scientific investigations. Accordingly, the classroom practice is shifted from the delivery of curricular material to inquiry and project-based approaches centered on content. This study proves that such a shift will help students develop a deeper and longer lasting understanding of the foundational concepts of calculus.

The sample sizes were 11 for the TP and AB students while it was 13 or 14 for the PP students. This is a considerable limitation which could have created power issues, however, this study was indeed a pilot one, carried out to assess the effectiveness of the *Precalculus: A Pathway to Calculus* program. Nevertheless, the relatively smaller sample sizes did not hinder the ability of the authors to come to statistically significant conclusions about the superior aspects of the program.

Based on the results of this study, the *Precalculus: A Pathway to Calculus* is likely to create a positive impact on the precalculus knowledge of students prior to entering calculus.

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PRECALCULUS CONCEPT ASSESSMENT: A PREDICTOR OF AP CALCULUS AB AND BC SCORES

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This study establishes a theoretical framework for predicting the American College Testing (ACT) Mathematics and AP Calculus AB and BC scores from the Precalculus Concept Assessment (PCA) exam results and suggests a total of 16 different regression based models to actually perform the prediction. The strong positive correlation between the actual and predicted values confirm that the PCA is a powerful tool for identifying students who are at the risk of not passing AP Calculus AB or BC tests and thus help teachers, parents and students to take the necessary measures in a timely manner.

Introduction

Assessments are a major practice in the K-12 educational system as well as post secondary. High school students are required to take more and more assessments to demonstrate what they have learned. Most states require high school students to take either an end of course (mathematics) exam and/or a graduation exam (with a portion being mathematics) to complete a course or to graduate from high school (Teuscher, Dingman, Nevels, & Reys, 2008). In addition, most colleges require students to take a mathematics placement exam to direct students into the appropriate first year mathematics course. Even though students are required to send official transcripts and take one of the college entrance exams, they are being asked to demonstrate their knowledge of mathematics on multiple assessments.

Math placement exams vary in mathematics content, the number and type of questions (multiple choice, open-ended, etc.), use of calculators, and time limits. The number of different mathematics placement exams used by institutions across the country continues to increase. However, there are only two commonalities among these placement exams (1) the focus of exam items is on content and procedures taught in remedial mathematics classes, which satisfy general education requirements or serves as prerequisites such as college algebra and precalculus; and (2) the results are not used to inform student or teachers of the possible deficiencies in student knowledge.

This article reports research results on how high school students performed on the AP Calculus AB or BC exam, the Mathematics portion of the standardized *American College Testing* (*ACT*) and on the *Precalculus Concept Assessment* (*PCA*), a research developed instrument based on college students' common misconceptions of functions (Carlson, Oehrtman, & Engelke, 2010) after completing four years of college preparatory mathematics and AP Calculus.

Theoretical Framework

Precalculus Concept Assessment

The *PCA* is a 25-item multiple choice exam that helps researchers and instructors learn what students think and understand about the foundational concepts of precalculus and beginning calculus (see Carlson et al., 2010, for released items). The *PCA* was developed based on research with collegiate level mathematics classes, and was piloted and revised over the past 15 years.

The *PCA* is based on a taxonomy developed to determine student's understandings and reasoning of foundational concepts learned during a precalculus course (Carlson et al., 2010).

Although the *PCA* was not created to be used as a placement exam for Calculus, Carlson et al. (2010) reported that 77% of college students who scored a 13 or higher on the *PCA* passed a first semester calculus course with a C or better. The correlation coefficient for college student *PCA* scores and their calculus grades was 0.47.

The *PCA* was validated with college students who enroll in College Algebra and Precalculus. The study reported in this paper provides a different sample of students, those who are in high school and enrolled in Advanced Placement (AP) Calculus. Students took the *ACT, PCA* and the AP Calculus AB or BC exams during their high school years. Although one might assume that precalculus at the college level is equivalent to precalculus at the high school level, the *PCA* had not been used to analyze student thinking at the high school level.

Content of AP Calculus Courses and Exams

The AP Calculus AB course focuses on limits, derivatives, and an introduction to integrals, which is typically taught in a first semester calculus course in college. The AP Calculus BC course focuses on the topics studied in AB; however, more depth is given to integration and students are introduced to sequences and series as well as parametric and polar functions, which is typically taught in a two semester calculus sequence in college. The AP Calculus exams award students with a score of one to five inclusive with five being the highest score. On each of the AB and BC exams, those students who score four or five pass the exam, and may use their scores to receive college credit for Calculus courses when they enter college. It is evident that a strong foundation of precalculus is absolutely essential for success in Calculus; therefore the *PCA* can be a valuable tool for identifying the students' weaknesses in precalculus if administered to students prior to entering Calculus.

Content of ACT Mathematics Exam

The *ACT mathematics* exam is a 60-question, 60-minute test designed to measure the mathematical skills students have typically acquired in courses taken by the end of 11^{th} grade (ACT, 2011). Students receive an overall score between one and 36 inclusive and three sub-scores based on six content areas: pre-algebra (23%), elementary algebra (17%), intermediate algebra (15%), coordinate geometry (15%), plane geometry (23%) and trigonometry (7%). All of these topics are highly correlated with the content of the *PCA* and if used with PCA, can be employed as a diagnostic tool for predicting how students are likely to succeed in the AP Calculus system.

Regression Analysis

In statistics, regression analysis includes techniques for modeling and analysis of several variables, when attention is focused on the relationship between a dependent variable and one or more independent variables. More specifically, regression analysis helps us understand how the typical value of the dependent variable changes when any of the independent variables is varied, while the other independent variables remain fixed. Usually, regression analysis estimates the conditional expected value of the dependent variable given the independent variables (i.e. the mean (average) value of the dependent variable when independent variables are kept fixed). Regression analysis is widely used for estimation and prediction. It is also used to explore and comprehend the causal relationships that exist among the independent variables in relation to the dependent variable. In this study, regression analysis is the primary means of inquiry in order to explore the relationships between the AP Calculus AB and BC exam scores, *PCA* results and *ACT mathematics* test scores.

Research Questions

In light of the scope of the *PCA*, the *ACT mathematics* test as well as the AP Calculus AB and BC exams, this study specifically seeks to answer the following research questions: (1) How are high school students' *PCA* scores and AP Calculus AB or BC scores related? (2) Can students' *PCA* scores be used to predict their performance on the AP Calculus AB or BC exams? (3) Can the prediction be improved when the *ACT* scores are available?

Methodology

In this study, the 16 different regression schemas are built upon three regression based models, namely, Multiple Linear Regression Model, Multinomial Logistic Regression Model and Cumulative Odds (CO) – Ordinal Regression Model.

Regression Models

Regression models were used in this study to predict students' AP Calculus AB or BC scores (i.e. an integer between 1 and 5 inclusive); in other words, students' AP Calculus AB or BC scores were the dependent variables. The independent variables were student responses to the 25 individual questions in the *PCA*, each being a 1 (that represents a correct answer) or a 0 (that represents an incorrect answer); these will be referred to as *PCA results* in the rest of the paper. In some of the regression models students' *ACT mathematics* scores were used as another independent variable.

The Multiple linear regression model. This model assumes that a linear relation exists between the dependent and the independent variables where the random errors are assumed to be independent and normally distributed random variables with zero mean and constant standard deviation, (i.e., assumptions of normality, linearity, and homogeneity of variance are met). The dependent variable is students' AP Calculus AB or BC score and whereas the independent variables are the responses to the 25 *PCA* questions with or without the *ACT mathematics* scores as; thus, depending on the regression model, there are 25 (without the *ACT mathematics* score) or 26 (with the *ACT mathematics* score) independent variables.

The multinomial logistic regression model. Multinomial logistic regression (Kutner et al., 2005) does not require any assumptions of normality, linearity, and homogeneity of variance for the independent variables. Because this regression model is less stringent it is often preferred to discriminant analysis when the data does not satisfy these assumptions.

Suppose the dependent variable has M nominal (unordered) categories. One value of the dependent variable is chosen as the reference category and the probability of membership in each of the other categories is compared to the probability of membership in the reference category. For the dependent variable with M categories, this requires the calculation of $M – 1$ equations, one for each category relative to the reference category, in order to describe the relationship between the dependent and the independent variables. Please note that multinomial logistic regression model ignores the ordinal nature that might exist within the levels of the dependent variable and treats each category in a similar manner.

The cumulative odds (CO) – ordinal logistic regression model. The CO – ordinal regression model (Kutner et al., 2005) calculates the probability of being at or below category m of an ordinal dependent variable with M categories. Ordinal logistic regression is different from multinomial logistic regression in that it takes into account the ordinal nature inherent within the levels of the dependent variable, which might be useful in some cases.

For the two logistic regression models (multinomial or CO – ordinal) each of the AP Calculus AB or BC scores had five levels (i.e. an integer between one and five inclusive). For multinomial the logistic regression model, the last level (AP Calculus AB or BC score being equal to 5) was selected as the reference category.

The dependent variables were again the *PCA* results used as categorical variables (factors). The *ACT mathematics* test scores could be used as both categorical and ordinal variables. When the *ACT mathematics* test scores were used as categorical variables (factors), each level inherent within the score was a separate independent variable; when they were used as ordinal variables (covariates), they constituted a single independent variable.

Participants

At the end of a school year, 193 high school students from two high schools in a mid-western town were administered the *PCA* to assess their understandings and reasoning abilities prior to entering AP Calculus (Teuscher & Reys, in press). Of the 193 students who took the *PCA* and enrolled in AP Calculus AB or BC the following year, 143 students took the AP Calculus exam at the end of the school year; 80 of these students were enrolled in the AP Calculus AB course while the remaining 63 were enrolled in AP Calculus BC course.

The AP Calculus exam is scored and then students are given a grade of one to five. Typically students who receive a four or five grade receive college credit for at least one semester of calculus. Those students who take the AP Calculus BC exam receive a BC grade and an AB sub-grade. It is possible that a student who takes the BC exam many not receive a four or five on the BC exam, but receives a four or five on the AB portion of the test, which can be interpreted as having passed the calculus AB exam, but not the BC exam.

Prediction Process

The models that used the *PCA* results to predict the AP Calculus AB of BC scores are based on the three regression models (Multiple Linear Regression, Multinomial Logistic regression and the CO – Ordinal Logistic Regression). The predictors in all three regression models were the actual results of each of the 25 questions in the *PCA* test, (i.e. each test question was associated with one of two categorical values, 1 if it was answered correctly and 0 if it was answered incorrectly).

The *ACT mathematics* score can theoretically take ordinal values between 1 and 36 inclusive (ACT, 2011). In statistics, higher level variables can always be downgraded to lower level ones, such that, a metric scale variable can be downgraded to an ordinal or a nominal variable; this process sometimes requires defining categories within the data and/or creating discrete values based on the continuous scale variables (Kent, 2001). The *ACT mathematics* score already takes discrete values, which is reason alone why it can be treated to be ordinal or nominal as well. While it is theoretically possible for a student to score between one and 36 inclusive (ACT, 2011), this is usually not the case in practice; for instance the scores of a group of high school students attending the same school may exhibit a certain pattern. The scores of the group of students subject to our analyses were between 19 and 34 and none of the students scored 22. This is another justification for the fact that *ACT mathematics* score can be treated as an ordinal or categorical variable.

Dependent Variable	Independent Variable(s)
AP Calculus	*PCA*
AP Calculus	*PCA* and Actual *ACT mathematics* scores

Table 1. The variables used for the three linear regression models to predict students' *ACT mathematics* and AP Calculus (AB or BC) scores.

Two different linear regression models used students' *PCA* results with or without the *ACT mathematics* scores to predict students' AP Calculus AB or BC scores; when the *ACT* *mathematics* scores were used, they were treated as ordinal metric variables. The variables used for the linear regression models used in this study are summarized in Table 1 above.

The Logistic Regression models (Multinomial and Ordinal) predict a categorical or an ordinal dependent variable using categorical predictors as factors with or without ordinal variables as covariates. These two models were employed to predict the AP Calculus AB or BC scores separately using students' *PCA* results with or without the actual *ACT mathematics* scores; the ACT mathematics scores were used as categorical variables (predictors) or as ordinal variables (covariates). The logistic regression models used in this study are summarized in Table 2.

Model Specification	Dependent Variable	Independent Variable(s)	
		Categorical Variables (Factors)	Ordinal Variables (Covariates)
A	AP Calculus AB or BC Score	*PCA*	
B	AP Calculus AB or BC Score	*PCA* Score and Actual *ACT mathematics* Scores	
C	AP Calculus AB or BC Score	*PCA* Score	Actual *ACT mathematics* Scores

Table 2. The logistic regression models (multinomial or CO-ordinal) used to predict students' AP Calculus AB or BC scores.

Results

As it might be expected, students enrolled in AP Calculus BC scored higher (mean of 17.51 and standard deviation of 3.18) than students in AP Calculus AB (mean of 15.69 and standard deviation of 3.21) on the *PCA* (where the total score possible was 25) prior to entering AP Calculus. Eighty-one percent of the students in this study who took one of the AP Calculus exams pass it with a four or five.

A positive Pearson correlation existed between students' *PCA* scores and the AP Calculus exam grades and it was statistically significant (r = 0.40, p = 0.000). This can be interpreted as students who receive a high *PCA* score are likely to receive a high AP Calculus exam grade. Then again, a positive Pearson correlation existed between students' *PCA* scores and the *ACT mathematics* test scores and it was statistically significant (r = 0.28, p = 0.02). This can be interpreted as students who receive a high *PCA* score and/or a high *ACT mathematics* test score are likely to receive a high AP Calculus exam grade. The AP Calculus AB scores were available for 80 students; the mean score was 4.00 and the standard deviation was 0.95. Whereas the AP Calculus BC scores were available for 63 students; the mean score was 4.13 and the standard deviation was 0.96.

The AP Calculus AB scores were predicted using the two multiple linear regression models given in Table 1 and the Pearson correlations were calculated between the actual and predicted values. The actual values of students' *ACT mathematics* scores were also used to assess whether or not their inclusion while predicting the AP Calculus AB and BC scores would in fact improve the prediction. The results indicate strong positive correlations and are summarized in Table 3 which can be interpreted as follows: AP Calculus AB scores can be predicted with 48% ($100 \times 0.69^2 = 48\%$) accuracy when using students *PCA* results alone or 75% ($100 \times 0.87^2 = 75\%$) accuracy when using students' *PCA* results along with their *ACT mathematics* test scores.

Model	AP Calculus AB Scores Predicted from the *PCA* Scores	AP Calculus AB Scores Predicted from the *PCA* and Actual *ACT mathematics* Test Scores
Pearson Correlation	0.69	0.87
N	80	48
M (SD)	3.74 (0.83)	3.84 (0.91)

Table 3. AP Calculus AB test scores predicted using the multiple linear regression models.

The AP Calculus AB scores were predicted using the three distinct model specifications for the multinomial logistic regression models given in Table 2 and the Pearson correlations were calculated between the actual and predicted values. The results indicate strong positive correlations and are summarized in Table 4. Model specifications B and C yielded perfect correlations with 100% accuracy in predicting the AP Calculus AB test scores. The results summarized in Table 4 can be interpreted as follows: AP Calculus AB scores can be predicted with 91% ($100 \times 0.95^2 = 91\%$) accuracy when using students' *PCA* results alone or 100% ($100 \times 1^2 = 100\%$) accuracy when using students' *PCA* results along with their *ACT mathematics* scores using the *ACT mathematics* scores as factors or covariates depending on the model.

Model Specification	A	B	C
Pearson Correlation	0.95	1.00	1.00
N	80	48	48
M (SD)	3.80 (0.99)	3.99 (0.99)	3.97 (0.95)

Table 4. AP Calculus AB test scores predicted using the multinomial logistic regression models.

The AP Calculus AB scores were also predicted using the three distinct model specifications for the CO – ordinal logistic regression models given in Table 2 and the Pearson correlations were calculated between the actual and predicted values. The results indicate strong positive correlations and are summarized in Table 5. Model specifications B yielded a perfect correlation with 100% accuracy in predicting the AP Calculus AB test scores. The results summarized in Table 5 can be interpreted as follows: AP Calculus AB scores can be predicted with 40% ($100 \times 0.63^2 = 40\%$) accuracy using the *PCA* results alone; with 100% ($100 \times 1^2 = 100\%$) accuracy using both the *PCA* results and *ACT mathematics* scores as factors or with 70% ($100 \times 0.84^2 = 70\%$) accuracy using *PCA* results along with the *ACT mathematics* scores as covariates.

Model Specification	A	B	C
Pearson Correlation	0.63	1.00	0.84
N	80	48	48
M (SD)	3.76 (1.07)	3.97 (1.06)	3.94 (1.01)

Table 5. AP Calculus AB test scores predicted using the CO – ordinal regression models.

The AP Calculus BC scores were predicted using the two multiple linear regression models given in Table 1 and the Pearson correlations were calculated between the actual and predicted values. The results indicate strong positive correlations and are summarized in Table 6. The results can be interpreted as follows: The AP Calculus BC scores can be predicted with 57% ($100 \times 0.75^2 = 57\%$) accuracy using the *PCA* results alone; or 95% ($100 \times 0.97^2 = 95\%$) accuracy using the *PCA* results along with the *ACT* mathematics test scores.

Model	AP Calculus AB scores predicted from the *PCA* scores	AP Calculus AB scores predicted from the *PCA* and the actual *ACT mathematics* scores
Pearson Correlation	0.75	0.97
N	63	25
M (SD)	4.15 (0.71)	3.92 (0.92)

Table 6. AP Calculus BC test scores predicted using the multiple linear regression models.

The AP Calculus BC scores were predicted using the three distinct model specifications for the multinomial logistic regression models given in Table 2 and the Pearson correlations were calculated between the actual and predicted values. The results indicate strong positive correlations and are summarized in Table 7. Model specifications B and C yielded perfect correlations with 100% accuracy in predicting the AP Calculus BC test scores. The results can be interpreted as follows: AP Calculus BC scores can be predicted with 69% ($100 \times 0.83^2 = 69\%$) accuracy using the *PCA* results alone; or 100% ($100 \times 1^2 = 100\%$) accuracy using the *PCA* results along with the *ACT mathematics* scores which are used as factors or covariates depending on the model.

Model Specification	A	B	C
Pearson Correlation	0.83	1.00	1.00
N	63	25	25
M (SD)	4.10 (0.98)	3.89 (0.93)	3.89 (0.93)

Table 7. AP Calculus BC test scores predicted using the multinomial logistic regression models.

The AP Calculus BC scores were predicted using the three distinct model specifications for the CO – ordinal logistic regression models given in Table 2 and the Pearson correlations were calculated between the actual and predicted values. The results indicate strong positive correlations and are summarized in Table 8. Model specifications B and C yielded perfect correlations with 100% accuracy in predicting the AP Calculus BC test scores. The results can be interpreted as follows: The AP Calculus BC scores can be predicted with 49% ($100 \times 0.70^2 = 49\%$) accuracy using the *PCA* results alone; or 100% ($100 \times 1^2 = 100\%$) accuracy using the *PCA* results along with the *ACT mathematics* scores as factors or covariates depending on the model.

Model Specification	A	B	C
Pearson Correlation	0.70	1.00	1.00
N	63	25	25
M (SD)	4.23 (0.94)	3.89 (0.93)	3.89 (0.93)

Table 8. AP Calculus BC test scores predicted using the CO – ordinal regression models.

Please note that each of the 16 regression models summarized above as well as the Pearson correlation values reported were statistically significant at the 0.01 level.

Discussion

Assessments are the standard for which teacher and institutions judge students' knowledge. With the *No Child Left Behind ACT of 2001* (NCLB, 2001) K-12 students are taking assessments each year to demonstrate adequate yearly progress. However, the majority of these exams were not developed with the intention of providing students or teachers with feedback on deficiencies in student's knowledge. The college mathematics placement exams were also not developed with the end goal of assessing relevant and connected concepts that are foundational for calculus.

The results of this study provide evidence that the *PCA* may be an exam that could be used for multiple settings across high schools and colleges in the US. The *PCA* was found to be significantly correlated with the AP Calculus AB and BC exams and correspondingly the *PCA* scores was a statistically significant predictor of the scores on these exams. The results verify that multiple linear, multinomial logistic and CO-ordinal logistic regression models can successfully be used in one or more of these predictions. As for the generalizability of the results obtained, the mean and standard deviation values calculated for each of the actual and predicted AP Calculus AB or BC scores were very close meaning that the results were indeed generalizable.

While predicting the AP Calculus AB and BC scores, using the actual *ACT mathematics* scores as factors or covariates improved the results of the prediction; particularly using the actual *ACT mathematics* scores as ordinal variables (or factors) while performing logistic regression yielded very strong positive and sometimes perfect correlations between the actual and the predicted values.

These findings are consistent with research reported by Carlson et al. (2010) who found that the *PCA* was a predictor of college students ability to receive a passing grade in calculus at the college level. The *PCA* was specifically created to provide feedback to instructors on what their students understand and do not understand about functions. Instructors could use the results from the *PCA* to determine what prior knowledge or more importantly what misconceptions students have when entering a course that may cause them to not understand and grasp the new material they encounter. The *PCA* could also be used to provide instructors with diagnostic feedback on the specific precalculus topics that students did not understand during their precalculus classes and then make modifications to their curriculum for future classes.

A considerable amount of time and taxpayer money is spent every year on students who retake calculus in college because they are not able to pass the college mathematics placement test. There are also a vast number of students who drop out of calculus in college and feel the urge to change their majors simply because of having the prejudice that they are unable to succeed in mathematics; in fact this is not a new problem and it has not been solved as of yet (Ma et.al., 1999). Thus, an early detection system could be part of the solution and be of immense assistance to students, parents and teachers to take the necessary measures early (i.e. when the students are still in high school). This is why a powerful tool like the *PCA* can be used to identify students who need to spend more time on precalculus and are likely to have a hard time in AP Calculus class or college level calculus, by predicting their AP Calculus AB or BC test scores even before they enter the AP Calculus system.

However, it must be noted that the timing of the *PCA* test is an important factor to produce the results which will enable the prediction of the scores on the AP Calculus AB or BC tests. The *PCA* test should ideally be administered immediately after completing the precalculus content courses prior to the students starting the AP Calculus AB or BC courses.

In closing, it is important to realize that without a purpose, assessments will become something that students *do* and not something that is *useful* to them or instructors. The *PCA* is a practical focused examination that can provide students and instructors with important feedback to improve students' understandings of the common mathematical topics that are necessary for students to be successful in calculus.

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MAKING THE FAMILIAR STRANGE: AN ANALYSIS OF LANGUAGE IN POSTSECONDARY CALCULUS TEXTBOOKS THEN AND NOW

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Three calculus textbooks covering a span of about 40 years were examined to determine whether and how the language used has changed given the Calculus reform movement and the impetus to make mathematics accessible to all. Placed in a discourse analytic framework using Halliday's (1978) theory of functional components, ideational, interpersonal and textual, and using the exposition of the concept of a function as a unit of comparison, the study showed that the functions and forms of language are integral indicators of the author's view of mathematics and the reader's extent of agency and construction of identity.

In the late 1980s, the Calculus Consortium at Harvard (CCH) comprised of eight institutions including universities, a high-school, and a community college, was funded by the National Science Foundation to redesign the Calculus curriculum with a view to making Calculus more applied, relevant, and accessible. The intent was to re/think and re/present the content so as to focus on real-world applications, to emphasize concepts and graphical representations, and to take advantage of the increasingly sophisticated technology. This initiative has been extensively embraced leading to the calculus reform movement. As a result, Calculus is now mostly presented in a manner radically different from the traditional approach which relied heavily on abstraction, formal notation and symbolism, and algebraic conventions. Besides the 'regular' Calculus courses, Calculus courses and materials have been developed for specific disciplines (such as Biology and Economics) and for different modes of delivery (such as Calculus with Computer Explorations).

The goal of this research is to see whether and how calculus textbooks designed for the postsecondary level in 'regular' Calculus courses have changed over the years with respect to the language used in the exposition and by inference, the view of mathematics manifested. One concept, that of a function and in particular its definition, is chosen and used to trace the dimensions of the language over the years and the consequent shifts in the view and presentation of mathematics in calculus textbooks. The questions that arise for me are: Has the language of calculus textbooks changed over time and if so, in what ways? From the language, how are the authors' views of mathematics characterized and how have they changed over time? Has the language changed from one that is exclusive (mathematics as an elite subject with an elite community) to one that is inclusive and accessible to all?

The three textbooks I have chosen are *Calculus* by Spivak (1967), *The Calculus of a Single Variable with Analytic Geometry*, 5th edition by Leithold (1986), and *Single Variable Calculus: Early Transcendentals*, 5th edition by Stewart (2003). Spivak and Leithold were both Mathematics Professors from American universities (Brandeis and Pepperdine, respectively) writing for an American audience while Stewart is an Emeritus Professor in Mathematics at a Canadian university (McMaster) writing for a North-American audience. Each of these textbooks was well-known and well-used in its time. I chose the first two because they were the ones that I

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

still have after the many moves in my life and the third because it is one that I use in my teaching at this time.

Textbooks may be studied subjectively to describe the interaction between the student and the written material or to describe teachers' use of textbooks and the subsequent effect on the teacher (Remillard et al, 2009). However, following Herbel-Eisenmann (2007), I seek to examine the '*voice'* of calculus textbooks over the years as 'objectively given structure' (emphasis in the original, p.396). This examination will be placed in a discourse analytic framework which attends to the aspects of text relating to language, voice, agency and identity. In particular, linguistic markers such as the use of pronouns, imperatives and modality will be traced as a means of addressing the above questions.

Analytic Framework

Starting with the ideas in Pimm (1987)'s *Speaking Mathematically: Communication in Mathematics Classrooms*, discourse analysis of text (written and oral) has now acquired a firm footing in Mathematics Education. Discourse analysis seeks to investigate how language frames the ongoing conversation about the origins and constructions and knowledge and their relation to action and empowerment. Critical discourse analysis recognizes the power relations in this activity and seeks to make space for alternate epistemologies and ideologies. In Mathematics Education, discourse analysis and critical discourse analysis have been used to study written student investigations (Morgan, 1998), silence and voice in mathematics classroom discourse (Wagner, 2007), the 'voice' of a mathematics textbook (Herbel-Eisenmann, 2007), and the pragmatics (the ways in which context communicates meaning) of mathematical discourse (Rowland, 2000).

Language has been increasingly seen as an important issue relating to mathematics teaching and learning. Rowland (2000) emphasizes two principles in studying language: the linguistic principle ('language as means of accessing thought') and the deictic principle (language as a means of communication and a 'code to express and point to concepts, meanings and attitudes') (p. 2). In his *Language as a Social Semiotic*, Halliday (1978) identifies three functional components or functions of language, the ideational, the interpersonal, and the textual, from which meaning is apprehended.

The ideational functional component of the text answers the questions: What is the view of mathematics as presented in the text? How is the subject of mathematics envisioned in the mind of the author of the text and in what style is it rendered? The ideational function describes the nature of the subject matter from the ideological and epistemological stance of the author. It is 'the component through which the language encodes the cultural experience and the speaker encodes his own individual experience as member of the culture' (Halliday, 1978, p. 112). It also names the objects, concepts, and processes involved in mathematical activity and indicates agency on the part of the author and reader. The ideational function is composed of the experiential function (dealing with transitivity and agency) and the logical function (relating to continuity and modes of argument).

The interpersonal functional component describes the social and personal roles and relationships among the authors and readers and the ways in which the readers interact with the written text and the textbook itself as a whole. Evidence of this function is discerned by considering the use of personal pronouns (first person, I/we/us/our, and second person, you/your), imperatives, and modality. The interpersonal function is the 'participatory function of language, language as doing something' (Halliday, 1978, p. 112).

The textual functional component describes the content matter or the mathematics presented in the text, the theme and modes of reasoning, the arguments and their forms, and the narratives of mathematical activity. Halliday describes it as 'the component which provides the texture: that which makes the difference between language that is suspended in vacuo and language that is operational in a context of the situation' (pp.112-113). Evidence of the textual function is seen in the cohesive devices the text uses to preserve consistency and continuity.

Halliday also introduces three concepts which shed more light on these three functional components, namely, field, tenor, and mode, respectively. The field refers to what is going on in the context of the situation and what the participants are doing, the tenor to the roles of the author and the reader and how they stand in relation to one another, and the mode to the channel or wavelength that the author has chosen to use depending on 'what function the language is being made to serve in the context of the situation' (p. 222).

I will examine each of the textbooks with respect to these three functional components and compare and contrast them as to the "voice" that emerges, the extent of agency, and the construction of the identity of the reader by the text.

Method

The data consists of the pages from the three Calculus textbooks that cover the exposition of the concept of a function. I chose this concept because it is fundamental and key to the subject. By exposition, I considered the preliminary introductory commentary and the definition (or definitions) of a function. In each textbook, there were many more pages devoted to the important classes of functions such as polynomial, exponential, logarithmic, and trigonometric functions, but I chose to limit the analysis to only those pages relating to development of the concept and the definition of a function.

I mined the relevant pages (about 20 pages from each textbook) carefully with respect to the markers for the three functions as articulated by Halliday and elaborated by Morgan (1996). I paid close attention to the use of personal pronouns, imperatives, and modal auxiliary verbs. I also considered the use of questions and conditionals, if, if…then, given, and given that, as evidence of forms of reasoning and modes of argument.

Findings and Discussion

The results of the comparison of the textbooks across markers for the functional components of language with respect to the concept of a function are given in Table 1.

	Spivak (1967)	Leithold (1986)	Stewart (2003)
Pronouns - 1st person	we/us/our 32 instances	we/us 5 instances	we/us 24 instances
Pronouns – 2nd person	you 9 instances	None	you 3 instances
Imperatives Inclusive	let's 1 instance	call, compare, let, note, observe, recall 6 instances	consider, determine, let, notice, remember 7 instances
Imperatives Exclusive	None	find, read 4 instances	draw, find, sketch, use 6 instances
Modal verbs	may 2 instances	None	None
Questions	2	None	1

Conditionals	if 6 instances if … then 10 instances	given 3 instances given that 2 instances	if 3 instances if … then 4 instances

Table 1. Comparison across Markers for the Functional Components.

The Interpersonal Functional Component

The most easily-detected functional component of language is the interpersonal which describes the roles and relationships of the author and the readers. This component can be discerned from the incidence of personal pronouns, imperatives, and expressions of modality.

The most striking occurrence is that of 32 instances of first-person pronouns in Spivak as compared with five in Leithold and 23 in Stewart. In Spivak, there were 29 uses of we, two of us and one of our. My reading of this is that Spivak views the reader as someone who is part of the community of people doing or studying mathematics. From the opening paragraph in his liberal use of we and us, Spivak sets the tone of including the reader in his deliberations. He concludes his opening paragraph with 'Let us therefore begin with the following:' (p. 37). An alternative reading of we is given by Pimm (1987) who questions the we that authors use and wonders how he personally is implicated in the proceedings as to responsibility for what may ensue. Another possible reading is that the use of we, us, and our suggests a more general form indicative of the register of mathematicians, especially in written mathematics.

In comparison to this substantial use of first-person pronouns, the five occurrences of we in Leithold read clinically as in 'we see that' or 'we observe that'. There is an implied us in the following sentence from Leithold: 'In Definition 0.5.1 the restriction that no two distinct ordered pairs can have the same first number assures that y is unique for a specific value of x' (p. 45), for whom does the restriction assure, if not us, the readers?

In keeping with Spivak's view of the reader as a thinking and feeling partner in the endeavour, there are nine instances of you as he recognizes the presence of the reader, such as 'It will therefore probably not surprise you to learn that …' (p. 37) and 'you may feel that we have also reached the point where…' (p. 45). However, Spivak does write at one point: 'You should have little difficulty checking the assertions that…' (p. 39), which is an assumption that may leave the reader a little frustrated if some difficulty is encountered. Later, he redeems himself with his regard for the reader in 'If the expression $f(s(a))$ looks unreasonable to you, then you are forgetting that $s(a)$ is a number like any other number, so that $f(s(a))$ makes sense' (p. 40). This is an early indication of affect in mathematics learning, in recognizing the role of emotions and feelings. There are no instances of address to the reader in Leithold while Stewart has three: 'as you can see', 'You can see that…', and 'when you turn on a hot-water faucet…'.

The use of personal pronouns indicates the presence or absence of humans in the activity and the implied distance and degree of formal relationship between the author and the reader (Morgan, 1996). Spivak and Leithold are at opposite ends of the continuum in this regard. Spivak even employs the construction, 'Lest you become too apprehensive about…' and 'let us hasten to point out that…' which is by far the greatest consideration an author can give to a reader. Here again Spivak is demonstrating his recognition of the phenomenon of affect, despite his quaint sentence construction. Leithold deploys his words in a detached 'scientific' manner, the very opposite of the kind of writing that Burton and Morgan (2000) exhort mathematicians to adopt.

The frequency of imperatives in a text indicates the degree to which the author wishes to draw the reader's attention to a point in the text (note that, observe that), to encourage the reader to reflect (consider, compare, recall, remember), or to give a simple command (find, sketch, use). Both Leithold and Stewart use a similar number of imperatives that indicate the usual textbook framing (consider, notice, observe, recall) and that signal the power of the author (determine, evaluate, find, sketch, use) to tell the reader what to do. The former are examples of inclusive imperatives that characterize the reader as 'thinker' while the latter are examples of exclusive imperatives that characterize the reader as 'scribbler' (Rotman, 1988). The use of imperatives shapes the relationship between author and reader and serves to construct the reader as a potential member of a community (Morgan, 1996). It is note-worthy that Spivak does not use any of these imperatives but still manages by his use of personal pronouns to convey a sense of introducing the reader to and including the reader in the community of mathematicians and the activity that mathematicians undertake.

The imperative 'let' occupies a special place in mathematics (as is commonly found in arguments and proofs, for example, let □ be…). Spivak uses it once in 'Let us therefore begin with…' (p. 39), which is more of an invitation rather than a call for consideration. Leithold uses the construction, let □ be…, three times. There are no instances of let in Stewart in the pages under consideration but there is a variety of other imperatives that are roughly equally inclusive and exclusive (Table 1).

Modality, as a feature of language, enables authors and speakers to express their feelings, values, attitudes, and judgments about the propositions in their texts. Halliday (1978) expresses a preference for the term, modulation, rather than modality in that the text is modified or nuanced in some way. Demonstrations of modality include modal auxiliary verbs such as 'may' and 'can', adverbs relating to the uncertain state of knowledge such as 'possibly' and 'maybe', the use of moods and tenses, and the use of hedges (Rowland, 2000, p. 65). For these three textbooks there was little or no evidence of modality. There were two instances of 'may' in Spivak ('You may feel that we have also reached…' and 'Two consolations may be offered', p. 45). These have nothing to do with the mathematics involved but indicate concern for and offer solace to the reader. Leithold and Stewart offer no suggestion that that there is any uncertainty related to mathematical activity and by their lack of use of modality, indicate a view of mathematics that strongly holds to an absolute, ideal perspective.

As seen from these markers for the interpersonal functional component, the tenor of the language in the three textbooks is marked differently. Leithold and Spivak are diametrically opposite in the use of the first and second person pronouns and imperatives in engaging and addressing the reader with Stewart striking a moderate note in this regard.

The Textual Functional Component

All three authors use the mode of discourse characterized by exposition (evident of the raison d'être of the textbook) in laying out a clear and concrete treatment of the subject matter. Questions as evidence of a conversational or dialogic style of exposition were barely used; there were two questions in Spivak, none in Leithold and one in Stewart. The one instance in Stewart is a perfect example of the question-and-answer cohesive form: 'The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by …' (p. 17).

One instance of a question in Spivak is a particularly engaging example of language that recognizes and cements the roles and relationships in the text: 'By what criterion, you may feel

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impelled to ask, can such functions, especially a monstrosity like (12), be considered simple?' This question manages to capture pronoun use, affect, and modality, all in one fell swoop. With respect to forms of reasoning, there are similar numbers of instances of conditionals in all three textbooks such as if and if…then which are widely used in mathematical arguments. Leithold has 5 instances of given and given that but these are not used as conditionals in an argument. Instead they are used in examples such as, Given that $\square$ and $\square$ are defined by $\square$ and $\square$. The current usage is more informal: 'Suppose that …' which, as a shortened form of 'Let us suppose that…', is more of an invitation.

The Ideational Functional Component

The ideational functional component in each of the three textbooks is very nearly identical in that the authors' content and meaning are very similar. Each author is interested in communicating the content of the concept of a function. Further, each author conveys the weightiness of the subject matter and the experience of being part of the culture of being mathematicians and doing mathematics. Each is writing of the objects and relations that are under consideration when introducing and discussing the concept of a function. By the degree of use of the linguistic markers analyzed above, each encodes in the text his individual vision of mathematics. The view of mathematics evinced in all three is fixed, absolute, and formal.

Besides the content, the ideational component, in describing which actors carry out which processes, speaks to the concept of agency as it is invited or suppressed. Morgan (1996) elaborates on the use of nominalization in order to suppress or mask agency. Clear examples of suppression of agency occur in Leithold: 'Equation (1) defines a function' and 'This equation gives the rule by which …' (p. 45). There are no similar constructions in Spivak or Stewart.

In summary, the three textbooks are similar in their theme and message but differ considerably in the interpersonal component with Stewart capturing a moderate position between what may be considered the extremes of linguistic markers by Leithold and Spivak.

Conclusion

The language of mathematics is often seen as foreign with its own lexicon, grammar, and modes of argument. More than being able to negotiate the language, students of mathematics must become fluent in it in order to be successful. Bakhtin declares that '[e]ach text presupposes a generally understood (that is, conventional within a given collective) system of signs, a language (if only the language of art)' (1953/1986, p. 105). Hence the mathematics textbook, as text, has a conventional system of signs which is part of a language that must be understood in order to be able to participate in the community involved in mathematical activity.

While there is a core of conventions that must be observed in written mathematics such as the infallibility of written mathematics, the analysis above shows that the forms of language used encode the ideological stance of the author. Spivak wrote at a time when mathematics in textbooks was more theoretical and formal. While he is careful to draw the reader in and attend to how the reader is managing, he plots a watchful thoughtful course through the content. For Leithold, mathematics is austere and closed. His language is methodical and authoritative. Stewart begins with an applied focus with multiple representations of functions from the real world. His language is readable and brisk with many examples, sidebars and cautionary notes. Further, the linguistic devices, such as nominalization and cohesive forms, indicate the extent of agency afforded the reader and the construction of the identity of the reader as a potential member of the community of mathematicians.

The differences in language in a textbook account for much of the reader's regard for the textbook and for the teacher's ability to mediate the textbook with her students. In this paper I have teased out the subconscious linguistic markings in the text and have shown that there is more to the text than meets the eye; that what we have taken as familiar is indeed strange, and that the language of mathematics does reveal beliefs and ideas about mathematics which we adopt and perpetuate without realizing the implications and consequences. This analysis suggests that it behooves us as teachers to look again at language and its signs and to re/examine our textbook choices, while attending to the functions and forms of language that subtly maintain hegemonic practices in the teaching and learning of mathematics.

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MIDDLE LEVEL TEACHERS' PERCEPTIONS OF WHOLE-CLASS MATHEMATICAL DISCOURSE

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Whole-class discourse is an important component in developing relational understanding in mathematics. Teachers' perceptions of this instructional and assessment strategy can greatly influence and affect its use, implementation, and overall success. However, facilitating whole-class discourse with adolescents can be difficult as they are often reluctant to participate in situations that are potentially embarrassing. This study identified strategies that middle level mathematics teachers used to promote whole-class discourse and also compared teachers' perceptions with their actual classroom practice. Several areas of concern were identified due to the discrepancies between teachers' perceptions and their actual practice.

The learning experiences students encounter within schools is fundamental to their academic achievement and attainment (Marzano, 2004). However, the selection, nature, and design of these experiences are largely left up to the teacher which means that teacher preferences in the design and implementation of these experiences will often determine how the curriculum content is presented to the students. Learning experiences that also include whole-class discourse (WCD) are important for middle level students as students' understanding of content is strengthened through communication (Gose, 2009; National Council of Teachers of Mathematics [NCTM], 2000; Pugalee, 2001; Roberts & Billings, 2009). Yet, generating mathematical discourse amongst adolescents can be especially challenging, particularly within a whole-class setting as social acceptance by adolescents' peers is paramount; they wish to avoid embarrassment or appearing incompetent (Van Hoose, Strahan, & L'Esperance, 2001).

The purpose of this qualitative multi-phase case study was to better understand teachers' perceptions towards WCD as it relates to the facilitation of WCD, the utility of WCD as an instructional strategy to support all students in the learning of mathematics, as well as to better understand the methods that teachers tend to implement to support the facilitation of WCD. By understanding teachers' perceptions of WCD, more informed decisions on how to support teachers within the classroom and improve effective instructional and assessment strategies can be further developed. Furthermore, knowing how teachers view their practice today can help educational leaders redefine and shape future classrooms.

Research Questions

The research question that guided this study focused on teachers perceptions of WCD in relation to their actual classroom practices. Specifically, in what ways were middle-level teachers' perceptions towards whole-class mathematical discourse, as a means of supporting all students in learning mathematics, aligned with their actual classroom practice?

Relevance to Current Literature

Mathematical discourse is certainly not a new phenomenon and there is an abundance of literature that supports various aspects of WCD and the benefits it provides students and teachers (Falle, 2004; Hoffman, Breyfogle, & Dressler, 2009; Huang, Normandia, & Greer, 2005; Kuhn, 2005; Larsen & Bartlo, 2009; Mercer, 2008; NCTM, 2000; Roberts & Billings, 2009; Stein, M. K., Engle, R. A., Smith, M. A., & Hughes, E. K., 2008). Two of these benefits support students' higher level thinking and provide teachers with insightful and timely assessment opportunities.

Helping students develop as creative and complex thinkers is important in any discipline, but part of the beauty of mathematics lies in the creative elements that are needed to be effective problem solvers; skills that are dependent on students' higher level thinking. When teachers foster and promote WCD, they also develop more creative and higher level thinking skills in students (Mercer, 2008; Roberts & Billings, 2009). Kuhn (2005) also links discourse to the development of thinking, and thus learning, and stresses that thinking is something that is typically done collaboratively. Kuhn's classroom discourse model, the theoretical model used for this study, depicts the collaborative nature of discourse (see Figure 1). In this model, students are identified by a number; e.g. S1, S2, and S3. The arrows and corresponding numbers indicate to whom the question was directed and the order in which these interactions were made.

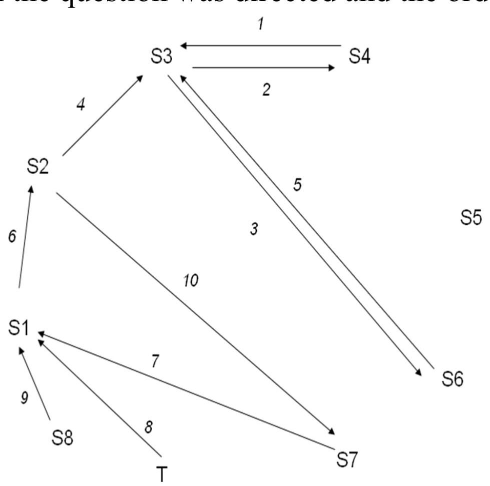

Figure 1: Kuhn's Authentic Classroom Discourse model

Providing timely and appropriate feedback on student learning is important in supporting student learning. During WCD, when students verbally construct their responses, teachers can more accurately assess students' understandings or misunderstandings (Clarke & Sullivan, 1990). Furthermore, mathematical WCD provides assessment opportunities that may not be evident in other forms of assessment (Falle, 2004). Misconceptions, be they general or specific, can be addressed in a timelier manner and addressed immediately. As an assessment strategy, few other strategies can provide more real-time information than WCD.

Methodology

Since teachers' perceptions were the focus of this study, a phenomenon that is naturally dynamic and flexible, I used a four phase qualitative case study with a grounded theoretical approach (Guba & Lincoln, 2005). A grounded theoretical approach was used because I placed significant emphasis on the interpretations and meaning that teachers had towards discourse as evident in their interviews and observations. Also, as a result of the teachers' observed interactions in an authentic and naturalistic setting, during which thick descriptions were created, theory was inductively generated (Bogdan & Biklen, 2007).

Phase 1 of this study surveyed 30 middle level public school mathematics teachers to determine their beliefs about WCD. These teachers came from four different schools that had similar demographics in academic mathematical achievement, ethnic and cultural diversity, and size of the student populations. Nine teachers from Phase 1 who indicated that they believed WCD was a highly effective strategy for teaching mathematics to adolescents were chosen for Phase 2. Phase 2 consisted of semi-structured interviews to better understand their perspectives and the strategies the teachers used to facilitate WCD. For Phase 3, three teachers were selected from the Phase 2 group for classroom observations. These teachers were selected based on the complexity, richness, and diversity of their responses during the semi-structured interviews. In essence, these three teachers seemed to exemplify WCD as a method of supporting mathematics learning for all students. Phase 4 consisted of follow-up interviews with the same teachers from Phase 3 so that I could better understand their perspective as it related to their actual practices.

Findings

Phase 1

For Phase 1, I administered a four item survey which identified the degree to which teachers valued WCD. The first three items of this survey were: 1) It is important for students to talk about mathematics as an entire class; 2) When the whole-class has a discussion about mathematics, it can be more beneficial than in small groups; and 3) I feel that I can comfortably facilitate discussions with the entire class. Using a one-to-five scale, with one meaning they strongly disagreed, a five meaning they strongly agreed, and a three was listed as being neutral teachers indicated their perceptions. The fourth item read, "I plan for whole-class discussions" and gave four options. The four options were: A) nearly daily, B) about one to two times a week, C) about once every two weeks, and D) once a month or less. The results from the first three items indicated that nearly every teacher agreed or strongly agreed with the items. Finally, for the fourth item, 20 of the 30 teachers indicated that they planned for WCD either one to two times a week or nearly daily.

Phase 2

For the second phase, semi-structured interviews helped clarify the perceptions teachers' had towards WCD. All of the teachers believed that WCD was a useful instructional strategy in some manner even though many of the teachers misunderstood. Many thought discourse was related to something negative; "discourse is something off track or off tangent," or "discourse to me would mean something is not going very well or varying opinions on a topic, I guess. Discourse to me sounds bad." However, when asked to describe whole-class discussions, one teacher who initially misunderstood the term discourse described it as "talking, taking turns talking. Maybe I would start them on a topic and it's an organized talking that has one spokes person. It is just not craziness going on." This is similar to the types of responses by the teachers who conveyed an understanding of WCD. One such teacher described WCD as “ways of getting all of the students actively engaged in contributing to verbal expressions of a specific topic.”

The interviewed teachers also believed that there were specific benefits for teachers and students alike. For the students, these benefits included a perceived increase in understanding and greater individual accountability in learning. The benefits for the teachers were the ability to check for understanding in a timely manner and the ability to adjust instruction and intended learning opportunities based on students’ understanding.

In addition, the teachers believed they had specific responsibilities in order to reap these benefits; the most frequently reported responsibility was being a “facilitator.” One teacher said, “I feel like a facilitator. I literally sit back and just let them debate themselves. And that is what I think is what we should be doing; letting the kids talk but you can kind of steer them too.” Other teachers indicated that they needed to create supportive and organized classrooms so students would feel comfortable participating; “to eliminate that fear of giving the wrong answer by showing them that you can give the right answer.” Teachers also believed that being a facilitator meant that they needed to probe student thinking by asking different types of questions and gradually shift more of the questioning responsibilities to the students; students should be the ones asking questions to each other without the teacher needing to prompt others to do so.

The teachers were also quite aware of the specific social, physical, and cognitive needs of adolescents and tried to create learning environments that were sensitive to these needs. Several teachers stated that adolescents often do not want to be put in situations that could compromise their social or emotional security. For these “shy” students, the teachers believed it was best to not press the student for a response but rather leave them be. One teacher even indicated that for, “the ones who are really shy it is not healthy to try and get them to say anything. I get this sense that they are not comfortable, they are not able to speak; they are not sure about themselves.”

Finally, many of the teachers described meaningful WCD in terms of increased participation and/or clear and observable physical cues such as more students raising their hands, a sense of excitement to share responses, and expressions on faces. Only three of the nine teachers described meaningful WCD in terms of an increase in students understanding. Carrie, one of the teachers observed during Phase 3 believed meaningful WCD was evident when “they no longer look at me to determine the validity of a student’s response; they just keep going, they ask each other instead.” Another teacher thought that meaningful WCD in their classroom was apparent in students’ higher quality response.

When asked about the techniques or strategies they used to promote meaningful WCD, their responses related to four common types. These included the use of questions (seven responses), small groups (nine responses), various technologies (four responses), and impromptu instructional modifications (six responses). These eventually became part of the teacher-generated framework (TGF) used in Phase 3.

Phase 3

In the third phase of the study, three teachers were selected for observations; they were Carrie, Erin, and Samantha. After analysis of the Phase 2 data, an initial theoretical framework of how teachers perceived WCD was created. This TGF was composed of two separate facets: 1) Required roles and duties of the teacher and instructional strategies frequently used by teachers to facilitate WCD and 2) the desired student traits that would be evident as a result of implementing WCD. The first element for Facet 1 is grounded in the teachers’ beliefs that their primary duty was to facilitate WCD and this meant to keep the discourse flowing and on topic by asking the class different types of questions. Facet 1 also included their strategies for promoting WCD; using small groups, technologies, and/or impromptu lesson modifications. The second element of Facet 1 was based on the teachers desire to increase engagement and participation to include many students. For Facet 2, teachers perceived that there would be specific and observable traits demonstrated by the students. They believed that through their efforts in facilitating WCD (Facet 1) there would be at least two traits that would aid in promoting the learning of mathematics. First, participation would increase because all students would be actively engaged. Next, students would provide higher quality responses; an indication that they understood the mathematics being learned.

Based on the TGF, not all teachers' perceptions of WCD corresponded with their actual practice. Although, two of the teachers, Erin and Carrie, used several of the strategies identified in the TGF and thus had perceptions that closely mirrored their practice in most instances. Carrie's perception of WCD seemed to be identical to her practice though there were few student-to-student discursive interactions because all student responses were first funneled through her. Erin's perceptions were also relatively close to her practice but she relied heavily upon voluntary responses. Even though she believed WCD was important for all students, it was typically the same few students who provided responses and many students were never selected or asked to participate. Carrie did not include every student in the WCD each class period either, but she used name cards to randomly select students to participate. Furthermore, Erin also had limited student-to-student discursive interactions because she also used the same wagon wheel (Okolo, Ferretti, & MacArthur, 2007) or funneling effect used by Carrie (See Figure 2).

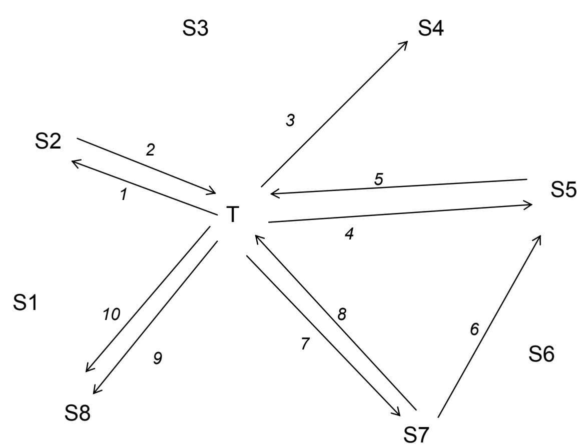

Figure 2: The wagon wheel structure most commonly used.

This technique more closely aligns with a traditional initiate, response, and evaluation (IRE) form of communication. Finally, Samantha's perception of WCD did not appear to reflect her practice at all. Even though students were in groups, these groups were not used to promote meaningful WCD. Furthermore, her use of voluntary responses and IRE structures limited the responses to the same few students in nearly every discursive interaction.

Phase 4

In this final phase of the study I wanted to learn more about two emergent themes framed around the following questions: 1) How did the observed teachers come to their understanding of

WCD and 2) can WCD support adolescent students' affective development? These two questions arose from the interviews and observations with each of the respective teachers.

Each teacher reported arriving at their understanding of WCD through different ways. Erin came to her understanding of WCD through her experiences in the classroom but also mentioned that her understandings had changed based on the initial semi-structured interview. As a result of this interview she decided to think about how she could implement WCD more often as she did not previously view WCD as an effective strategy for teaching and learning mathematics. Carrie stated that she came to her understanding of WCD through observations of other teachers along with frequent reflections on her own practice. Observations were important to her because she was able to witness various strategies and then incorporated or modified them as needed. Whereas Samantha believed strongly that learning was a social endeavor and this was based on her previous college course work. During this interview Samantha also revealed that she believed that WCD was a naturally occurring phenomenon in her classroom and thus she did not need to plan for, and in fact, never planned for WCD during the observations.

Each of the three teachers talked about how WCD supported adolescents during this time of their life as they grow and develop socially and emotionally. The teachers said that WCD helped students gain and foster a critical sense of belonging within the classroom. The teachers were aware that adolescents constantly seek acceptance from their peers and thus WCD provided an avenue through which students gained recognition as being part of the classroom and validation of their ideas. Carrie believed that by selecting students that did not typically participate and asking them to contribute they would feel more connected to the class; this was counter to many of the thoughts and beliefs of the other teachers interviewed.

Implications and Recommendations

After careful analysis of the findings from each of the four phases of this study, there were four areas of concern that were identified that could have an impact on students' learning of mathematics with respect to WCD. These four areas were: 1) Connecting pedagogy to the nature and needs of adolescents, 2) creating cultures of participation that are inclusive to all students, 3) assessing for learning through collaborative and formative methods, and 4) shifting responsibilities from the teacher to the students to promote relational understanding.

With respect to connecting pedagogy to the nature and needs of adolescents, teachers also clearly expressed a need to be mindful of students' affective development. Unfortunately, this led to numerous students not being included in the mathematical learning experiences. In essence, students were frequently, and unintentionally, being marginalized. However, discourse between peers is an essential element in students' learning and development thus all students should be involved (Grennon, Brooks and Brooks, 1999).

Next, with respect to creating cultures of participation that are inclusive to all students, it is important to remember the tenets echoed by the NCTM (2000); meaningful interactions are built upon cultures of collaboration. Classroom cultures of participation are collaborative, value all individuals' comments and ideas, and maintain accountability for all members within the learning community for contributing to and shaping the understandings that arise through shared discursive interactions. This also means that strict adherence, deliberately or otherwise, to the use of voluntary responses combined with IRE structures of discourse need to be reevaluated. Reliance on voluntary structures cannot effectively support classroom cultures of participation. Employing random selection techniques like the ones Carrie used could be one way to begin to shifting from IRE structures to more inclusive practices.

When it comes to teachers using WCD as an assessment tool, Stiggins (2002) reminds educators that the primary purpose of assessments is to assess for learning rather than an assessment of student learning. What this means is that teachers need to be using assessment strategies that intentionally and deliberately aim to improve student learning. An implication of such deliberate assessment strategies is that teachers need to be continually probing students' thinking to learn what misunderstandings may exist. While each of the three teachers observed indicated that WCD was a beneficial method of assessing for learning, rarely were student responses challenged. In addition, when teacher-centered IRE approaches are combined with voluntary responses, as they were in these cases, only a select few students' learning will ever be assessed at any given time. Therefore, the class discourse could not be accurately relied upon as a formative method of assessment to gauge all students' learning.

Questioning plays an important role in the assessment process of WCD. Stein et al. (2008) suggest focusing on developing deliberate and meaningful questions prior to the implementation of the lesson based on anticipated difficulties students may have. In doing so, teachers will be better prepared to facilitate WCD. Furthermore, teachers will not need to think of questions to ask in the moment and can then focus on the quality of the responses provided and thus help students link the student-directed WCD to the intended learning outcomes.

The final area of concern dealt with shifting the primary responsibility of facilitating WCD from the teacher to the students. WCD should be a student-centered phenomenon, which means that the students need to be the leaders within the classroom; each of the three observed teachers stated as much during their interviews. But, because of an over use of IRE structures, teachers maintained the intellectual authority during the WCD. As a result, there was no apparent shift in responsibility to the students.

Despite these concerns, promising change was found in the classrooms that strove to create cultures of participation. In the observed classrooms that created cultures of participation, or had elements that supported a culture of participation, students were invited, encouraged, and expected to participate and contribute to the mathematical understandings that were generated.

Conclusion

This study was based on Larson and Bartlo's (2009) recommendation that classrooms should shift from structures that are teacher-centered, wherein the teacher is the primary source of knowledge, to student-centered that require all students to be active participants in the learning of mathematics. Mathematics should be a collaborative social endeavor that is inclusive of all adolescent learners; it should be explored, digested, and realized within a learning environment that places students at the forefront (Okolo et al., 2007; Roberts & Billings, 2009).

Within this context, this study examined middle level public school mathematics teachers' perceptions of WCD and revealed surprising discrepancies between teachers' beliefs and their actual practices. This study also identified a Teacher Generated Framework in Phase 3 that supported the facilitation of WCD so that more students could be actively engaged in generating mathematical understandings.

While this study was limited, cases of this sort assist in constructing theory and methods that support future research into the perceptions and instructional techniques used to facilitate inclusive whole-class discourse. Further research into the interaction and relationship between teachers' perceptions of WCD and the instructional tools and strategies they use to implement and facilitate meaningful WCD will help create a more precise understanding of this phenomenon.

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PUPILS' ARTICULATION OF MATHEMATICAL THINKING

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Brunei Primary mathematics curriculum (2006) advocates teachers on the use of questions to promote mathematical thinking in the pupils. When the pupils are asked and probed of their thinking, meaningful learning will take place. Research has made it clear that probing questions is one of the effective ways to promote thinking among the pupils. This paper describes the study which reports on the pupil's responses to teachers' questions supported by the pupils' responses in the interviews. It was observed that the pupils' responses in the lessons observed were mainly choral and consist one to three word answers. The pupils' answers were mainly repeated reasons and repeated procedures. This finding is consistent with the pupils' responses in the interview where the responses were mainly learned procedures and reasons that did not show explicit mathematical thinking.

Introduction

With reference to the Brunei new upper primary mathematics syllabus published in 2006, one of the aims is to develop "the ability to solve mathematical problems and think clearly and logically" (p. 3). In line with this aim, a few specific objectives of the new mathematics curriculum stated in the document are to "develop the ability to estimate, develop spatial awareness, develop basic skills in collecting and interpreting data, develop the ability to communicate mathematical ideas clearly, tackle non-routine problems systematically and apply what has been learned to solve real problems" (*Mathematics Syllabus for Upper Primary Schools 2006*; p. 3). These developments, in particular the development of mathematical thinking and problem solving, are stipulated to be developed through the interaction between mathematical content and mathematical processes. Mathematical thinking is identified as one of the mathematical processes. In the document, mathematical thinking has been described in several sub-processes such as "Guessing and checking, Drawing diagrams, Making lists, Looking for patterns, Working backwards, Classifying, Identifying attributes, Sequencing, Generalising, Verifying, Visualising, Substituting, Re-arranging, Putting observation into words, Making predictions, Simplifying the problem and solving part of the problem" (p. 5). These sub-processes clearly represent explicit views of mathematical thinking in terms of behaviour and articulated thinking.

In terms of behaviour, Wertheimer (1961), a gestalt psychologist, described the components of thinking from a phenomenological point of view. He mentioned that thinking processes can be noticed in terms of the behaviour. Such terms can be in the form of comparing and discriminating (identification of similarities and differences); analysing (looking at parts); induction (generalisation); experience (collecting facts or grasping structure); experimentation (looking for differences between possible hypotheses); associating (items together and recognising relationships); guessing and checking, trial and error and learning on the basis of success (with or without appreciating structural significance). This view suggests that mathematical thinking assists pupils to realise the necessary knowledge or skills for solving a particular problem. It should also be seen as the driving force behind such knowledge and skills. This is referred to as the "mathematical attitude" (Katagiri, 2006). This view of mathematical thinking seems to be comparable with the current conceptual framework of the Brunei mathematics curriculum which shows interwoven attitudes, processes and content (Khalid, 2007).

In terms of articulated thinking, Mason, Burton and Stacey (1982) identified four fundamental mental processes that show the progress of mathematical thinking: specialising – trying special cases, looking at examples; generalising – looking for patterns and relationships; conjecturing – predicting relationships and results and convincing – finding and communicating reasons why something is true. This view is related to the ability of the primary pupils that can be developed. When mathematical thinking is viewed in terms of articulated thinking, patterns of thinking develop (Fennema & Carpenter, 1996). The ability of the pupils to think by themselves, accompanied with conceptual understanding, strategic competence, adaptive reasoning, and productive disposition are the new goals in education (Kilpatrick, Swafford & Findell, 2001). The Brunei mathematics curriculum implemented in 2006 has similar goals and as previously mentioned, emphasises understanding through the development of mathematical thinking.

A study by Cheeseman & Clarke, (2001) investigating articulation of mathematical thinking, found that young children (aged 5 to 7 years old) could give an account of mathematical events from their perspective. They also found that through interviewing them, the children could recall at least part of their conversations with the teacher during the mathematics lesson. They concluded that when teachers used interaction that challenged children to think about their mathematical understanding, they could reconstruct thinking. In the study, they reported that 59% of the children could talk about their learning as a result of the lesson — some at a factual level, some at a procedural level and some at a conceptual level. The ability demonstrated by these children suggested that mathematical thinking can be fostered even at an early age.

Research Problem and Research Question

The objective of the study is to investigate the mathematical thinking of the pupils by observing the types of pupils' responses and supports the findings with the information from the interviews to assess whether year 5 pupils showed any development of their articulated mathematical thinking. Hence, the study seeks to answer the question: What are the pupils' responses to teachers' questions that show mathematical thinking in learning mathematics?

Research Methodology

Sample

The participants in the study were year 5 pupils who ages between 10 to 11 years. A total of 40 pupils were randomly selected from five year 5 classes situated in Tutong and Brunei Muara districts. These pupils were seen asked by the teachers in the respective classes during the observation.

Instruments

The research utilised two codes- the first code was used to analyse the pupils' responses in the observation sessions and the second code was used to analyse the responses of the pupils in the interviews. Table 1 and Table 2 show the two codes mentioned.

Table 1
Pupil Response Codes

Pupils' Responses	Description
No response	No answer to teacher's question.

Chorus	Whole class responds orally to teacher's question.
Individual pupil	Denotes which pupil answers (enables link with probing question).
Correct	Correct response given.
Partially correct	Partially correct response or incomplete response.
Incorrect	Incorrect response given.
Explanation	Requires pupil to give information.
Justification	Requires pupil to explain answer or strategy.
Question	Pupil answers the teacher's question with another question.

Table 2
Categories of Responses for Aspects of the Interviews

Aspects of Interview	Categories of Responses
Recall of events	• no recall • could talk about the learning with video • recall spontaneously
Description of events	• no description • describe the action • describe the outcome • describe the event from their perspective • describe their reasoning / justify their thinking
Explanation of their learning	• no explanation • learned nothing • learned a behaviour not mathematics • remembered factual information • learned how to do something
Explanation of their thinking	• no explanation • simple explanation of thinking such as "compare the digits" • explicit description of thinking • explain/reconstruct thinking, reasoning, justifying, evaluate thinking

The interview data provided qualitative accounts of the pupils' mathematical thinking in terms of the following areas:

- recall of a task or an activity;
- description of events;
- explanation of their learning;
- explanation of their thinking.

The areas above followed the "complementary accounts methodology" used by Clarke (2001) to capture pupils' accounts of their mathematical thinking.

Data Analysis

The pupils' responses in the observations were coded, counted and converted into average to highlight which type of responses was the most. Similarly, the pupils' responses in the interviews data were also analysed by counting frequency of mention or recurring regularities which then were categorised. After the responses from the observation and the interview were collected, the responses were entered into the SPSS package for quantitative analysis.

Result and Discussions

a) Analysis of the Observation of the Pupils' Responses

Table 3 illustrates the total frequency of each category of pupils' responses in the ten lessons observed. The table is divided into two. The first three rows illustrate the frequency counts of "No response", "Chorus" and the "Individual pupil" categories. The other rows show the breakdown of the individual responses which were coded as "Correct", "Partially correct", "Incorrect", "Explanation", "Justification" and "Question". Some of these responses were classified under two categories which explain the discrepancy between the total frequency counts and the individual frequency counts. Only the chorus and the individual responses were added to give the total frequency of pupils' responses. The "No response" frequency counts were not included because the study focused only the pupils who responded to the teachers' questions to demonstrate mathematical thinking.

Table 3

Total Frequency of Pupils' Responses in Primary 5 (10 lessons)

Pupils' Responses	T1	T2	T3	T4	T5	Average
No response	0	2	0	1	1	
Chorus*	27 (64.3%)	41 (65.1%)	38 (76%)	42 (85.7%)	56 (70%)	72.2%
(Individual pupil)*	15 (35.7%)	22 (34.9%)	12 (24%)	7 (14.3%)	24 (30%)	27.8%
Total*	n = 42	n = 63	n = 50	n = 49	n = 80	
Correct (I)	13 (31%)	21 (33.3%)	12 (24%)	5 (10.2%)	22 (27.5%)	25.2%
Partially correct (I)	2 (4.8%)	0	0	0	0	1.0%
Incorrect (I)	1 (2.4%)	1 (1.6%)	0	2 (4.1%)	2 (2.5%)	2.1%
Explanation (I)	6 (14.3%)	3 (4.8%)	0	0	1 (1.3%)	4.1%
Justification (I)	0	0	0	0	0	0
Question (I)	0	0	0	0	0	0

The frequency counts of the pupils' responses revealed that the prevalent response in each lesson observed was the whole-class choral response. On average, 72.2% of the responses made to each teacher were choral. Consider the following discussion between the pupils and teacher T2 on identifying the numerator and the denominator of 3/4:

T: "Which is greater, three or four?"
P: "Four." (Chorus 1)
T: "Which is less numerator or denominator?"
P: "Numerator." (Chorus 2)
T: "Yes, the numerator is less than the denominator."
P: "Yes." (Chorus 3)
T: "So the numerator is smaller than the denominator. It is called proper fraction."
T: "Why do we call proper fraction [sic]*?"*
P1: "Numerator is small." (Explanation 1)
T: "Yes the numerator is smaller than the denominator."
(An episode of verbal exchanges in one of T2's lessons.)

The episode revealed that the chorus responses 1, 2 and 3 were one word answers because the questions posed by the teacher were closed questions. When the teacher asked for the reason why it was called a proper fraction, which was an open question, the explanation given by P1 was a repeated reason provided by the teacher earlier in the exchanges. The responses given by individual pupils were the other category of pupils' responses with an average of 27.8% per teacher. Individual responses were important in two ways. First the coding assisted the writer to trace for probing questions and second, to analyse whether the responses were reflecting thinking or not. Consider the following verbal exchange between a pupil named Abdul Mateen and his teacher, T1, on conversion of 3 centimetres into millimetres.

T: "Can you convert 3 cm into mm?"
P: "Yes."
T: "So, what is the answer?"
P: "30 mm."
T: "Why not 40 mm?"
Abdul Mateen: "Because it is 3 cm."
T: "Why is that?"
Abdul Mateen: "Because 1 cm is 10 mm."
T: "So what do you think?"
Abdul Mateen: "So 3 cm is 30 cm."
T: "Good."

(An episode of a verbal exchange between Abdul Mateen and T1.)

In the verbal exchange between Abdul Mateen and his teacher, T1, the responses given by the pupil demonstrated development of thinking as his teacher asked him subsequent sets of questions. However, the responses were one to three word answers only and were not explanatory or justifying in nature. Further information on the nature of the pupils' thinking is discussed in the interview section.

Eighty individual responses were coded with an average of 25.2% correct responses per teacher, indicating that the individual pupils had answered what the teachers intended. On the other hand, per teacher, an average of 2.1% of the pupils' responses was incorrect. Among the individual responses were explanatory type answers with an average of 4.1% per teacher. As mentioned earlier, the explanatory answers were repeated reasons which proved to be parroting and regurgitating the teachers' reasons or explanation.

b) Analysis of the Interviews with the Pupils

The interviews were conducted with individual pupils to collect data on their mathematical thinking developed through the teachers' questions posed in the lessons observed. The interviews were necessary to support the pupils' responses coded in the lessons observed. The interviews were conversational in style. The writer conducted the interviews in a bilingual manner which was a mixture of Brunei National Language (Malay) and English because most of the pupils had difficulty expressing themselves in English. Another reason was that English was the second language spoken by these pupils. The data generated from the pupils' interviews were quantitatively and qualitatively analysed. All of the interviews were recorded, transcribed in full and coded following the study done by Clarke (2001) on pupils' accounts of mathematical thinking.

Recall of Events

In Table 4, about 85% of the year 5 pupils interviewed were able to recall the learning that occurred in their respective lessons without the need to use the video recording as a prompt for the recollection of the learning. Only one pupil (5%) did not respond to the "unstructured" question "Do you remember when the teacher asked you questions during class?" and the other 10% could give responses when the video recording was played for a short period of time.

Table 4
Categories of Responses of Recall of Events

Category of response	**Year 5**
No recall	2 (5%)
Could talk about their learning with video	4 (10%)
Recall spontaneously	34 (85%)
Total	n = 40

Description of Events

In Table 5, the pupils' recall of events were categorised in two categories. The first was that some of the pupils described the events in terms of what they had done in the lesson. This made up 70% of the pupils' responses. The other description described the outcome (30%).

Table 5
Categories of Description of Events

Category of Response	**Year 5**
No description of interaction	2 (15%)
Describe the action	28 (70%)
Describe the outcome	12 (30%)
Describe the event from their perspective	0
Describe their reasoning/justify their thinking	0
Total	n = 40

Three examples of the interview scripts from each level showing description of actions are presented as follows:

Interviewer: "So what did you say or do here?"
Abdul Mateen: "I use ruler and a measuring tape to measure lengths."
Siti Nur: "I multiply using vertical format."
Mohd Adi: "I counted 2/2 as 1 and 2/2 as 1 to make 2 and 1/2."

Three examples of the interview transcripts from each level describing the outcome are shown as follows:

Interviewer: "So what did you say here?"
Abdul Azli: "We compare fractions. Which one is bigger or smaller."
Amirah : "Change times from minutes to seconds."
Lailan: "Comparing fractions less than and more than."

There was no evidence of the pupils in either group describing the events from their perspective nor describing their reasoning or justifying their thinking.

Explanation of Their Learning

Table 6 shows 80% of the pupils in year 5 were able to specify their learning and they stated that they had learned something in the lessons. However, 5% of the learning was behavioural learning such as drawing, shading and colouring which was not mathematics. In addition, one of the twenty pupils in year 5 was unable to specify mathematical learning.

Table 6

Categories of Explanation of Their Learning

Category of Response	**Year 5**
Unable to specify learning	2 (5%)
Learned nothing	0
Learned a behaviour not mathematics	2 (5%)
Remembered factual information	32 (80%)
Learned how to do something	32 (80%)
Total	n = 40

An example of the description of doing something is cited below:

Interviewer: *"What did you learn today in mathematics?"*
Siti Juliana: *"Fractions."*
Interviewer: *"What about fractions?"*
Siti Juliana: *"Divide an apple into half and quarters."*
Interviewer: *"Can you give me an example of fraction from that?"*
Siti Juliana: *"One out of two parts because there are four equal parts"*

Explanation of Their Thinking

With reference to Table 7, the number of pupils who were able to explain their thinking for year 5 was significantly low at 5%.

Table 7
Categories of Explanation of Their Thinking

Category of Response	**Year 5**
No explanation of thinking	38 (95%)
Simple description of thinking	2 (5%)
Explicit description of thinking	0
Explain/reconstruct thinking, reasoning, justifying, evaluate thinking	0
Total	n = 40

Only four of the total number of pupils in year 5 who were interviewed was able to give a description of their thinking. Their thinking was a simple description instead of a more explicit manner. However, it is difficult to ascertain whether these pupils have been trained to think because the time constraints of the study did not allow the writer to follow the pupils in greater lengths. Therefore, the category of simple description of thinking was appropriate to code instances of thinking among the pupils interviewed. Unlike some pupils in Clarke's (2001) study, none of the forty interviewees offered an explicit description of thinking or reconstructed, justified or evaluated their thinking.

Conclusion

In conclusion, the pupils' ability to articulate their thinking can be attributed to some extent, to the quality of teachers' questions. Hence in this light, it is recommended that the pupils should be asked more open ended questions particularly probing questions to provoke thinking. While it is a fact that English is not their first language, the language use to formulate the questions should be common in all subjects for familiarity and for easy understanding. This is important so that the pupils can not only articulate but also use the language for discussion and argumentations. Discussion and argumentations can be cultivated from problems which are familiar, open and non routine. The Brunei primary mathematics syllabus has provided problems that are routine and non routine embedded in every topic. Hence, it is crucial for teachers to use these as a platform for lessons to inculcate discussion and argumentation. Further study can be done to consider mathematical thinking in terms of behaviour and attitude. Such studies may also include bigger sample size and covers the four districts in Brunei.

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MECHANISMS FOR SCIENTIFIC DEBATE IN REAL ANALYSIS CLASSROOMS

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This study uses a constructivist model of scientific debate to analyze two accounts of fruitful debate from separate real analysis classrooms. Our findings highlight the importance of such debate for helping students reconcile differences between their mathematical conceptions and others held in the community of practice through various forms of "conceptual triangulation" on shared mathematical tasks.

Cobb (1994) argued that mathematics instructors must coordinate the dual processes of promoting individual student's construction of mathematical understanding and enculturating students into standard mathematical meanings and processes. However, in many university classrooms, student conceptions remain tacit to the extent that incompatibilities between individual notions and standardized meanings go unrecognized or unresolved. This study investigates scientific debate as a pedagogical means by which members of the community of practice gain more faithful access to one another's mathematical conceptions such that both the construction of meaning and enculturation to standard mathematics can proceed in tandem. Moreover, we highlight mechanisms by which scientific debate can arise and proceed in the undergraduate classroom.

This study presents two accounts of successful scientific debate from different undergraduate, real analysis classrooms. In each, we identify the means by which the debate began and proceeded toward resolution. The studies came from the independent research of the two authors, and this paper represents one product of our collaboration to combine observations from separate instances of non-traditional, real analysis instruction.

Theoretical Framework

We draw the language of our framework for analyzing scientific debate from radical constructivism as outlined by Von Glasersfeld (1995). Von Glasersfeld posits that concepts are developed by individuals as tools for making sense of their ongoing experiences. As people cannot have access directly to one another's thoughts or concepts, all communication stands as new experiences for a participating individual. The need for scientific debate arises when the conceptions held by one or more parties engaged in a discussion are incompatible for the purposes of the task in which they are collectively engaged. The end goal of scientific debate is for the parties involved to adjust their respective conceptions as related to the given task until they achieve mutual "fit" or compatibility. Thus the basic constituents of scientific debate are (1) two or more communicating parties who each have constructed (and evoked) (2) one or more conceptions as related to (3) a given task (or set of tasks) being discussed.

The resolution of scientific debate depends upon the ability of the individuals involved to first recognize incompatibility between their respective conceptions and negotiate the process of altering their conceptions until compatibility is achieved. Once the incompatibility is recognized, the parties involved must reflect upon their conceptions to adjust toward mutual fit. Freudenthal (1991) described several modes of reflection under the umbrella of "shifting one's standpoint" (p. 105). The two most pertinent for the present analysis are shifting the vantage from which a task is being viewed or transferring the concept under consideration into an alternate task. Either form of shifting extends the discussion in new directions and allows constituent parties to identify the incompatibility(ies) in their thinking and adjust toward achieving mutual fit. We call these methods of shifting viewpoints "conceptual triangulation."

Study 1

The first study was conducted during a real analysis course at a large American public university. This course consisted of two class meetings per week (80 minutes in length) for 15 weeks. The first author of the paper observed the classroom as a researcher, and interviewed the instructor and 5 students most weeks of the semester (usually 7-9 interviews total) regarding the course content and classroom discussions. The flow of classroom discussion and all communication on the board were logged in the form of written notes.

The classroom differed from traditional Definition-Theorem-Proof instruction (Weber, 2004) in that the instructor expected the students to make determinations regarding the mathematical body of theory, and at least some students adopted this role in their own deliberations (Dawkins, 2009). The instructor always guided the classroom discussion, but consistently involved members of the class by presenting them with choices (of alternate definitions or possible theorems) about which the class voted and generated examples, arguments, and counter-arguments. In order to involve students in defining, the instructor only introduced definitions after the class discussed sets of examples and/or less formal ideas to which definitions correspond (e.g. continuity and connectedness).

The episode in this study occurred during the class session leading up to the definition of the limit of a function at a point. The instructor's intended to present examples that did and did not have limits due to restricted domains so as to motivate the standard requirement that limits only be defined at cluster (accumulation) points of the domain of the function. We divide the episode according to the instructor's introduction of various examples to guide the class to an agreement regarding those cases in which the limit can and cannot be defined. We particularly trace the viewpoints expressed by two of the most vocal students in the discussion, Cyan and Locke.

Results from Study 1

The instructor began class by writing the expression $\lim_{x\to p} f(x)$ on the board and asking, "When does this have meaning?" She introduced the function $f(n) = .5n$ defined only on the natural numbers and assigned the task of finding the function value and function limit at $x = 3$. Cyan asked about the domain restriction, and the instructor replied that this was a choice they must make. When she requested the value of the function at 3, both Cyan and Locke offered the value 3/2. Cyan explained that the image of $f(x)$ approaches 3/2 as x approaches 3.

In line with the instructor's original intention of highlighting what "approaches" means, she introduced a second task involving a function f defined as x^2 on the interval $[-1,1]$ and $f(5) = 2$, but not defined elsewhere. She asked, "What does it mean for x to approach 3?" directing their attention to the sequence $x = 1 - 1/n$. One student suggested that x must be "arbitrarily close" to 3. The instructor affirmed that "x approaches 3" means "x must get arbitrarily close to 3." No one expressed disagreement regarding the existence of this function's limit. The instructor then asked the class about the image of the first function at the point 2.75. Locke suggested that they should "pretend that it is in the domain." Initially the instructor did not directly respond to this assertion and pursued the topic of cluster points. She pointed out that there are no points in the domain arbitrarily close to 3 other than 3, but Locke expressed further confusion on this point and Cyan argued that x-values near to 3 should be considered. The instructor asked the class whether they want to "pretend" and asked Locke and Cyan directly whether they agreed that $f(5/2) = 5/4$. She articulated her surprise at their willingness to “define analysis” according to pretending. Locke protested that, “Just ‘cause it’s not in the domain doesn’t mean that the function doesn’t apply to it.” Though Locke and Cyan agreed in concert to evaluate the function at 5/2, other students articulated disagreement because of the original assumptions of the task.

To extend the discussion, the instructor introduced another task with a function defined as $(x^2-9)/(x-3)$ when $x \neq 3$ and as 0 when $x = 3$. She pointed out that $f(3) = 0$ and by constructing a table of values argued that the function limit at 3 equaled 6. The instructor persisted in questioning the class whether their answers depended upon “pretend games.” When students in the class expressed agreement that the function value was 0 while the limit was 6, she asked why the limit in the prior example had to equal 3/2. Cyan protested that, “We know the behavior of [the previous] function.” Locke, seeking to clarify the source of the disagreement, explained, “We were thinking of [the first example] on R to R.”

The instructor then introduced a final task with a function defined as $.5n$ on the natural numbers, but which had a vertical asymptote at $x = 3$. She pointed out that she could “pretend” that the original function was extended this way. Cyan continued to disagree with her because, “We have more information.” Locke revised his claim that the original function was defined everywhere. The instructor asked the class, “Do you see how un-mathematical this is?” Later, when she summarized their discussion of cluster points, the instructor wrote on the board: “In order to make sense of $\lim_{x\to p} f(x)$, there must be points in the domain of f arbitrarily close to p. Otherwise $\lim_{x\to p} f(x)$ becomes a ‘pretend game’.” In later interviews, both Locke and Cyan expressed understanding and agreement with the instructor’s argument that functions defined only on the natural numbers cannot have limits.

Discussion of Study 1

Cyan’s assertion and connected explanation regarding the first function limit made the instructor aware that their conceptions of the task were not fully compatible. She initially assumed there was a difference between their conceptions of limits, though it turned out that Cyan and Locke’s conception of domain was incompatible with hers. Locke and Cyan’s conceptions of domain, which were formed at lower levels of instruction, may only exclude points at which functions are not well-defined (dividing by zero, roots of negatives, etc.) such that restricting a linear function to the natural numbers created confusion.

Once the first task revealed that there was some conceptual incompatibility, the instructor proceeded to introduce three related tasks intended to help Cyan and Locke to adjust their disparate conceptions toward standardized conceptions. The second task emphasized the “approaches” aspect of the function definition. The third task was intended to separate the function limit from the function value. When Cyan and Locke agreed with her assessment of the second and third task, the instructor understood that neither of these issues related to limits housed the disparity between Locke and Cyan’s thinking and her own.

After the instructor repeatedly addressed aspects of function limits, Locke called her attention directly to the issue of domain (though it had been mentioned repeatedly). He explained that he understood the original function to be defined for all real numbers. The final task the instructor introduced addressed the ambiguity of assuming the value of functions outside their stated domain. This example displayed the instructor’s conception of domain for Locke and Cyan, namely that it is equally mathematically valid to assume any possible function values for inputs outside of the stated domain. At the time of interview after this class meeting, both students reported agreement with the instructor’s assertion meaning that they had adjusted their conception of domain to “fit” the instructors’ (standardized) conception of domain relative to the tasks discussed in class. The differing assessments of the first task led the instructor, Locke, and Cyan to reflect upon their constituent conceptions related to the limits of functions. The instructor guided the students to triangulate their conception of the first task by viewing various constituent conceptions ("approaches", function values and limits, function domain) in alternate tasks. Ultimately they were able to resolve the scientific debate after three standpoint shifts.

Study 2

The second study was conducted as part of a teaching experiment (Steffe & Thompson, 2000) in a real analysis course at another large, American public university. The real analysis course consisted of two regular sessions (75 minutes in length) and a recitation (50 minutes in length, mainly for cooperative proof writing) for 15 weeks. The second author served as an instructor of the course. Eleven mathematics or secondary mathematics education students volunteered to participate in this study, working in three small groups. Although some content was provided in a traditional lecture style, instruction in the course mainly followed an inquiry approach, in which students were often asked to make and justify conjectures and to evaluate arguments. Each day, one student from each group was assigned rotationally as a group facilitator of the day. The facilitators were instructed to ensure that all members of their group share their ideas and thoughts freely with one another and help them attend fully to one another's perspectives.

On the day of the study, the students were asked to work with the following task: (1) Does there exist a real number x satisfying that for any $\varepsilon > 0$, $|x| < \varepsilon$?, and (2) If there does exist an x, is there any other real number x satisfying that for any $\varepsilon > 0$, $|x| < \varepsilon$? Prior to group discussion, the instructor gave the students a short time to think individually about the given task. The facilitator of each group then initiated group discussion by giving individual students time to present their initial thoughts about the task; group discussion followed in order to share ideas and form a consensus. After the group discussion, two students from each group were sent to the other two groups such that each group had a representative from all three original groups. Students in a newly formed group presented their original group's ideas and listened to students from the other groups. At that stage, students asked or responded to questions from other groups to understand the various ideas, without debating the validity of the ideas. After sharing ideas from different groups, everyone returned to their original group to compare and contrast their ideas with those of the other groups.

Student activities were videotaped, and students' written work was pen-casted (synchronized with their voice). We focused on individual students' shifting standpoints regarding various approaches to the task and how those approaches were expressed by different people. We then examined how this type of practice allowed constituent parties to identify and reconcile the incompatibilities in their thinking.

Results from Study 2

Our analysis in this paper focused on a group of four students: Alan, Andy, Mary, and Oliver. When the students received the task, Andy was assigned to be the facilitator of the group. After a short time to think individually, Andy asked his group to share their thoughts on the task. Oliver first suggested choosing $x = 0$ and then $x = \varepsilon - 1/\varepsilon$. Mary expressed that she did not figure out what she was supposed to do for the task, but liked Oliver's idea. Alan also said that he did not understand what the task was about. Oliver reinterpreted for Alan that the task was questions about the existence and uniqueness of x, to which Alan assented. Finally, Andy agreed with Oliver to consider 0 as a value for x, and claimed that 0 would be the unique value for x. At this point, the students did not address the fact that Andy and Oliver's answers to the task were incompatible. After everyone presented his or her initial thoughts, Oliver went on to explain:

Oliver: Choose x to be x = ε – 1/ε since ε > 0 instead of going to be equal to 0. And so, ε – 1/ε means that no matter how small you get [uses two fingers for a small increment], there's something slightly smaller. That [x] will be less than ε and the absolute value of that [x] would be less than ε.

Alan and Mary immediately accepted Oliver's idea of choosing x in terms of ε, and suggested 1/ε as another value for x. At that point, no one in this group noted that their values for x might not satisfy the inequality $|x| < \varepsilon$ unless $\varepsilon > 1$, however, Andy realized that his answer was distinct. He insisted that 0 was the only value for x because the inequality should be satisfied for any positive value of ε, but the other group members did not accept Andy's claim. Alan contended that there were plenty of other ways to choose x such as 1/ε.

Andy: As just ε gets smaller and smaller, that puts more and more of a restriction on what x can be. And ε will never actually reach 0, but it will get very, very close to 0, and could basically be any number other than 0. [...] So, the only value of x that would actually work is 0, as far as I can tell.

Alan: Actually, I think this is not true. In other words, that x is not unique. [...] What I'm thinking is along the line of what Oliver talked about. 1/ε is always gonna be smaller than ε because ε is greater than 0. [...] Since we're trying to show the existence of x, choose x = 1/ε. Alright? So, then, you can choose, I mean, there are certainly plenty of other choices, [...] if you choose x = 1/ε, ε can be anything. ε is arbitrary as long as it's, uh, it's arbitrary, 'cause [ε is] an arbitrary real number greater than 0.

Andy: Um, well, x has to be a fixed value though. You can't put it in terms of ε.

Oliver: I believe, I believe you can because ε is arbitrary.

Mary: [...] It's already part of the inequality. So, I don't understand why you don't think we can set x as 1/ε.

At that point, the students were all aware of the arbitrariness of ε in the task. However, Andy's understanding of how the arbitrariness would work in the task was quite different from the other students; Andy treated the arbitrariness of ε to imply the decrease of ε towards 0. This created a restriction on choosing x independently from ε, whereas other students allowed the dependence of x on ε. Alan, Mary, and Oliver seemed to perceive the given statement as, "for any $\varepsilon > 0$, there exists a real number x such that $|x| < \varepsilon$," in which the order of the variables x and ε was reversed. (It is a similar result to Dubinsky and Yiparaki (2000) and Roh's (2010) studies, in which students tended to perceive ∃∀ statements as ∀∃ statements.) Nonetheless, the students did not yet recognize that the difference between the ways they were interpreting the given statement were incompatible. Alan was first to adopt a standard interpretation of the statement, and Oliver adjusted his viewpoint because x preceded ε in the given task. By the end of the group discussion before exchanging groups, Alan, Andy, and Oliver agreed that due to the order between x and ε in the given task, x must be defined prior to ε.

Alan: x comes first, right?

Alan: Yeah, so we can set x to be ... Oh, wait! ε doesn't exist when x comes into existence. So, we can't set x to be 1/ε.

Andy: Yeah, 'cause you need to pick a value of x but if you haven't defined what ε equals –

Oliver: Yeah, I would guess that we're saying that due to the set up of the problem, 0 was the only choice that satisfies the first part. And, [reading part 2]. So, I would say due to the set up of the problem which is that x is identified first.

Andy: Yeah, I think that is probably the best way to go about it.

When the instructor asked the class to exchange groups, Mary still did not understand why 0 was the only value and 1/ε could not be chosen for *x*. Since she was reluctant to go alone to a different group, Oliver decided to stay with her when exchanging group members. It was Elise who joined from a different group and helped Mary understand why 0 should be the only value for *x*. Elise reported that her group concluded that 0 is the unique value for *x* in the task, and presented how her group came to this conclusion. First, Elise balled her right hand into a fist to represent *x* and stated, "'there exist' is an existence where you choose it [*x*]. There it's just one number. You have to choose *x* and that [*x*] can't change." She moved her left hand horizontally, back and forth over her fist, to represent 'for any $\varepsilon > 0$, $|x| < \varepsilon$.' Tapping the fist, which was not moving, Elise also stated that, "It says 'for any ε,' that means no matter what you choose for ε, this number [*x*] has to work." She then demonstrated contradictions in cases where nonzero real numbers were chosen for *x*.

Elise: If I choose, […] like say I choose .5 [for x], I can choose an ε greater than 0 but less than x. […] So, the only thing that works for x is 0. […] But because 0 is the only number that works, there is not anything else. Because no matter what number you choose [x] bigger than 0, there will be an ε smaller [than x].

Elise's presentation led Mary and Oliver to reflect on their group's conceptions of the task. Oliver perceived that his conception and his group's prior explanations were compatible with Elise's explanation. He treated his conception that "*x* cannot be defined in terms of ε" as being compatible with Elise's conception that "*x* can't change" Mary perceived a distinction between her group's explanations and Elise's, and strongly preferred Elise's explanation.

Mary: We found the same thing for the first part. And the second part we thought of other ideas, but came to the same conclusion that there does not exist any other x's that satisfy the thing –

Oliver: For the exact same reasons. So if you change the order of how ε is in front, so 'for any ε > 0, does there exist an x that's element of the reals?', then we would be able to choose something but otherwise we aren't able to define something [in terms of ε].

Mary: and how you explained it is exactly how I needed to hear it 'cause they weren't saying it, anything like how you're saying it.

Elise: Yeah, that one. Because you have to go in order, it's like you do there exist –

Mary: They said the order part, and then I still didn't get it until you [said] –

When the instructor asked students to return to their original groups, Andy recapitulated what he learned from Elise's group as well.

Andy: The group [Elise's] that I went to basically had the same train of thought that we did. […] they came up with the fact that 0 would work, and they also came up with that it was the unique solution to the problem. They were also thinking about being able to put it [x] in terms of ε but realizing that you can't do that because x comes first. […]

Mary: The girl [Elise][…] explained it exactly how you just explained it. I didn't understand that concept before […]but now it makes sense. Like, I heard you guys saying it. I just couldn't understand why, and now I get it.

The group discussion after students returned to their original groups affirmed the sense of mutual compatibility between all of the group members' assessment of the given task, where *x* must be defined before ε. The group exchange provided students with an opportunity to compare various perspectives by attending to alternate interpretations made by different people. Consequently, students in this group, Mary in particular, were able to shift their standpoints toward the one shared by their group members.

Discussion of Study 2

The need for scientific debate in Study 2 arose when Andy's answer to the task was not the same as that of Alan, Mary, and Oliver. The students constructed different conceptions about the arbitrariness of ε and the relationship between ε and x in the given task. The classroom environment introduced iterative stages into the scientific debate. Time for individual thinking followed by articulation enabled the students to share various answers to the task in the group and recognize the incompatibility between Andy and the other students' conceptions. Alan's explanation that, "ε doesn't exist when x comes into existence," helped Alan and Oliver adjust their conception regarding the logical dependencies between ε and x in the task. The group exchange allowed Mary to view the task in terms of Elise's gesticulatory explanation and adjust her conception to fit that expressed by her group mates previously. The final exchange with the original group affirmed the sense of mutual fit.

Conclusions

Analyzing the structures of the two studies reveals parallel mechanisms that initiated the scientific debates and allowed them to proceed. In both studies, a *shared task* presented the means by which different parties in the classroom could assess the relative compatibilities of their conceptions of key mathematical topics (e.g. domain, limits, interdependence of quantities in a logical proposition). The phrase "shared task" should be distinguished from other mathematical activities in the classroom in the sense that questions which are presented and promptly solved by the instructor (for instance) do not provide the opportunity for different parties to bring their conceptions to the attention of other members of the class. By a shared task, we mean a mathematical task that is introduced into the class discussion with the expectation that multiple parties can or will share their understanding thereof with other members of the class. In both studies, the shared task revealed conceptual incompatibilities, but the scientific debate was advanced by means of conceptual triangulation. In Study 1, the introduction of alternate shared tasks with the same constituent conceptions provided the means of triangulation. In Study 2, the stages of class discussion introduced other students' conceptions expressed in terms of alternate explanations as a means of triangulation on the same task. These two processes of triangulation are represented in Figure 1.

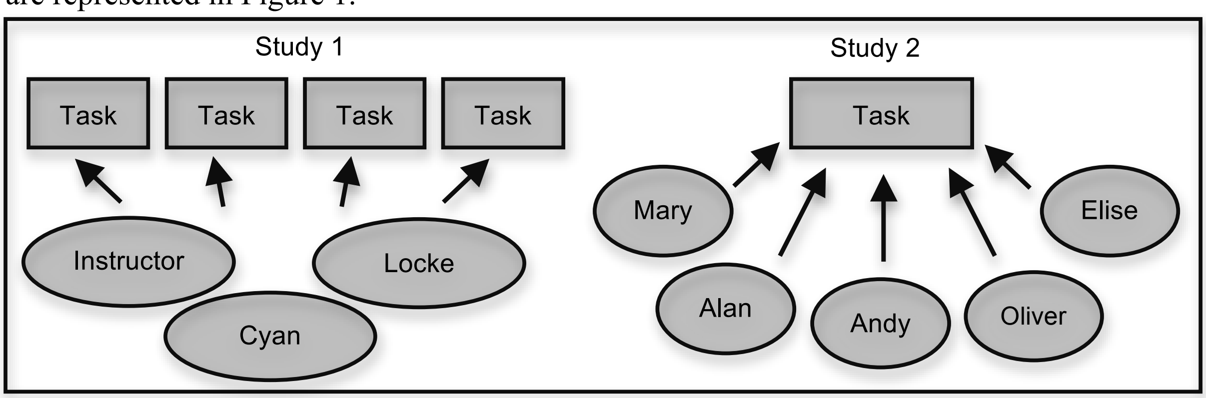

Figure 1. Alternate means of conceptual triangulation in the two examples of debate

Though the participant members of both debates achieved acceptable compatibility in their conceptions relative to the original shared task, the structures of the courses, which facilitated this progress, differed meaningfully. We classify both of the classroom environments in question as non-traditional proof-based instruction in large part because they involve shared tasks in which the students are expected to form and express their conceptions about the topics being discussed. However, in the first study the classroom discussion filtered through the instructor and she was able to leverage her knowledge of both analysis and of student difficulties in analysis to introduce tasks that would likely move the debate forward. In the second study, no alternate tasks were introduced, but students heard a variety of explanations of the same task. Thus the second debate progressed and reached conclusion without instructor intervention beyond setting up the parameters and responsibilities of group discussion. These studies highlight the importance of fostering scientific debate in the classroom; the confusion that each debate overcame could easily have remained latent in an alternate pedagogical setting. However, such exchanges take time. These studies describe two points in the middle of a continuum of real analysis pedagogy between presentation as in Definition-Theorem-Proof instruction and guided reinvention of standard mathematics (Gravemeijer & Doorman, 1999; Swinyard, 2008). Each yields benefits and drawbacks. The second study's structure empowered students to reach conceptual fit themselves; the former opened the door for a secondary discussion, prompted by the "pretend" term, on a meta-mathematical norm of operation that mathematicians must reason within the constraints of stated assumptions.

Our central claim about scientific debate mirrors that of Cobb, Boufi, McClain, and Whitenack (1997) when they said, "We… conjecture that children's participation in [reflective] discourse *constitutes conditions for the possibility of mathematical learning*" (p. 264). We do not claim that every student in the classrooms we described came to the same conclusions. We posit that classroom environments in which shared tasks arise and scientific debates are appropriately pursued enhance the probability that larger portions of the class will achieve higher degrees of conceptual fit. The process of adjustment toward conceptual fit between the members of the community of practice constitutes a means of tandem progress toward the dual classroom goals of student conceptual development and enculturation to standard mathematics.

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EXPLORING THE DISCOURSE ON CONTINUITY IN RELATION TO LIMITS IN A BEGINNING-LEVEL CALCULUS CLASSROOM

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Students' informal views of the notion of continuity can present challenges as they attend to the precise aspects of various calculus concepts. This paper explores continuity in relation to the limit concept in a beginning-level calculus classroom. The study uses a discursive approach developed by Sfard (2008) to investigate the instructor's discourse on continuity and the realizations of limit it supports. This is followed by the comparison of the instructor's discourse on continuity with those of the students. The findings indicate that, despite the instructor's explicit focus on the relationship between continuity and limits by means of the precise definition of continuity, students mainly focused on the intuitive definition in the instructor's discourse. Therefore, students in this study realized continuity through dynamic motion but not through limits.

Introduction

In many undergraduate calculus courses and textbooks, a precise definition of *continuity* is often given after the discussions on the concept of limit (Hughes-Hallett et al., 2005; Weir, Hass, Giordano, & Finney, 2008). However, earlier assumptions about continuity may underlie learners' realizations of several calculus concepts. For example, the realization of infinity as a process that goes on indefinitely is based on the assumption of natural continuity (Lakoff & Núñez, 2000). Students' incorrect considerations of functions as rules given by a single formula also rely on an implicit assumption of continuity (Tall & Vinner, 1981). Similarly, research has identified many obstacles students have about limits resulting from a dynamic view of the concept, which is based on continuous motion (Tall, 1980; Tall & Schwarzenberger, 1978; Williams, 1991). In particular, Bezuidenhout (2001) argued that *limit implies continuity* and *limit as the function's value* are among the incorrect realizations of limit that are based on students' informal views of continuity. These studies suggest that students' basic informal realizations of continuity may later create obstacles as they attend to the precise aspects of various calculus concepts.

This study explores continuity in relation to the limit concept in the context of undergraduate mathematics education and addresses the following questions: (a) How is the notion of continuity introduced by the instructor in a beginning-level undergraduate calculus course?, (b) Which realizations of the limit concept does the instructor's discourse on continuity support?, and (c) How do the elements of the instructor's discourse on continuity compare and contrast with those of the students?

Theoretical Framework

The study uses a discursive lens, namely, Sfard's (2008) *commognitive* framework. The term *commognitive* combines the terms *cognition* and *communication* by considering thinking as an individualized form of communication. This lens considers discourse as the main unit of analysis, where discourse is defined as "the different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). From this view, mathematics is a particular type of discourse that can be distinguished by its word use, visual mediators, routines, and narratives.

Word use refers to the ways in which words are used in a mathematical discourse. *Visual mediators* are the visible objects that are created to enhance mathematical communication. *Routines* refer to the set of metarules that characterize the repetitive patterns in the activity of participants of a mathematical discourse. *Narratives* refer to the set of utterances describing mathematical objects and their relationships that are subject to endorsement or rejection. The narratives of a mathematical discourse that are endorsed by the majority of the experts of the community are considered as true (e.g., definitions, theorems).

A critical feature of word use is *objectification*, which occurs through *reification* and *alienation*. Reification "is the act of replacing sentences about processes and actions with propositions about states and objects" (Sfard, 2008, p. 44), whereas alienation refers to "using discursive forms that present phenomena in an impersonal way, as if they were occurring of themselves, without the participation of human beings" (Sfard, 2008, p. 295). Said differently, objectification changes the talk about processes to the talk about end-states or mathematical entities. This study focuses on the participants' word use and routines in the context of continuity and its relationship to the concept of limit.

Methodology

This work is part of a case study that investigates the development of discourse on limits. The participants included one instructor and his section of undergraduate students taking a beginning-level calculus course in a large Midwestern university. For the instructor's discourse, the data consisted of field notes and eight video-taped classroom observations in which the instructor discussed limits and continuity. For the students' discourse, the data consisted of four students' responses to a task-based interview and the field notes taken during the interviews. The interviews were audio-taped and lasted between 53-76 minutes. Participation in the interviews was voluntary. The class observations and the interview sessions were transcribed with respect to the participants' utterances and actions.

Although the main focus of the interviews was the limit concept, in four of the six interview questions, students also had the opportunity to talk about continuity. There were two questions consisting of piecewise functions given in algebraic form. Students were asked to find the limit of the functions at the break points, and at positive and negative infinity. They were then asked to discuss the continuity of the functions. In another question, the students were given the graph of a function and determined the limits of the function at various points. They then discussed continuity of the function at those points. The final continuity-related question was an adaptation of Bezuidenhout's (2001) task, where students were given the statement $\lim_{x \to 2} f(x) = 3$. They were then asked if the following are true given the statement: (a) f is continuous at the point $x = 2$, (b) f is defined at $x = 2$, and (c) $f(2) = 3$. The purpose of this question was to investigate whether students had the difficulties *limit implies continuity* and *limit as the function's value*, which are based on assumptions about natural continuity.

Participants' word use was analyzed with respect to the degree of objectification in their discourse. Particular attention was given to the realizations of limit the instructor's and students' discourse on continuity supported. Consistent with Sfard's (2008) terminology, participants' word use was considered *operational* if they talked about limit as a process based on natural continuity; *objectified* if they talked about limit as a distinct mathematical object or a number that is obtained at the end of the limit process.

The operational word use on limits relies on words signifying continuous motion (e.g., [the function values] "approach", "get closer and closer to", "go to" [the limit L]). The objectified word use on limits, on the other hand, either relies on words signifying proximity (e.g., [the function values are] "arbitrarily close to", "within ε distance of" [the limit L]) or refers to limit as a particular number (e.g., "the limit is equal to five") or a mathematical object (e.g., "the limit exists", "we can find the limit by plugging in").

The following pseudonyms will be used for the participants: Dr. Brenner (the instructor); Amy, Jessica, Harry, and Keith (the students).

Results

The instructor's discourse on continuity

Dr. Brenner introduced continuity at the end of his discussions about limit and before the review session he conducted for the mid-term exam. He mentioned that he was going to give one intuitive and one mathematical definition for continuity and asked students to use the latter in the exam. Table 1 shows Dr. Brenner's intuitive definition of continuity. Since he did not utter any limit-related words in this definition, his word use on limits was not coded.

	What is said	What is done
[1]	So what is a continuous function?	He writes continuous functions on the board.
[2]	A continuous function is a function, and I am just going to say it in words, that I can graph without taking the chalk off the board.	He states these verbally.
[3]	So this is really all that we need to know, from an intuitive perspective, about continuous functions.	He states these verbally.
[4]	Continuous just means I can graph without taking the chalk off the board.	He states these verbally.

Table 1. Dr. Brenner's utterances about the intuitive definition of continuity

After introducing the intuitive definition of continuity, Dr. Brenner drew graphs of a variety of functions and discussed their continuity based on whether he could graph them "without taking the chalk off the board" (Table 1, [2], [4]). While doing so, he moved his hand along the graphs of the functions and mentioned that "approaching and graphing like this" could be thought as a "limiting process". Therefore, he relied on the limit process and whether he took the chalk off the board when determining the continuity of a given function from its graph. Dr. Brenner's discourse on the intuitive definition of continuity was based on dynamic motion where limit was considered as a process.

Dr. Brenner then introduced the precise definition of continuity and, unlike the intuitive definition, he wrote it on the board. Figure 1 and Table 2 show his precise definition of continuity.

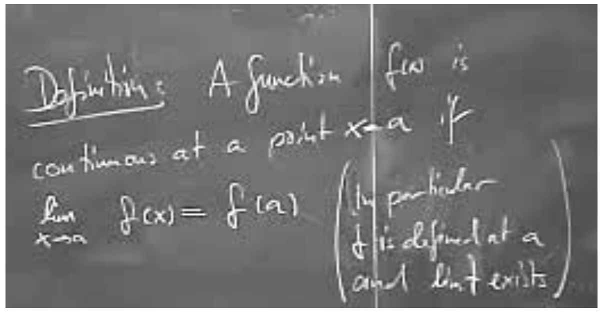

Figure 1. Dr. Brenner's precise definition of continuity

What is said	What is done	Type of limit-related utterance
[1] So let's try to make this precise.	He states these verbally.	
[2] A function f of x is continuous at a point x equals a if…	He writes these on the board but does not finish his sentence.	
[3] So what I want to say is what does it mean to be continuous at this point?	He states these verbally.	
[4] The limit has to exist and it has to equal to the function value.	He states these verbally.	Objectified
[5] So I need the limit to exist.		Objectified
[6] So the limit as x approaches a of f of x has to exist and equal to f of a.	He writes $\lim_{x \to a} f(x) = f(a)$ on the board.	Objectified
[7] Just writing this already implies that it [the limit] exists because it equals a number.	He states these verbally.	Objectified
[8] So, in particular, f is defined at a and the limit exists.	He writes these in parenthesis near the definition.	Objectified

Table 2. Dr. Brenner's utterances about the precise definition of continuity

Dr. Brenner's precise definition of continuity considered limit as a number (Table 2, [7]), which is equal to the function value at the limit point (Table 2, [4], [6]). His utterances about limits were objectified when he talked about the precise definition. In fact, 97% of his 90 utterances about limits in this lesson on continuity were coded as objectified.

It should be noted that Dr. Brenner did not explicitly talk about continuity prior to the lesson on continuity although his operational word use on limits in earlier lessons was based on continuous motion. The lesson on continuity was the first context in which he connected the notions of continuity and limit. Later in the lesson, he also said that "a question about continuity is a question about limits" and "the definition of a continuous function is given in terms of the limit" to emphasize the connections between the two concepts.

Dr. Brenner talked about *plugging in* as one of the possible ways of computing a limit in prior lessons. However, he put extra emphasis on plugging in as a means of computing the limit of a *continuous* function during the lesson on continuity. For example, while showing

$f(x)=\frac{x+1}{x^2+1}$ is continuous, he showed for every number a, $\lim_{x\to a} f(x)=\frac{a+1}{a^2+1}=f(a)$ and mentioned that he found that limit by plugging in. He then turned back to the precise definition of continuity and described the relationship between continuous functions and plugging in as illustrated in Table 3.

It is possible for students in Dr. Brenner's class to generalize this routine of plugging in for continuous functions to functions in general. However, Dr. Brenner explicitly highlighted that a function's value at the limit point is not related to the limit of the function at that point. He said: "To compute the limit as x approaches zero, what the function does at the point x equals zero is irrelevant. What only matters is what happens nearby". Similarly, he noted in an earlier lesson that

> The function value is something completely different. And here we see again that the actual value of the function at the point where we are looking at has very little to do with the limit. In fact, it has nothing to do with the limit.

What is said	What is done	Type of limit-related utterance
[1] In fact, we found this limit by plugging in.	He shows $\lim_{x\to a} f(x)=\frac{a+1}{a^2+1}=f(a)$	Objectified
[2] A continuous function is a function where we can always find limits by plugging in.	He states these verbally.	Objectified
[3] That is really what this definition says.	He goes back to the precise definition of limit he wrote on the board (See Figure 1).	
[4] f is continuous if you can find the limit at a by plugging in.	He first shows $\lim_{x\to a} f(x)$ and then $f(a)$ in the definition (See Figure 1).	Objectified
[5] That is exactly what the definition says.	He states these verbally.	
[6] In other words, function f is continuous at a if we can find the limit of f at x equals a by plugging in.	He writes these on the board.	Objectified
[7] These are great functions.	He states these verbally.	
[8] Continuous functions are exactly the ones that we can find the limit by plugging in.	He states these verbally.	Objectified

Table 3. Dr. Brenner's utterances about the relationship between finding the limit of a continuous function and plugging in

In summary, Dr. Brenner's intuitive definition of continuity supported the realization of limit as a process whereas his precise definition supported the realization of limit as an end-state: a number. In the context of the informal definition of continuity, his word use on limits was operational. In the context of the precise definition of continuity and following applications, his word use on limits was objectified.

The Students' Discourse on Continuity

When exploring continuity of functions at particular points, students did not talk about continuity through limits during the interviews. Amy talked about points of removable discontinuity as "jumps" and attended to the "open circles", or holes in graphs when exploring continuity of functions: "it [the function] is discontinuous because it has an open circle [shows the *y* value where the function does not attain its limit], which means that the function jumps essentially [moves her hand along the graph]".

Jessica's arguments about continuity were related to "connections" and "links". When discussing the discontinuity of one of the piecewise functions from the graph she drew, she said:

> It is discontinuous because we cannot find a connection or link between these two graphs [shows the piecewise graphs on the right and the left side of the *y* value where the function does not attain its limit]. You just need to focus on the point [shows the *y* value again] that connects both continuous functions.

In addition to her arguments about connectedness, Jessica also attended to the instances where the functions jumped: "It [shows the function value] jumped here [moves her hand along the graph]. It is not continuous".

Harry also used the word "jump" when talking about continuity of a function at a given point: "it is not continuous because the value [shows the *y* value of the function] jumps". He later mentioned taking his hand off the paper: "So the way I see if a function is continuous…Am I taking my hand off the paper? I had to take it off now [moves his hand along the graph] so it is discontinuous".

Keith's main routine when determining the continuity of a function was to find the instances where he took his pencil off the graphs of the functions: "It [the function] is continuous because you would not have to pick up your pencil. I know there is a more mathematical reason but I can't remember it now". Keith was aware that this explanation lacked mathematical precision but could not think of a way other than the intuitive approach of tracing graphs and reporting the points where he picked up his pencil as points of discontinuity. He also talked about jumps: "It [the function] is not continuous because there is a jump".

None of the students talked about continuity in relation to limits when examining the continuity of the functions in the interviews. This was also evident by the fact that none of the students uttered the word "limit" when investigating the continuity of a function. Instead, they often used the word "jump", which was not a word Dr. Brenner used when talking about continuity. Their approach was in accord with Dr. Brenner's intuitive definition of continuity in that they moved their hands along the graphs and attended to the instances where they took their hand or pencil off the graph. Note that, Dr. Brenner moved his hand along the graphs and determined functions' continuity based on whether he could graph them "without taking the chalk off the board" (Table 1, [2], [4]) in the context of the intuitive definition of continuity. He also noted that approaching and graphing in that manner could be thought as the "limiting process". So, by referring to limit as a process, his word use on limits was operational in the context of intuitive definition of continuity. Although the students also used dynamic motion, it was not clear how, or whether, they realized limits when exploring the continuity of functions since they did not talk about limits in that context.

When working on the adapted version of Bezuidenhout's (2001) task, which particularly investigated the student difficulties *limit implies continuity* and *limit as the function's value,* one student (Amy) talked about limits in relation to continuity.

Limit implies continuity refers to the idea that if a function has a limit at a point, then it must be continuous at that point (Bezuidenhout, 2001). Amy's arguments supported this idea during the interview. She said:

> Limits are only defined for continuous functions I believe. I am going to say it [the limit] does not exist because the point is open [shows the graph of the function where there is a hole] so it [shows the function] is not continuous.

When working on Bezuidenhout's (2001) task, Amy mentioned that if $\lim_{x \to 2} f(x) = 3$, then f must be continuous at the point $x = 2$ and said "so what I know about limits is that they can only exist if the function is continuous at that particular point [shows $x = 2$]". Therefore, Amy clearly thought that limit implies continuity. The other students did not show any sign of the difficulty throughout the interview. They drew graphs as counter examples in which the functions were discontinuous but had limits. However, none of them explicitly referred to continuity when reasoning about their graphs. Instead, they just showed their graphs and said the limit existed.

Limit as the function's value refers to the idea that when finding the limit of a function at a given point, it is sufficient to look at the function's value at that point (Bezuidenhout, 2001). This view of limit is in close relation to the routine of plugging in the limit point to the function to find the limit of the function at that point. Amy also showed signs of this difficulty. She argued that the statement $\lim_{x \to 2} f(x) = 3$ implied the function is defined at $x = 2$, and $f(2) = 3$. Her main routine when finding the limit of a function at a given point was to plug in the limit point to the function. If she obtained a number, she reported it as the limit value; if the function was not defined at the limit point or if the function did not attain its limit value, she mentioned that the limit did not exist. Jessica, Harry, and Keith consistently attended to the right hand and the left hand limits and whether they were equal to each other when finding limits. Even though they utilized plugging in when applicable, they did not generalize this routine to all functions and showed no sign of considering limits as function values during the interviews. They all drew graphs as counter examples in which the limit of the function at the limit point was not equal to the function value. Yet, they did not explicitly refer to continuity when reasoning about their graphs but used phrases of the form "limit exists but the function value is different".

Note that considering *limit as the function's value* through plugging in is consistent with the assumption of continuity in that, in order for the routine of plugging in to work, the function must be continuous at the limit point. Therefore, it is not surprising that Amy agreed with both of the ideas *limit implies continuity*, and *limit as the function's value*. According to her, if limits existed only for continuous functions, then the limit value had to be equal to the function's value.

Dr. Brenner explicitly mentioned that a function's value at the limit point was irrelevant to the limit of the function at that point. However, he also frequently mentioned one could find the limit of continuous functions by plugging in. Although he used different function types (e.g., piecewise, trigonometric, rational, etc.) for which the routine of plugging in could not be used when determining limits, the routine supported Amy's focus on the function values when computing limits. On the other hand, his work with a variety of functions rather than everywhere continuous polynomials may be why the other students did not show signs of the difficulties. Nevertheless, those three students' realizations of continuity were still limited to the intuitive rather than the precise definition.

In summary, students in this study did not talk about limits when exploring continuity. Students' discussions on continuity were based on the graphs they drew and the informal definition of continuity rather than limits even when their exploration of continuity immediately followed their computations of limits for the functions. Amy's discussions of continuity when exploring limits resulted from her incorrect views about limits. The other students could provide accurate counter examples for the statements investigating the relationship between continuity and limits but could not elaborate on their reasoning through the precise definition that connects these two notions. Despite the instructor's extensive focus on continuity in relation to limits, students' discourse on continuity was based on dynamic motion but not on limits. Therefore, although their routines supported the realization of limit as a process, their lack of utterances about limits did not enable the categorization of their word use on limits as operational or objectified.

Discussion

While a lesson devoted to continuity may highlight the mathematical definition and how continuity is related to the limit concept, students' previous informal experiences with continuity may interfere with their development of mathematically rigorous realizations of the concept. Although the lesson on continuity was the first time students in this study were formally introduced to the notion in this classroom, it was clear that they had some previous realizations of continuity. First, they often mentioned the word "jump" when talking about discontinuity, which was not a word the instructor uttered in the lesson. Second, Dr. Brenner did not write his intuitive definition on the board and so did not leave a written trace (e.g., in the form of class notes) for students to refer to. Yet, the intuitive approach dominated the students' discourse on continuity. Further, the word "continuous" is a commonly used everyday term. Therefore, students' use of the word "jump" may be related to their early learning experiences (e.g., with teachers and textbooks using the word "jump" as well as with daily uses of the term continuity).

Some elements of the instructor's discourse seemed to help the students identify the mathematical features of continuity that were different than their previous realizations. For example, Dr. Brenner used a variety of function types that challenged the common assumptions about continuity. Moreover, he explicitly highlighted that a question about continuity was a question about limits, and the function's value was irrelevant to the limit value. On the other hand, some other elements in his discourse (e.g., plugging in and his "taking the chalk off the board" routine) may have perpetuated the dynamic view in the students' discourse on continuity.

The dominance of the dynamic view in students' discourse on continuity has some implications for their discourse on limits. Besides the difficulties mentioned here (*limit implies continuity,* and *limit as the function's value*), research has identified other difficulties about limits that are based on tacit assumptions regarding natural continuity through dynamic motion. Therefore, students' common assumptions about dynamic motion need to be challenged. It might be important for instructors to use the lesson on continuity during which they relate continuity to limits as an opportunity to highlight the *qualitative* differences between an intuitive, motion-based approach and a precise, static approach to the notion.

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MATHEMATICALLY IMPORTANT PEDAGOGICAL OPPORTUNITIES

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The mathematics education community values using student thinking to develop mathematical concepts, but the nuances of this practice are not clearly understood. For example, not all student thinking provides the basis of productive discussions. In this paper we describe a conceptualization of instances in a classroom lesson that provide the teacher with opportunities to extend or change the nature of students' mathematical understanding—what we refer to as Mathematically Important Pedagogical Opportunities (MIPOs). We analyze classroom dialogue to illustrate how this lens can be used to make more tangible the often abstract but fundamental goal of pursuing students' mathematical thinking.

Research in mathematics teacher education suggests the benefits of instruction that builds on student thinking (e.g., Fennema et al., 1996), but such instruction is complex and difficult both to learn and to enact (Ball & Cohen, 1999; Feiman-Nemser, 2001; Sherin, 2002). Often opportunities to use student thinking to further mathematical understanding either go unnoticed or are not acted upon by teachers, particularly novices (Peterson & Leatham, 2009; Stockero, Van Zoest, & Taylor, 2010). Despite a growing number of teachers who are convinced of the value of student thinking and the need to encourage it, neither teachers nor those who educate them have a clear understanding of how that thinking can best be used to develop mathematical concepts (Peterson & Leatham, 2009; Van Zoest, Stockero, & Kratky, 2010). We address this issue by providing a conceptual framework for thinking about the mathematically important pedagogical opportunities provided by student thinking.

Although skilled teachers and teacher educators often intuitively "know" when important mathematical moments occur during a lesson and can readily produce ideas about how to capitalize on them, the literature reveals a construct that is not well-defined. Ideas related to these instances are mentioned in many different ways. For example, Jaworski (1994) refers to such opportunities as "critical moments in the classroom when students created a moment of choice or opportunity" (p. 527). Davies and Walker (2005) use the term "significant mathematical instances" (p. 275) and Davis (1997) calls them "potentially powerful learning opportunities" (p. 360). Schoenfeld (2008) refers to such moments as "the fodder for a content-related conversation" (p. 57), as "an issue that the teacher judges to be a candidate for classroom discussion" (p. 65) and as the "grist for later discussion or reflection" (p. 70). Schifter (1996) spoke of "novel student idea[s] that prompt teachers to reflect on and rethink their instruction" (p. 130).

It is clear from this literature that these instances, whatever they are called, are important to mathematics teaching and learning. It is also clear that there are some critical mathematical and pedagogical characteristics of such moments. In particular, references to them allude to important mathematics, pedagogical opportunities, and student thinking. We consider these three criteria and focus on their intersection as being the location of Mathematically Important Pedagogical Opportunities (MIPOs). A better understanding of the MIPO construct can inform the work of facilitating and researching teachers' use of students' thinking in mathematically productive ways. One difficulty in learning to use students' mathematical thinking is that there are so many different ways it can interpreted and acted upon. As Lewis (2008) observed, "The 'real' classroom experience is elusive: each moment is experienced differently by the actors involved and their perceptions of those experiences change with time and reflection. The choices of what to focus on, which story to follow, are endless" (p. 5). One critical role that mathematics teacher education can play is to provide lenses, informed by research and advocated by the community at large, for teachers to use both as they teach and as they reflect on and learn from their teaching. The conceptual framework put forth in this paper is designed to be such a lens. In the following, we carefully define and describe the MIPO construct, and initiate a discussion about how teachers and teacher educators might profitably use it to support students' mathematical learning.

Defining Mathematically Important Pedagogical Opportunities

We define a Mathematically Important Pedagogical Opportunity (MIPO) as an instance in a classroom lesson that provides the teacher with an opportunity to extend or change the nature of students' mathematical understanding. To be considered a MIPO, an instance needs to meet two important criteria: it needs to involve *important mathematics* and be a *pedagogical opportunity*.

Mathematically Important

To be *mathematically important* in a given classroom, the instance must be centered on an idea related to mathematical goals for student learning. In the narrowest sense, this would be a mathematical goal for the lesson in which it occurs, but more broadly, it could also be related to the goals for a unit of instruction, an entire course, or for understanding mathematics as a whole (see Figure 1). In the first case, the instance may focus on a particular mathematical idea or connections among ideas within the lesson, while in the latter cases, the instance might involve making connections to other areas of mathematics or developing mathematical ways of thinking.

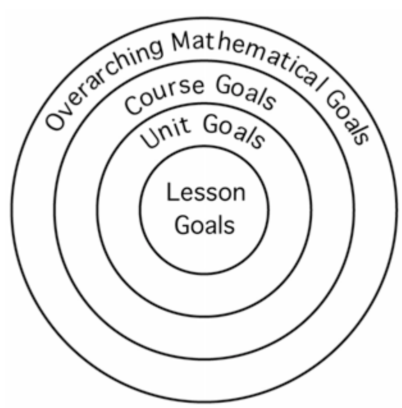

Figure 1. *Layers of mathematical goals to which a MIPO may relate.*

There appears to be an inverse relationship between the distance of the underlying mathematical idea in the instance from the goals of the day's lesson and the needed power of the mathematical idea for it to meet the mathematically important requirement of a MIPO. That is, the threshold for mathematical importance increases the further one moves away from the center in Figure 1. Furthermore, the mathematical importance of an instance is relative to the context in which it occurs. For example, an instance considered mathematically important in a calculus class because it highlights a subtlety in the topic being studied would not qualify as mathematically important in the context of an introductory algebra course if the students did not have the background knowledge to make sense of it. On the other hand, an instance that highlights something crucial about the nature of mathematics could qualify as mathematically important in any classroom in which it was accessible to the students and would help them to better understand mathematics as a whole.

When considering whether an instance is mathematically important, it is necessary to distinguish between its relationship to mathematical goals and to other goals for student learning. For example, helping students work more productively in small groups is a goal in many classrooms. Although this goal may support students' mathematical learning, the goal itself is not mathematical in nature, and thus, an instance related only to this goal would not be a MIPO—a MIPO must be firmly grounded in important mathematics.

Pedagogical Opportunity

In addition to involving important mathematics, a MIPO requires a *pedagogical opportunity*. Pedagogical opportunities are observable student actions that provide an opening for working towards an instructional goal. As such, pedagogical *opportunities* can be cultivated by the teacher, but cannot be created independently of the students. Teachers routinely make pedagogical *moves* that are designed to create opportunities for students to learn mathematics, such as posing quality tasks, asking probing questions, assessing students' progress and modifying their instruction in response to additional information. Well-executed pedagogical moves can, in fact, increase the likelihood of pedagogical opportunities in a teacher's class, but the opportunities themselves come from the students, not the teacher. For example, if a teacher were to introduce a theoretical student error into the class discussion to help clarify an issue that she felt her students were struggling with, it would remain a pedagogical move and not an opportunity until a student or students in the class interacted with the error publicly. A pedagogical opportunity must be grounded in an observable student action. Student actions, by providing insight into student engagement with an instructional goal, provide an opening for the teacher to work toward achieving that goal.

The Intersection of Mathematically Important and Pedagogical Opportunity

Mathematically Important Pedagogical Opportunities (MIPOs) occur at the intersection of *important mathematics* and *pedagogical opportunities*. In this intersection, observable student actions provide pedagogical openings for working towards mathematical goals for student learning. Although simple yes/no answers or other utterances may provide evidence *that* students are thinking, these openings only occur when student actions provide insight into *what* students are thinking about mathematical ideas. Thus, *observable student thinking* underlies the MIPO construct. Our conception of the relationship among important mathematics, pedagogical opportunities and student thinking is shown Figure 2. In this section we elaborate on each region in Figure 2 to further clarify the MIPO construct.

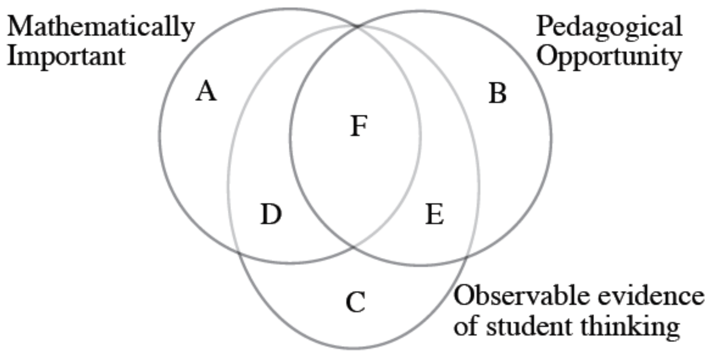

Figure 2. *Relationship among important mathematics, pedagogical opportunities and student thinking.*

Region A represents situations that are mathematically important, but neither provide evidence of student thinking nor a pedagogical opportunity. A teacher presenting important mathematical information would fall into this region, as would situations in which a teacher makes a pedagogical move to engage students with the mathematics, but students fail to provide observable evidence of having done so. In general, these are situations where important mathematics is present, but observable evidence of student thinking is not. For example, if a teacher were to make a mistake on the board related to important mathematics in the lesson, it would be a mathematically important moment. If the teacher corrects the error and moves on without student engagement with the error, this moment would not provide an opportunity to extend or change the nature of students' mathematical understanding. Student actions in response to the error that reveal their mathematical thinking, such as asking questions that illuminate the key mathematics behind the error, would provide an opening for working towards an instructional goal. This would put the moment in the intersection of the three areas in Figure 2, thus classifying it as a MIPO.

Region B represents situations where student actions do not provide evidence of student thinking and are not mathematically important, yet provide inroads for important pedagogical goals. For example, if a student were to get up to sharpen his pencil in the middle of a class discussion, the action would not provide insight into his thinking, but it could provide a pedagogical opportunity to talk about important classroom norms. In fact, pedagogical opportunities that neither provide evidence of student thinking nor relate to important mathematics seem to relate to general pedagogical rather than content-specific goals.

Region C represents student actions that provide evidence of their thinking, but the thinking is neither about important mathematics nor related to instructional goals. For example, a student might make a comment about the length of a homework assignment or reiterate a memorized fact. Although these comments give the teacher information about the student's thinking, they neither connect to important mathematics nor provide an opening for working towards an instructional goal, and thus, do not meet the criteria for a MIPO.

Region D represents situations that involve important mathematics and evidence of student thinking, but do not provide an opening for working towards an instructional goal. For example, a student in an algebra class could eloquently summarize why adding a constant to a linear equation corresponds to a vertical shift of the graph. While this comment could make a positive contribution by summarizing what students already know, it would not create the opportunity to extend or change the nature of students' mathematical understanding.

Region E represents pedagogical opportunities that provide insight into student thinking, but are not related to important mathematics. For example, a student might say, “I don’t see why I need to think by myself for one minute before I talk with my group.” This is not related to mathematics, but it does provide observable evidence of student thinking and would provide an opening for discussing the instructional goal of allowing individuals time to formulate their own thoughts before being influenced by others.

Region F represents situations in which student thinking about an important mathematical idea provides an opening for working towards a mathematical goal for student learning. This is what creates a MIPO. In this region, a student might, for instance, question or comment on a mathematical idea, verbalize their incomplete thoughts as they try to make sense of a mathematical idea, express incorrect mathematical thinking, make an error, or notice a mathematical contradiction. In all these instances, what is important is that the student thinking provides an opening for the teacher to make a pedagogical move that will extend or change the nature of students’ understanding of important mathematics.

Because student thinking is at the heart of every MIPO, there is no region in Figure 2 that includes both important mathematics and pedagogical opportunity without involving observable evidence of student thinking. It is when student thinking is made public that teachers have an opportunity to use that thinking to further students’ mathematical understanding. Although MIPOs can occur in any classroom environment, they are more likely to occur in classrooms that provide ample opportunity for students to make their thinking public.

An Example from the Literature

In this section we illustrate the MIPO construct by using it as a lens to analyze a piece of transcript taken from Leinhardt and Steele (2005, pp. 107-108). The episode comes from a 5th grade class discussion about finding output values for the rule 3x + 1 given different input values. The teacher, Magdalene Lampert, added ¼ as an input value to a table of input-output values and asked for the output. All of the previous input values had been whole numbers.

Soochow: One and three fourths.
T: How would you explain it please?
Soochow: *Because one-fourth times three is three-fourths and then you just add o– add a one*.
T: Okay, so first you times by three and then you add one.
T: Who can explain why one fourth times three is three fourths? Sun Wu?
Sun Wu: *One fourth, like one fourth of a pie and then somebody brings two more and one times three is three—three pieces of pie that came out of four pieces of pie?*
T: Okay, are they all the same size? Those three pieces of pie? Lisa?
Lisa: Yes
T: How do you know?
Lisa: Because if you’re adding one fourth times three you’re going to [—] [—] equal parts
T: Okay. Cause I’m, I’m taking three things that are all the same size. They’re all the size of one fourth. Ali?
Ali: *It could be one fourth [—] could be a whole one.*

T: Can you explain what you mean?
Ali: Can I come to the board?
T: Yes, here take this, [chalk] it's easier to see.
Ali: Here's like a big pie [draws circle and divides it into fourths]
T: Um-hum.
Ali: And then you could divide it into fourths, four pieces. *And then one fourth could be one (points to one segment of circle) and then would be like this one (points to the 1 on the input side of the chart).*
T: I don't understand what you mean. Does anybody else understand what Ali means? Bridgette?
Bridgette: Me-, he means that if you ha-, if you have one fourth and you make say you color in three of the four pieces [—] equal one whole.
T: Is that what you meant?
Bridgette: Yeah.
T: Okay, what do you think about that? Ali is saying three times one fourth is one fourth [*sic*]. Add one fourth and you'd get four so it would be just like here [points to the 4 beside the 1 in the function chart]. But the input number here was one [writes faint 1 in input column beside the 4] and now the input number here is one fourth [points to the ¼ in new chart]. What do you think Sun Wu?
Sun Wu: *He thinks the um, the one is like one fourth. But it's really one, another, four.*
T: What do you think about that Ali? [draws another circle]. How many fourths are there in one whole?
Ali: Four fourths [T draws new circle divided into fourths].
T: Four fourths? So if I was going to put a number in here I could put one and a fourth [*sic*] [writes in column]
T: Is there anything I could put in there besides one and a fourth? Elsie?
Elsie: Wouldn't it be one and three fourths?
T: Oh, I'm sorry. It should be one and three fourths like that anyway [changes chart]. Is that what you meant?
Elsie: Yeah.

We now use Figure 2 to analyze excerpts from this dialogue. In doing so we highlight examples from each region of the figure in order to help the reader distinguish instances that are MIPOs from those that are not. Lines 17 and 22 are examples of Region A, as in both cases the teacher emphasizes the same size of the pieces—an important mathematical principle. There is no student thinking involved and no pedagogical opportunity, but it is mathematically important. The student's inquiry in Line 26 about coming to the board provides an opportunity to address the teacher's expectations for sharing one's work and using tools in the class, but neither provides insight in the student's thinking nor involves important mathematics; thus, it falls in Region B. Lines 7, 24, 28, 36 and 51 provide evidence of student thinking, but do not involve important mathematics or provide pedagogical opportunities, thus they are examples of Region

C. Note that this dialogue also contains several student utterances that fell short of being observable evidence of student thinking (Lines 18, 40 & 59). Although these utterances suggest that the students are thinking, they don't provide insights into what they are thinking. In Line 56 a student provides a correction to an error made by the teacher. This is evidence of the student's thinking and also involves important mathematics, but does not provide a pedagogical opportunity, thus it falls in Region D. In Line 20, we see evidence of a student's thinking, but it isn't clear exactly what is going on mathematically. Because of this it isn't possible to determine if it involves an important mathematical issue or if it is merely a misspeaking. It does, however, provide a pedagogical opportunity to discuss the importance of the words that we use, thus it falls in Region E. Lines 9, 14, 30, and 47 all deal with observable student thinking, important mathematics, and pedagogical opportunities, thus fall in Region F. In Line 9, for example, Soochow's explanation of how to find the output value provided an opportunity to review how to multiply a fraction by a whole number. Sun Wu's explanation at Line 14 shifted the unit from the whole pie to a piece of the pie, providing a pedagogical opportunity for the teacher to engage her students in further discussion of this important mathematical idea. In each of these cases, the teacher expertly incorporated the MIPO into her instruction and it is possible to see how the discussion supported students in learning about important mathematics.

Discussion and Conclusion

Researchers and practitioners in mathematics teacher education advocate the use of student thinking as a means of improving mathematics instruction. Many teachers we have observed, particularly novices, seem to interpret this call to mean that *all* student thinking is equally valuable and, thus, should *all* be pursued in similar ways. We argue, however, that this is not the case. While teachers certainly need to carefully listen to all student ideas, this listening must be followed by thoughtful consideration of whether a particular idea or comment is worth pursuing.

By highlighting three critical components of instances in a classroom that provide opportunities to advance students' mathematical understanding—important mathematics, pedagogical opportunity and student thinking—the MIPO construct can be used as a tool to help teachers learn to distinguish moments that provide opportunities to further students' mathematical learning from those that do not. In addition, it provides a tool for helping teachers make sense of classroom situations—what Levin, Hammer and Coffey (2009) call *framing*. From this perspective, whether a teacher notices the value in an event depends on how he or she frames what is taking place during instruction. If, for example, a teacher views a student error as something that needs to be corrected, he or she is unlikely to consider the mathematical thinking behind the error or whether the error could be used to highlight a specific mathematical idea. On the other hand, a teacher who views an error as a site for learning is more likely to consider both the mathematics underlying the error and how it could be used to develop mathematical understanding. In considering whether an instance is a MIPO, teachers need to frame instances of student thinking in terms of both mathematical importance and the pedagogical opportunity they provide. Framing classroom events in this way has the potential to change the way teachers analyze and act upon instances in the classroom.

The conceptualization of MIPOs we have described provides both teachers and teacher educators a lens for analyzing the complexity of classroom mathematics discourse and a vocabulary for discussing the mathematical and pedagogical importance therein. We believe such a lens is significant because MIPOs are high-leverage instances of student thinking that have the potential to change the nature of mathematics instruction if incorporated well into a lesson. This conceptualization of MIPOs provides a tool that can help make more tangible the often abstract but fundamental goal of building on students' mathematical thinking.

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THE ROLE OF LISTENING IN TEACHER TRANSFORMATION AND PEDAGOGIC PRACTICE

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This case study research examines the transformation of one middle school mathematics teacher's pedagogic practices who has envisioned and enacted a problem centered learning approach to teaching and the factors he believes influenced and supported his transformation. The findings reveal that listening for this teacher infused the unfolding process of his transformation and is now an integral dynamic of his emergent mathematics curriculum.

Envisioning and enacting alternatives for teaching mathematics is a difficult and complex task often met with challenges from outside forces and influences and frequently hampered by our own inability to imagine that something else is possible. Our inability to see beyond our current and traditional ideas and ways of doing are often the greatest challenges we face in transformation. This study examined the pedagogic practices of one middle school mathematics teacher who has "seen" an alternative and who has enacted a problem centered learning approach to teaching mathematic (Wheatley, 1991) and further considered the factors he believed influenced and supported his transformation. He has envisioned and enacted and emergent mathematics problem solving curriculum in his classroom with and for his students. As his story of transformation unfolded the important role of listening for this teacher became apparent. Listening to his students became an inherent and integral part of the process of his transformation as he initially tried to transition from traditional pedagogic practices to a problem centered learning approach to teaching mathematics.

Theoretical Framework

Strengthening students' understanding of mathematics through communication is well established (Gose, 2009; Pugalee, 2001; Way, 2008) and is clearly emphasized in the National Council of Teachers of Mathematics (2000) *Principles and Standards for School Mathematics*. An important facet of communication for teaching mathematics is the discourse that occurs in the classroom between teachers and students but also between students and students when they articulate their own ideas and consider their peer's mathematical perspectives as a way to construct meaningful mathematics understandings. Developing a classroom environment and a curriculum wherein mathematical discourse is orchestrated and sustained requires of teachers to re-envision their role in the classroom as one of a participant in an ongoing conversation with mathematics at the center.

Carpenter and Fenema (1992) suggest that listening to their students may be a critical behavior for teachers wanting to develop more effective mathematics instruction. Davis (1997) further suggests that "attentiveness to how mathematics teachers listen may be a worthwhile route to pursue as we seek to understand and, consequently, to help teachers better understand their practice" (p. 356). Davis (1994) believes that listening is not just hearing what someone has said but rather an engaged physical activity.

> When two persons converse, for example, it can be *seen* that they are listening to each other as the actions of their bodies become bodily *interactions*. They lean toward and

> reach out for each other, momentarily unaware that they are violating the Western taboos on proximity, touch, and extended eye contact. They seem to focus in a way that suggests they are oblivious to the noise around them; they attend to each word and to each action as though nothing of importance had occurred prior to the discussion and nothing of importance awaits them at its end. They are unconcerned that their voices are perhaps too loud, their bodies too animated. (p. 268).

So listening is neither "motionless or silent" (p. 269). Listening, as Davis describes it, is an activity in which one is engaged and participating. Traditional approaches to the teaching of mathematics are often centered around the teacher's telling and presenting predetermined methods and procedures to an audience of quiet, motionless students who are assumed to be attending simply because they are quiet and motionless. Davis claims that this assumption cannot be made, that it is something other than the motionless silence of students that can make us aware that they are listening.

> In the classroom … as the novel is read or as the mathematical principle emerges, the teacher knows the students are listening not because they have ceased to move but because a certain rhythm or harmony is established – there is an awareness that each is immersed in and conducted by the same subject matter. The gazes are fixed not on the teacher nor on one another, but on that which is among them (1994, p.269)

Davis extends his consideration of listening as a lens from which teacher practice can be reinterpreted with his discussion of evaluative, interpretive, and hermeneutical listening. Evaluative listening "tends to forget its own responsibility in interactions (1997, p. 360). This type of listening occurs when the teacher is not listening *to* the students and the listening is focused on the elicitation of correct responses or "guiding" the students to a pre-determined method or solution path. Interpretive listening is described by Davis as listening *to* what learners are saying and perhaps even trying to understand what they are making sense of but still "listening *for* particular responses" (1997, p. 363). The kinds of questions that arise from this type of listening may allow for a greater range of responses than those accompanying evaluative listening but they may not foster much diversity. Hermeneutical listening is the third type of listening that Davis (1997) describes. It is the

> sort of listening [that] is an imaginative participation in the formation and transformation of experience. Hermeneutical listening demands the willingness to interrogate the taken for granted and the prejudices that frame our perceptions and actions. (p. 369-370)

Hermeneutical listening occurs when the teacher becomes "a participant in the exploration of … mathematics" (p.369). A teacher listening hermeneutically frees herself to participate with her students in the negotiated and evolving nature of interaction rather than simply listening to "take" the students to a certain predetermined point. The alternatives, evaluative and interpretive listening, engage the teacher in attempting to draw students into "*front-end* mathematics" where the "mathematics concepts [are] treated as though they … are independent of learners and their experience" (p. 339).

Methods

Data Collection and Analysis

This study was approached from the perspective of an in-depth case. This case was purposively selected (Schwandt, 1997) because it was thought to present insights into the impact of and interrelationships among changing ideas about mathematics and the enactment of a reform mathematics curriculum. Wesley, as he is called for this study, is a middle school teacher, who teaches seventh grade mathematics, pre-algebra and algebra I in a middle to upper income suburban public school in the Southwest region of the United States. Wesley is known throughout his state as an innovative teacher and has received some notoriety as a result of his innovative approaches and nontraditional teaching. While some may criticize and challenge his non-traditional approach to teaching mathematics, which includes problem centered learning and not using textbooks regularly, many praise his classroom approaches to teaching as being among the best and most effective in his district and state.

The purpose of this research was to identify and explicate a variety of aspects of this teacher's beliefs and classroom practices and gain insight into his story of transformation by addressing the following two focus questions:

1. What factors does this teacher consider to have influenced the transformation of his beliefs about mathematics teaching and learning?
2. What pedagogic practices does this teacher include in his implementation of a problem centered learning environment?

This study involved interviewing Wesley who had adopted and implemented an approach to mathematics curriculum grounded in problem centered learning (Wheatley, 1991; Wheatley & Reynolds, 1999) and constructivism (Ernest, 1996; von Glassersfeld, 1995) and observing and analyzing his classroom pedagogic practices for several months during a school year. The focus of these interviews was to explore the experiences Wesley believed were instrumental in developing his alternative view of the curriculum and in implementing a problem-centered learning approach in his classroom. Questions about his changing beliefs about mathematics and how those beliefs informed the curricular decisions he made were particularly probed.

The data collection for this study included audio-taped formal and informal interviews, field notes, transcripts of video-taped classroom interactions, and documents collected which were all transcribed as collected throughout the study. The transcripts were carefully read and coded for key words, and ideas, looking for themes or categories to emerge. Since one of the goals of the study was the telling of Wesley's teacher story, his interview transcripts were read and reread to determine if certain aspects of his experience needed to be further elaborated so other people reading about him might have a an understanding of him. In this way the data pertaining to Wesley's story were organized using an interpretational approach which is a "process of examining data closely in order to find constructs, themes, and patterns that can be used to describe and explain the phenomenon being studied" (Gall et al., 1996, p. 562). A reflective or interpretational approach to data analysis is described by Gall et al. (1996) as involving "a decision by the researcher to rely on intuition and personal judgment to analyze the data rather than on technical procedures involving an explicit category classification system" (p. 570). Thus, data were coded and as certain patterns or themes became apparent, new segments of data were continually and constantly compared within and across these themes and categories (Strauss & Corbin, 1998). This constant and continual comparison of data and emergence of themes guided the interviews with Wesley subsequent to the initial interview conducted.

Setting for the Study: Wesley's Classroom

To conduct research for Wesley's case two of his seventh grade classes, one pre-algebra class and an algebra I class were observed on a weekly basis for one semester. The first class each day was the algebra I class which was comprised of twenty-four students, thirteen of whom were boys and eleven were girls. The second period class was the pre-algebra class which was comprised of twenty-seven students, fifteen of whom were girls and twelve were boys. The students in both these classes would not be considered widely diverse in their abilities since for this district students are placed in these classes based on their standardized test scores and previous grades. Additionally, the racial background of the students in both these classes would not be considered diverse with only two or three students in each class being non-Caucasian.

The classroom was arranged with approximately thirty student desks divided into six rows that faced the front of the room where a white board was hung on the wall with two overhead projectors and screens. Along one side of the room there were seven computers placed on built-in desks with seven accompanying chairs. On the other side of the room was the teacher's desk with a computer and storage, all of which were built in and attached to the wall. Along the back wall of the room there were windows that extended the full length of the room under which were storage cabinets without doors that held a variety of manipulatives and materials for student access and use. On the surface of the cabinets there were containers with a variety of mathematics games, challenge puzzles, and transparency sheets with pens for student use. A Texas Instrument graphing calculator was attached with Velcro to each of the student desks for the students to use at their discretion. The walls of the room were decorated with a variety of posters, some motivational in nature and others presenting mathematical ideas, and with the many newspaper articles and awards that Wesley had won. The tops of the cabinets were lined with many trophies won in a city-wide stock market game.

Each of the class periods lasted fifty-five minutes and typically began with either a warm-up activity and discussion or some discussion regarding homework assignments from the previous night. Occasionally, the class period would begin with discussion about an assessment from the previous day. Regardless of the focus of the initial discussion in class, this time typically lasted about ten minutes until there was a shift of activity to the problem or task for that day. Following the opening discussion students were often given a choice of two or three tasks or problems that they could address during class. Students often voted on which problem they would consider first; this would also be the problem they would try to discuss prior to the end of class if they were ready. Many times the students would ask if they could continue to look at a problem they had first looked at maybe the day before or in some cases maybe a week or longer before. There would be some discussion about this and most often Wesley would agree that they could use their class time to continue looking at that problem if most everyone agreed. During problem solving students worked with their "partner" by quickly moving their chairs to face their partner or move across the room to an open seat near their partner. Also during the problem solving sessions students would frequently discuss their thinking with members of other groups and move around the room freely, to and from the white board "showing" another student what they were thinking or to retrieve manipulatives or a transparency sheet and pen. As students worked in small groups, Wesley moved around the room listening to students and infusing the conversation with questions, re-directing them or providing them with further challenge as needed. There were other times, however when the desks remained in rows and Wesley presented the problem solving tasks at the overhead. During these times, students would often be using their calculators or manipulatives to explore certain ideas or problems. Even when the desks remained in rows with Wesley primarily in the front of the room, students were free to work with one another and to discuss with those around them their thinking and ideas about the problem.

Typically, following problem solving where students had worked with their partners, but not always, class ended with student presentations of their solutions using the overhead and their transparency sheet. During this time the students presenting would discuss ways in which they thought about the problem that perhaps did not arrive them at a solution and then also ways in which they felt they had derived a solution for the problem. The students not presenting, those at their seats, would raise their hands and wait to be called on by those presenting to ask questions about the solution being presented or to share perhaps a different way they had in looking at the problem. Often this time would result in only two or three groups presenting their ideas with several students going to the board to explain other ideas they had regarding the problem. There were many days that the "end of class" discussion did not occur at the end of class but rather the next day or perhaps even a week later but they always came back to each problem for a time of discussion. All instructional activities emphasized student thinking, whether a warm-up activity, a problem solving task, or problem solving specifically with manipulatives, Wesley encouraged the students to complete the tasks in ways that made sense to them rather than providing them with specific procedures or algorithms for finding a solution (Wheatley, 1991; Wheatley & Reynolds, 1999).

Results

Wesley describes his pedagogical methods as beginning teacher as teacher-centered and a traditional type of approach during his first several years of teaching mathematics. "It was I'll explain or demonstrate from the overhead … the students would do problem sets and then they would come in the next day, we would grade that." It was not until a change in building administrators in the late 1980's that Wesley was encouraged to try something that varied from his traditional approach.

Trying New Things

In the mid-eighties the district where Wesley teaches experienced a change in superintendents. With a new superintendent whose focus was on "changing the direction of the district" and "innovation" several new building administrators were hired over a few years including a new principal at the middle school. This new principal, Wesley recalls, "was talking about cooperative learning and said to me 'when you get done with the book maybe you could go back and teach it without using the book, teach it with manipulatives, maybe some cooperative learning'." Describing himself as a "risk taker," Wesley believes that he was approached and encouraged by his administrator to try some new teaching approaches because he was team leader and because he was willing and open to consider new things. "So I started messing around with looking for some stuff I could do, some stuff we could do with cooperative problem solving and letting kids talk." Wesley continues, referring to urges by his principal to explore cooperative learning and the use of manipulatives with his students, "that kind of started this whole, my getting into this, my messing around with different things." It was these beginning experiences, trying some new and different things, letting his students talk and him listening to them, that created a sense of wanting or searching for more in Wesley.

Transformation Begins

During the summer of 1991, the school district's new superintendent, having been there only a couple of years, brought in Allan, a university professor whose primary research focus was children's mathematical thinking, for an elementary and middle school mathematics professional development workshop. Wesley recalls that something significant happened that day. "What happened that day that was different is that Allan started me thinking about mathematics, and what it is, what is mathematics, and so that's been an ongoing thing." While believing that the questions asked of him that day impacted him greatly, Wesley does not feel that it was this in isolation that began a process of change in the way he now teaches mathematics.

> I think I was impacted some by a middle school conference I attended with my principal. Some of the people there were talking about a learning pyramid that says something like you learn only 10% of what you read and then more if you have to teach it to somebody. I can remember wondering, "well, what about if you interact with others to help each other." So I know that I was really beginning to question some things and beginning to see the benefits of students working together to learn mathematics at that time. [Interview data]

Wesley also recalls that as he has reflectively considered how and perhaps why he has changed so significantly as a teacher that there were many things that impacted him, influenced his thinking, or helped him become more open to other ways of teaching mathematics.

> When I look back I have really asked those questions like "what's going on with children" and "what are we doing to children with traditional approaches" but I don't think I was really dissatisfied with what I was doing in my classroom, you know when my approach was more traditional, I just know that I was open to trying things and seemed to be searching for something and I started to explore things. [Interview data]

As Wesley continued to explore and implement new things over the course of a few years in his classroom, such as students working in groups more often and using manipulatives during the more frequent problem solving opportunities he was providing for them, he in turn was afforded more opportunities to listen to his students. It was this listening to his students, trying to understand and make sense of what they were understanding that infused his desire to continue to provide more of these kinds of opportunities as a part of his teaching.

Over a period of six or seven years Allan continued to be brought back to Wesley's school district to conduct mathematics professional development for both middle school and elementary school teachers. During these years the process of change and transformation in Wesley's ways of thinking about mathematics teaching and learning continued as did the emergence of new issues to be addressed and a variety of constraints that had to be considered.

Since those early years of perturbation and trying new things, Wesley has developed a classroom environment and community wherein students share their ideas, listening to one another, and are encouraged to come to consensus. Listening to students for Wesley not only played a major role in his transformation as a teacher but is now a natural and inherent part of his interaction with his students.

> One of the things that was really important and still is, I just don't think about it as much anymore because I just do it, I think, is listening to the students. Back when I was really trying things out and searching for problems it was really helpful and important to listen to the students while they worked on problems. I would try to understand what they were understanding and what they didn't have figured out. [Interview data]

For Wesley this kind of listening, listening to determine what the students are making sense of, is more than just listening for assessment purposes, for asking questions, or for even selecting the next task. It is listening to participate, to be engaged in the problem solving and in the community with his students. Wesley believes that this kind of listening, an active or engaged listening, is essential for his students to fully participate in the community he strives to develop in his classroom.

Discussion

Rather than being the end product of his attempts to change from one pedagogic approach to another, what happens in Wesley's classroom today is the expression of his beliefs about mathematics teaching and learning wherein a way of communicating with his students about mathematics is the learning environment in his classroom. While Wesley states that he follows a problem centered learning model what he does with his students is more than a model or a set of steps that someone else could follow to create the same thing, it is an orientation about what is mathematics and what it means to teach and learn mathematics, one that views the classroom as a community and the teacher as co-learner.

This study shed light on the inherent and integral role of listening in Wesley's transformation. Wesley is a participant in exploring mathematics with his students. Hermeneutical listening is part of his orientation about teaching and learning. He views teaching not as the act of telling and learning as a not as a sequence of actions but rather understands and views learning "in terms of an ongoing … dance" (Davis, Sumara, and Kieren, 1996, p.153), wherein "individual and collective meanings are seen to evolve in the course of classroom interactions" (Cobb, Jaworski, and Presmeg, 1996, p. 15). Teacher and students are involved in the unfolding of the curriculum as knowledge emerges as meaning as their focus shifts from teaching or the learning of "things" to "that which is among them." Wesley hermeneutically listens to his students: the listening occurs so that he can become "a participant in the exploration of … mathematics" (Davis, 1997, p. 369). Listening for Wesley infused the unfolding process of his transformation and is now an integral dynamic of his emergent mathematics curriculum. Each day Wesley actively listens as one who is participating in a conversation; a mathematical conversation and discourse that evolves and emerges in his classroom with his students.

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SUPPORTING STUDENTS' DEVELOPMENT OF MATHEMATICAL REASONING THROUGH PRODUCTIVE DISCOURSE

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We noted fifth grade students' progress in making generalizations that related the independent and dependent variables and justifications that considered the global context of the pattern finding tasks during a 3-day teaching experiment. Since such development is at odds with previously reported research, we focused on teachers' discursive practices that co-constructed students' development of reasoning. Using Stein et al.'s (2008) framework, we found that anticipating, monitoring, connecting and purposefully sequencing students' responses co-constructed students' development of mathematical reasoning.

Introduction

Literature shows the need for students to develop fluency in mathematical reasoning (Martin & Kasmer, 2010; Yankelewitz, Mueller & Maher, 2010). We operationally define mathematical reasoning in this study as making and justifying inferences in pattern finding activities. Students' inferences according to Lannin (2005) can be classified as recursive or explicit. Recursive generalizations use the term-to-term change in the dependent variable to find unknowns. Explicit generalizations relate the input and output values and enable calculation of outputs given n inputs. Sowder and Harel (1998) classified justifications as externally based schemes, empirical and analytic. With externally based schemes, students do not show ownership of the justifications but instead refer to an external source. Empirical schemes show students' ownership of the justifications but do not regard the generality of the context. Analytic justifications consider the generality of the mathematical task's context.

Fluency in mathematical reasoning serves a lot of purposes in mathematics education. For example, it supports conceptual understanding and retention of students in advanced mathematics classes (Martin & Kasmer, 2010; Horn, 2008). Despite such importance, research has shown that most students have difficulties reasoning mathematically (Ellis, 2007; Lannin, 2005). These difficulties were observed in Lannin's (2005) study with sixth grade students and Ellis's (2007) study with seventh grade students where by students tended to make recursive generalization with ease in contrast to explicit generalizations. They also tended to make justifications that did not consider the general context of mathematical problems.

In this study, we observed that students developed their mathematical reasoning with more ease than reported in previous research (Lannin, Baker & Townsend, 2006). We classified the generalizations as explicit or recursive (Lannin, 2005). The percentages of students using each class of generalization are presented in Table 1 and indicate that students tended to use explicit generalizations by the third day of our teaching experiment. They also progressed to using empirical and analytical justification schemes (Sowder & Harel, 1998), with more students using the latter as in Table 2. Since it is a challenge to support development of mathematical reasoning, the objective of this study is to explore teacher's discursive practices that co-produced such development of mathematical reasoning as presented in Tables 1 and 2 that is at odds with previously reported studies.

Table 1: Percentage of Different Generalizations Expressed by Fifth Grade Students

Generalization strategy	**Task 1**	**Task 2**	**Task 3**
Recursive	69.6	30.7	8.9
Explicit	30.4	69.3	91.3

Table 2: Percentage of Different Justification Schemes Used by Fifth Grade Students

Justification strategy	**Task 1**	**Task 2**	**Task 3**
Externally based schemes	13.0	6.2	0
Empirical schemes	39.1	43.8	7.7
Analytical schemes	47.8	50.0	91.3

Theoretical Framework

Our perspective on learning which informs this study is that individuals learn as they participate in communities that enable them to internalize the cultural norms and appropriate the cultural tools that may modify their abilities and dispositions as learners (Claxton, 2002). The theoretical and methodological implication of this view is that, we needed to pay attention to students' assessed mathematical reasoning and the context that may explain those outcomes (Carlone, Frank-Haun & Webb, in press). As such, we addressed the research question, what are the discursive practices that co-constructed students' development of mathematical reasoning during this teaching experiment? We employed Stein et al. (2008) framework as our analytic lens to study the classroom elements that co-constructed student's development of mathematical reasoning. Stein et al.'s framework afforded us the discursive tools with which to discuss the teachers' pedagogical practices and reasoning. According to Stein et al., the following practices support productive discourse.

Anticipating Students' Responses

Anticipating is the expectation a teacher needs to have about students' possible interpretation of the problem, representations, solution strategies and their levels of sophistication. It is therefore necessary for teachers to have knowledge of students' thinking and different approaches to the tasks they give to students.

Monitoring Students' Responses

To monitor students' responses, the teacher moves around the classroom as students work on their tasks to note strategies and solutions and to listen to assess students' understanding. The teacher might also ask probing questions at this phase to help connect students' reasoning to the major concepts.

Purposefully Selecting Student Responses for Public Display

A teacher selects strategies, ideas and representations to display during class discussion. The decisions might be based on the mathematical ideas that are in the selected students' work. It is important at this phase for teachers to note their patterns of selection to ensure all students and different ideas are given attention in the class.

Purposefully Sequencing Student Responses

After selecting the responses that will be used during class discussion, a teacher also needs to make decisions of the sequence the responses will be presented. This sequence might be dependent on the teachers' knowledge of the students' thinking and the objective of the lesson. The sequence of students' responses affects coherence of the lesson, students' depth of understanding and has ability to clarify misunderstandings.

Connecting Student Responses

Class discussion presents a chance for teachers to guide students to connect ideas and strategies so as to modify their own ideas for better understanding. This can be accomplished if students evaluate each other's arguments and ideas. Teachers who carefully use questioning strategies and sequencing of tasks also provide helpful connections. These connections assist students' identification of the consequences of using different strategies.

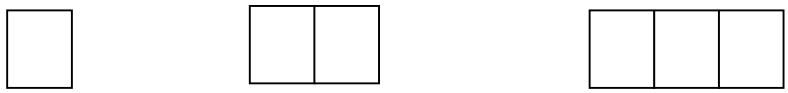
According to Stein et al. (2008), this model integrates the teachers' practices that were discussed separately by different authors. Stein and colleagues also argue that the ability to use these five practices is dependent on teachers' understanding of students' reasoning and the teachers' current thinking. This study contributes to this model and mathematics education by discussing how these practices, integrated as in this model, are enacted in an elementary mathematics classroom and the teachers' thinking that might take place in making the decisions at each of these phases. This study also associates this model to students' development of mathematical reasoning. Additionally, it contributes to ways in which teachers may balance the support of discourse and mathematical reasoning, which has been reported as challenging (Baxter & Williams, 2010; Sherin, 2002).

Methodology

Data for this study were collected during a three-day teaching experiment. The lesson for each day took about 90 minutes. The participating students were 23 fifth graders at a science magnet elementary school in the South Eastern US. One of the researchers, referred to in this study as teacher/researcher, replaced the classroom teacher during the teaching experiment. Other researchers assisted in collecting video and audio data. Field notes and student artifacts were collected at the end of each lesson as sources of data. The teacher/researcher's reflections on her reasoning behind the observed discursive practices were also part of the data collected.

Students worked on pattern finding tasks. Task 1 involved trains of square tables as modeled in figure 1. Task 2 involved trains of triangle tables and task 3 involved trains of hexagon tables. They were asked to find the maximum number of people that would sit around a train of one, two, three and 100 tables given that only one person can sit on one side of each smaller table. They organized their data using t-tables, observed the patterns, made and justified their generalizations. Small group discussions were followed by whole class discussions. We used Stein et al. (2008) framework to analyze the data.

Figure 1: Task 1



Results

Anticipating Student Responses

The teacher/researcher explained in episode 1 what she anticipated students' strategies and reasoning to be.

Episode 1

Teacher/Researcher: I have used this task in two previous teaching experiments and several graduate classes. I have learned from the first teaching experiment that triangle tables are difficult for students to find an explicit rule so I like to start the pattern block tables with squares. Generally students at all levels start off with a recursive strategy to find the pattern…

From this episode, her previous experience with teaching the task helped in anticipating what the students' responses might be. What she anticipated influenced her planning stage as she made decisions on how to sequence the tasks for optimal reasoning by students. She also anticipated that the students might not have had a lot of experience with finding patterns. As such, she planned to spend enough time to introduce pattern finding activities and the language (e.g. building models, input/output tables, and justification, and rules or generalizations) associated with such activities.

Monitoring Students' Responses

The teacher/researcher walked around the classroom to note students' responses as students worked on the tasks in pairs. This gave a chance to students to ask for clarifying questions about the tasks and for the teacher/researcher to ask probing questions to further students' reasoning. Episode 2 is one example where the teacher/researcher was walking around the classroom as students were working on justifying their rules about how many people can sit around a train of triangle tables.

Episode 2

Teacher/researcher [to Jane and Freddy]: So, have you figured this out yet (how to justify t+2 = p as a rule for finding number of people who can sit around a train of n triangle tables)?

Jane: The two comes from the two more people.

Freddy: From the two sides of the –

Teacher/researcher: Which two sides?

Freddy: Actually from the two sides of the triangle that you add to.

Teacher/researcher: Okay, build a model and show me where those two are.

By asking questions to the students, the teacher/researcher got a chance to note students' reasoning and responses. Other questions were meant to challenge students to move beyond the recursive rules they expressed to developing explicit rules. Monitoring students' responses also informed the teacher/researcher on how to effectively progress with the lesson. The following episode 3, whereby the teacher/researcher was thinking aloud during the lesson on the first day of the experiment is one example of such a case.

Episode 3

Teacher/researcher: I'll wait until they build their models and then we'll talk about the T table or the input/output table.

Although she planned to spend ample time discussing how to collect data into input/output table, monitoring students' responses redirected that plan. In her reflection she explained "I decided to abandon giving directions on building the t-table. Brenda and Dan already understood how to collect their data." After noticing that some students already knew how to build input/output tables, we observed that she briefly discussed the tables. This was done to make sure each student in the classroom understood how to collect data using T charts and for the students to "learn how to collect data in a systematic manner."

Purposefully Selecting Student Responses for Public Display

Responses were selected for different purposes. The teacher/researcher repeatedly mentioned that she was selecting responses from students so as to give each student a chance to display their work and explain their thinking. Students' generalizations and justifications that were expressed differently or were different from or conflicting each other were selected too. She did this as she communicated that reasoning tasks can be approached using different strategies. This gave a chance to students to evaluate strategies and responses. It also created opportunities for students to adopt strategies and responses that made more sense to them.

Purposefully Sequencing Student Responses

During whole class discussions, she generally ordered presentations of students' reasoning from less to more efficient. According to the teacher/researcher, this was purposefully done to include more strategies and solutions for public display while simultaneously staging the students to notice how the strategies and tasks were related to each other. This approach supported mathematical argumentation, conceptual understanding and co-construction of ideas. An example of this was on the second day when students were justifying their rules. Becky explained that the rule n+2=p was valid for a train of triangle tables because every time 2 tables were being put together; the train was losing 2 sides that were previously available. The following argumentation and thinking in episode 4 followed.

Episode 4

Sam: … speaking of the way she said, you're not subtracting two, you're adding two.

Teacher/researcher: Oh, okay. Becky?

Becky: Yes?

Teacher/researcher: He said that you're (your rule is) not taking away, you're adding two to the number of blocks.

This argumentation based on students' perceptual schemes went on, bringing in new understandings. The teacher/researcher finally asked another student, Benny, with a different justification who explained that the rule t+2=p holds because "the plus 2 comes from the end sides of the triangle table (train)" and number of people is constantly equal to number of tables in that pattern.

In general, students' generalizations were also purposefully sequenced from recursive to explicit during whole class discussions. This was achieved by having whole class discussions in multiple phases. There were mainly three phases. Phase 1 discussions were after or when students built the models and collected data into the input/output tables. Phase 2 was after students explored patterns and were prompted to respond to questions that could be answered using recursive rules. Finally, whole class discussions were held after students were challenged to develop explicit rules. This practice created a context in which recursive rules were presented before explicit rules.

Connecting Student Responses

Connecting different students' responses. Whole class discussions at each critical phase of the tasks (modeling and data collection, generalizing recursive rules, generalizing and justifying explicit rules) afforded students opportunities to connect their reasoning to other students' reasoning. Similarly, teacher/researcher constantly encouraged students to evaluate each other's argument. Episode 5 is one instance on the first day of the teaching experiment as students were working on task 1 where the teacher encouraged students to reason as to why their classmates' rules were valid or not. In this episode, Brenda presented a rule for finding number of seats available on a train of n square tables and the other students evaluated why that rule could be valid. We observed that this approach encouraged students to incorporate other students' strategies into their own. For example, after one student used a geometric representation to justify the rule 2n+2 =p for a train of square tables, most students incorporated this representation into their explanations.

Episode 5

Brenda: Multiply the tables times two and then add two to find the number of chairs.

Teacher/researcher [to the whole class]: Think about that (rule) one more time, does that work?

Stan: Yeah.

Teacher/researcher: How do you know it works?

King: Because T (input) times two every time is two less than C (output) and if you add two plus T it works.

Teacher/researcher: And Sam, what did you say?

Sam: It does work.

Connecting responses across tasks. Several attempts were made by the teacher/researcher to connect students' reasoning across tasks. Her choice of isomorphic tasks was one attempt. Furthermore, during the lessons she asked questions that encouraged students to reflect on previously worked out tasks (as in episode 6) as they reasoned about the tasks they were working on. This approach supported development of reasoning as evidenced in students' reasoning in episode 7 that followed the teacher/researcher's prompt.

Episode 6

Teacher/researcher: Alright, most of you have some great ideas. Most of you are thinking very hard and working very hard on this and I'm going to give you a hint…. Remember on the first day when we did squares (Task 1)? And you said there would be a hundred tables here (showing a model developed by a student)? And that meant that there were hundred (seats) on the top and a hundred (seats) on the bottom and one on each end? That was a hint. (Use it) to think about the hexagons.

Episode 7

Jane: So, a hexagon has two sides (on the top), so it has two hundred on each side (top and bottom).

Teacher/researcher: Right.

Jane: And two hundred times two is four hundred plus the two on the ends is four hundred and two. Is that right?

Discussion

Stylianides and Ball (2008) recommend that teachers need to have content knowledge for teaching mathematical reasoning. Such knowledge includes knowledge about reasoning tasks and the type of reasoning that tasks can support. To recap, the teacher/researcher anticipated generalization and justification about a train of triangle tables to be more challenging since she used this task with undergraduate and graduate students. Knowledge about the tasks and students' reasoning were important in her expectation of students' responses and consequently planning. Teachers can prepare at the anticipation stage by trying out different strategies that students might use or keep a variety of student work to review when tasks are repeated with other students. Keeping such a record helps the teacher to be flexible if student responses are different from what the teacher anticipated.

Since past research (Lannin, et al., 2006) reports that students develop recursive generalizations with relative ease, discussing recursive generalizations is one way to include responses from more students during whole class discussions. Discussing recursive generalizations before explicit generalization potentially supports students' ability in using both types of generalizations while simultaneously appreciating the usefulness of explicit generalizations. Also, some mathematical tasks in their curriculum may require students to use recursive generalizations.

There is substantial research (Cobb, Gresalfi & Hodge, 2009; Mueller, 2009; Store, Berenson & Carter, 2010) that supports the view that discussions nurture mathematical reasoning. We add to this body of research, that teachers need to purposefully select responses for public display. Selection of responses may be based at including all students' thinking in the discussions and sequencing responses to connect to the goal of the lesson. Selecting responses that are expressed differently and correctly may support students' appreciation of mathematical reasoning as a discipline that can be approached using different strategies. Purposeful selection of responses that are contradictory also creates opportunities for the classroom members to reflect on their thinking and develop their reasoning (Komatsu, 2010).

It is important to emphasize the nonlinearity of the sequence of these practices. As mentioned, the discussions were held in different phases. On each phase, the teacher/researcher walked around the classroom to monitor students' responses and made decisions of what student responses should be displayed publicly, and how to sequence and connect displays of student responses.

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DOES IT STILL "WORK": CGI IN A CONTEMPORARY CA HIGH-POVERTY DISTRICT

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Given the dynamic landscape of mathematics instruction, programs that were once deemed effective need to be periodically examined to insure that their benefits continue to be realized. This study focused on an implementation of Cognitively Guided Instruction (CGI) in an economically disadvantaged school district serving a large percentage of English Language Learners. This setting and instructional context was considerably different from those of earlier studies of CGI's effectiveness. Results of a written test administered in the spring of each school year showed that the more CGI training a teacher had, the more story problems their first graders could solve.

As Slavin and Lake (2008) pointed out "the mathematics performance of America's students does not justify complacency. In particular, schools serving many students at risk need more effective programs." In their review of research on effective elementary school mathematics programs, they claimed, "programs found to be effective with any subgroup tend to be effective with all groups." On the other hand Raudenbush (2005) cautioned that interventions do not always "work" for every population of children so research should include clarifying the subsets of children who benefit from specific interventions. In Slavin and Lake's review a large proportion of the studies they examined were conducted before 2000. The demographics of the country have changed dramatically in the last ten years especially with the influx of English Language Learners into schools, and the conditions of mathematics instruction have changed drastically for teachers. The programs found to be effective in the 1980s and 90s may be more difficult to implement with the advent of the stress on standardized testing in recent years. Given the dynamic landscape of mathematics instruction, programs that were once deemed effective need to periodically be examined to insure that their benefits continue to be realized, and they need to be researched with different populations of students to determine their generalizability.

Theoretical Framework

Our study focused on one of the programs, Cognitively Guided Instruction (CGI) (Carpenter, Fennema, Peterson, Chiang, and Loef, 1990), that Slavin and Lake(2008) deemed to be effective, and we explored the degree to which this program was effective in 2007 – 2010 in an economically disadvantaged school district serving a diverse population of students including a sizeable number of English Language Learners. While descriptive studies indicate that CGI has been implemented with great success in diverse settings including high-poverty predominantly African-American schools in Prince Georges County, Maryland (Carey, Fennema, Carpenter & Franke, 1995) ; Oneida Indian Reservation schools in Wisconsin (Henkes, 1998), and predominantly Hispanic schools in the Los Angeles area (Franke & Kazemi, 2001), these studies did not include student achievement data. The only study with achievement data and a comparison group documenting CGI's success with economically disadvantaged students of color is over 15 years old (Villasenor & Kepner, 1993). We felt the need to re-establish CGI's effectiveness with the kind of research-based evidence valued by decision makers at this point in time.

The CGI project demonstrated that when teachers learn about children's mathematical thinking, teachers change their mathematics instruction to address the mathematical development of the individuals in their classroom, and their students outperform children from control classrooms on measures of problem solving (Carpenter, Fennema, Peterson, Chiang & Loef, 1990). In CGI workshops teachers learn about a framework describing the relationship between the difficulty of word problems and the level of sophistication indicated by children's solution strategies. The framework assumes that teachers feel free to choose problems for their students to solve and can take the time to discuss solution strategies with them. Teachers in our study were facing more constraints than teachers in previous CGI studies so it remained to be seen if their students would benefit from their participation in CGI professional development in the same ways as students in previous studies had. One of the findings from the longitudinal CGI study indicated that teachers' knowledge of CGI developed over time, with some teachers taking up to three years to become effective CGI teachers (Fennema, et al., 1996). Teachers with less freedom to experiment with using story problems in their classrooms, might take more time to become effective CGI teachers than teachers in previous studies, therefore we looked for a dosage effect to see how the number of years of a teachers' participation with CGI affected their students' achievement.

The research question guiding our analysis was:

What are the differences in first graders' problem solving performance according to the number of years their teachers have participated in Cognitively Guided Instruction professional development?

Methods

Professional Development Format

The project took place over a three-year period. The teachers attended meetings once a month afterschool. They also had three days of training each summer. The focus of all meetings was children's mathematical thinking. The CGI problem type and solution strategy frameworks were developed early on and were revisited frequently. Teachers were videotaped interviewing three of their students twice a year and these videotapes were shown and discussed in meetings. Teachers engaged their children in problem solving in their classrooms and brought student work to meetings to analyze and discuss. One of CGI the facilitators visited each teacher's classroom once a month to observe a problem solving session and discussed her observations of children's mathematical thinking at the meetings.

Setting

The professional development took place in one elementary school district with five elementary schools. The majority ethnic group in the district was Latino (37%) followed by Asian (19%), White (16%) and African American (15%). Eighty-six percent of the students received free or reduced price lunch and 43% of the students were designated as English Language Learners. Teachers in the district felt pressure to increase student test scores to meet the No-Child-Left-Behind Adequate Yearly Progress targets. Among the district norms were daily timed facts tests and close adherence to the mathematics textbook.

Design of Study

The data for this analysis came from the evaluation component of the CGI professional development project. As in many projects of this kind, during our three-year project the district had budget issues such that some of the teachers who began participating with us lost their positions by the third year of the project. We had to alter our research plan to accommodate this change. While the research design that we ended up with and describe here is not ideal, we felt that the data merited dissemination nonetheless.

To provide a comparison group we employed a delayed treatment design. Teachers at two of the schools began participation in the first year of the program and teachers from two other schools joined professional development in the second and third year of the program.

We collected two sets of data described in more detail below, a written test given to all first graders in the district and an interview done with a random sample of six children from each classroom. Timeframes and participants in these two data sets are distinct and will be discussed separately.

Written Test

In late May of each school year, all of the first graders in the district took a written test administered by their classroom teacher. The test consisted of 4 free-response story problems (see Table 1). Teachers read the problems aloud to students as needed, and children were told that they could draw a picture or use their fingers, blocks or equations to solve the problems.

Table 1. Story Problem Test Items.

Separate Result Unknown	Maria had 13 balloons. 5 balloons popped. How many balloons did she have left?
Join Change Unknown	Juan has 6 baseball cards. How many more baseball cards does he need to collect to have 14 altogether?
Multiplication	Sam has 3 packs of gum. There are 6 pieces of gum in each pack. How many pieces of gum does Sam have altogether?
Compare Difference Unknown	Lee has 12 pennies. Luis has 7 pennies. How many more pennies does Lee have than Luis?

*Participants***.** First graders taking the test had teachers with various levels of experience with CGI ranging from no experience to three years of experience. Table 2 shows the number of children and classrooms associated with each of the levels of CGI experience.

Table 2. Numbers of children and classrooms for each level of CGI experience

	No Experience with CGI	Teacher with 1 year of CGI	Teacher with 2 years of CGI	Teacher with 3 years of CGI
Number of classrooms	14	12	6	5
Number of children	284	238	115	106

Problem Solving Interviews

First graders worked one-on-one with trained interviewers in the fall and spring of the first grade in the second and third year of the project to solve story problems (see Table 3).

Interviewers read the problem out loud to the children and encouraged them to use any tool that they needed to solve the problem. The interviewer read the problem as often as the child needed. When a child stated a number, the interviewer asked the child if this was their answer, and if it was elicited an explanation by asking, “how did you get that answer?” All children were given the first three problems. If children were successful on two of these, they were given the other two problems.

Table 3. Story Problems for Interview

Separate Result Unknown (SRU)	The bear had 10 cookies. He ate 3. How many did he have then?
Multiplication	Robin has 3 bags of cookies. There are 5 cookies in each bag. How many cookies does she have?
Join Change Unknown (JCU)	You have $7. You want to buy a game that costs $11. How much more money do you need to earn so that you will have enough for the game?
Partitive Division	Four kids are sharing 12 candies. How many candies should each child get?
Compare Difference Unknown (CDU)	You have 12 balloons. Your friend has 7 balloons. How many more balloons do you have than your friend?

Participants. First graders participating in the interviews were in classrooms with teachers with various levels of professional development (see Table 4). Children were randomly selected in the fall of the school year and a few of these children left the district during the school year so were not available for interviews in the spring.

Table 4.

	No Experience with CGI	Teacher with 1 year of CGI	Teacher with 2 years of CGI	Teacher with 3 years of CGI
Number of classrooms	4	6	6	5
Number of children	24	27	30	26

Data Analysis

For both the Written Test and the Problem Solving Interviews, students received one point for every correct answer, one half if they used a correct strategy but miscounted, or zero points if they incorrectly solved the problem. Total scores were computed, analyzed, and descriptive and inferential statistics are discussed here. For the Written Test, a one-way analysis of variance (ANOVA) was conducted to determine if the mean student scores for each CGI experience group were statistically different from each other. Because there were four groups that represented the number of years of teachers’ CGI experience, a Scheffe Post Hoc test was performed to determine which groups performed significantly differently from other groups. For the Problem Solving Interviews, a one-way analysis of covariance (ANCOVA) was conducted to determine if the mean scores for each CGI experience group were statistically different while using the students’ pretest scores as a covariate.

Results

Written Test

The longer the teachers had participated in the CGI professional development, the higher their students' scores on the written test were. As is illustrated in Table 5, the mean scores for students whose teachers had three years of CGI were almost one point higher than the mean scores of children whose teachers had no CGI training.

Table 5. Mean Performance on Story Problem Written Test

Teachers' Experience with CGI	Mean number of items correct (4 possible)
0 years (n=284)	1.76
1 year (n = 238)	2.18
2 year (n = 115)	2.29
3 years (n = 106)	2.6

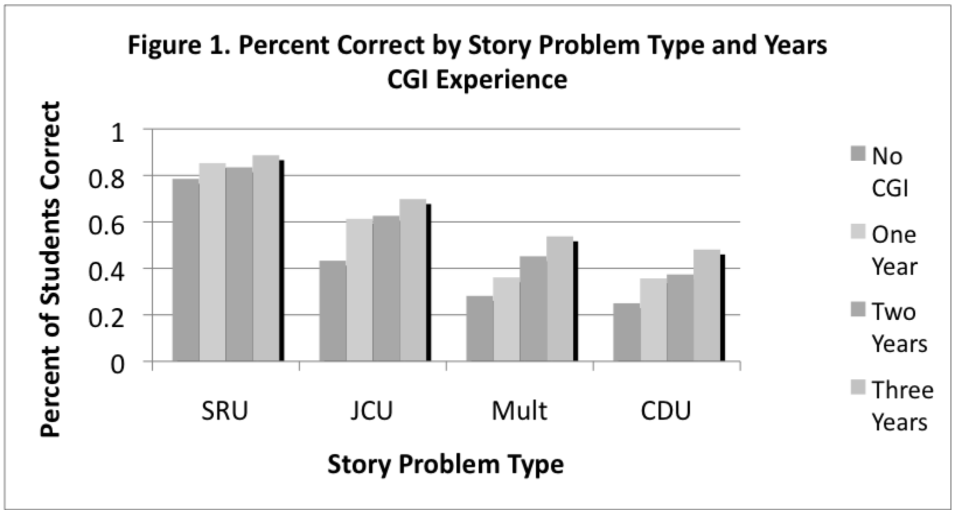
Table 6 summarizes the results of the Scheffe Post Hoc test of the differences in means for the written test. Students with teachers who had one or more years of CGI experience, scored higher than those students with teachers who did not participate in the CGI professional development program. Those students with teachers who participated in the CGI professional development for three years scored higher than those students with teachers who only participated one year in CGI experience. There were no differences found in student scores for teachers who participated one or two years and two or three years.

Table 6. Difference in students' mean performance on written test according to Teachers' Dosages (number of years of CGI training)

	0	1	2	3
0	X	-.43*	-.53*	-.85*
1	.43*	X	-.10	-.42*
2	.53*	.10	X	-.32
3	.85*	.42*	.32	X

* = statistically significant at $p < .05$ level

An examination of students' performance on the written test by problem types (see Figure 1) shows that teachers' experience made a small difference for students' performance on the Separate Result Unknown problem (SRU); while one year of experience made the biggest difference for the Join Change Unknown (JCU). For the other two problems [Multiplication (Mult) and Compare Difference Unknown (CDU)] the longer the teachers participated in CGI, the greater percentage of their students were able to solve these problems.



Problem Solving Interview

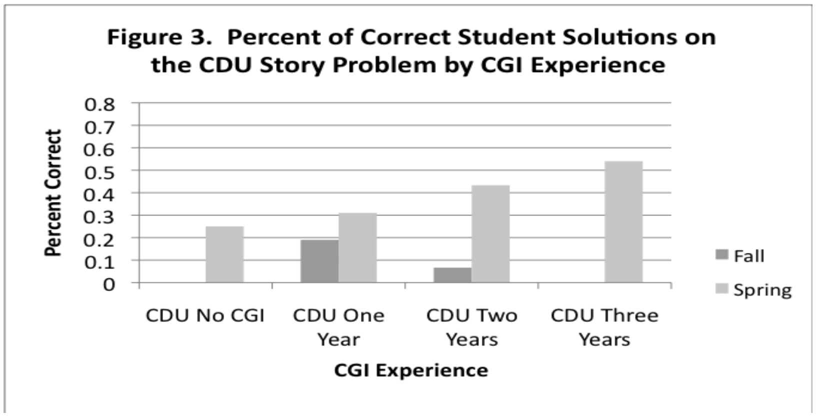
First graders' performance on the problem solving interview varied in the fall with children whose teachers' had no experience with CGI having the lowest performance (see Table 7). This can be explained by the fact that as kindergarteners these children had been with teachers without any CGI training while the children in the other groups tended to have been in kindergarten classrooms with teachers who had been involved in the CGI training. This fact does not account for the fact that in the fall children in first year teachers' classrooms outperformed children in the other classrooms, and we do not have an explanation for this beyond being due to chance. The difference between the fall and spring means were statistically significant for all four groups.

Table 7. Children performance on Problem Solving Interview (highest possible score 5)

Teachers' Experience with CGI	Fall Mean Performance	Spring Mean Performance	Difference in Mean (Spring – Fall)
No CGI n = 24 children	.75	2.38	1.68*
One Year n = 27 children	1.46	2.74	1.28*
Two Years n = 30 children	1.22	2.90	1.68*
Three Years n = 26 children	.94	2.96	2.02*

Table 7 shows that children in all the groups tended to improve in their performance from the fall to spring with the children in classrooms of teachers with three years of CGI experiences tending to improve the most. Although the students that changed the most were those who had teachers with three years of CGI training, none of the comparisons between groups proved to be statistically significant.

An analysis of students' performance by problem type showed similar trends as the written test for the Join Change Unknown problems with success rates being higher for students in teachers' classrooms with at least one year of CGI experience and a steady increase in the percentage of children successful on the Compare Difference Unknown problems with each year of CGI training (see Figure 3). The Partitive Division problem had the lowest success rate for all groups, with the greatest success rate going from 25% for the No CGI experience group to 38.5% for the 3 years CGI experience group.



Conclusion

The results of the written test show that for this implementation of CGI in this particular school district, children tended to be more successful solving story problems when their teachers had CGI training. Moreover their seemed to be a dosage effect because as children's teachers had more CGI experience, more of their students were able to solve problems. We propose two explanations for differences in student achievement. First, we posit that participation in CGI training led the teachers to do more problem solving with their students. This has been the case in previous CGI studies (Carpenter et al., 1990, Fennema et al., 1996). While we did not have the resources to observe the teachers on a regular basis, their reports to us in the workshops support this assertion. In addition, they brought student work to our meetings so they were engaging their students in problem solving at least once a month.

The second explanation relates to our analysis of particular problem types showing that differences in overall success rates were due to differences in performance on the less conventional problem types. During CGI training, teachers learn about these problem types and the ways that children think about them. For example, they learn that children can readily solve Join Change Unknown problems but children do so using additive strategies rather then subtractive strategies. In our workshops the teachers readily picked up on the distinction between Join Change Unknown and Separate Result Unknown problems, and we posit that their students encountered this problem type more frequently than children whose teachers did not have CGI training. In addition when their students solved these problems, we posit that their teachers supported them in solving the problems in ways that aligned with an additive interpretation rather than imposing a subtractive interpretation on to the children. Compare Difference Unknown problems tend to be the hardest problem type for students (Fuson, Carroll & Landis, 1996) and our data suggest that as teachers had more opportunities to ponder this problem type over time, they became more effective at facilitating their students in solving it.

While the data show that teacher participation in CGI training affected student achievement, first graders in this sample did not perform as well as kindergarteners in Wisconsin in 1993 (Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993). In that study the majority of the children were native English speakers and the teachers had a great deal more discretion over instruction than teachers in our study. Teasing apart which factors, language issues or instruction, might account for the difference in our students' performance as compared to that of the 1993 sample is impossible to do with the data that we have. The teachers in our group reported feeling limited in the amount of time they could devote to problem solving because of the emphasis on getting through their textbook. Their use of timed tests and other drill activities may also have affected children's beliefs about mathematics making them more hesitant to solve problems using manipulatives or drawings. Insuring problems were comprehensible to English Language Learners through careful wording and choice of context, added to the complexity of implementing CGI in this setting. Given these differences in the conditions of implementing CGI, we felt that the students' performance was encouraging. The added complexity of this setting suggests that teachers may take longer to become effective in employing CGI in their instruction than did the teachers in previous studies.

We continue to work with teachers in the district and are now working with the 3rd and 4th grade teachers who are teaching the cohort of students who participated in this study. If there are cumulative effects of having CGI teachers over time, we will be able to demonstrate these effects with case studies of some individuals, many of who are English Language Learners.

We acknowledge that our sample size is small and our study inadequate to make strong claims, nonetheless we do think the data show positive effects for CGI and suggest that its benefits generalize to economically disadvantaged English Language Learners in this time of heightened accountability. We hope that these tentative findings can be supported with a larger, more robust study of implementation of CGI with this student population.

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INVESTIGATING A FIFTH-GRADE MATHEMATICS TEACHER'S PRACTICES IN RELATION TO HYBRID SPACE

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This study adds to current literature on hybrid space with a focus one fifth-grade mathematics teacher. The researchers conclude that the teacher's instructional practices allowed her to create opportunities for a hybrid space to emerge in her classroom. The framework developed in Barton and Tan's (2009) work in science is applied to a mathematics classroom to investigate the use funds of knowledge and Discourses. Evidence from this case study found that two themes were present in the classroom: family funds of knowledge and Discourse and peer funds of knowledge and Discourse. The implications of this study could help teachers and researchers better understand how to motivate students concerning mathematics.

Purpose

The purpose of this study is to examine the instructional practices of one 5th grade mathematics teacher through the lens of hybrid space. Hybrid space is a classroom environment where teachers can engage students in learning by bridging the mathematics content with the knowledge and background of their student (Moje, Ciechanowski, Kramer, Ellis, Carrillo, & Collazo, 2004). Investigating the teaching methods of one teacher and funds of knowledge and Discourses in a classroom can inform researchers and practitioners on how to make learning applicable to students' lives (Barton & Tan, 2009). A case study was used to answer the research question: How can teacher practices create a hybrid space in a mathematics classroom?

Perspectives

The National Council of Teachers of Mathematics (NCTM) acknowledged the importance of discourse in teaching mathematics through principles and standards that call for teachers to build on discourse in the classroom (NCTM, 1991, 2000). Cobb and his colleagues (1997) discuss the concept of discourse and theorize that it can provide support to students' mathematical development that is more consistent with reform shifts to have students engage in verbalizing their mathematical reasoning. Discourse is also an important construct in Gee's (2000) work on identity, which defines Discourse as a "way of being 'certain kinds of people'" (p.110). Their "way of being" might entail being a math person or being a member of a sports team. Importantly, Gee's definition of Discourse encompasses the verbal discourse NCTM recommends but also expands to include how students interact with the content and express themselves in relation to the content (Barton & Tan, 2009). For example, students might interact with the content of mathematics by discussing what a meter is and how it is defined, but they may also choose to create representations of a meter with physical objects or use tools to assist them in their understanding of the mathematics content. Students may even become to see themselves as mathematics experts and an important contributor to mathematical understanding in the classroom.

Hybrid space is a classroom setting where a teacher fosters students' participation so that they become engaged with the content in a meaningful way (Moje et al., 2004). Barton and Tan (2009) further define a hybrid space as "a supportive scaffold that links traditionally marginalized funds of knowledge and Discourses to academic funds and Discourse" (p.52). Academic funds in that definition refers to the mathematical content. Funds of knowledge in their view refers to the base of knowledge that students have from their experiences and cultural backgrounds (Gonzalez & Moll, 2002). Students may enter a classroom with minimal knowledge of how to add fractions, but they may have used fractions in cooking or dividing candy among siblings. Teachers can use a hybrid space as a bridge between the students' previous funds of knowledge and academic funds of knowledge to build on the students' Discourse with mathematics. It is a combination of funds of knowledge and Discourse that is integral in creating a hybrid space. Flessner (2009) stressed the importance of creating an environment that was conducive for creating a hybrid space. His classroom allowed students to feel "safe, respected, and confident" in addition to being "contributing members of the classroom community." (Flessner, 2009, 432) In this view, it is important that a teacher allow students to contribute their own knowledge about content. A teacher need not be the sole supplier of expertise. Teachers can add contextual content that relates to student experiences to provide opportunities for a hybrid space in the classroom. For example, a teacher can present problems that relate to real life situations such as games the students play or provide materials that the students have experience with such as measuring cups or scales. This view is also discussed by NCTM which calls for the use of contextual methods of instruction so that students learn mathematics that is in relation to real world scenarios (NCTM, 2000). Whereas contextual learning is an important part of reform in education, Boaler (1993) found that contextual problems are not necessarily increasing student learning because they may not incorporate realistic, real world problems that students were able to make their own. This idea emphasizes that it is important for context to relate to the students' experiences and backgrounds so that they are engaged in meaningful Discourse.

The present study examines how creating a hybrid space in a classroom can allow students to build on their Discourse in mathematics through the use of funds of knowledge and Discourse. Barton and Tan (2009) created a framework for viewing a hybrid space in the context of a science classroom. This framework included four specific constructs: (a) family funds of knowledge & Discourse, (b) community funds of knowledge & Discourse, (c) peer funds of knowledge & Discourse, and (d) popular culture funds of knowledge & Discourse (Barton & Tan, 2009). Family funds of knowledge and Discourse refers to how students draw on their experiences of family life such as shopping trips or playing school with siblings to participate in classroom mathematics. Community funds of knowledge and Discourse relates to how students draw from their community such as their participation in their school or neighborhood community to participate in classroom mathematics. Peer funds of knowledge and Discourse is how students draw on their experiences with peers to help and support each other in learning mathematics such as validating each other's responses in class or showing concern for peer understanding. Popular culture funds of knowledge and Discourse refers to how students draw on their experiences with popular culture such as television or music to participate in classroom mathematics (Barton & Tan, 2009). This present study utilizes Barton and Tan's framework to examine a hybrid space in an elementary mathematics classroom.

Methods

This study came out of a larger study on student motivation in elementary mathematics (Linder, Smart, Cribbs, 2011). Based on the findings from the original study, it was determined that one teacher in an elementary setting in the Northwest region of South Carolina had students who displayed significantly higher levels of motivation than all other students in the school. This

5th grade teacher became the focus of the current study to determine what teacher practices might account for these results.

A case study analysis of this 5th grade teacher's instructional practices in mathematics was conducted in the spring of 2009. Data was collected through classroom observations and paired student interviews. Nine classroom observations were conducted, recorded, and later transcribed by a member of the research team. Student interviews were also conducted in pairs using a semi-structured protocol to get a better idea of how students perceived mathematics and their teacher in relation to mathematics. A member of the research team later transcribed the interviews. Data analysis included repetitive readings of all transcriptions and viewing of videos to create an initial theoretical explanation of what was occurring in the classroom. This theory was then strengthened by supporting evidence gleaned from the data collected (Yin, 2009).

The school that this study took place is a Title I school where 81% of the school population receives free or reduced lunch. The teacher observed taught four different classes through the course of the day, and interviews were conducted with 21 different students from these classes.

Results

Classroom Environment

It was apparent from the first observation that the teacher had a strong command of her classroom. She was firm and purposeful with her students and their activities, but she also nurtured students with a friendly and open demeanor. Even in the midst of noisy and mobile activities, the teacher maintained control and kept students on task. One of the defining characteristics to creating an engaging classroom was the teacher's constant questioning of students and challenging them to think about the content. She often engaged the students in discourse with her and with each other. The following interaction is an example of this discourse between the teacher and the students as they discuss weight. The students have been broken up into groups of 3 or 4. They have also been given a bag of popcorn, one for each group, to investigate together.

> Teacher: "…Now, I have a question for you, or several questions for you. What is a reasonable estimate for the weight of this bag of popcorn in pounds? What would be a reasonable estimate for that?" The teacher called on one student with her hand raised.
> Student: "One pound."
> Teacher: "Ok, Alice says one pound. Does anybody else have another…? You say one pound." The teacher pointed to a student at the front and then called on another student.
> Student: "Two pounds."
> Teacher: "Ok, two pounds …Ok, so we've got one pound, we've got two pounds, ok. What is a reasonable estimate for the weight of popcorn in ounces? Think about the estimate that you gave me and what would be a reasonable estimate in ounces? What would be a reasonable estimate in ounces?" Teacher called on another student who responded with 20 ounces.
> Teacher: "Ok, [student] says twenty ounces. What would you say?" The teacher called on another student. "He says sixteen. Why do you say sixteen?"
> Student: "Because it feels like a pound."
> Teacher: "It feels like a pound. Ok, why'd you say twenty?"
> Student: "Because it feels like a pound and it has some ounces left over."

The teacher continued the discussion by asking the students to give examples of what other objects might weight the same as the bag of popcorn. The teacher asked the students questions that were open ended and required them to explain their reasoning. The students were asked to interact with the content in the form of a bag of popcorn and also relate it to their own experiences. This type of interaction was seen repeatedly in classroom observations of the teacher.

When students were asked in interviews how they would describe their teacher, they stated that she was "smart", "funny", "helpful", and "a good teacher" among many other similar comments. In response to the question "Do you think [your teacher] thinks math is important?" two students stated the following:

> Student 1: "She like applies it to real life things…Like with geometry, like with like structures, how they need to be built. Like construction. How they need to be built just to stand."
>
> Student 2: "And sometimes in math like she compare (sic), like when she doing geometry, she compare it to real things in life. Like a rectangular prism as a cereal box. A sphere as a ball. She can explain it as to real life things."

It was through these methods of instruction along with the comments made by students in interviews that the researchers began to see a pattern that allowed for students to bridge the gap between the mathematics content being taught and their background. This concept will be discussed further in the next section through a framework of hybrid space.

Creating a Hybrid Space

Two of the four constructs from Barton and Tan's (2009) framework emerged through our analysis present in this study. These constructs are discussed below with reference to how they were present in the classroom with supporting evidence from student interviews. It is important to remember that the teacher was not intentionally creating a hybrid space but rather this space occurred spontaneously as a result of her teaching practices. Flessner (2009) discusses this idea by stating that it was only when he stopped trying to create a hybrid space that one occurred, and that it may be best for teachers to focus on creating the conditions that are conducive for a hybrid space.

Family Funds of Knowledge and Discourse

Family funds of knowledge and Discourse is related to the student's family life such as events or activities that were discussed by the students Barton and Tan (2009). In the first lesson that was observed, the teacher brought items from her house for students to compare and contrast. The essential question that was posted for the lesson was "What is capacity?" The teacher asked the students to make a decision as a class as to how items should be ordered from the smallest to largest capacity without looking at any labels. The students were presented with the following list of items: soda bottle, stomach medicine, marinating sauce, milk carton, throat spray, cooking oil, and olive oil. One student told the teacher that she thought the stomach medicine should go next as the teacher lined the items up from smallest to largest capacity on the desk. The teacher asked her to explain her rationale as detailed below.

> Student: "…because, like, when I take it, it's very thick."
> Teacher: "Ok, so, you based it on the thickness of it?" The teacher holds up the Pepto-Bismol
> Student: "Yeah. I based by how it looks."
> Teacher: "You didn't feel it."
> Student: "Like, yesterday I had to get some."

Teacher: “Oh, ok, ok, so, you’re using your knowledge that you might have used before by holding it, but let me ask you this…Ok, let me ask you this. You didn’t hold any of these other ones.”

Student: “No”

Teacher: “So how do you know that this [teacher holds up Pepto-Bismol] is heavier than this [holding up the throat spray]? “

Student: “I know the soda is heavier than all.”

Teacher: “So because it’s thicker, you think it is heavier? Ok, we’ll check all of that out.”

Another student is called on immediately after this discussion and relates her experience at home with capacity.

Student: “Um, in my house my sisters, they be trying to say that, well, they put two cups out. They put a big, it’s small and all thin and stuff, and they have a little small one. And they say the ones that’s small has more liquid inside it when it doesn’t.”

Teacher: “Ok, um, because it’s smaller, anyone have one of those, than something that’s taller?”

Student: “It’s wider.”

Teacher: “It’s wider. It could have more liquid than something that’s taller. And that’s true.”

The teacher discussed the idea of wide versus tall with the rest of the class, and the students began to order the items again. This interaction indicates that the teacher was open to the students sharing their experiences with capacity as well as discussing how it related to the current lesson. This helped the students to connect new concepts to previously developed understanding of the content.

During another classroom observation, the teacher put up the essential question “How are pounds and ounces used in the customary system?” Students were again asked at the beginning of the lesson to not only tell the teacher what they already knew about the topic but also encouraged to share how they used pounds in their lives. The discussion below began shortly after the lesson had started.

Teacher: “She said there are sixteen ounces in a pound. Where have you seen pounds and ounces used.” The teacher called on students to respond.

Student 1: “At the doctor.”

Student 2: “The store.”

Student 3: “Bi-lo”

Teacher: “Alright, at a store, Bi-lo, you said Bi-lo, and you’re weighing what?”

Student 3: “Like food.”

Student 4: “Like vegetables.”

Teacher: “Food, alright, wait a minute. One person at a time.”

Student 3: “Like food or vegetables.”

Teacher: “Ok, vegetables, ok. Where else?” The teacher called on another student.

Student 5: “At home.”

Teacher: “At home, how’d you use it at home?”

Student 5: “Like when you want to know how much rice you want to use.”

Teacher: “Ok” the teacher called on another student.

Student 6: “You can also use to how much milk you want to put in a cake.”

The teacher asked for a few more comments and then moved the discussion on to how to estimate weight by looking at an item. The teacher had brought bags of popcorn that was given to each group in the class. Students were asked to weigh the bag in their hands and later to compare it to other items in the class and in their book bags. Though conversations such as these can occur spontaneously in a classroom, the teacher fostered an environment where students could discuss mathematical concepts in relation to their own experiences. She also encouraged discussion and questions in the class. The brought household items for students to interact with and asked them to use their own items to help understand the concepts. Students were eager to participate and seemed to be excited about sharing their experiences.

Peer Funds of Knowledge and Discourse

In this study, peer funds of knowledge is related specifically to a subcategory that Barton and Tan (2009) refer to as a studenting Discourse. Studenting is how students provide peer support or specifically help each other with mathematics (Barton & Tan, 2009; Moje et al., 2004). This concept was evident in how students grouped together and helped each other to understand concepts. The following discussion between two students occurred during the lesson on capacity as described in detail in the previous section. The students are looking at a plastic measuring cup that the teacher has given them and asked the class to come up with inferences.

Student1: "Thirty-two equals is one quart and on this scale is always skipping fifty and one liter…"
Student2: "No, it's skipping like a hundred and fifty, I mean a hundred. Look."
Student1: "Yeah, yeah."
Student2: "It's skipping a hundred."
Student1: "One liter equals one thousand milliliters."
Student2: "It's skipping one hundred."
Student1: "Oh, I see. I already got confused with it."

Student 2 helps student 1 to understand what the intervals on the measuring cup indicate. During another observation, this type of studenting was observed again as one student struggled to understand a warm-up problem that was posted on the board at the beginning of class. The students are trying to determine what a flat diagram on the board will look like if it is folded to make a 3D image.

Teacher: "Ok, what shape does that make?" The teacher called on a student.
Student 1: "A cylinder"
Student 2: "The cylinder wouldn't look like that…one circle would be at the bottom. One circle is at the top." The teacher looked confused by the student's comment, so the student walked up to the board to show the teacher his explanation.
Student 2: "A cylinder would be like this. A circle would be here. The other circle would be right here, so when the wrap…" The student indicated with his hand what he was doing with the figure on the board.
Teacher: "Ok, what is it then?"
Student 2: "I thought it was a prism."
Student 1: "no"
Student 3: "Can I, can I…" This student goes up to the board to help the other student. The teacher lets them think through the problem together on the board.
Teacher: "We have a debate going on here."
Student 3: "If you put this side together with this side, it will make a round side like this. Then you pull this down like that right there and down there and it becomes a cylinder."

Student 2: "But the, but look, a cylinder it usually this circle's right here and this circle's right here."
Student 3: "These two circles up, and you fold it this way."
Teacher: "You probably, that's probably the way you're used to seeing it. Have you thought that this can actually be folded like this?" The teacher attempted to demonstrate the folding with her hands. "And then you're going to have this part here and that part here." The teacher is referring to the circles. "Can you see that?"
Student 3: "It's not a prism."
Student 2: "Oh, yeah"
Teacher: "Ok, guys. Ok, this is interesting. Do you see it now?"
Student 2: "Yeah, I see it."

This construct is repeated throughout the observations and supported in interviews. When two students were asked in their interview if they ever got to ask for help from friends in class, one student responded with "Like some people like didn't know how to do it. 'Cause I got a 100. And we had to help other people, so they could get a 100 too." In several of the interviews, students stated how they were concerned about their fellow peers understanding. When asked if they got to work in groups, one student stated the following.

"She [the teacher] placed us as in, like, she has like, I'm one of the high students. She has like certain high students with some people who don't learn as quickly and has some people who learns quickly. And then some people like for others to help."

It was clear from this comment that the students were engaged in helping each other through grouping. This pattern became even more apparent when the students were asked in an interview if their teacher would listen to them if they came up with a new strategy or idea in class.

Student 1: "Yes, she'll listen 'cause like we come up with new strategies to use in math. We come up with new strategies to see, try a strategy, um, like on a problem. And if she get it right, she'll say it's right. If she get it wrong, it's like, she'll tell you it's wrong. For some strategies, is like only on that problem. You a strategy on only that problem not on other problems like on the same subject, or the same thing."
Student 2: "She'll like accept it 'cause it might help…"
Student 1: "Other students."
Student 2: "Yeah, it might help other students. And like it might help her explain it better."

Not only were students supporting the idea of helping each other through sharing strategies, but they also felt that what they had to say was significant to the class. Both classroom observations and interviews revealed that students were concerned with each other's understanding and were willing to help each other. Their teacher often put them in groups or pairs in an effort to facilitate their learning and also encouraged them to explain their rationale to each other.

Discussion

This study suggests that a hybrid space can be created in a mathematics classroom and some of the constructs that have been found in other content areas are also present in a mathematics context. Two constructs from Barton and Tan's (2009) framework were evident after analyzing the data: family funds of knowledge and Discourse and peer funds of knowledge and Discourse. The reason why the other two constructs were not present could be that the teacher was not trying to create a hybrid space in her classroom. If the teacher had been intentional in her instruction for creating a hybrid space, all of the constructs in Barton and Tan's framework (2009) might have been present. The study was also limited in scope in that the lessons observed were over a short period of time. If observations were conducted over the course of the year, it is possible other constructs would be seen in the data. It is also possible that different constructs are more readily seen in different content areas as Barton and Tan's (2009) framework was based on a science classroom and that further research needs to be done to further investigate the framework for mathematics.

This study also presents the possibility that a hybrid space can contribute to student motivation in mathematics. The larger study that led to this case study was on student motivation. Students in this fifth-grade setting displayed higher levels of motivation for mathematics than all other K-5 students at the same school (Linder, Smart, Cribbs, 2011). The fact that students are active participants through Discourse in the lessons observed could contribute to their motivation. Sfard (2009) stated that "learning to speak, to solve mathematical problem or to cook means individualization of these activities, that is, a gradual transition from being an only marginally involved follower to other people's implementation to acting as a competent participant, with full agency over the activity" (p. 56). This statement could hint at the interplay that is occurring in the classroom.

The findings from this study suggest that teacher practices do impact the development of a hybrid space in a mathematics classroom. This study also suggests that a hybrid space in a mathematics classroom may provide a method of improving motivation. The Discourse that was present in this teacher's classroom shows a picture of how students are trying to make meaning of the mathematics content and add it into their funds of knowledge. Further research needs to be done to better understand the relationship between teacher practices and hybrid space as well as influence a hybrid space can have on student learning. It is useful to investigate these teaching practices in more depth and examine the long-term impact to student learning and motivation when they are members of a classroom that allows for a hybrid space to exist.

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QUALITATIVE PROPORTIONAL THINKING AND TECHNOLOGY

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The research reported in this paper has a bearing on qualitative proportional thinking, for which technology is used; specifically an interactive computational program which supports the construction of the concept of proportion in a qualitative way is developed. The indicators of this concept are constructed and the ones of the development of qualitative proportional thinking, the design of didactic activities and their insertion in the interactive computational program is shown, as well as its implementation, to finish with the analysis and discussion of results.

Introduction

The development of proportional thinking is important since basic educational levels, as from it depend that children be able to comprehend and face everyday situations which are linked with the concept of proportion. At the same time, as it is established by XXX (2002) in order for *the student at basic level to be able to assign sense and meaning to proportion it is fundamental to develop his proportional thinking, both the qualitative and the quantitative*. That is to say, for the development of proportional thinking it is required, among others that the subject constructs the concept of proportion, and in order for him to be able to construct this concept it is required to have proportional thinking. In other words, there exists a bidirectional relationship between the mathematical concept of proportion and proportional thinking.

In order for a child to identify the proportional, in accordance with Piaget (1978), this should be made starting from the concrete to reach the abstract. Similarly, the construction of this mathematical concept is represented both at a qualitative level and at a quantitative one, which determines the proportional thoughts both qualitative and quantitative, respectively.

From another perspective, nowadays, within the educational contexts there exists the possibility of designing and developing technological materials of an interactive type. Whatever classroom experience can use electronic technology as mediation way, but it is necessary to value dimensions such as: objectives to be reached, organization of the themes which enable the development of the preferred contents to be included, learning activities and the evaluation both of the learning and the overall process.

On the other hand, the increment of technological materials is amazing; however, didactic strategies designed with theoretical elements are not always included. In the research being shown, in this paper, the educational theory named *Mathematics within the Context of Sciences* is used, by means of which the mathematical themes and concepts should be treated is such a way that they are linked to the environment, to everyday activities, to labor and professional competencies, as well as to the other sciences studied by the pupil.

From what has been described, a research has been dealt with whose problem is the construction of the concept of proportion and the development of proportional thinking both qualitative and quantitative, by means of didactic activities with electronic technology, in sixth grade children from elementary education. Since the research is so vast, this presentation only reports one didactic activity for the development of qualitative proportional thinking, as well as its implementation and analysis, which constitutes a partial research.

The objective of the partial research which is reported is to design didactic activities for the construction of the concept or proportion (in a qualitative way) and the development of qualitative proportional thinking, by means of an interactive computational program.

The supposition of research emerges from the fact that upon constructing or reconstructing the concept of proportion, there will be, to some extent, a bearing on the development of proportional thinking, for this report, on the qualitative proportional thinking.

Methodology

The design of the activities for the development of qualitative proportional thinking, by means of an interactive computational program, uses a methodology provided in the following three steps: 1. To determine the indicators associated with the construction of the concept of proportion (in a qualitative way), as well as the indicators associated with the development of qualitative proportional thinking (by means of the mathematical concept). 2. To design the didactic activities on the concept of proportion and their insertion in the interactive computational program. 3. To implement the interactive computational program and, to analyze and to discuss the results.

Theoretical Foundation

Mathematics within the Context of Sciences

The theory of *Mathematics within the Context of Sciences* establishes that mathematics should be presented to the student, at any educational level, through contextualized events, (Camarena, 2005, 2009). This theory conceives the learning and teaching process as a system where the five phases of the theory intervene: curricular, cognitive, didactic, epistemological and educational.

Research Development

1. Determination of Indicators

The authors who are mentioned in this section also belong in the section of theoretical foundation, it is just that in order not to duplicate the information they have only been located in the present section. It is through them that the indicators for the development of the concept of proportion are identified, as well as the indicators for the development of qualitative proportional thinking. It is worth mentioning that the identified indicators have been highlighted in bold type.

As an antecedent of what has just been mentioned stands the research by Ruiz (2002) who designed and applied a teaching proposal dealing with the concept of proportion and who found out different difficulties presented by students in the last year of elementary education, among them we find the two following: 1. Qualitative thinking of elementary school students revolving around proportionality is barely developed. 2. Students showed confusion when establishing proportions in an intuitive way and explicitly in geometric figures, since they were not able to establish proportions when they compare it in figures of similar length with the width of another.

Piaget (1978) argues that between 11 and 12 years of age, it is possible to observe in the subject the presence of the notion of proportion in different fields, such as: spatial proportions (similar figures), the relationships between weights and lengths of the arms of the scales, the probabilities, etc. Piaget also mentions that through his experiments he points out that the child acquires the qualitative identity before quantitative conservation and makes a distinction between qualitative comparisons and the true quantification. In fact, for Piaget (1978) the notion of proportion always starts in a qualitative and logical way, before being quantitatively structured. He stresses that I order for the student to develop his *qualitative proportional thinking* it is necessary to start from the notions of ***enlargement and reduction (1)***, following the idea of a photocopy or a scale drawing, assuming that the student at a really early age manages to recognize what is proportional using perception and observation. One way to express his *qualitative thinking* is to use linguistic expressions such as “greater than…” and “lesser than…”, that is to say, ***using verbal categories (2)***

In accordance with Piaget and Inhelder (1978), after the student develops the perceptual part (qualitative proportional thinking) an ordering when making comparisons appears (which is located in the *shift from qualitative proportional thinking to quantitative proportional thinking*), this can be verified when the student compares figures superimposing them taking as origin the indicator ***to compare (3)***. In this respect, Piaget points out that in this *shift from the qualitative to the quantitative* the student can build a figure enlarging it or reducing it, thus constituting the indicators ***to enlarge and to reduce (5)***. Later on, the student uses the measurement when making comparisons, firstly confronting parts of the object and superimposing one figure over another and then using a measuring instrument, conventional or not. Thus, ***measuring with instruments (6)*** represents another indicator, allowing the development of his *quantitative proportional thinking.*

It is important that the student when developing his *qualitative proportional thinking* manages to ***use the rule of three (7)*** assigning sense to this and not merely in a mechanical way (Ruiz, 2000), with which one more indicator is defined.

In terms of Freudenthal (1983), in order to establish proportions, *both intuitively* (that is, qualitatively) *and explicitly* (that is, quantitatively), the comparisons are expressed in two modalities: direct and indirect. The direct modality of comparing is when and object is ***superimposed*** over another object, which defines the indicator ***to compare directly (8)***, whereas the indirect one is when there are two objects and an ***instrument*** to compare them, like the use of a ruler or simply by counting obtaining as a result the indicator ***compare indirectly (9).*** The child is able to compare two objects indirectly and is able to do it in a qualitative and/or quantitative way.

It is important to mention that authors such as Piaget (1978) and Streefland (1991) mention in a natural way qualitative proportional thinking is first developed, through the perception of the empirical. On the other hand, it has been demonstrated that in the educational practice the use of algorithm is given priority, thus students developing a quantitative proportional thinking in a mechanic way, when in many occasions they have not developed qualitative proportional thinking. Thus, the qualitative-quantitative sequence is not always present in students.

Freudenthal (1983) points out that the comprehension, in an intuitive way, (i.e. qualitative), of proportion can be guided and delved in by the use of visualization and the latter can be illustrated using detailed constructions, where the drawings are differentiated and show what points correspond among themselves in the original and in the image. Freudenthal also suggests that upon working on the proportion of lengths flat figures are used as means of representation, for its expressiveness is more global, in the sense that they make it easier for the student the qualitative and quantitative comprehension between magnitudes by means of visual perception. From this perspective of mathematics, it is also important that the student manages to express *proportion* ***as a fraction (10)*** which constitutes one more indicator. Moreover, for Freudenthal (1983), it is precise that in teaching it is precise to take into account both ***internal proportions (11)*** and ***external proportions (12)***, defining the former as the relationships which are established between different values of the same magnitude, and the latter as relationships between values of different magnitudes; both proportions express two indicators to be taken into account.

Resorting to what has been mentioned by researchers in the previous paragraphs, a table in which the indicators referred to proportion as well as to qualitative proportional thinking is shown next; the didactic actions associated to such indicators are established as well.

Objects of study	**Objectives**	**Indicators**	**Didactic actions**
The mathematical concept of proportion	To establish proportions in a qualitative way	Compares directly (8) Compares indirectly(9)	♦ To superimpose figures ♦ To use a measuring instrument
	To establish proportions in quantitative way	Compares indirectly (9) Uses internal and external proportions (11, 12) Expresses proportion as a fraction (10)	♦ To use a measuring instrument ♦ To use a table relating data and writing the proportion as a fractions
The development of qualitative proportional thinking	To contribute to the development of qualitative proportional thinking	Enlarges and reduces (1) Uses verbal categories such as “greater tan” or “lesser tan” (2)	♦ To select reduced figures or enlarged ones by means of ♦ To use linguistic expressions
	To contribute to the shift from qualitative proportional thinking to quantitative.	Compares (3) Counts (4) Enlarges and reduces figures (5)	♦ To superimpose figures ♦ To count the sides of squares in one grid ♦ To draw enlarged or reduced figures in a grid

Table 1. Indicators and their didactic actions.

It can be observed in table 1 the diverse didactic actions which students should work upon, and which they can carry out in the interactive computational program. These didactic actions are expressed in a generic way like: To superimpose figures, To use verbal categories* To use measuring instruments, To use tables*, To select figures, To draw figures in a grid, To count the squares in a grid.

The actions marked with an asterisk, require of a recorder, which is external to the interactive computational program. When the student carries out these didactic actions it is necessary to use the recorder to have evidence. The verbal categories are recorded in the audio, as well as the comments by the students. The didactic actions “to use tables”, allows the student to relate date from the same column, between two columns or that the table can be filled out by him. For the case of relating data, the audio allows to identify this indicator.

The link between didactic actions and computational actions. The computational program was developed in order for the student to be able to carry out the didactic actions. For this research the computational program which has been developed allows to incorporate actions such as to drag figures, to access the figures of a box, to use a virtual pencil as though it were real, to make use of a grid to count the squares or to draw upon it, to make measurements with instruments such as a virtual ruler, to use tables to be filled out; al the actions afore mentioned are named ***computational actions.***

In table 2 the link between didactic actions and computational actions, necessary for the constructions of the concept of proportion, as well as the concept of qualitative proportional thinking is established. That is, computational actions are the ones which allow that didactic actions associated with each indicator of the construction of the objects of study can be carried out.

Didactic actions	Computational actions
To superimpose figures	To drag figures
To use measuring instruments	To use a virtual ruler
To use tables	To fill out a table
To select figures	To access figures
To draw figures in a grid	To use a virtual pencil
To count the squares in a grid	To use a grid

Table 2. Link between didactic actions and computational actions.

2. Design of Didactic Activities and Insertion in the Interactive Computational Program

In this section are first described the pedagogical elements from the interactive computational program which was overtly designed for this project. Later one the didactic activity is shown, specifying its purpose, which in founded upon the theoretical elements previously shown; the contextualized event which identifies it, in accordance with the theory of Mathematics within the Context of Sciences; the didactic actions which can be used by the student when dealing with the activity and, the computational action associated with this event; to finish with the insertion in the interactive computational program.

We present next the pedagogical elements from the interactive computational program. The pedagogical elements form the computational program allow the representation and visualization of instructions, the activities to be carried out, the help and options which the user can resort to, etc., in general the elements which are used in the learning process.

1(a) Familiar context 1(b) Motivation 1(c) Follow up

Figure 1. Screens from the interactive computational program

In accordance with Mathematics within the Context of Sciences, a familiar environment as well as metaphors of the real world adapted to sixth grade elementary students are presented, see figure 1a. The program is user friendly as it allows the user to log out, log in and send. For instance, you hold a click on the door to log out, see same figure 1a. A visual style is presented by means of font types, buttons and general aspects focused on elementary school children. Brevity in the texts is used, incorporating sounds and graphics to substitute possible content in the text. The type, style and size on fonts are legible. Messages are simple and concise, so as to avoid the user being confused.

The user is given tolerance of three attempts to correctly solve the activities and when he carries out an activity correctly, he is warned as a motivating means, see figure 1b. In the interactive computational program the student can look up his advance, see figure 1c.

When the student does not accomplish to carry out the activity correctly, then, animations are used to illustrate the concepts by means of examples and thus the association between concepts is immediately generated. Next we present the activity and its insertion in the interactive computational program.

Purpose of the Activity

The activity pursuits the construction of the concept of proportion and in consequence the development of qualitative proportional thinking. We start from what has been pointed out by Piaget (1978), Streefland (1991) and XXX (2002), on the fact that early teaching of proportion should start from qualitative levels of recognition of them, for this reason, the activity does not require of the use of quantities for its solution.

Contextualized event of the Activity

“Ricardo is ten years old, he went to Veracruz and at the harbor he saw a ship which liked him very much, the child drew it in a sheet of pager. When he came back to school he showed it to his teacher. She asked for four versions of the ship at the photocopier, some of them enlarged and others reduced, in different sizes. After that she took the photocopies to her students and asked them to choose the figure of the ship which was enlarged twice the size. Help them to find the ship which is enlarged”.

Didactic and Computational Actions of the Activity

In order to solve this activity the student can make use of the following didactic action: *to select the reduced figure* or the enlarged one to the given one. This corresponds to indicator (1): *to enlarge and reduce*, from the development of qualitative proportional thinking and is associated with the computational action of *accessing figures*.

The student can also make use of the didactic action of *superimposing figures*, this didactic action corresponds to indicator (8) of *directly comparing* when we aim at establishing proportions in a qualitative way (intuitive), similarly, it corresponds to indicator (3) of *comparing* the development of the shift from qualitative proportional thinking to the quantitative one.

In order to superimpose the figures, the student has the option to drag, with the mouse, whichever of the four figures to superimpose them over the original one and revise, by the use of visualization, whether the figures is enlarged or reduced in all the sides in the same quantity, the computational action involved corresponds to *drag figures*.

The recorder is turned on all the time in order to have the comments by students and their *linguistic expressions* which allow us to identify the qualitative proportional thinking through indicator (2) of *using verbal categories*.

If, not even in a second attempt is the student able to select the correct figure, it means that his qualitative proportional thinking is scarcely developed; however, as it has been mentioned, some students develop more of a kind of thinking than another, that is why they are given the option to move to didactic actions from quantitative proportional thinking. Then, there is the didactic action of *using a measuring instrument*, associated with indicator (9) of *comparing indirectly*, which corresponds to the construction of the concept of proportion both in a qualitative, and in a quantitative way, it will depend on whether the proportion is established in a numerical way (quantitative) or it is only verbalized (qualitative). At the same time, this didactic action is associated with indicator (6) of *measuring with instruments* from quantitative proportional thinking. The corresponding computational action is a *virtual ruler*, which allows the student to measure each side of the original figure, as well as each side of the figures that appear to select the one which is enlarged. It is worth mentioning that this interactive computational program randomly generates the figures (it is not always a ship), as well as their enlargement or reduction in different scales (which are not always expressed in whole numbers), with the purpose that it does not become a mechanic activity and that it is not memorized. In general the computational program randomly generates the data, the figures, the contexts and the tables.

Insertion of the Activity in the Interactive Computational Program

In a first screen the figure of the ship and 4 figures which are similar are shown, but with Little differences among them, see figure 1b. For instance, they can be twice the size, thrice the size, reduced to half or to one third of the original size and the child is asked to choose from the 4 figures the one which is enlarged to twice the size of the original.

When the child has selected one of the figures, it is because he has visualized the enlargement (1) or has dragged the figures in order to superimpose them and then be able to compare them directly (8) and define which one is enlarged twice the size. After selecting the figure, the computational program analyzes the choice made and following it sends an answer with the result of the analysis: a) If it is correct, another activity is shown to the child. B) If the selection is incorrect, the program sends a written message saying "the selection is not correct" and it asks if the child wants to have another attempt. If the child answers affirmatively, then, the same activity with the same data appears, but now with an auxiliary tool which is a virtual ruler, as it has been mentioned, which enables him to measure and make comparisons (indirect 9,6). If even with this support of the virtual ruler the selection is again incorrect, then, the interactive computational program presents another version of the same activity, which is randomly generated with other figures and other data, in order for the child to try again.

If after these attempts the student has not been able to select the correct figure, then, he is presented with one simulation of the activity, with other figures and other date, in such a way that the simulator superimposes figures and drags the virtual ruler, appearing the data of the measurements of the figures, in this way it can be by the actions performed by the simulator which figure is enlarged or reduced, in accordance with what is being asked. This allows the child to associate these actions with the ones he should have made in the attempts he carried out.

3. Implementation, Analysis and Discussion of Results

For the implementation we had a group of 29 students from sixth grade of elementary education in Mexico, specifically from a public school in Mexico City. The ages of the students ranged fluctuated between eleven and twelve years old. Six sessions were devoted for the work with students, each one lasting two hours.

The analysis and discussion of results are carried out in relation to the indicators of the construction of the concept of proportion (in a qualitative way) and to the ones of the development of qualitative proportional thinking.

Seven out of twenty-nine students are identified who chose the enlarged figure (Ind. 1), by means of visualization, though; it is worth mentioning that two of them managed to do that in the second attempt. However, in general it can be said that 24% of students, by means of visualization are able to identify the proportions and select the correct figures, that is, they have developed their qualitative proportional thinking. This can be observed through the recordings of the verbal categories (Ind.2) by students, such as "this is greater than the other", "it appears that this is twice the size", which were expressed, by six out of the seven students, during this activity.

The students who have not developed this qualitative proportional thinking are unable to visually identify the correct figure and they need to resort to direct comparisons and indirect ones. In the direct comparison they make use of the didactic action of superimposing figures, whereas in the indirect comparison, the didactic action is the use of a measuring instrument.

During the activity, eight out of twenty nine students drag figures thus fulfilling the indicator of comparing directly (Ind.8), which provides evidence that they can establish proportions in a qualitative way. Note that the action of superimposing figures also determines the indicator of comparing (Ind.3), which favors the shift from qualitative proportional thinking to the quantitative one. Thus, 28% of the students start to develop their qualitative proportional thinking and shifting to quantitative proportional thinking. Moreover, comments by three out of the eight students are recorded, such as: "this side is twice the size of the other" or "this side fits twice in the other", denoting the use of internal proportions (Ind.11) see table 1, which provides evidence of an incipient use of proportions in a quantitative way.
The remaining fourteen out of twenty-nine students used the two attempts which are given by the system to make a selection by means of visualization or the dragging of figures, without success. Moreover, in the audio recordings they registered linguistic expressions that show their difficulty to make the selection such as "the tree figures which are greater resemble each other". It is possible to say that these students have not managed to develop their qualitative proportional thinking.

Upon not being successful, the interactive computational program provides them with a virtual ruler. Only six out of the fourteen students measured (Ind. 6) the sides of the figures and compared them using this instrument identifying the correct figure, with which the indicator of comparing indirectly (Ind.9) is verified. Upon being measuring, the students obtain numerical values which compare sides to corresponding sides, establishing internal proportions (Ind. 11); which gives rise to establish ratios in an explicit way; six out of the fourteen students when carrying out this actions show signs of the development of their quantitative proportional thinking.

The remaining eight students out of the fourteen who used the virtual ruler, only measured one of the sides and selected the figures which had twice the size of the homologous side, but they did not notice that they had to measure all the sides of the figure to compare them to the figure they selected. These errors led them to the unsuccessful selection of the correct figure, which shows us that these eight students have not developed their qualitative proportional thinking either.

Conclusions

The different computational actions, dragging with the mouse, and using a virtual ruler, have a bearing on the didactic actions of superimposing one figure over the other to make the comparison or using a measuring instrument. All that depended on the kind of proportional thinking which students had previously developed either the qualitative or the quantitative one.

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SEEING COMPLEXITY IN THE PROCESS OF CHILDREN'S BLOCK PLAY

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Nearly a century of research has established a relationship between the complexity of children's block play and mathematical thinking and reasoning in other contexts and in later years. However, most of this research has evaluated complexity by examining children's completed structures. This video-ethnographic study looks at mathematical thinking and reasoning during *block play as a way of contributing to learning trajectory research in early childhood mathematics that values diverse ways of thinking. The paper highlights opportunities for mathematical reasoning that emerged during the construction of "less" complex structures.*

The importance of children's block play for developing spatial thinking and geometric reasoning has been established through multiple studies (e.g., Casey et al, 2008; Guanella, 1934) These studies, dating back almost a century, have looked at ways to increase the complexity of children's block structures as well as ways that this complexity can support more complicated mathematical thinking, particularly in reference to spatial visualization skills, such as imagining flips and rotations (e.g., NRC, 2009). In addition, this body of work has documented relationships between block play and mathematical thinking more broadly (e.g., Caldera et at, 1999; van Nes & van Eerde, 2010). For example, recently Wolfgang, Stannard and Jones (2003) demonstrated a relationship between complex block play at age 4 and mathematics achievement in the 7th grade.

One emerging line of research, developed in response to increased interest in young children's mathematical learning trajectories (NRC, 2009), is the role that teachers can play in scaffolding children's spatial and geometric learning through block play (Clements & Samara, 2007; Kersh, Casey & Young, 2008; NRC, 2009). Gregory, Kim and Whiren (2003) demonstrated that trained educators could support students in creating more complicated block structures by talking with children as they played with blocks. Their study, like most others, evaluated the complexity of the block building through the examination of the qualities of children's completed structures.

The purpose of the current study is to add to this emerging line of research on children's learning trajectories in block play by evaluating children's thinking and reasoning as they go about the process of building with blocks. In doing this work, we hope to contribute to diverse ways of evaluting and scaffolding children's mathematical thinking during block play. Nearly all previous researchers have based their analysis of complexity on the completed block structures. For example, in 1934 Guanella described block structures as ranging in complexity from "pre-organized," where no structure is built, to piles and rows, to solid forms that include closed spaces, to three-dimensional structures (Guanella as cited in Casey et al, 2008). In their recent study, Gregory, Kim and Whiren (2003) expanded on the categories for analysis developed by several previous blocks researchers to evaluate student structures based on three broad categories: the complexity of the building, the complexity of arches, and number of dimensions (use of points, lines and planes) in the finished structures.

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We would like to consider the possibility that complex mathematical thinking might not be entirely represented by increased complexity of completed block structures. In other words, a child might set a task for herself in work with blocks on a single plane that is more mathematically complex than one involved in creating a structure on multiple planes. We believe that the documentation of children's learning paths in their work with blocks ought to include an examination of the various cognitive demands and opportunities made throughout the building process as well as documentation of progression in creating more and more complex structures.

Guiding our study were the following research questions:

- What kinds of mathematical thinking and reasoning emerge during children's play with various kinds of blocks in both formal and informal contexts?
- How do interactions with others shape the mathematical thinking and reasoning that occurs?

Theoretical Framework

This study draws on Vygtoskian (1978) traditions that see play as central to children's development and see learning as an outcome of engagement in social activity. In his work, Vygotsky emphasized the tools that children draw on in their learning. These tools include the physical – such as Lego or wooden blocks – and also the social – such as ways of thinking or playing that are learned from other humans in social settings. Although Vygotsky's work around scaffolding is most often taken up to analyze interactions between children and adults, in this study we would like to acknowledge the role that children can play as the knowledgeable others in their peers' play. Particularly, during free time where some children may spend far more time than others with certain materials and thus develop expertise, children themselves may be able to scaffold the block play of their peers toward more complexity. Vygotsky also described the role that a cultural tool like "self-talk" can play in scaffolding a difficult task. He describes a child standing on a chair to reach a cookie, vocalizing each step in the process as a way of managing the difficult task. In a similar way, many children talk to themselves as they play with blocks. Following Vygotsky, we believe it is important to capture and analyze this talk as a way of gaining further access to children's mathematical thinking and reasoning.

Methods

This project is embedded in a larger ethnographic study of children in a rural community as they move from kindergarten to first grade, which seeks to explore their mathematical learning during this time span. As a small, rural school, Taylor County Public School is an ideal site for a longitudinal study because there is only one classroom for each grade. Nearly all of the students who attend the school are eligible for free lunch and about 90 percent are African American. These characteristics make this an important site for broadening the research base on learning trajectories around block-building because much of the reseach on blocks has occurred in settings where a majority of the children are European American and come from middle-income families. Data collection for the larger project includes weekly classroom observations supported by video- and audio-taping, individual video-taped assessment interviews with each child, audio-recorded parent interviews and focus groups, and video- and audio-taped observations of Parent Math Nights. Data collection and analysis for this paper has been informed by the multi-year relationship the first author has with the school.

As an ethnographic study, the research team seeks to understand children's experiences through both observation and participation. For this aspect of the project, this means that rather than comparing students' play and their block structures to pre-existing criteria, we attempted to document the mathematical interactions students engaged in during the process of play. In the preschool classroom, we took fieldnotes, audio-recorded conversations, and documented interactions with video cameras during both formal mathematics lessons and free-choice time, when children were allowed to choose among the many materials in the room, including Lego blocks, wooden blocks, and unfix cubes.

Data analysis was supported by the qualtiative data analysis program NVivo9. Using this program, the research team was able to code excerpts of field notes as well as segments of video clips for analysis. For example, a 30-second video segment could be identified and coded separately from a longer video clip. The observer who recorded the video also used the Nvivo9 program to describe the action going on in the video and transcribe some speech. Initially, data were coded based on the setting, the type of interaction, the materials used, and the mathematical concepts. The current codes are listed in the text box to the right; however, coding, particularly around describing the block play and mathematics, is being continually refined. At the time of writing this proposal, we still have another three months of scheduled observations in the prekindergarten. As a result, the analysis reported in the following section is tentative and still evolving. Over time, we hope to create a map of the opportunities for thinking and reasoning possible in the midst of block play.

Results

In this section, we describe several representative episodes of block play captured by videotape or fieldnotes over the course of the study. These episodes were chosen to demonstrate the mathematical thinking and reasoning that occurred as children interacted with different kinds of blocks, in different settings, and with different people. While all are unique, the kinds of thinking and interactions that occurred in each episode can be found across our data record.

Episode 1: Carter and Wooden Blocks

One afternoon, Carter, who frequently played with all kinds of blocks during free choice time, chose to build with wooden blocks. The block set available in the classroom was relatively traditional, with a variety of rectangular and triangular prisms, cylinders, and arches. On this occasion, Carter decided to build a road out of blocks, announcing his intention to no one in particular. Carter began by laying down a 6-by-3-inch block. He then put together four 3-by-1.5-inch blocks, choosing from dozens of blocks spilling from the tub around him. For the next road segment, he chose two 6-by-1.5-inch blocks, putting them side by side. He then used another 6-by-1.5-inch block and then two 3-by-1.5-inch pieces. Finally, he chose two 3-by-3-inch blocks for the last segment in the road.

The wooden blocks, which were designed to make many equivalent shapes possible, allowed Carter to experiment and practice with composing and decomposing shapes. Carter seemed to embrace this activity as part of his play. After all, several of the large 6-by-3-inch blocks were available and Carter could have chosen to create a road out of identical blocks. Instead, he seemed to go out of his way to make as many equivalent combinations as possible in the building of his road. This ability to compose and decompose shapes has been highlighted as an important mathematical concept for young children because children "who can compose shapes develop better understanding of composing and decomposing numbers" (NCTM, 2010, p. 56) as well as a foundation for geometric concepts. In addition, play like this afforded Carter the opportunity to practice visualization because he needed to orient blocks in the proper direction to make equivalent shapes and to choose the desired block from a pile in the tub.

It is also important to note Carter's intention to build a "road." Traditional rubrics for scoring the complexity of block structures would categorize the road as less complex than other structures because of the lack of vertical building, arches and enclosures. However, this analysis does not take into account Carter's intentional construction of an object that he likely visualized in his head before he created it as well as his efforts to create equivalent segments. In considering complexity, it is important to consider not just the finished product (in this case, a straight line), but also the particular tools used to construct the line (blocks of various sizes and shapes rather than all identical blocks and equivalent block units rather than varyingly-sized units). In this episode, Carter played by himself with little scaffolding from others; however, the available tools contributed to his mathematical thinking. This includes both the opportunities for visualization, identification, composing, and decomposing provided by the blocks, but also the story scaffolding provided by other toys. Casey and colleagues (2008) found that telling stories contributed to children building more complex block structures, and in many ways available toys served this purpose for children in the classroom. For example, in this case, it is unlikely that Carter would have chosen to build a road if cars and trucks had not been available to drive on it.

Episode 2: Markus and Unifix Cubes

One morning toward the end of free-choice time, Markus came upon a bucket of unifix cubes scattered across the carpet. Initially, he bent and began to stack as many as he could into a tower nearly as tall as himself. After holding his tall tower against himself, he sat and began breaking the tower into smaller chunks. He then picked up one of these smaller towers and counted each cube, touching it as he said the number. After counting seven cubes, he reached out grabbed a loose cube and added it on. He then repeated the process, making four other towers of eight cubes each. He then laid these towers against each other on the floor. When he made his fifth tower, he laid it down by the others to check its length and then took off two blocks when he realized it was too tall. At this point, Dahlia and Trevor approached and began to stack the blocks into tall towers. Markus corrected them, saying "We need to make eight! For Bingo." Trevor and Dahila accepted this and began to make towers of eight blocks, each of them counting the blocks in their finished towers one at a time, touching each block and laying their completed towers down next to the pile Markus had made. Other children started to approach, making various lengths with the cubes. Markus reiterated that the towers had to be eight long. At this time the teacher approached and asked Markus, "How many are in your tower?" He picked one up and counted for her, touching each block, "1,2,3,4,5,6,7,8." She replied, "Very good."

Unifix cubes are not often considered in block research; however, as we watched children play with them during the informal moments in the classroom, it seemed odd to exclude them. Because these cubes could only be used as isolated ones or as towers, the mathematical thinking they were most likely to elicit involved counting and comparison. Students compared towers to each other (as Markus did when he was making units of eight), compared them to themselves (as Markus did when he made his first tall tower), and compared them to objects in the room. Students also often counted these blocks, practicing one-to-one correspondence as they did so. Sometimes, this counting seemed to serve a particular purpose, as it did for Markus, who knew that the blocks were passed out in groups of eight for Bingo because the boards had eight spaces. However, more often students seemed to count simply because they had the blocks. This may have been because the teacher frequently used these blocks for counting in formal lessons and also tended to ask children to count the blocks whenever she saw them being used during free-choice time. The unifix cubes seemed to offer fewer opportunities for spatial reasoning than other blocks, although opporutnities to compare lengths did encourage students to talk about more, less, longer and shorter, and to make adjustments to their towers as a result of these analyses. Markus's use of his already constructed towers of eight to measure his new towers demonstrated an ability to reason about length and quantity as he was able to recognize that towers of the same length would have the same amount of cubes without having to count again and was also able to adjust towers by taking off or adding on cubes to get the correct lengths. The task Markus set for himself and the way he set about it reveal aspects of his thinking and reasoning that would not be apparent in the completed pile of towers of eight blocks.

In this episode, the blocks themselves and the potential of a Bingo game later provided scaffolding for Markus to explore length and comparison and to practice counting. Markus, himself, provided this scaffolding for other students, by repeatedly naming the task at hand. Although the other students did not talk about the Bingo game, it is likely that this real-world purpose, served to persuade them to engage in the counting activity Markus set up. On the other hand, the teacher's intervention in this case did little to scaffold mathematical thinking. Markus had already counted the blocks in multiple towers by the time the teacher asked him to count. Indeed, he already knew there were eight blocks in the tower he was holding when she asked him to count the blocks. He performed the task anyway and her response did little to either acknowledge the thinking he had already done or to push him toward new mathematical engagements.

Episode 3: Xavier, Carter and Legos

At the beginning of free-choice time, Carter pulled out the Dulpo Lego blocks. He placed a large blue mat (25 by 25 nodes) on the floor and began to attach blocks along the perimeter of two sides. Nearby, Xavier began to build a tower of 2-by-4 blocks. Once the tower was 10 blocks tall, Xavier attached it to the mat on which Carter was working. Carter rejected this effort, saying "No, I'm making something right here." He reached out and removed Xavier's tower. Xavier accepted this correction and moved away from Carter, who followed him asking, "You want to help me?" Xavier tried one more time to make a tower and attach it, but Carter again removed it from the board. He gestured to the blocks he had placed along the perimeter and said: "I'm doing this." Xavier began to join Carter in filling up the board with a single layer of blocks. This went smoothly until the entire board was nearly full. The boys then began to search for pieces that would exactly fit the few remaining holes. Carter dug through the box to find a 2-by-2 square, which he placed in a matching open space in the center of the board. Xavier turned a rectangular block to fit a remaining hole. The last space in board required a 2-by-8 block; however, one of the surrounding blocks had plastic googly eyes projecting from the side, which made it impossible to fit an appropriately sized block in the adjacent space. After both Carter and Xavier tried a number of correctly sized blocks with little success, they appealed to the researcher observing. She asked them to feel the side of the block with the raised eyes. Xavier did so and then took that block out, replaced it with an identically sized one, and then filled in the last hole before beaming up at the camera. In all, this process took Xavier and Carter 25 minutes.

The Dulpo Legos, with their defined working spaces on mats, primarily rectangular shapes, and ease of fitting together, scaffolded different kinds of mathematical thinking than wooden blocks or unifix cubes. Children were able to build much taller structures with the Dulpo Legos than with the wooden blocks. In addition, the concepts of area, perimeter, and rotations were frequently explored during Lego block play. As Carter did in this example, students frequently created a perimeter first and then filled it in. This required choosing blocks that would be exactly the same length as the mat and negotiating the tricky business of turning corners. Carter, who spent a lot of time in the block area, performed these tasks with certainty, rarely needing to remove a block and replace it with one that had a different length or width, while other children frequently needed to make such adjustments.

These differences in the ways children went about building similar structures reveal differences in development of visualization skills and understandings of length and width and would be unlikely to be captured in the evaluation of a final product. Similarly, in this case, many of the scoring systems for complexity would rank a structure as more complex if it had the tower that Xavier proposed adding; however, Carter's sustained commitment to filling the board demonstrated perseverence with a task and also created an opportunity for both him and Xavier to think about how to fit geometric pieces together exactly, which required anlysis of shapes, lengths and widths and work with rotations. Typically, when children added towers to constructions the play became focused on creating cities or sometimes parking garages, rather than on a task like filling all of the available space on the mat. Thus, Carter scaffolded a more mathematically complex task for Xavier through his commitment to working only in one layer. \

Episode 4: Dahlia, Her Mother & Legos

At the Parent Math Night, one of the stations consisted of several boxes of Legos, including special pieces, such as wheels and trees, and written directions for making specific projects. Dahlia came to the event with her mother and her six-year-old sister, Tamara. Her mother took the written directions for a house, held them in front of Tamara, and quietly talked her through the process of building. Dahlia looked on for a moment and then began to build a tower with 2-by-8 blocks. After watching Tamara for a moment, she searched through the box to find window and door pieces the same size as the blocks. She pulled on her mother's sleeve and said "See my house." Her mother looked over, saying, "That's pretty, Dahlia. What you going to do next?" Dahlia pressed her tower onto a Lego board and began to add blocks to her board. After a moment her mother smiled told her to look at Tamara's house, which had a roof, doors, and other accessories, as well as four walls creating the structure. Dahlia looked on for a moment and then began to add additional blocks to her construction.

In this episode, Dahlia, with her mother's help, used her older sister's construction as a model for her own work so that while Tamara constructed a house that mirrored the one pictured in the Lego directions, Dahlia turned her tower into a "house." In someways, viewing the tower as a house was merely cosmetic, such as in Dahlia's decision to add the clear bricks to stand in for windows. However, Dahlia's building of a tower was her independent translation of Tamara's four-walled enclosure. In her attempt to emulate her older sister, she needed to watch what Tamara was doing and decide how to replicate it with the blocks available to her, a process that encouraged the development of visualization skills and manipulation of geometric figures. Her mother encouraged her to add complexity to her structure by prompting her to think about what she should do next and presenting Tamara's house as a model. Unlike the teacher and parapro, both in the classroom and at the Math Night, the parents we observed rarely asked their children to count or to name shapes. Rather, parents scaffolded their children's work and thinking through engagement in the same or similar tasks. For example, the father and son at the table with Dahlia both made their own airplanes. This sort of scaffolding seemed to allow more opportunities for spatial thinking and reasoning to unfold than the teachers' efforts, which continually redirected children back toward traditional preschool mathematics, such as counting.

Discussion

Given the concerns expressed in many recent reports (e.g., NAEYC & NCTM, 2002; NRC, 2009) about the quality of mathematics experiences many young children receive, particularly poor and minority children, we believe it is important to consider ways that opportunities for complex thinking and problem solving can be incorporated into preschool and kindergarten school days in both formal and informal spaces. In the current climate of standardized testing, many early childhood teachers have become focused on basic skills, which has led to a reduction in free play. However new research has shown that these play opportunities are critical for complex mathematical thinking later (NRC, 2009).

This ethnographic analysis of block play raises a number of issues for mathematics educators to attend to as they develop early childhood mathematics curricula, engage in professional development with early childhood teachers, and do further research on young children's learning trajectories. First, it is important to recognize that the complexity of mathematical thinking young children engage in during block play may not be entirely captured in the finished structures. The research community would benefit from increased attention to creation processes, and children and teachers would benefit from increased opportunities to talk about their thinking during and after their work. Second, children's play with blocks needs to be scaffolded to maximize opportunities for mathematical thinking; however, this scaffolding does not always need to come from a teacher in either a formal or informal setting. Well-chosen materials can scaffold particular kinds of thinking, and other children and parents can also support children in problem solving. More research needs to be done on how to build on the strengths parents and children bring to this process. In addition, teachers need to learn both how to watch children during block play to figure out the mathematics that may be present and to recognize the more complex process skills that young children need to develop along with counting and shape recognition. Finally, we would be remiss not to note that in our study, as in much previous work (e.g., Kersh, Casey & Young, 2008), gender mattered in block play. Nearly all of our classroom clips with Legos, wooden blocks, and unifix cubes show the work of boys. However, during Parent Night and in the assessment interviews, girls engaged enthusiastically with the blocks, as Dahlia did in the described episode. Creating spaces where girls were expected to build with blocks and where they did not have to struggle to maintain access to the materials seemed important in giving girls access to these learning opportunities. Guided play as part of formal lessons in classrooms could potentially provide another welcoming space for girls to engage in block play.

Developing learning trajectories around blocks that include both images of what finished structures may look like and descriptions of processes that children go through as they engage with these materials is important to the study of children's early learning broadly and, more particularly, to the study of the relationship between formal and informal learning. Teachers will be able to use these trajectories to learn about the deeper mathematics embedded in block play and to make connections between formal and informal lessons. It is important to both theory and practice to develop these trajectories by looking at the thinking that goes on during block building in a variety of social settings not simply at a progression of completed structures. This theoretical view is likely to be more inclusive of diverse mathematics and of diverse children.

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THE SOCIOCULTURAL CONTEXTS OF PRE-K CHILDREN'S MATHEMATICAL EXPERIENCES IN SCHOOL

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This study is conducted to understand the sociocultural contexts of pre-k children's mathematical experiences in their school lives. Through interpretive ethnographic research, I observed pre-k children's natural mathematical experiences in the classroom focusing on the sociocultural contexts embedded in their mathematical learning. This study was drawn from Bruner's idea about the culture of education, and Bakhtin's ideas of language were employed in the analysis. This study discusses the influence of teachers' directed mathematics activities on children's informal experiences and cultural dynamics in mathematical experiences in pre-k classrooms. The implications of pre-k children's meaningful mathematical learning are provided.

Children's lives are always related to mathematics, and many mathematical attributes are derived from everyday life. Through their ordinary life experiences, children naturally engage in significant levels of mathematical activity (Trafton & Schulte, 1989; Clements, Sarama, & DiBiase, 2004). Considering children's early ordinary experiences not only as significant practices for present mathematical learning but also as a cornerstone for later mathematical learning, many mathematics education scholars have been paying more attention to informal mathematics. With a heightened awareness of values of informal mathematical experiences, a number of researchers of mathematics have explored children's mathematical experiences while considering the social and cultural contexts in which the children reside (Ginsburg and Russell, 1981; Stigler and Baranes, 1988; Nunes, Schliemann, & Carraher, 1993; Tudge and Doucet, 2004; Seo and Ginsburg, 2004). However, many studies have been primarily concentrated on the children's social and cultural backgrounds in generic ways. For example, in many of those studies, artificial factors, such as X, representing children's social backgrounds were used to compare or examine their influences on or their associations with children's mathematical learning. Graue and Walsh (1998) note, "researchers who look too widely risk losing the chance to look very carefully" (p. 10); a more careful analysis that pays attention to social and cultural contexts within children's mathematical experiences in school and the relationship between the local context and the larger context is necessary. The purpose of this study is to understand the sociocultural contexts of pre-k children's mathematical experiences in their school lives. Before zooming in on the main foci of this study, I first position the perspective of the sociocultural contexts adopted here by mainly referring to Bruner's idea about the culture of education. Second, I introduce Bakhtin's idea of language which was employed in the analysis of this study.

Theoretical Framework

The Sociocultural Context of Mathematical Experience

In his book, *Culture of Education* (1996), Bruner states that "school is a culture itself" (p. 98). He also presents the classroom as a "living context" (p. 44) where "teachers and pupils come together to effect that crucial but mysterious interchange that we so glibly call *education*" (p. 44). A school classroom itself is a sociocultural context in which mathematical knowledge and experiences are shared, negotiated, and constructed. Bruner (1996) considers the influences of the broader society on the school classroom as well as the impact of school culture on the individual child:

> [S]chool can never be considered as culturally "free standing." *What* it teaches, what modes of thought and what "speech registers" it actually cultivates in its pupils, cannot be isolated from where the school is situated in the lives and culture of its students. For a school's curriculum is not only about "subjects." (p. 28)

According to Bruner, the classroom is, rather than the physical setting as an instructional environment for young children, a living context which reflects implicit cultural values of the larger society where shared and negotiated ways of thinking and collective cultural activities are produced. Bruner (1996) adds that "education is never neutral, […] education is always political" (p. 25); therefore, social values reflected in school curricula and classroom culture cannot be free from moral-political considerations of social class, gender, race, and other prerogatives of social power. Bruner's view of the classroom as a culture itself is consistent with Stigler and Baranes' (1988) account of the cultural attributes of mathematics learning. They argue that mathematics by its very nature is cultural:

> The practice of mathematics is not seen as the discovery of truths existing outside of the realm of human activity; rather, it is considered to be as domain-specific, context-bound, and procedurally rooted as are other forms of knowledge. Mathematics learning is not uniquely immune to the influences of culture, but rather, it is as cultural as is learning in other domains. (p. 258)

I consider the sociocultural context of children's mathematical experiences not only as an important influential factor on their learning but also as a mirror reflecting cultural values about mathematics in our society.

Another concept discussed by Bruner is important to the purpose of this study: "mental life—and what we call 'intelligence'—is interactive, not just within the individual's head. Intelligence depends hugely on interaction, on understanding what others have in mind" (Bruner, 2008, p. 101). Graue and Walsh (1995) also note that "the context is the world as realized through interaction and the most immediate frame of reference for mutually engaged actors" (p. 9). Even though mathematics can be conceived as an entirely mental activity, it is socially constructed and shared through reciprocal interactions. In order to understand children's mathematical experiences in schools, it is necessary to consider the sociocultural contexts which the children create as well as reside in. That is, children's mathematical experiences need to be understood within their sociocultural contexts, which include individual and social meanings derived from their interactions and relationships with people and objects.

Bakhtin's Idea of Language

Heteroglossia vs. Unitary Language

Bakhtin's view on language provides valuable insights on social, cultural, and political phenomena of education. By contemplating the social nature of language, he shows the dynamics of world views and socio-ideological phenomena in which we are influenced by, and sometimes controlled by, the means of language. Introducing the concepts of *heteroglossia* and *unitary language*, Bakhtin (1981) explains that a living language is in tension-filled dynamics between *centripetal forces*, "verbal-ideological centralization and unification," and *centrifugal forces*, "decentralization and disunification," which "intersect in the utterance" (p. 272). Bakhtin's notion of *heteroglossia* can be referred to as a social diversity of speech types, individual voices, or different dialects, which is *centrifugal force* that opposes *centripetal force* of *unitary language*—socially, historically, and culturally constructed norms of language. Therefore, each utterance undergoes and participates in this process of centralization and decentralization and is stratified. In addition, to Bakhtin, language is not just a means of conversation, but a pathway of social and ideological dynamics. Even though he employs linguistic terms, he expands his ideas of language to sociology, affirming that language is "conceived as ideologically saturated, language as a world view, even as a concrete opinion" (Bakhtin, 1981, p. 271). Based on this concept of language, the nature of language is broadened to include "processes of indexical meaning that anchor utterances to their linguistic and nonlinguistic contexts and to unspoken background assumptions" (Shweder, et al., 2006, p. 742).

Dialogic Living Language vs. Monologic Dead Language

One of the Bakhtinian scholars, Volosinov (2000)—a Russian linguist and also Bakhtin's close friend, whose work largely coincides with Bakhtin's work—introduces the concept of *dialogic living language* as opposed to *monologic dead language.* According to Volosinov, *isolated, finished, monologic utterances* are "divorced from its verbal and actual context and standing open not to any possible sort of active response but to passive understanding" (p. 73). Volosinov (2002) considers *dialogic living language* as "the actual reality of language-speech" (p. 94) and argues that "language is a continuous generative process implemented in the social-verbal interaction of speakers" (p. 98). According to Volosinov (2000), a word is "the product of the reciprocal relationship between speaker and listener, addresser and addressee", and "the immediate social situation and the broader social milieu wholly determine" (p. 86) the structure of an utterance and the meaning of a word.

Bakhtin (1981) also places an emphasis on word's orientation "toward the listener and his answer" (p. 280). In living dialogue, any speakers, speech subjects, or addressers mutually reflect one another. Individual utterances are not "indifferent to one another, and are not self-sufficient" (Bakhtin, 1986, p. 91). Bakhtin (1986) elaborates this idea that "each utterance is filled with echoes and reverberations of other utterances to which it is related by the communality of the sphere of speech communication" (p. 91). Volosinov's emphasis on reciprocal relationship between addresser and addressee and immediate social situation of communication has a resemblance to Bakhtin's illustrations of a living word and dialogue. Consequently, *dialogic living language* can be resumed in the words of *continuously generative process implemented in reciprocally reflective social-verbal interactions*.

I appropriate Bakhtin's idea of the dynamic nature of language because his idea of language as social phenomenon provides the view on language as a culture and informs cultural dynamics in the school classroom. Based on these two conceptual contrasts regarding language as discussed by Bakhtin, this study explores the sociocultural contexts of children's mathematical experiences in the pre-k classroom. Research questions are as follows:

1. What mathematical practices and concepts do pre-k children experience in school life?
2. How are the mathematical practices and concepts that pre-k children experience in the classroom related to the sociocultural contexts of their experiences?
3. What are the educational meanings which the children's mathematical experiences imply?

Methods

I appropriate interpretive ethnographic research, and research site and participants are as follows:

Site	Participants
- A rural area public school located in central Georgia - Total population of county: 1,863 (U.S. Census Bureau, July,2008) - High poverty rate (Nearly all students, free lunch) - A single public school in a county which serves PK-12 grades with fewer than 300 students	- One pre-k classroom, 2 teachers - Pilot study (2009-2010): 20 children - New school year (2010-2011): 18 children (mostly African American children, 2 White girls, 1 Mexican American ESL learner)

Multiple data were collected though the method of participant observation. As part of a larger study, I observed the pre-k classroom for 6 sessions from February 1 to April 19, 2010, for a pilot study and for 8 sessions from September 9, 2010, to February 10, 2011. For the pilot study, I observed calendar time, breakfast time, whole group lessons, and free play time in order to get familiar with the research site and recognize various situations in which mathematical practices occurred. After the pilot study, I observed this pre-k classroom with newly enrolled children for 8 more sessions while focusing more on large group activities, small group activities, and children's free play. The collected data include jotting notes, expanded field-notes, voice records, and video records. While recording videos and audios, I took jottings and tried to record all my momentary thoughts, initial impressions, or additional information about the situations. After each observation, I wrote the expanded field-notes. Data analysis was an inductive process. Reading through the large amounts of collected data, I made preliminary notes of what seemed relevant. Then, I repeatedly investigated the collected data and made more focused coding. These were mostly based on the research questions and classified under the specific themes of children's social contexts in mathematical experiences focusing on the relationship between their learning and cultural experiences.

Results: Pre-K children's Mathematical Experiences in Context

Adult Directed Mathematical Experiences

Even though the pre- k teacher's roles were different depending on the class time schedule in each day, the teacher played a director's role by introducing materials and activities, asking questions, and reading books to the children during large group circle time, small group activity time, and free play time. Regardless of the different forms of activities, the teacher-children interactions were limited to the typical form of closed question-answer type.

> The teacher starts reading the title of the book, *Run, Bunny, Run*. After reading the first page, she points at the number on the shirt that the character Bunny wears and asks a question:
> Teacher: What number is Bunny? / Children: Three. / Teacher: Three. What number is bear?
> Children: Two. / Teacher: Two. What number is beaver? / Children: One.
> Teacher: One. What number is chicken? / Children: Four.
> Teacher: Who is in the lead? (She points at the first animal Bunny.) / Children: Bunny.
> Teacher: Bunny. Bunny, first. Can you … say, he is first. / Children: He is first.
> Teacher: Bunny is first. O.K.? Now, who's last? (She points at the last animal in the race.)
> Children: Bear. / Teacher: Bear. Bear is in the end; he is last. Bunny's first.
> - From observation field-notes, Nov. 14, 2010

The teacher's questions related to mathematics predominantly consisted of counting, reading the numbers, and matching the quantity of objects to the numbers: What is this number? How many dots? Can you count? What comes after ten? Can you find the number 8? During whole group time, the teacher's questions were directed to the whole group, and sometimes one of the children was called on and asked to answer the teacher's question. When the teacher asked questions to the whole group of children, only a few of them answered. The teacher always gave the children cues by gesturing to indicate that she wanted the children to stop counting or by making first sounds of the answers. Then, the children just stopped when the teacher said or showed stop cues. Therefore, many of the children counted the numbers without paying much attention to what they were saying. In small group, the children took turns one by one in a group. The children were required to accomplish mathematical tasks individually in a group activity. Compared to large group activities, in a small group activity, the children were more restricted because the teachers could manage their behaviors more efficiently with a small number of children. Teacher's questions were almost same with those in whole group activities. During free play time, the teacher intentionally asked the children to count objects they had in that moment when she found any situations related to counting, or sometimes without any consideration of children's play contexts. Her intentional questions sometimes promoted the children to practice counting and recognize how many objects they had. Nevertheless, when the teacher's question was out of context, these questions cut off the children's play and did not stimulate the children's meaningful counting. Likewise, the teacher's interaction with the children primarily centered on her main goals of subject area such as counting and naming numbers when it comes to mathematics.

Mathematically Interactive Experiences among Peers

During the large group activities or the small group activities, the children's interactions were limited and sometimes rather restricted. Mostly, they took turns to accomplish same mathematical tasks. They were required to quietly see what other children did and to answer the questions that the teacher asked to all the members in a group. In free play time, relatively more peer interactions related to mathematics were observed, particularly in the block area. When the children played with Lego blocks, they shared the shapes of Lego pieces and explored the form and the shapes of blocks. Even when they did not verbally interact with their peers by using mathematical language, there were mathematical interactions among children, in which they shared geometric information and their thoughts about others' mathematical works. The most common pattern of "playing numbers" is that they reported to peers how many objects they had, for instance, shouting "Look! I(we) got two!":

> Joseph and Ted are playing with the magnetic numbers. They are putting them on the wall of the teacher's desk. With the magnetic numbers, they create some images on the wall. Ted picks up the number four and puts it on the wall. He says, "I got a four! You want me four!" (Do you want my '4'?") He puts it by the number four previously placed on the wall. He makes these two "four" closer and says, "We got two fours." - From observation field-notes, Nov. 18, 2010

"I got (number)!" game encouraged children to count the number of objects or name the numbers. Playing with other peers, the children not only shared mathematical concepts but also promoted each other to engage in mathematical play. However, their interactions were still limited in a variety of mathematical concepts. Usually, verbal interactions among children that contain mathematical concepts were concentrated on the matter of how-many. There was no interaction related to measuring, comparing size or quantity, or sorting. Moreover, such mathematical interactions were observed only occasionally.

Silent, BUT Engaged Learning Experiences

Through building blocks and Lego, playing with puzzles and toys, drawing, and making artwork, the children naturally engaged in mathematical practices. They counted a quantity of materials, measured amount and space, identified numbers and shapes, compared size or quantity, and sorted objects based on the characteristics such as colors, shapes, and uses. When they played alone, the children sometimes paid longer and higher attention to their play and spontaneously repeated mathematical practices. For example, Dayton, who was very good at counting and naming numbers during the formal mathematics lesson, played with a clock puzzle. He repeated counting and reading numbers, putting puzzle together, and building tower with puzzle pieces and continued his play longer than 30 minutes:

> Dayton is playing with a clock puzzle. (…) After finishing the puzzle, he takes out all the puzzle pieces one by one in an order. He makes a tower with the pieces from the number one to seven in order. From the number eight, he makes another tower beside of the first one. Now, he has two puzzle piece towers: one 5-story tower and one 7-story tower. When he was about to move the piece on the top of the 7-story tower, this tower fell down. He gathers the fallen puzzle pieces. With seven gathered pieces, he makes one 3-story and one 4-story tower. He takes all three puzzle pieces from the 3-story tower. Grabbing these puzzle piece with his right hand, he moves the pieces of the 4-story tower one by one to the side of 5-story tower. Finally, he has three puzzle pieces in his right hand, a 4-story and a 5-story tower. He puts one more piece on the 4-story tower and makes two same-story towers. He carefully puts the last two pieces on each of the towers. He shouts, "Look! Look! I made towers!" - Observation field-notes, Nov. 14, 2010

While managing the height of the towers in order to make same stories of two twin towers and building towers in this way and that, Dayton did mathematical work such as comparing heights, dividing numbers, and making in order and generated his own mathematical meanings from that task. Playing with this puzzle, even though this child worked very silently and independently without verbal and social interactions with others, he was more deeply engaged in his mathematical work than any other experiential contexts. He used and experienced a variety of mathematical concepts, spontaneously repeated these mathematical practices, and generated mathematical meanings in play.

Discussion: Educational Meanings in Children's Mathematical Experiences

First of all, this study found that children's mathematical experiences were more limited when the teachers directed the mathematical lessons or the children's free play than when the children played alone or with peers. During free play, their experiences with and explorations of mathematical factors were relatively more diverse. They used informal mathematical knowledge, such as sorting, ordering, measuring quantity or sizes, and comparing, and practiced these mathematical concepts spontaneously. Meanwhile, the teachers' concentration on the children's numerical abilities had an influence on the children's use and employment of their mathematical knowledge and skills in their free play. Teachers' mathematical interactions helped the children frequently recognized the numbers and were autonomously engaged in counting the numbers of objects during free play. However, even though the invisible interactions among the children included some mathematical concepts other than numeracy, their verbal interactions mostly referred to naming numbers and counting.

Mathematics in this pre-k classroom reflects the tension-filled cultural dynamics between predominant, centralized, and *unitary* values and decentralized, disunified, and *heteroglot* values, which are discussed by Bakhtin. Children's *heteroglot* informal mathematical experiences appeared during free play time. However, unitary mainstream school mathematics centralizes children's *heteroglot* mathematical experiences and informal mathematical knowledge and values, closing the diversity of mathematics by practicing them mainstream *unitary* rote methods. Current schools' and teachers' inclinations to conventional school curriculum reflect Bakhtin's (1981) discussion of *unitary language* as a utopian philosopheme. Schools have concentrated on utopian *unitary* knowledge, which "makes its presence felt as a force for overcoming this heteroglossia" (Bakhtin, p. 270) and have ignored their own *heteroglot* nature. In school, pre-tailored and pre-established conventional school knowledge is generally transmitted to students as a universal and singular language by teachers without inner meditations. While emphasizing *unitary* mathematics curriculum, schools might neglect or devalue other potentials of children's informal math experiences. Bakhtin (1981) explains underlying political aspects of language by stating that "[a]s a result of the work done by all these stratifying forces in language, these are no 'neutral' words and forms" (p. 293). Likewise, the tension-filled cultural dynamics between *unitary* and *heteroglot* cultural values in mathematics can all be associated to social and political stratification of powers, distinctions, and rewards.

This finding puts more emphasis on the importance of adults' roles in amplifying children's informal mathematical knowledge and experiences. Wheatley (2003) indicates that teachers do not make use of children's informal mathematical knowledge and skills for their meaningful learning experiences. According to Wheatley, teachers have either too little content knowledge to recognize mathematical concepts, which are embedded in children's ordinary lives, or are unable to naturally expand these children's informal mathematical knowledge to their learning. Clement, Sarama, & DiBiase (2004) also point out that pre-k children bring their informal mathematical knowledge to school and that teachers should help them to "mathematize" their pre-experienced knowledge. Gutstein, Lipman, Hernandez, & Reyes (1997) argues that "helping teachers build on children's informal knowledge in mathematics classrooms helps children use their intellect well, make meaning out of mathematical situations, learn mathematics with understanding, and connect their informal knowledge to school mathematics (p. 711)".

Second, children's mathematical interactions with peers rarely occurred. In particular, during teacher's directed lessons, not only were children's peer interactions restricted, but also, the teachers' closed questions did not promote the children's discussions related to mathematical knowledge and expected them to work independently. During free play time, mathematically interactive conversations were not frequently observed among the children. This result reflects Hatano and Inagaki (1998) discussion of "American teachers' inclination toward individualism" (p. 90), which is opposed to Japanese teachers' sharing students' experiences and ideas. These authors explain that American teachers tended to believe "students benefit most from individualized lessons, [and the American teachers] tried to optimize their instructions by individualizing it" (p. 90). Rogoff (2003) also demonstrates this prototypical structure of dyadic interaction in American schools by affirming, "U.S. classrooms are commonly structured with the teacher taking a speaking turn between each child turn" (p. 148).

Moreover, teacher's guided repeated questions as "what is this number?" and "how many?" are deplete of contexts of classroom discourse. While tossing about the simple, rote, and monologic questions and answers, the teacher and the children give and take abstract words without any deeper contextual understanding and without openness to possible diverse responses. This classroom discourse's abstraction of the contents of number, without consideration of the placed temporal and spatial contexts, is comparable to Bakhtin's notion of *monologic dead language*. According to Volosinov (2000), *monologic dead language* is "already an abstraction"

(p. 72), and the abstract objectivism neglects "the dynamics of speech," "living multiplicity of meaning and accent," and "the inner generative process of language" (Volosinov, p. 77). Volosinov (2000) notes, "In reifying the system of language and in viewing living language as if it were dead and alien, abstract objectivism makes language something external to the stream of verbal communication" (p. 81). In this classroom, the contents of speeches are externalized from the stream of verbal communication while decontextualizing the contents and knowledge which each utterance infers. Therefore, someone else's hackneyed postulations are handed down from individual to individual, from people to people, from generation to generation, in the shape of formalized and systemized speech, which has depleted its histories and contexts. Therefore, through this *monologic dead language* of classroom discourses, to which children are routinely exposed, they are socialized to the school language uses and conceptual and ideological meanings of mathematical knowledge. Schools make children concentrate only on predominant school knowledge dehumanized, decontextualized, and abstracted and disregard the *heteroglot* nature of the world, their living world, without meditation.

Finally, I question, "What are meaningful mathematical experiences to pre-k children?" I first argue that a school should be where *heteroglot* cultures, world views, and diverse voices are welcomed, respected, and included. When the formal mathematical activities only emphasize a certain kind of mathematical concept, children might lose opportunities to articulate a variety of informal *heteroglot* mathematical experiences and concepts, even though they are exposed to this *heteroglossia* of mathematics in ordinary experiences. Children are subordinated in pre-tailored school mathematics, while losing their abilities to vitalize their meaningful *heteroglot* mathematics in their lives. This does not mean there should be no *unitary language*—centralized and unified school mathematics curriculum. The tension-filled dynamic of culture is rather a natural phenomenon. It means that schools should revive their nature of *heteroglossia* and be open to children's diverse world views, informal experiences, and voices. When schools are opened to diverse *heteroglot* cultural voices, the centralized *unitary language* would be able to overcome possible risks of cultural narrowness and stagnancy and of alienating potential values. Second, children should be invited to join in the *dialogic living discourses* in the classroom in order to revive meaningful learning and their rights to share their thoughts and to respond to others' words. More specifically, classroom discourses should be reflective of the concrete temporal and special contexts and have a certain story-line. This discourse includes not the decontextualized but the contexualized knowledge. Then, children attain their agencies in school discourse, and school discourse would gain "the dynamic flow of generative process" (Volosinov, 2000, p. 78). Through a true dialogic living discourse, "[school] language acquires life and historically evolves precisely here, in concrete verbal communication" (Volosinov, 2000, p. 95).

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WHOLE NUMBER AND OPERATION CONTENT KNOLWEDGE OF PRESERVICE K-8 MATHEMATICS TEACHERS: A SYNTHESIS OF RESEARCH

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In this report we summarize part of a recent effort to synthesize the state of knowledge of preservice elementary teachers (PSTs') subject matter knowledge and the development thereof. This research was conducted by a recent PME-NA Working Group across all content areas, in this report we focus on whole number and operation. Our review of the literature resulted in only 12 papers drawn from research and practitioner journals and 4 conference proceedings. The scarcity of research on PSTs' subject matter knowledge and the development thereof was characteristic across content areas. The few studies that we found reported that PSTs enter and exit our teacher education programs with weak conceptions in whole number and operations and illuminate the need for further research to (a) better understand PSTs when they enter our programs and (b) better understand how PSTs' conceptions develop.

Introduction & Rationale

Consider a PST solving 527 – 135 (see Figure 1) using the standard algorithm and explaining regrouping as follows:

> You put a 1 over next to the number and that gives you ten. … I don't get how the 1 can become a 10. One and 10 are two different numbers. How can you subtract 1 from here and then add 10 over here? Where did the other 9 come from?"

This PST clearly solved the problem correctly and got the right answer, but was not able to provide an explanation for why this solution method results in the correct answer.

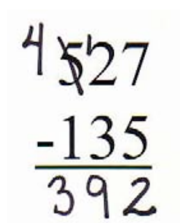

Figure 1: One PSTs' explanation of regrouping in 527 – 135.

Now consider another PST's reflection on her inability to explain regrouping:

> I learned [at the beginning of my elementary mathematics methods class] that there was a lot more to the concept [of number and place value] than I was aware of. I am able to use math effectively in my everyday life, such as balancing my checkbook, but when I was presented with questions as to why I carry out such procedures as *carrying*[1] and *borrowing* in addition and subtraction, I was stuck. I could not explain why I followed any of these procedures or rules. I just knew how to do them. This came as a huge shock

[1] Note that students in the United States often term regrouping in the context of addition *carrying*

> to me considering I did well in most of my math classes. I felt terrible that I could not explain simple addition and subtraction.

Both of these PSTs have determined that they want to teach children, however at this point neither of them would be able to conceptually help an elementary-aged child make sense of why regrouping works when using the standard algorithms taught in the United States.

At the core of elementary school mathematics is the teaching of number concepts and operations. The National Council of Teachers of Mathematics (NCTM) stressed that all pre K-12 students should: (a) understand numbers, ways of representing numbers, relationships among numbers, and number systems; (b) understand meanings of operations and how they relate to one another; and (c) compute fluently and make reasonable estimates (NCTM, 2001, p. 32). A conceptual understanding of number and operation underlies all future mathematics (and other STEM fields) learning. “Number pervades all areas of mathematics. The other four Content Standards [other than Number and Operations] as well as all five Process Standards are grounded in number.” However, research has shown that elementary teachers and PSTs in the United States and Australia continue to lack a conceptual understanding of number concepts and operations (Ball, 1988/1989; Ma, 1999; Southwell & Penglase, 2005; Thanheiser, 2009a, 2010). To be in a position to help PSTs develop more sophisticated conceptions we (mathematics educators) need to understand the conceptions with which the PSTs enter our classrooms so we can build on those conceptions (Bransford, Brown, & Cocking, 1999). As the authors of *The Mathematical Education of Teachers* suggest, “The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know” (CMBS, 2001). Recent efforts have highlighted the need for teachers to have a deep and multifaceted understanding of the mathematics they teach (Hill, Ball, & Schilling, 2008; Ma, 1999). Less clear, however, is how this teacher knowledge is established.

In our synthesis work we thus examined the current state of knowledge in the field of mathematics education on (a) what we know about PSTs conceptions and the development thereof from research (focus on whole number and operation for this paper) and (b) identify further research needs on what we still need to know about PSTs conceptions and the development thereof.

Methods

We began our synthesis effort by breaking into content groups. Each content group followed the same search criteria. We began by searching ERIC with the search terms listed in *Figure 2.* We limited the search to the years 1998 – 2010 to get an overview of what research occurred during those years. Each combination of search terms was entered into the ERIC data base. All results were checked for a focus on preservice elementary teachers’ content knowledge of whole numbers and operations. We read the title and abstract to determine whether the paper fit our criteria. If the title and abstract did not suffice to make a determination of fit, we read the whole paper. We included all papers that:

- Focused on our target group – *preservice elementary teachers*.
    - We included preservice middle school teachers as well as some certification programs that focus on K-8, since not all countries follow the same school system. By doing so we also determined a student’s age range from 5 years to 12 years (possibly 14 years). As a result, we excluded papers focusing on high school teachers.

  - We included papers focusing on both pre- and in-service teachers (i.e. mixed groups) but excluded papers focusing only on in-service teachers.
- Focused *on content knowledge* of *whole numbers*. We included papers that focused on whole numbers as well as other topics. We excluded papers that focused on beliefs or general content knowledge.
- Published research studies in peer reviewed research journals [for the purpose of this report we are including papers published in PME-NA conference proceedings and research papers published in practitioner journals]. We included all research studies published in peer-reviewed research journals. In the end, we excluded all papers that were not research studies.

Search Terms Round 1		**Search Terms Round 2**
preservice + whole number	preservice + algorithm	elementary education + whole number
prospective + whole number	prospective + algorithm	teacher + number sense
preservice + operation	preservice + multidigit	student teacher + mathematics + number
prospective + operation	prospective + multidigit	teacher + whole number
preservice + place value	preservice + whole number	
prospective + place value		

Figure 2. Search terms for whole number group.

Round 1 initially resulted in 22 studies. Round 2 of our search resulted in no additional studies. After exclusion criteria were applied, we were left with 16 studies. These studies were entered into a database listing the reference, content area, research questions, study type, research design, lens or approach used, selection criteria, description of participants, conditions of and procedures for data collection, data analysis, findings, and conclusions/implications. At least 2 researchers met to discuss in-exclusion of papers and determine the entries into the database. All disagreements about in-exclusion into the database were resolved through discussion.

Results

With Number being such a pervasive topic in elementary school mathematics, surprisingly few papers have focused on PSTs' conceptions of whole numbers. A review of the research literature on preservice teachers' understanding of whole numbers resulted in 12 papers drawn from research journals and 4 from conference proceedings.

Of the research papers reviewed, three categories emerged. The first contained two research studies focused on conceptual aspects of understanding numbers (Thanheiser, 2009a, 2010) and four on the development thereof (McClain, 2003; Mills & Thanheiser, 2010; Thanheiser, 2009b; Thanheiser & Rhoads, 2009). The focus of the second group of research papers consisted of three studies on alternative computational algorithms (Harkness & Thomas, 2008; Lo, Grant, & Flowers, 2008; Nugent, 2007). The final grouping included two studies on number sense (Tsao, 2005; Yang, 2007), two studies on understanding operations and order of operations (Chapman, 2007; Glidden, 2008), one study about PSTs' struggle with number concepts (Southwell & Penglase, 2005), and one study on manipulatives usage (Green, Piel, & Flowers, 2008). Below we briefly describe each of these categories beginning with papers focusing on PSTs' conceptions of number and the development thereof.

PSTs' Conceptions of Number and the Development Thereof

Two papers focused on PSTs' conceptions of number. Thanheiser (2009, 2010) examined PSTs' conceptions of number and identified and categorized PSTs' conceptions of multidigit whole numbers into four major groups: thinking in terms of (a) *reference units*, (b) *groups of ones*, (c) *concatenated-digits plus*, and (d) *concatenated-digits only*. See Table 1 for the definition and distribution of the conceptions among the PSTs in that study.

Table 1 *Definition and Distribution of Conceptions in the Context of the Standard Algorithm for the 15 U.S. PSTs in Thanheiser's Study*

Conception	# of PSTs
1. *Reference units.* PSTs with this conception reliably[2] conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 *hundreds* or 30 *tens* or 300 ones, the 8 as 8 *tens* or 80 *ones*, and the 9 as 9 *ones*. They can reconceive of 1 *hundred* as 10 *tens*, and so on.	3
2. *Groups of ones*. PSTs with this conception reliably conceive of all digits correctly in terms of groups of ones (389 as 300 *ones,* 80 *ones*, and 9 *ones*) but not in terms of reference units; they do not relate reference units (e.g., 10 *tens* to 1 *hundred*).	2
3. *Concatenated-digits plus.* PSTs with this conception conceive of *at least one* digit as an incorrect unit type at least on occasion. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 ones but the 8 is only 8 ones).	7
4. *Concatenated-digits only*. PSTs holding this conception conceive of *all* digits in terms of *ones* (e.g., 548 as 5 *ones,* 4 *ones*, and 8 *ones*).	3

Thus, Thanheiser (2009a) found that 2/3 of the PSTs in that study saw the digits in a number incorrectly in terms of ones at least some of the time. In follow up studies Thanheiser (2010) further examined the PSTs' interpretations of regrouped digits and refined the *concatenated-digits plus* conception into three further categories:

a. Regrouped digits consistently explained as 10 (regardless of whether it is in the context of addition or subtraction),
b. Regrouped digits explained consistently depending on context (i.e., 10 in subtraction, 1 in addition or vice versa)
c. Changed interpretations of the regrouped digit depending on the question posed (i.e., regrouped 1 in the ten's place in the context of addition as 10 or 1 in different tasks).

In this study, 3 of 33 PSTs were able to correctly explain the values of the regrouped digits in the in both addition and subtraction contexts and 5 PSTs saw the values of all regrouped digits as 1 – consistent with the concatenated digits conception. The distribution of the remaining 25 PSTs who fell into the concatenated digits plus category can be seen in Table 2.

Three studies focused on the development of PSTs' conceptions (Mills & Thanheiser, 2010; Thanheiser, 2009b; Thanheiser & Rhoads, 2009). Thanheiser & Rhoads (2009) found that exploring the Mayan numerals (understanding/reading numbers and inventing algorithms for addition and subtraction) helped PSTs explicate the structure of the base-ten system. This allowed PSTs who typically used the terms "tens" and "hundreds" etc. as labels rather than

[2] *Reliably* in these definitions means that after the PSTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

quantities to move beyond labels and consider the quantities each digit represents. By working with a number system with which they are less comfortable, PSTs were able to explore the meaning of a place-value system. In another study Thanheiser (2009b) explored the development of PSTs' conceptions in a teaching experiment setting and found that simply exposing students to correct conceptions (telling them the meanings of the digits) does not necessarily help them develop such conceptions. And even though PSTs may sit in a classroom in which alternative methods are explored and made sense of, they may do so in a perfunctory manner. They may cling to the standard algorithms (possibly without making sense of it) as superior methods of solving problems. To allow those PSTs to move beyond the standard algorithms we need to expose them to problems with the standard algorithms and with seeing each digit as representing ones. This can be done by introducing artifacts of children's mathematical thinking to highlight these shortcomings (Thanheiser & Mills, 2010).

Table 2 *Conceptions of the 33 PSTs in the Context of Standard Algorithms (Detailed)*

<table>
<tr><th>Conception across addition and subtraction tasks</th><th>Number of PSTs</th></tr>
<tr><td>One of the two correct conceptions (reference units or groups of ones</td><td>3</td></tr>
<tr><td>Concatenated digits plus
<table>
<tr><td>Refined Conception</td><td>Number of PSTs</td></tr>
<tr><td>Regrouped digits consistently explained as 10 (regardless of whether it is in the context of addition or subtraction)</td><td>7</td></tr>
<tr><td>Regrouped digits explained consistently depending on context (i.e., 10 in subtraction, 1 in addition or vice versa)</td><td>10</td></tr>
<tr><td>Changed interpretations of the regrouped digit depending on the question posed (i.e., regrouped 1 in the ten's place in the context of addition as 10 or 1 in different tasks).</td><td>8</td></tr>
</table>
</td><td>25</td></tr>
<tr><td>Concatenated digits only</td><td>5</td></tr>
</table>

McClain (2003) examined the development of PSTs' conceptions of place value and found that research on children's development of conceptions can support the PSTs' development. She stated, "This finding also has broader implications—that the broad base of research conducted in elementary classrooms can feed forward to inform efforts at supporting the development of preservice teachers' content knowledge" (p. 301). As a result, PSTs also came to a realization that in order to teach for conceptual understanding, they themselves would need to possess this type of understanding.

PSTs' Working with [Alternative] Algorithms for Basic Operations

Three papers focused on exploring PSTs' working with alternative algorithms. Harkness and Thomas (2008) found that PSTs' understandings of invented algorithms is more procedural than conceptual. They found that getting at the mathematics is difficult and learned that if they expect PSTs to write about the mathematics they must explicitly ask them to do so. Lo, Grant, & Flowers (2008) found that teachers' ability to develop and justify reasoning strategies for multiplication develops slowly. They found that PSTs struggled recognizing the difference between procedural and conceptual descriptions of solutions to multiplication problems. They also found that PSTs struggled coordinating the area/array interpretations as well as the equal groups interpretation with various strategies for multiplication. It was not clear whether students needed more time or different kinds of experiences to continue to develop this understanding. As a result, the authors suggest more research be conducted to investigate this. They also emphasized that we need to highlight the "ineffectiveness of memorizing and applying rules/procedures without understanding why they work" (p. 20). Nugent (2007) also concluded that teachers need to understand why an algorithm works and how the arithmetic extends to algebraic procedures while exploring the lattice methods.

PSTs' Understanding of Number Sense and Number Concepts and Addressing Misconceptions with Manipulatives.

Two papers focused on number sense (Tsao, 2005; Yang, 2007) and one on number concepts (Southwell & Penglase, 2005), one on operations (Chapman, 2007) and one on order of operations (Glidden, 2008).

Tsao and Yang both found that PSTs, especially the ones who struggled, focused on procedures rather than number sense when solving problems. Tsao found that many PSTs are not ready to be immersed into a curriculum that reflects the vision of less emphasis on paper-and-pencil computation and more emphasis on number sense and mental arithmetic, as stated in the NCTM Standards. The data indicates that the high ability students were more successful on each type of number sense item than the low ability students. Compared to high ability students the low ability students in this study (a) tended to use the rule-based method more frequently when answering interview items and (b) preferred the use of standard written computation algorithms rather than the use of number sense based strategies. The high ability students tended to use benchmarks, to apply "number sense" based knowledge. Yang examined the following four categories: (1) understanding the meanings of numbers, operations, and their relationships, (2) recognizing relative number size, (3) judging the reasonableness of a computational result by using strategies of estimation, and (4) developing and using benchmarks appropriately. He found that for each category, about two thirds of participants relied on rule-based methods to answer the questions. Thus, procedural thinking seemed to be a powerful method of approaching problems especially for low ability students.

Southwell and Penglase focused on number concepts. In the whole number area they examined basic concepts (numeration), basic facts, four basic operations, order of operations, and word problems. They found that place value concepts seem to have caused the most difficulty in their survey of various number concepts, however, PSTs performed poorly across tasks. Chapman focused on operations and found that PSTs' initial knowledge of arithmetic operations was inadequate to teach conceptually and in depth. Glidden focused on order of operations and found that PSTs hold superficial knowledge of the order of operations.

Thus PSTs tend to approach problems procedurally because they lack a conceptual understanding of place value, operations, and order of operations.

One paper focused on addressing PSTs' misconceptions with the use of manipulatives. Green, Piel, & Flowers (2008) explored the use of manipulatives and found that manipulative-based instruction resulted in statistically significant decreases of arithmetic misconceptions and statistically significant increases of arithmetic knowledge, with respect to the basic operations. They state that manipulatives can effectively reverse most arithmetic misconceptions of PSTs and that the same activities used to reverse misconceptions can also improve the accuracy and depth of arithmetic knowledge. Thus, they conclude, manipulatives can and should be used effectively in PST classrooms.

Discussion/Conclusions

So What Can We Say?

In our synthesis of research, three distinct categories examining PSTs' content knowledge of whole numbers emerged. The categories include PSTs': (a) conceptions of whole number concepts and development thereof; (b) working with alternative algorithms for basic operations; and (c) understanding of number sense and number concepts. The results are promising. However, given the dearth of research published in peer-reviewed research journals during the last ten years, it has been determined there is still a need to examine PSTs' development in this content area. As exemplified by the two prospective teachers highlighted in the introduction of this paper, prior research has documented that PSTs heavily rely on memorized procedures in the content area of whole numbers and operations. In addition, many PSTs struggle to conceptually explain why the procedures work. Whereas, some research has been conducted on examining how we (mathematics educators) can help these PSTs develop more sophisticated conceptions, there is still much work is left to do for us (mathematics education community) to fully understand how PSTs concepts develop and how this development can be facilitated. This may be accomplished by conducting research in our teacher preparation courses to address the conceptions and misconceptions preservice teachers have when learning whole number concepts. For example, having PSTs explore whole number concepts in numeration system other than base-10 may provide the cognitive dissonance needed for more mathematics educators to research PSTs' development of whole number concepts in a meaningful way.

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INTEGERS: HISTORY, TEXTBOOK APPROACHES, AND CHILDREN'S PRODUCTIVE MATHEMATICAL INTUITIONS

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We juxtapose the history of integers with current pedagogical approaches. We also present findings concerning children's reasoning about negative numbers prior to instruction. We see alignment between children's intuitions and the avenues that afforded progress for mathematicians, while the textbook approaches tend to run counter to lessons learned from history.

Introduction

The introduction of the notion of signed numbers is a critical point in children's learning of mathematics. As with fractions, the advent of integers widens the domain of numbers in children's mathematical worlds and, as such, presents notorious difficulties. Some of the challenges that children experience in coming to grips with the notion of negative numbers parallel those that mathematicians encountered historically. On the other hand, the experience of most children today as they learn about integers is quite distinct from that of mathematicians of old. After all, at the time when Diophantus was denying the possibility of negative solutions to linear equations – a position that was perfectly reasonable based on his conceptions of number – he had no teacher or textbook telling him otherwise (Gallardo, 2002; Henley, 1999).

We looked to the history of integers to guide our review of current textbook approaches to integer instruction. The juxtaposition of these two bodies of evidence raises questions about the utility of prevalent textbook approaches. At the same time, we have seen children in interviews engage productively with ideas concerning positive and negative numbers prior to formal instruction. These findings suggest potentially fruitful alternative directions for textbook instruction concerning integers.

Theoretical Framework: A Historical Perspective

Before there were positive and negative numbers, there were simply numbers. The notion of *positive* is only meaningful in contrast with *negative*. Thus, the history of integers is largely the history of negative numbers. Negative numbers arose from the operation of subtraction, and they came about in at least two ways:

1. There is sense in which a subtrahend can be regarded as a *subtractive number* (Henley, 1999), which is to say that the subtraction operation is subsumed into the quality of the number. For example, instead of interpreting the expression 9 – 5 as meaning that we subtract 5 from 9, we might instead think in terms of 9 and "subtract 5" being combined additively. Under this interpretation, "subtract 5" becomes an entity that we can think about, even in the absence of any particular minuend. We can reason about -5 without

having a notion of this entity as a negative number, but simply as an amount to be taken away. Thus, 3 – 8 = -5 since we still must take 5 more away.

2. The natural numbers are not closed under subtraction. We must either restrict subtraction to cases in which the minuend is greater than or equal to the subtrahend, or we must confront situations in which the subtrahend is greater.

Negative numbers, or at least their ancestors, arose for mathematicians in these ways. For centuries, mathematicians struggled to make sense of such numbers by relating them to quantities. Some real-world situations were amenable to interpretation in terms of positives and negatives, while in other cases negatives were nonsensical (Colebrooke, 1817; Henley, 1999).

The history of negatives is a history of progress in the face of resistance. The mathematical machinery continued to be developed even as mathematicians held negatives in dubious regard, and some objected vehemently to their use. Resistance derived from legitimate paradoxes: How is it possible to take something from nothing? How could -7 be “smaller” than 3, let alone 0? Such questions plagued great thinkers and stymied progress (Hefendehl-Hebeker, 1991). Nonetheless, over time, more and more mathematicians used and came to embrace negatives because of their usefulness. The resistance to negatives stemmed from attachment to concrete or quantitative interpretations of number. Acceptance followed from an alternative, purely mathematical stance: Negative numbers were legitimate because they solved uniquely mathematical problems, not because they were necessary for solving problems concerning money, elevation, temperature or any such thing, and not because they made sense in all the same ways that that regular (non-negative) numbers made sense. Ultimately, it was the advent of abstract algebra that led the mathematical community to fully accept negatives as numbers, on par with positives. The integers came to be recognized as a domain, distinct from other domains such as the natural numbers, and it was unproblematic for different domains to have different properties (Henley, 1999).

Children’s Introduction to Integers

The extension of the natural numbers poses challenges for both teachers and students. Children are expected to expand their mathematical worlds to include negative integers and (non-negative) rational numbers, on the way to real numbers (Bruno & Martinón, 1999). These extensions challenge students’ previous conceptions, which often involve overgeneralizations of their experiences with the natural numbers. There is a wealth of research concerning teaching and learning in the rational number domain (e.g., Empson & Levi, 2011; Fosnot & Dolk, 2002; Lamon, 1999; Sowder, 1995). However, few reports exist that address the challenges of students’ learning about integers (Kilpatrick, Swafford, & Findell, 2001).

Integers and integer operations present conceptual difficulties for students (Janvier, 1983; Vlassis, 2004). These difficulties can be appreciated in light of the history of mathematics, wherein mathematicians struggled with counterintuitive notions associated with negative numbers (Henley, 1999; Gallardo, 2002; Thomaidis & Tzanakis, 2007). Mathematicians of old struggled with the distinction between magnitude and order that arises with negatives: There is a sense in which -7 is more than 3; however, it comes before 3 on the number line, and we use the term “less than” to refer to this relationship. Children encounter these same issues when introduced to negative numbers. However, our world has changed such that mathematically literate adults today can take negative numbers for granted. Thus, teachers may treat the introduction of negatives matter of factly, whereas for children it involves a conceptual revolution.

It was only after considerable conceptual struggles that great mathematicians such as Descartes and Newton came to accept negatives as numbers. It is no wonder, then, that children would face difficulties in making sense of negative numbers and operations involving them. At the same time, however, researchers have found children to be capable of reasoning about integers in relatively sophisticated ways, even in the lower elementary grades (Behrend & Mohs, 2006; Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Hativa & Cohen, 1995; Wilcox, 2008). For example, Bishop et al. found that first graders who had never heard of negative numbers nonetheless began to invent and reason productively about them in the contexts of playing a number line game and solving open number sentences.

Methods

There are three branches of this study: historical review, textbook analysis, and analysis of children's thinking. In this section, we discuss our methods of data collection and analysis with respect to each of these three branches.

Historical Review

We reviewed sources concerning the history of mathematics with a focus on ideas associated with negative (and positive) numbers, especially the integers. We identified in the literature aspects of the extension to negatives that troubled mathematicians, as well as trends and insights that led to the acceptance of negatives as numbers. These findings further informed our analysis by providing a lens through which both the textbook approaches and children's reasoning could be interpreted.

Textbook Analysis

We analyzed the treatment of integer topics in 18 fifth- and sixth-grade mathematics textbooks currently adopted by the State of California. Our methods of analysis were both quantitative and qualitative. We counted the numbers of textbooks employing particular approaches or types of problems, so that descriptive statistics would provide a sense of the frequency of these approaches. We focused our analysis on those chapters/sections of textbooks devoted to integers directly – those lessons concerning what integers are and how to operate with them. Our qualitative analysis of the textbook content began with a process of open coding (Strauss & Corbin, 1998). We used principles of grounded theory (primarily the constant comparative method) to identify emergent, distinguishing themes and features of how integers are presented in textbooks (Strauss & Corbin, 1998). We balanced emergent codes from the data itself with historical and theoretical findings from the literature. We report on trends seen across the sample of textbooks and highlight pedagogical issues that stand out to us. In particular, we examine these textbook approaches through a lens informed by the history of integers.

Analysis of Children's Thinking

We conducted interviews with 55 elementary school students from Texas and California. Twenty-nine of these children were in grades K-2 at the time of the interviews; the rest were in grades 3-5. The responses that we discuss in this paper were to open number sentences, such as the problem $3 - 5 = \square$. These were used with interview participants at all grade levels because, whether or not K-5 children have been introduced to negative numbers, the question is at least sensible. They are familiar with the natural numbers and with subtraction. Those children who did have some familiarity with negative numbers were posed problems that explicitly involved negatives, such as $-5 + -1 = \square$. The interviews themselves were conducted at the children's school sites, during the school day. They lasted between 30 and 50 minutes and were videotaped.

Our analysis of the videotaped interviews was qualitative, focusing on children's solution strategies as well as their underlying ways of reasoning. Through constant comparative analysis, we identified various themes concerning students' reasoning about integers. We then related these to the history of mathematics, identifying parallels in the form of obstacles to progress in dealing with negative numbers, as well as ways of thinking that afforded innovations. In this short paper, we highlight certain productive ways of reasoning.

Results: Textbook Approaches to Integer Instruction

A hallmark of current textbook approaches to the teaching of integers is the use of contexts and models. The most typical contexts used in textbook presentations of the integers themselves or of addition or subtraction involving integers are the following: money (used in 94% of the textbooks sampled), elevation (89%), temperature (89%), and movement forward and back (61%). The typical models employed in instruction involve (1) number lines and (2) colored chips (or, in some cases, "charged particles") that are used to represent positive ones and negative ones. An example of the use of a money context would be a person's bank account balance and how it is affected by one or more transactions. The following story problem and example solution appear in the integer addition section of a sixth-grade textbook:

> The Debate Club's income from a car wash was $300, including tips. Supply expenses were $25. Use integer addition to find the club's total profit or loss.
>
> 300 + (-25) *Use negative for the expenses.*
> 300 – 25 *Find the difference of the absolute values.*
> 275 *The answer is positive.*
> The club earned $275. (Burger, Bennett, Chard, Jackson, & Kennedy, 2007)

Perhaps the most poignant theme concerning story problems related to integers is that these tend to be quite accessible without the use of negative numbers. In fact, problems in real-world contexts can be solved without invoking negatives, and this generally seems to be more natural. For example, in the Debate Club problem above, the use of integers is entirely unnecessary. The debate teamed earned $300 from a car wash, and they spent $25 on supplies. So, their profit is $300 - $25 = $275. The only reason to represent this situation as involving addition of a negative, rather than whole-number subtraction, is the book's instruction to do so. (In fact, the same story problem would be identified as a subtraction problem if it appeared in a second- or third-grade textbook.) Furthermore, even when the problem is framed as involving the addition of two integers, the way that this "addition" is performed is to subtract!

The character of the role of the context here is not that integers are useful for solving problems in certain contexts. Rather, the message we get is that there are contexts than *can* be described in terms of integers if we choose to do so. In other words, integers do not appear as tools that are necessary for solving problems in the world, but as an artificial, purely mathematical lens that we can apply to real-world contexts when that is the game to be played. We note that this is not merely a matter of poorly chosen story problems. Two years into a project concerning integers, our team has yet to identify a real-world context that actually requires one to use negative numbers. When one considers that Greek mathematicians, for example, solved a wide variety of real-world problems without the use of negative numbers, the lack of appropriate contexts for which negative numbers are necessary seems less surprising.

In most textbooks, students are expected to translate a story problem into an expression involving integers (as in the example above) and then to evaluate that expression by appealing to sign rules or to a procedure associated with a prescribed model. That is, we do not see students' intuitions about the contexts being tapped to guide solution strategies (beyond the task of translating to a number sentence). Furthermore, it is difficult to find power in the rules that students are expected to use to evaluate integer expressions. While algebraic equations may enable students to solve problems that they could not solve otherwise, number sentences involving integers represent extra work required of students in the name of solving otherwise readily accessible tasks.

We do not mean to suggest that the only reason to relate integers to contexts would be that they provide indispensable tools for solving problems. There could be legitimate pedagogical purposes for doing so. Certainly, experts in reasoning about integers are capable of relating them to real-world contexts. However, in our review of textbook treatments of integer topics, we have found that these contexts tend to be used in a contrived fashion that may undermine the utility of the integers *within* mathematics.

Children's Intuitions about Integers

We look to interviews with elementary children to reconsider opportunities for learning about negative numbers. We have seen powerful examples of children reasoning about positive and negative numbers prior to formal instruction about integers. We discuss the reasoning of two children, a first grader (Jackson) and a fourth grader (Roland), in the context of solving open number sentences. (We use pseudonyms for the children's names.)

Jackson's Reasoning

Jackson, a first grader, had heard of negative numbers and had seen a number line that included some negatives in his classroom. He had received no formal instruction concerning negative numbers. This minimal exposure to the notion of negatives was enough to enable him to extend his counting strategies to the left of zero. For example, Jackson solved $-2 + 5 = \square$ by starting at -2 and counting up five: -1, 0, 1, 2, 3. Although he had not been taught to solve addition problems involving negatives, Jackson's previous experiences with addition enabled him to solve this novel problem.

Early in his interview, Jackson had said that he could not solve problems such as $6 + -3 = \square$. He had no way of making sense of starting with six of something and then adding negative three. However, following Jackson's solution above ($-2 + 5 = 3$), he was presented with $5 + -2 = \square$. Based on his answer that $-2 + 5 = 3$, Jackson concluded that $5 + -2$ also equaled 3, explaining (in the language of a first grader) that the results must be the same due to the commutative property of addition. When asked what $5 + -2$ meant, Jackson decided that it meant the same as subtraction since $5 – 2$ was also 3. He went on to solve the problem $6 + \square = 4$ by thinking about how much he would have to subtract from 6 to get 4.

Roland's Reasoning

Roland was in fourth grade. Like Jackson, he had heard of negative numbers but had not received formal instruction concerning operations involving them. Nonethless, Roland reinvented common sign rules for addition and subtraction of integers by reasoning about negative numbers as "the opposite" of positives. For example, Roland solved $-5 + -1 = \square$ by reasoning that adding two negative numbers "would make it further from the positive numbers." Since he knew that the sum of 5 and 1 was 6, he reasoned that -5 plus -1 should equal -6. Roland was also able to solve $-5 – -3 = \square$ by building on his previous reasoning. He said, $-5 – -3$ "would probably be… minus two, because if you use addition… it would be farther from positive numbers, so if you do the opposite it should be closer." Roland reasoned that adding -3 to -5 would result in moving futher into the negatives, so subtracting -3 from -5 should result in moving toward the positives instead. The opposite operation had to have the opposite effect.

Mathematical Insights. Jackson had productive insights concerning the properties of integers. He made these by drawing on basic understandings of the natural number domain, together with simple familiarity with the notion of negative numbers as numbers less than, and to the left of, zero. He made progress by reasoning within the realm of mathematics, not by looking outside of it. Similarly, Roland used logic, together with an understanding of + and – as opposites, to effectively reinvent sign rules for addition and subtraction of negative numbers. Middle school textbooks go to elaborate lengths to present these rules in such ways that hopefully students will remember them. Roland, a fourth grader, figured them out for himself.

Roland and Jackson did not make the leaps described above by relating integers to a context or by applying a chip model. Rather, they leveraged logic and familiar mathematical properties to solve novel problems. These moves opened up new possibilities for them. At an elementary level, these children's productive insights echo the work of mathematicians such as Newton who overcame difficulties associated with negatives through a pragmatic approach, driven by mathematical utility and logical necessity.

Implications

In light of the history of the integers, it seems worthwhile to reconsider the ways that contexts and models are typically used in integer instruction. In the history of mathematics, attempts to relate integers to contexts and models both facilitated and constrained progress. The notion of negative numbers arose not out of real-world situations involving directed magnitudes, but from within mathematics itself. Ultimately, mathematicians made sense of and accepted negatives as numbers not because of any relationship to real-world quantities, but on pure mathematical terms. Resistance to negatives was alleviated by a shift in perspective. Mathematicians let go the search for clarifying models and came to view numbers as abstract entities. They recognized that the natural numbers and the integers are distinct domains with different properties, and that the integers are useful for solving mathematical problems (as opposed to story problems).

Our purpose is not to denigrate contexts, models, manipulative materials, or textbooks. On the contrary, when it comes to many topics, we support approaches that relate numbers to quantities. In the case of young children learning to reason in the natural number domain, we see real-world contexts and materials such as Unifix cubes and base-ten blocks as extremely helpful resources. However, integers are different! The notion of a negative number is a strange idea and one that does not correspond well with the experiences of people in the world. Even individuals who have come to accept negative numbers within mathematics tend to regard them as fictitious entities with respect to the physical world. One child that we interviewed described negatives as belonging to a "ghost world" in which faint images of objects could be seen, although the objects were not really there.

It is no wonder that attempts to relate integers to contexts often appear contrived. Why do we insist on such an approach? Consider instead an honest, purely mathematical alternative whereby students are asked to confront issues that were previously swept under the rug: *Why can't we subtract bigger from smaller? What if we could? We can add and multiply without restriction. Suppose that we wanted to be able to subtract and divide with as little restriction as necessary.*

How could we do that, and how might we interpret the results? These are fair, mathematical questions that children are capable of exploring. Rather than think up elaborate ways of getting students to learn and remember the sign rules for operations with integers, we could invite children to establish for themselves what those rules ought to be. Preliminary evidence from interviews suggests to us that children could come to construct the integers from the natural numbers, given appropriate support for doing so.

Endnote

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FROM OPPOSITIONAL CULTURE TO MANIFESTING GOALS: A FRAMEWORK OF RESILIENCE AND GROWTH

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Using the psychological construct of resilience, we analyze research on mathematics identity, race and ethnicity. Findings in the literature show the existence of a meta-framework, a trajectory from a framework of violence to one of positive growth. The foundational frameworks of Ogbu elucidate the lived adversity whereas later works, such as Nasir, illustrate lived positive responses that promote academic success. Illustrating and supporting studies are included.

Purposes of the Study

The construct of identity has become an important tool for mathematics education research. Many researchers have used identity to make sense of questions about differences in students' mathematical thinking and learning (Author 2, 2008; Boaler & Greeno, 2000; and Oslund, 2010), and especially to consider unyielding questions about differences in mathematical performance across racial and ethnic lines (Jilk, 2007; Martin, 2000; and McGee, 2009). Identity, as a tool, has been used to frame differences in academic performance since at least the 1980's when John Ogbu first wrote about voluntary and involuntary minorities, suggesting that external factors associated with identity contribute to student success. As researchers continue to study race, mathematics, and identity, some have built upon the work of Ogbu (and others), crafting new research trajectories and constructs of identity. The purpose of this paper is to examine changes in these constructs, seeking patterns in the changes and drawing conclusions about them.

As we explored the history and uses of identity that explore mathematics education and race/ethnicity, we were struck by a common trajectory that began with notions of adversity or violence then moved towards a positive adaptation. Foundational frameworks (like Ogbu's) tended to invoke adversity and violence to describe the relationship between racial identity and success and failure in academic mathematics. Subsequent authors seem to build from the premise of violence toward more non-violent framings. Intrigued by this trajectory, we sought framings that might clarify these trends and settled on a framework of resilience. In the next section of this paper, we will elaborate the framework of resilience and then, in our findings, demonstrate how that framework plays out in research on racialized mathematics identity.

Theoretical Framework

Resiliency is a psychological construct that refers to the process of positive adjustment to adverse circumstances. This definition implies two important conditions: exposure to adversity and a positive response (Luthar, Cicchetti, & Becker, 2000). We recognize that it is unusual to draw upon a psychological construct to analyze research; our purpose is not to comment upon the resiliency or psychological condition of any of the authors of the articles we cite. However, we are purposeful in drawing upon the *conditions of resiliency* to make sense of our findings.

Psychological researchers have broadly defined the first condition, exposure to adversity: Adversity can range from traumatic, short-term life events (such as a moment of physical violence) to long-term stressful conditions (such as living in poverty) (Luthar, et al., 2000). This adversity is experiential and includes the history of slavery and other oppressive acts against non-majority peoples as well as current physical and psychological violence resulting from the persistence of racism and segregation in the United States. We use the notion of adversity to examine how research on math identity acknowledges the *lived adversity* (physical and mental) faced by racial and ethnic minorities in the United States. We are also interested in the second condition of resiliency: the *lived positive response*. The positive response suggests the existence of productive activities of racially/ethnically minoritized students as they engage in learning mathematics. For example, as students draw upon the mathematics in their daily lives to make sense of school mathematics, they frame their lived experiences in a positive light and use these experiences as a resource for mathematical learning.

Both *lived* components of the definition of resiliency give rise to *theoretical* constructs. In other words, researchers use *theoretical adversity* to metaphorically frame identity in ways that focus on violence and adversity. One example is Herzig's reference (2004) to the term "egotistical suicide". This theoretical construct is used to describe graduate student attrition in mathematics departments: Herzig suggests that students who voluntarily withdraw from the mathematics community are committing "suicide". By purposefully using this term, Herzig situates her theoretical framework in the context of violence and adversity. Herzig's use of suicide is an example of what we are calling *theoretical violence*.

The *lived positive response* also has a theoretical counterpart, the *theoretical positive response.* Researchers who use a theoretical positive response frame race and identity in ways that focus on positive, productive adaptations. For example, the work on Funds of Knowledge (Moll, Amanti, Neff, & Gonzalez, 1992) starts with an orientation toward experiences and background as cognitive resources that can support academic success.

It is our assumption that researchers use the lived adversity and lived positive responses of involuntary minorities to inform their theoretical framings of identity in their study of student achievement in mathematics. Although these researchers acknowledge the violence experienced by minority communities, some decentralize the lived adversity within these communities and celebrate the potential therein. In our findings, we offer a limited exploration of how adversity appears in research on race, identity, and mathematics education and explore an evolution in the literature that demonstrates a positive response in the portrayal of the minority student.

Methods

We performed a comprehensive search for articles on mathematics identity. We utilized the following search engines: JSTOR, ERIC, EBSCO, PsycINFO, Social Sciences Citation Index, ProQuest, and Sociological Abstracts. Approximately 1,200 citations were compiled using search terms such as "mathematics identity, education" and "math*, identity, education". The term "education" was included to eliminate extraneous articles that focus on the mathematical use of the word identity. A hand search completed this phase with the help of Google Scholar.

These 1,200 citations were refined to remove duplicates and irrelevant articles. For example, we removed articles on STEM (science, technology, engineering and mathematics) majors, cognitive science, and articles on academic identity with little mention of mathematics. Finally, we removed articles in languages other than English. This reduced the list to 303 papers.

Next, the articles were categorized into groups such as gender, race/ethnicity, and teacher identity. Sixty-nine articles were categorized under race/ethnicity. Articles where identity was not central were then eliminated. For example, we did not include articles that addressed the mathematical achievement gap among ethnicities but did not expressly invoke identity. A clear framework or definition of identity was not required since some authors' definition of "efficacy" or "goals" resembled other's definition of "mathematical identity". Finally, dissertations were removed. This left a final count of sixteen articles to analyze.

We examined these articles by mapping and exploring commonalities and differences across theoretical frameworks, the definition of mathematical identity, problem construction/resolution, and citations. We noticed violence used as a theme in some foundational articles and observed that violence or nonviolence arose as researchers built upon these foundational pieces. Our findings elaborate on the use of violence and the movement away from it. We include the articles that best illustrate our observations on the theoretical construct of resilience and its implications.

Findings

John Ogbu, Involuntary Minorities, and a Culture of Opposition

Several mathematics education researchers build from the foundational frameworks of John Ogbu (e.g. Martin, 2000; Nasir, 2002; and Stinson, 2008). Ogbu's work (1987, 1992, and 1998) acknowledges the lived violence of minority people and reflects that violence in his theoretical framework. His theoretical differentiation of involuntary and voluntary minorities arises from differences in their experiences of lived violence in the United States. Involuntary minorities are non-immigrant minorities whose ancestors have been enslaved, conquered, or colonized and who have been made members of American society against their will. Thus, the category of involuntary minority is predicated on this history of violence. In contrast, voluntary minorities emigrated from their homeland to the United States in search of greater opportunities (Ogbu & Simons, 1998). Because of their recent arrival, voluntary minorities lack the same legacy of violence (in the United States) as involuntary minorities and they see their presence in the United States optimistically and not dictated by the government or the White[1] hegemony. This does not imply that voluntary minorities have had no exposure to violence – Ogbu's framework emphasizes the contrasts between the lived experiences of minorities in the "adopted" country.

Ogbu draws upon the differences in lived violence to account for variation in academic performance between these two groups. The violent sociohistorical legacies and lived adversities of involuntary minorities cause them to develop an identity opposing that of White America's middle-class. This *oppositional culture* counters the hegemonic group by redefining the values placed on role models, paths to economic, social and academic success and government-controlled institutions. As a result, based on the rejection of school, a White institution, involuntary minority parents give their children contradictory signals on the value of knowledge; they value education but reject the institution of school (Ogbu & Simons, 1998).

Ogbu also develops a framing of three influencers that contribute to academic failure among minorities: society, school, and community (Ogbu, 1987). These influencers are affected by and contribute to the lived adversities of involuntary minorities. Societal traditions of denying lucrative jobs and equal opportunities (in vocation and education) teach involuntary minorities that personal investment in education will not yield the expected results, and thus contribute to lived adversities. Likewise, schools, a reflection of the larger society, contribute to lived adversities when they prevent minority students from receiving an equitable education. Finally the communities who take up the oppositional culture contribute to failure by sending the message that academic success is success in a White institution and therefore undesirable.

The lived adversity seen in the language of opposition and conflict is felt in some studies that build on Ogbu's theoretical framework. Lim (2008) analyzed the school mathematics experiences and illustrated the lived adversity of two Black sixth-grade girls. One student, Stella, had a naturally boisterous, outgoing spirit but became shy, subdued and deeply self-alienating in her mathematics class. This was due to her knowledge of the simultaneous existence of two conflicting worlds within the school, the academic world and the social African-American world. Stella's parents reinforced this conflict by encouraging her to succeed but sharing anecdotes of failure in mathematics based on theirs and others' lived adversities. The conflict between socially conforming and learning mathematics led to indecision between studying pre-algebra and rejecting social alienation. Stella's understanding of how these diverging cultures clashed, suppression of her academic desires and opposition to the academic, White, world returns us to Ogbu's framing of identity based on adversity and oppositional cultures. Stella not only regarded the culture of her mathematics class as oppositional and violent, but she labeled it dangerous.

Lim's second student, Rachel, offers a slightly different enactment of oppositional culture. Rachel did not compromise her identity in the mathematics classroom and excelled in class. She routinely completed her class work early but was never made aware of challenging mathematics tracks. Rather than consider a "White", hegemonic track to academic success, Rachel looked to the defined oppositional path, sports, as the best way to meet her life's goals. Although Rachel discussed the importance of education, she visualized her world through a lens of opposition that does not see the value and potential in her work as a scholar. To both girls, school was not an area to grow and thrive; it was an arena of conflict. Both girls were candidates for pre-algebra but neither placed in that course and thus were limited by responses to their lived adversity.

Ogbu's frameworks respond to the violent lived adversities of African-Americans. In violent or threatening academic settings, students isolate themselves and do not consider positive effects of the academic world. Ogbu's frameworks are critical because they clearly acknowledge and proclaim the role of violence and adversity in minority students' (lack of) educational opportunities. This acknowledgement provides the essential foundation for subsequent frameworks that, to borrow from Dr. Martin Luther King, Jr., "hew out of the mountain of despair a stone of hope" (1963). The next two sections explore how researchers have leveraged Ogbu's affirmation of violence to "hew… a stone of hope" and construct new insights into the mathematical learning of minority students.

Moving from Violence toward Positive Adaptation

While researchers who study ethnicity and race in mathematics education cannot ignore the lived adversity of involuntary minorities, some developed frameworks that recognize but lead away from violence. Martin's work (2000) on African-American students' mathematical identity uses Ogbu's work to link identity and mathematics learning and establishes the importance of racialized experiences in the mathematics classroom. Martin expands Ogbu's school and societal influences (Ogbu, 1987) on the failure in academic mathematics among Black students to include success. Martin acknowledges the opposition and lived adversity of African-Americans but also adopts a stance that moves away from violence by analyzing Ogbu's influencers for success factors and lived positive responses, shifting his framework towards resilience.

Martin finds that African-American students who succeeded in academic mathematics resisted or opposed influences seen by them as negative, whether from Black or White society. They took ownership of their academic development but experienced social pressures as a result. Teasing, ridicule, and severe taunting by their African-American peers ostracized students for their achievements. This tension reflects Ogbu and Simons' ideas (1998) on the discord that occurs when a student thrives in the White academic world. Yet these students react to their lived adversities by succeeding academically, a move, which can be seen as a positive response.

Martin notes the lack of research on the successes of African-Americans and indicates that much can be learned from understanding how positive responses can develop from lived adversities. Inspired by Martin's call to increase this research, Stinson (2008) wrote about four Black boys and their mathematical success by analyzing the sociocultural and sociohistorical discourses outside the mathematics classroom that affected their agency. Stinson found that the students understood the importance of academic success. Yet, because sociocultural discourses identify African-American men in certain ways, these students were compelled to renegotiate perceived adverse images of them with teachers and community members in order to succeed in the classroom. They responded positively to adverse images either by controlling others' perceptions of them or by not thinking of success as being racialized "the term *nerd* could be applied to all races" (p. 998); they resisted Ogbu's notion of an oppositional culture.

The students also developed a double-consciousness; an awareness and understanding of the positions of both marginalized, excluded populations and hegemonic groups. This positive response proactively avoided violence and minimized the lived adversity. Stinson argued that an individual, whose identity is formed outside the dominant discourse and develops a double-consciousness deliberately or inadvertently, tends to be motivated to be self-empowered or to transform society. By being aware of the hegemonic discourses surrounding Black men, these students understood, and worked to control and alter, others' perceptions of them. They appropriately tailored their behavior in and out of the classroom. Furthermore, the students either rejected or renegotiated Ogbu's ideas on African-Americans' rejection of academic excellence.

These counter-stories contradict theoretical constructs on the failure of African-American boys in mathematics. Two generations after Ogbu's work, Stinson's work looked at positive responses and the success stories of African-American boys to bring the intellectual conversation to a new level. He considered an example, where students are not limited by violence, but are empowered by the positive responses in understanding how society labels them. By controlling the Discourse, these students gained greater control over their destiny. Ogbu's frameworks set the stage for these realizations, since these students found success in both of their worlds by simultaneously accepting and rejecting the lived adversity that comes from both.

Stinson's ideas are illustrated in Jones (2003), which used action research to help a third-grade girl develop a "hybrid identity" in an after-school math club. Similar to the two girls in Lim's work, Patti experienced lived adversity because she enacted two conflicting identities. Her personal, home identity was that of a boisterous, active, African-American girl who was encouraged by her mother to adopt an academic identity (White, middle-class) in school by excelling in mathematics. In contrast, she did not successfully conform to the limited notions of a girl's appropriate behavior in the classroom, which limited her mathematical success. In fact, her teacher did not recognize Patti's mathematical talent and held low expectations for her. Jones used an after-school math club as a third space where Patti and others were encouraged to develop a hybrid identity, a positive response to their lived adversity which included teachers' discouraging remarks. In the club, academic and cultural and personal experiences were valued and encouraged to be part of the discourse, learning was stimulated, and the lived adversity was minimized. Because the math club was a safe space where Patti's identity did not need to be compromised or suppressed, Patti enjoyed mathematics, developed her skills and a strong mathematical identity. Jones' intervention may have prevented results mimicking Lim, showing how the positive response can help a student move away from lived adversity.

Further adding to the success stories of African-American men in mathematics, Berry (2008) documented how a group of Black middle-school boys' mathematics learning experiences' were racialized and how they persisted and succeeded to enroll in Algebra I. Four boys were tested for the academic program at the insistence of their parents or another adult. Their teachers did not recognize the potential and cognitive abilities of these boys; others rejected those notions and advocated on behalf of the child. The parents each believed their child was not considered for the gifted program because of race. These parents did not use what they perceived to be a teacher's racism to develop an oppositional worldview; they did not subscribe to the violence of racism to reify Ogbu's theories. Instead, these parents understood that teachers and principals hold the key to knowledge in the classroom and worked to change their views. The parents developed a positive response to their lived adversity by being involved with school activities to reject any notion that they did not care about their child's academic progress. The parents taught their children that being a Black boy required them to work harder for less recognition, as indicated by the struggle to have school personnel recognize intellectual talents and potential.

As researchers step away from violence to focus on the academic successes of involuntary minorities, they deemphasize violence and consider positive responses. Martin and Jones expand the canon of student experiences from what Ogbu and Lim portray. Stinson conveys a theoretical framework on dual-consciousness, a positive response to one's lived adversities. Berry shows that parents did not develop a segregated worldview that opposes the culture of power; rather based on their lived experiences these parents challenged the hegemonic powers in the schools to respond positively and ensure their children the best education possible.

Beyond Violence: The Positive Response

Ogbu's influence extends to researchers who use his work to frame the positive response. Looking at learning through a cultural lens, Nasir (2002) theorizes how mathematical learning, identity, and goals interact within a community of practice. Nasir, like Martin, builds on Ogbu's framework by suggesting that race and culture influence academics and that identity and learning are connected. Nasir advances Ogbu's theories by supplementing identity and learning with goals. However, rather than emphasize the lived adversity of minority students and their opposition to learning, Nasir portrays minority students as advancing their own learning (albeit in nonacademic contexts) through goal setting and changes in identity.

Nasir examines two practices of African-American culture: dominoes and basketball, and illustrates how mathematics is pervasive and valued in these practices and thus in Black culture. She emphasizes that minority students who are sometimes seen by their teachers as mathematically weak are, in fact, mathematical beings, eager and willing to learn mathematical ideas. Unlike Lim's work (2008), Nasir does not require students to redefine themselves in order to succeed. She does not see sports as a rejection of academics. Instead, she embraces athletics as a way to reach the academic potential of African-American students and places the burden for change upon schools, suggesting that students will excel in learning when learning is well designed. Thus Nasir, while acknowledging and building from Ogbu's work, has, in this article, moved from an emphasis on adversity and violence to a focus on the resiliency of Black youth, showing their celebration of learning in spite of unsuccessful school contexts.

The move toward resiliency is also illustrated in Thompson and Lewis' (2005) case study of a Black high school boy, Malik, in an urban school. Malik's goal was to be a pilot. This motivated him to enroll in upper-level mathematics courses in high school. Such coursework would provide knowledge and strengthen his transcript and applications to competitive colleges. Malik began high school defining success through athletics. Living in a violent neighborhood with high crime rates, drug activity, and low participation in school, Malik witnessed first-hand the detrimental effects of the culture of opposition and rejected it completely. Malik's witnessing of the lived adversity of his community initiated the positive response of setting his personal goal and motivating himself to accomplish this goal in multiple ways, including comparing his academic development with national averages to minimize a false sense of confidence.

Malik's goal redefined his identity as a dedicated student and he consequently petitioned his principal to offer a Pre-Calculus / Calculus course. The principal was hesitant; he felt there was a lack of interest in upper-level mathematics courses. However, he was willing to establish the course if a certain number of students expressed interest. In the end, 16 percent of the graduating students at the school did so, contradicting the notion that African-Americans are not interested in education. This principal did not decide the students' fate, he left it to Malik's agency and the students' motivation, laying the foundation for a positive response and reducing the possibility of contributing to the culture of opposition by denying students their intellectual desires.

Using Ogbu's foundational work, Nasir deemphasizes adversity and develops a framework of positive responses in mathematics learning. Thompson and Lewis illustrate how the lived adversity of the culture of opposition triggered the positive response in a student so strongly that they petitioned and succeeded in having their school offer advanced mathematics classes.

Conclusion

We applied a non-traditional meta-framework to research on mathematics identity and race. This meta-framework starts with violence as a basis for the theoretical constructs that draw from the lived adversities of involuntary minorities. The pervasive experiences that contribute to adversity cannot be ignored when explaining achievements in mathematics. However, they are the beginning and not the ending of frameworks that explore mathematical achievement gaps. These frameworks give way to frameworks that draw upon positive responses. For example, Martin acknowledges that racializing mathematics and recognizing adversity is essential. However, he also contends that race does not have to limit one's mathematical potential. Nasir moves one step further from adversity in her work. She formulates a theoretical positive response based on a lived positive response and celebrates possibility. Her theoretical stance emphasizes strength and power among involuntary minorities while minimizing violence.

Although Nasir and Martin and others do not describe their work in terms of resilience, their acknowledgment of Ogbu's work as a foundation and their work to establish positive theoretical framings suggests that resilience is an appropriate label for the trajectory of their work.

Our meta-framework of resilience suggests that the acknowledgment, proclamation even, of adversity, might be important to the trajectory of research on situations of violence and adversity. It also suggests particular ways in which research that explores lived adversity might develop over time. Finally, it embraces the inspirational call of many nonviolent protestors like Dr. Martin Luther King, Jr.; Nelson Mandela and Mahatma Gandhi who each enacted the belief that injustice could be resolved by first recognizing the multiple kinds of violence and then moving through a positive response toward a more just world.

Endnotes

1. The literature is inconsistent on the use of racial terms and their capitalization. We will use "Black" and "African-American" interchangeably and capitalize all references to race.

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STUDENT PERSPECTIVES TOWARD UNDERGRADUATE MATHEMATICS COURSES

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This research was an investigation of 208 undergraduate students' perspectives toward undergraduate mathematics instruction. Survey ratings and limited student written commentary were analyzed as a whole and by gender and race/ethnicity. Overall, students tended to report somewhat favorable affect and experiences. However, some students, particularly females, expressed negative perspectives. Participants noted the role of both the instructor and themselves as important factors in defining the undergraduate mathematics experience. The results suggest improvements for undergraduate mathematics instruction.

Purpose of the Study

Most undergraduate students must successfully complete at least one mathematics course to obtain a bachelor's degree (Gupta, Harris, Carrier, & Caron, 2006). Mathematics is important preparation for many fields of study. Mathematics-oriented fields tend to be lucrative, offering workers secure positions that can impact life quality. For example, in 2009 Computer and Mathematical Science Occupations ranked third in mean annual wages ($76,290) out of the U.S. Bureau of Labor Statistics' (2007) 22 major occupational groups.

Nevertheless, many students enter college with insufficient preparation for or poor dispositions toward mathematics courses (e.g., Gupta et al., 2006; Schmidt, 2007). They may fail to see the utility of the subject. They may avoid mathematics coursework beyond that which is required and even avoid careers with substantial mathematics demands. Some research findings associate attitudes and beliefs with mathematics performance and participation (e.g., Watt, Eccles, & Durik, 2006). Perspectives on mathematics may differ by students' "social identities," such as gender and race/ethnicity. For this reason, some researchers (e.g., Lubienski, 2008) encourage mathematics education researchers to study gaps among different student groups as a springboard from which to launch equity efforts. More research is needed to examine the undergraduate mathematics experience as a starting point for improving undergraduate mathematics courses. Accordingly, the survey research reported here investigated 208 undergraduate students' perspectives toward undergraduate mathematics instruction. The data were analyzed as a whole and by race/ethnicity and gender.

Theoretical Framework

The following topics related to undergraduate mathematics learning are discussed briefly below: occupational importance of mathematics, mathematics anxiety, instructional strategies and mathematics performance, and the relationship of race/ethnicity and gender to mathematics.

Occupational Importance of Mathematics

Many jobs call for mathematics skills, as does daily life itself (e.g., Bureau of Labor Statistics, 2007; Stubblefield, 2006). According to Bureau of Labor Statistics (2007) projections, many jobs that once required little mathematics background will call for specific skills in algebra, geometry, measurement, probability, and statistics. Prevot (2006) found that many fields, such as economics, commerce, and medicine, are taking on mathematical aspects, so that individuals unfamiliar with algebraic symbolism and higher mathematics may struggle to read some of the current literature in these fields. Goodman and Stampflii (2001) report that many employers prefer mathematically trained candidates for secretarial and administrative assistants, especially where statistical or accounting work may be required. They also note that life insurance companies and industrial research organizations provide numerous positions for mathematicians.

Mathematics Anxiety

Most students resist taking mathematics courses due to anxiety toward the subject. This anxiety is so strong that many students defer taking mathematics courses until their last semester of undergraduate studies (Liljedahl, 2005). Tobias and Weissbrod (1980) define mathematics anxiety as the panic, helplessness, paralysis, and mental disorganization that arise in some people when they are required to solve a mathematical problem. Mathematics anxiety, a common phenomenon among college students, exceeds a dislike of mathematics (Perry, 2004; Vinson, 2001). Many researchers have documented that mathematics anxiety stems from students' fear of failure and feeling of inadequacy. In particular, Krantz (1999) describes this extreme form of mathematics anxiety as the inability by an otherwise intelligent person to cope with quantifications and, more generally, mathematics. If students suffer from mathematics anxiety, their willingness to enroll and succeed in mathematics courses is diminished (Stubblefield, 2006). Consequently, many students at the undergraduate level tend to view mathematics with negative attitudes and feelings, which have been found to explain a significant portion of the variance in student learning (Gupta et al., 2006; Papanastasiou, 2003). Students who do well in mathematics have more positive attitudes about the subject; they are thus likely to take more courses and may perform better (Tapia & Marsh, 2004). However, many students enter undergraduate programs with an already developed attitude toward mathematics instruction (Gupta et al., 2006).

Instructional Strategies and Mathematics Performance

An instructor's method of teaching can impact students' attitudes toward mathematics. According to Walczyk and Ramsey (2003), students complain that instruction that is primarily lecture is boring and hard to relate to, grading does not reflect achievement, competitive rather than cooperative learning is emphasized, memorization receives greater focus than understanding, concepts are not linked, demonstrations are rarely used, little class interaction takes place, and faculty are indifferent. On the other hand, students report that good teaching practice consists of consciously constructing a supportive learning environment that can increase students' interest and success in mathematics and show students various ways to approach the same mathematics problem (Carlan, 2001).

The Relationship of Race/Ethnicity and Gender to Mathematics

Mathematics performance and attitudes have been found to differ by race/ethnicity and gender, thus making these demographic factors important to consider in conducting research (e.g., Acherman-Chor, Aladro, & Gupta, 2003; Ma, 2006). These "social identities" intersect in ways that further complicate simple data interpretation (Riegle-Crumb, 2006). Tapia and Marsh (2004) posit that among college students, feelings about mathematics mainly relate to individual personal experiences. Ma's (2006) research found that racial/ethnic differences in mathematics were more evident among students from socially advantaged than from socially disadvantaged backgrounds. For example, Hispanic students from socially advantaged families failed to do well in mathematics courses. Thus, Ma (2006) cautions that a family's socioeconomic background may not predict participation and success in mathematics course work; rather, "It actually depends on the racial-ethnic background of students" (p. 145).

In their research on a cohort of college algebra students, Acherman-Chor, Aladro, and Gupta (2003) found that ethnic minorities received more encouragement and motivation from their parents than White students. In terms of role models, African American teachers can help shape how African American students perceive themselves as learners and doers of mathematics, as well as stimulate a sense of belonging in the mathematics classroom and eventually in the field of mathematics (Moody, 2004). Acherman-Chor, Aladro, and Gupta (2003) found, "Contrary to the established assumption that minority parents do not understand the value of higher education, both Hispanic and Black students reported higher levels of parental educational expectations than did their White counterparts" (p. 141).

Most students, in general, tend to have positive attitudes about the pursuit of mathematics when they feel inspired and determined to succeed (Liljedahl, 2005; Tapia & Marsh, 2004). Although women exhibit high levels of math anxiety, they perform well in relation to men in college mathematics (Acherman-Chor et al., 2003; Van Nelson & Leganza, 2006). As noted, Tapia and Marsh (2004) posit that among college students, feeling good about mathematics relates to individual personal experiences more than the social construct of gender.

Methodology

Participants and Data Collection

The study included 208 undergraduate students enrolled in mathematics courses at a large western U.S. public university during the spring semester of 2007. Participants whose instructors consented to participate in the study were enrolled in Intermediate Algebra, Number Concepts for Elementary School Teachers, Precalculus and Trigonometry, Calculus I, Calculus II, or Calculus III. The participants, who ranged from freshmen to seniors, consisted of 99 females (47.6%) and 106 males (51.0%) from 11 undergraduate mathematics classes. (Three students' gender, accounting for the remaining 1.4%, was not disclosed,) Of the 208 participants, 1.9% were American Indian or Alaskan Native, 1.9% Black, not of Hispanic origin, 8.2% Hispanic, 9.1% Asian, Native Hawaiian, or Pacific Islander, 73.1% White, not of Hispanic origin, 3.4% other, and 2.4% racially undisclosed.

The research was a quantitative study based on survey-item ratings that were supported by limited qualitative data in the form of written comments. Students' written comments helped the researchers interpret the ratings. The quantitative and qualitative data were collected at the same time and were both considered integral to the data analysis. According to Hanson, Creswell, Plano-Clark, Petska, and Creswell (2005), "Using both forms of data allows researchers to simultaneously generalize results from a sample to a population and to gain a deeper understanding of the phenomena of interest" (p. 224).

A 29-item, author-developed questionnaire was used to assess participants' perspectives toward undergraduate mathematics. Items were based on undergraduate students' experiences with and perspectives toward mathematics, as gleaned from the professional literature and the authors' professional experiences as college instructors in mathematics and mathematics education. Questions addressed such areas as perceptions of the difficulty level of mathematics and expectations for students' performance in mathematics. Participants responded to the first 28 items on a five-point Likert scale with choices of "strongly disagree," "disagree," "unsure," "agree," or "strongly agree" for each statement. Participants had the option to provide comments after each of the 28 items to explain their ratings. The final item was an open-ended invitation to provide any other comments participants might want to make. The survey was administered to ten of the eleven classes at the beginning of class and one at the end of class.

Data Analysis

Survey ratings were assigned scores ranging from 1 for "strongly disagree" to 5 for "strongly agree." These quantitative data were analyzed using SPSS statistical software. Descriptive statistics were figured for the data as a whole and the data disaggregated by gender and race/ethnicity. The number of participants in each racial/ethnic minority group—American Native or Alaskan Native; Black, not of Hispanic origin; Hispanic; Asian and Native Hawaiian or Pacific Islander—were small compared to White participants, so race/ethnicity was put into two groups, non-White (n = 51; 24.5%) and White (n = 152; 73.1%), for the purpose of data analysis. A Mann-Whitney test was used to compare mean scores for survey items by gender and race/ethnicity with an alpha level of .05 established for determining significance.

Student written comments were analyzed by performing multiple readings of the data to construct conceptual categories that appeared to reflect participant perspectives (cf. Bogdan & Biklen, 2007). The authors repeatedly coded student comments separately and together until they agreed on the coding categories used, classification of student comments, and general themes observed. Written commentary was analyzed as a whole and separately by gender and race/ethnicity.

Findings

Quantitative Analysis

The ten most extreme mean ratings, those farthest from a rating of "unsure" (3.0), are weighted more heavily toward positive portrayals of the undergraduate mathematics classroom, as shown below. (Scores above 3 tend toward "agree" and those below 3 toward "disagree.") One possible exception is the tendency to perceive classroom instruction as mainly lecture, if presumed that lecture is construed as an undesirable method (which does appear to be the case based on some written comments). However, as worded, the item is descriptive rather than evaluative.

- I believe that I am capable of doing well in mathematics (4.1).
- I am provided with challenging, thought-provoking problems in math class. (4.0)
- Tutoring or other help outside of math class would help me improve my math performance. (3.9)
- My math instructors encourage me to do my best in mathematics. (3.9)
- I am encouraged to ask questions in math class. (3.8)
- The main teaching method for my math classes is lecture. (3.8)
- My math instructors hold high expectations for my performance. (3.7)
- My math instructors give me extra time and assistance in and out of class when I need it. (3.7)
- Some students are expected to do better in mathematics based on their race/ethnicity and/or gender. (2.3)
- Classmates often intimidate me in math class. (2.3)

Ratings for some items tended toward more negative positions but not as strongly as the more positive perspectives indicated above. The most salient is, "Mathematics is a difficult subject area" (3.5). Gender differences, and to a much lesser degree race/ethnicity, are evident. Significant differences according to these participant subgroups are shown below. Effect sizes also appear in parentheses, with the magnitude characterized as small (.01), medium (.06), or large (.14).

Gender. Males assumed a more favorable score for each rated comment below. In most cases, both mean scores tended to be favorable or unfavorable but were nevertheless significantly discrepant.

- I fear mathematics. ($p < .001$; .66, large effect)
- I feel fear/anxiety when taking mathematics tests. ($p < .001$; .67, large effect)
- Instructors often intimidate me in math class. ($p < .01$; .04, medium effect)
- Classmates often intimidate me in math class. ($p < .001$; .055, medium effect)
- Mathematics is taught in a way that matches the way I think and learn. ($p < .01$; .04, medium effect)
- I would like to take more math courses than what is required for my major. ($p < .01$; .04, medium effect)
- I believe that I am capable of doing well in mathematics. ($p < .05$; .02, small effect)
- My parent(s) hold high expectations for my math performance. ($p < .05$; .02, small effect)
- My parents have actively encouraged me in mathematics (e.g., said I was good at it or encouraged me to take more/higher math). ($p < .05$; .02, small effect)

Race/Ethnicity. White students rated the first two items below significantly more favorably than non-White students. Non-Whites rated the final two comments more strongly. These four differences have small effect sizes.

- I believe that I am capable of doing well in mathematics. ($p < .05$; .02, small effect)
- Classmates often intimidate me in math class. ($p < .05$; .02, small effect)
- Tutoring or other help outside of math class would help me improve my math performance. ($p < .05$; .02, small effect)
- Mathematics is taught in a way that matches the way I think and learn. ($p < .05$; .02, small effect; .02, small effect)

Qualitative Analysis

One-fourth of the sample (52 students) provided a written response to the final, open-ended response: “Please add any other comments you would like to make.” Despite the influence of gender, and to a lesser degree race/ethnicity, noted above, written comments indicated that students’ perspectives on mathematics also bear a strong association with individual personal experiences apart from demographic characteristics. Individual experiences include, for example, presence or absence of strong role models, parental or self-expectations for performance, and personal work ethic. Three dominant themes, each described briefly below, appeared across students’ written comments: the role of mathematics or test type, instructional factors, and student preparation and effort.

Mathematics/test type. As noted earlier, the survey ratings show that students tend to believe they can do mathematics despite a somewhat lesser tendency to find mathematics difficult. Written comments indicate that these perceptions depend, however, on the type of mathematics and mathematics problem or exam given. Typical student responses were, “I have trouble with Integral Calculus more than any other kind of math,” “It can be really easy and it can be extremely difficult, often depends on the question,” and “It depends. I do better in some math topics than others.”

Instructional factors. In general, students credited instructors with a strong role in the undergraduate mathematics experience. Many claimed that the difficulty level depends on the instructor. For example, one said, "If teacher is good it can be easy" and another the opposite: "Some teachers make it harder than it really is." Students expressed a desire for good, comprehensible explanations that include examples and meaningful applications. They indicated that they want a slower instructional pace and greater engagement. Several students suggested smaller class sizes through comments such as the following:

- I strongly believe math classes should have way less students. The classes are too big and the instructor can't engage with everyone and answer every single question.
- Smaller class sizes are much more effective in teaching because students will actually ask questions. It's intimidating to ask a question in a large lecture hall.

Student preparation and effort. In their commentary, participants tended to acknowledge their own responsibility in exerting effort and investing sufficient time to be properly prepared for and successful in their mathematics classes. They made such comments as, "Math is not difficult if you attend class," "It only depends on how much work and effort you are willing to put in," and "I can do anything… I'm just lazy." In accepting the relationship of their work effort to performance and evaluation outcomes, several students described associated stress and time constraints, as the following comments illustrate:

- I have to work really hard, and it's really stressful.
- Math takes a lot of time and self-discipline, if given the time I think I can do well.
- I would have to spend time I don't have to do better.

Student effort also includes seeking help outside of class. As gleaned from students' written comments, the three most common sources where students attained help were, in order, the university mathematics center (which provided tutoring), peers/friends, and course instructors. Some students contend that they can perform well if the instructor provides extra help outside of class or they seek tutoring. Females reported a slightly greater tendency than males to need or seek mathematics help outside of class, especially through structured academic services such as tutoring and mathematics centers.

Significant differences by student subgroups. Student comments support ratings that portray a somewhat less positive experience for females than males in mathematics. For example, females made seven of the nine negative comments displaying negative attitudes or beliefs in relation to survey item "I fear mathematics". They wrote statements such as "I am just not good at math." and "I go in knowing I [am] not going to do well." Likewise, for the item "My parent(s) hold high expectations of my math performance," females provided all three written responses that indicated a lack of high parental expectation for their mathematics performance, for example: "They understand that it's not a strength of mine." In reference to whether they would take more mathematics courses than required for their major, females tended to make much stronger assertions. Two said, "No way!" Others said, "No more than necessary please!," "No thank you," and "Once my major requirements are fulfilled…I am done with math. Period." Male students' comments were much less dramatic, often indicating that they might take more mathematics and sometimes suggesting practical reasons for not pursuing more, such as "Would like to, but major requirements don't leave room for extra coursework" and "I would love to learn more, but…pure math will only hurt my GPA." No prominent differences by student race/ethnicity are evident in the written commentary.

Limitations

A minor limitation of this study is that one class completed the survey at the end of class, whereas the others were completed at the beginning of class. It is possible that students in this class spent less time and conceivably less thoughtfulness in providing survey responses. Another limitation is that race/ethnicity was divided into two categories for the purpose of analysis. Although this categorization is convenient for analysis and bears merit in the general sense that White students are presumed to have greater privilege than students of color on the whole, important distinctions among racial/ethnic groups are ignored in this type of classification. Further, the subsample size for non-White students was somewhat small. Finally, performing multiple Mann-Whitney comparisons increases risk of Type I error and thus interpretations must be considered tentative.

Closing Discussion

Somewhat contrary to what one might presume based on research regarding students' mathematics anxiety, this sample's strongest ratings tended to be somewhat favorable in relation to the undergraduate mathematics experience, although this varied by type of mathematics and instruction. Nevertheless, a sufficient number of students, in particular, females, report unpleasant experiences in the subject, calling for continued efforts to improve undergraduate mathematics instruction. The method of delivering instruction as it relates to such factors as the instructor's role and class size appeared to be the most important component in crafting classroom climate. A number of students suggested, for example, a slower pace of instruction. Within our math-phobic society, this is something we can control and should give due consideration. One female urged, "This study should be seriously considered."

The results of this study indicate that gender is still an important issue in mathematics instruction. Regardless of student achievement levels, student dispositions continue to matter in a number of ways, such as their potential impact on future course and career participation. In addition to showing less favorable attitudes and beliefs than males, females reported less parental support. This resonates with other research that females are significantly less likely than boys to receive maternal encouragement to participate in out-of-school STEM (science, technology, engineering, and mathematics) activities or parental provision of activity-related materials (Jacobs & Bleeker, 2004; Simpkins, Davis-Kean, & Eccles, 2005). This points to the continued importance of educating parents and the public at large of the value of mathematics for *all* students.

Gender appeared as a greater concern in these study results than race/ethnicity. Nevertheless, racial/ethnic minorities were less likely than Whites to believe in their mathematics abilities and were more susceptible to classmate intimidation, although they tended not to experience such intimidation in general. It is not surprising that females were less likely than males to rate mathematics instruction as matching the way they think and learn, but it seems unusual that students of color found classroom instruction a better match for them than Whites. This small effect may be real or a product of a relatively small subsample size and the fact that all racial/ethnic minority groups were combined into one group for statistical comparison, blurring distinctions among groups. Nevertheless, this finding would be worthy of further study.

It is encouraging that many students seemed to recognize the importance of their own effort and preparation. In addition to an externalized focus on instructional methods, this means that students also accept a substantial amount of responsibility for their own learning and view their degree of success as something they can influence, an important belief. Some students expressed frustration with a lack of time to sufficiently apply this work ethic. This returns to the idea of the fast pace of instruction that may hinder understanding and foster hopelessness in students. In conclusion, it seems that undergraduate mathematics instruction would benefit from a balanced approach of improving mathematics instruction (e.g., using a slower and more conceptual approach), seeking to influence high societal expectations for all students in mathematics, and reinforcing students' understanding of the importance of personal effort in learning mathematics, which includes seeking available outside resources.

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MRS. THOMAS: A CASE CONCEPTUALIZING A TEACHER'S KNOWLEDGE OF EQUITY IN TEACHING MATHEMATICS

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Mathematics educators constantly discuss equity in the classroom, but do we know what equity in mathematics looks like? Establishing an equitable classroom environment that results in student learning for African Americans requires specialized teacher knowledge, productive dispositions and beliefs, and effective teaching practices. In this case study, I use the Knowledge of Equity in Teaching Framework to unpack a successful elementary mathematics teacher's knowledge of equity in teaching mathematics to African American students. The results of the study reveal the components of the framework influence each other.

W. E. B. Du Bois, one of the founders of The National Association for the Advancement of Colored People (NAACP), envisioned that the education of African Americans would equip leaders to protect the political and social rights of the Black community. Realizing Du Bois' vision requires more effective teacher education as well as support structures for teachers. Unfortunately, the majority of teachers are not fully equipped to educate a diverse population of students. Currently, in the best of circumstances we have mathematics teachers who have strong mathematical content knowledge and high levels of pedagogical content knowledge, yet African American students are still underperforming in mathematics (Martin, 2009).

Research has described different knowledge bases required for teaching including knowledge of mathematics content that relates directly to the mathematics they teach, typical student errors and misconceptions and how to deal with them, and how to work in diverse school settings, including an understanding of the conditions of their students' lives and of the social factors that affect schooling (Ball, Hill, & Bass, 2005; Ma, 1999; Shulman, 1986). Yet, no research has involved understanding teacher knowledge related to equity in teaching mathematics to African American students.

The purpose of this case study is to understand an elementary teacher's knowledge as it relates to equity in the instruction of mathematics to African American students. The research questions underlying this study are:

1. What knowledge related to equity in teaching African American students does a successful elementary mathematics teacher draw upon?
2. What beliefs related to teaching mathematics to African American students influence the mathematics teaching of a successful elementary mathematics teacher?

Conceptual Framework

Teachers must develop and draw from numerous knowledge bases to effectively teach mathematics to all students. Although mathematics education researchers have studied knowledge for teaching over the past two decades (e.g., Carpenter, Fennema, Peterson, & Carey, 1988; Hill, Ball, and Schilling, 2008; Hill, Rowan, & Ball, 2005; Ma, 1999), the conceptualization and research related to the knowledge of equity has not been pursued. As a result, I developed a conceptual framework related to teachers' knowledge of equity issues, beliefs, and teacher's knowledge of equity pedagogy based on a review of the literature (see Figure 1).

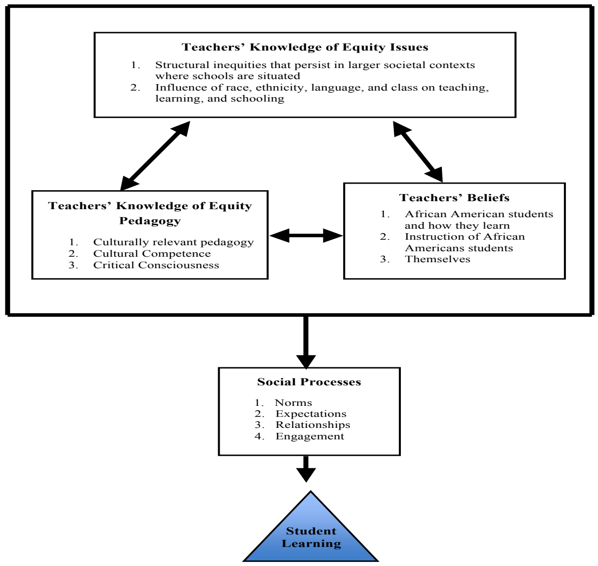

Figure 1. Knowledge of equity in teaching framework.

The top section of the framework emphasizes that teachers should develop knowledge about structural inequities that persist in larger societal contexts where schools are situated as well as the influence of race, ethnicity, language and class has on teaching and learning. Pai (1990) suggests:

> Our goals, how we teach, what we teach, how we relate to children and each other are rooted in the norms of our culture…In a society with as much sociocultural and racial diversity as the United States, the lack of this wonderment about alternative ways often results in unequal education and social injustice. (p. 229 as cited in Gay, 2000, p. 23)

The rectangle on the right side of the framework emphasizes teachers' beliefs. Philipp (2007) asserts beliefs are "psychologically held understandings, premises, or propositions about the world that are thought to be true…beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action" (p. 259). Teachers need productive beliefs specifically related to African American students, how African American students learn, and effective teaching practices for African American students. Moreover, they need productive beliefs related to themselves as teachers and the roles they play as they teach mathematics through an equitable lens.

On the left-hand side of the conceptual framework, I include three components within knowledge of equity pedagogy: culturally relevant teaching, cultural competence, and critical consciousness. Ladson-Billings (1994) defines culturally relevant teaching as a "pedagogy that empowers students intellectually, socially, emotionally, and politically by using cultural referents to impart knowledge, skills, and attitudes" (p. 17). Gay (2000) extends this definition by stating culturally responsive teaching is "using the culture knowledge, prior experiences, frames of reference, and performance styles of ethnically diverse students to make learning outcomes more relevant and effective for them" (p. 29). Culturally relevant teaching is designed to teach the "whole" child through acknowledging and addressing the child's race, language, ethnicity, and class through teaching. Students' "identities" are not ignored or subsumed in the deficit model of teaching and learning, but are contributors to the teaching and learning of students. For example, culturally relevant teachers use the strengths, experiences, and accomplishments of students in their instructional approaches.

Cultural competence includes teachers' knowledge about their own culture and the role it has in their lives; knowledge of students' cultures; the necessity to *study their students* (Ladson-Billings, 2001); and the ability to communicate (both verbal and nonverbal), understand, and interact with people from different cultures. A culturally competent teacher has the capacity to function in other cultural contexts.

Critical consciousness engages both teachers and students to critically question, reflect, participate in meaning making, and act in their sociopolitical environment. Ladson-Billings (2001) suggests teachers who promote critical consciousness: (a) have knowledge of the larger sociopolitical context; (b) invest in students and understand students are important for who they are and who they can become; (c) plan and implement academic experiences that connect students to the larger social context; and (d) believe the success of students has consequences for their quality of life.

The conceptual framework outlines the critical components of knowledge of equity issues, beliefs, and knowledge of equity pedagogy. It guided the design and implementation of a larger study (see Jackson, 2010). In this article, I report the findings of the three components of the framework for one teacher.

Methods

Participants

Participants were identified based on the following criteria: (1) Teach mathematics in grades 3-5; (2) Composition of mathematics class is at least 20% African American students; (3) Have a minimum of 4 years of teaching experience; (4) Recommended by knowledgeable mathematics educators based on competence in teaching African American students; and (5) Recommended based on mathematics achievement of African American students on state and/or local assessments.

I contacted the Hartville School District's Superintendent (all names are pseudonyms) who placed me in contact with the mathematics coordinator. The mathematics coordinator identified seven teachers whose African American students had high average achievement scores on the district assessment (75% and above). She also confirmed with the building principals that these teachers have a positive rapport with their African American students. Through their recommendations, Mrs. Thomas was asked, and agreed to participate.

Data Collection

The data sources included semi-structured initial and final interviews to determine Mrs. Thomas' knowledge related to equity in teaching mathematics to African American students. Each interview lasted approximately one hour. In addition, I observed and videorecorded her teach eight mathematics lessons during the months of November-January. I wrote field notes during each observation and then followed up with four unstructured stimulated-recall interviews, which lasted 35-50 minutes each.

Data Analysis

After conducting audio-recorded interviews, I transcribed the data. The transcripts were then uploaded into QSR NVivo 8 software and coded using a microanalysis with nodes related to the Equity in Teaching Framework. I used the coding categories shown in Table 1. After the initial coding, other themes emerged. I then conducted a second layer of coding with these emerging themes. Two doctoral students each coded an interview transcript using the coding dictionary.

All discrepancies were resolved and clarified. Finally, I mapped out Mrs. Thomas' knowledge of equity. Reliability was confirmed with another faculty member.

Table 1

Coding categories for teachers' knowledge of equity in teaching

Teachers' Knowledge of Equity	• knowledge about structural inequities that persist in larger societal contexts where schools are situated • knowledge related the influence of race, ethnicity, language and class on teaching and learning
Teachers' Beliefs	• productive beliefs specifically related to African American students • how African American students learn • effective teaching practices for African American students • productive beliefs related to themselves as teachers and the roles they play as they teach mathematics

Teachers' Knowledge of Equity Pedagogy	• Culturally relevant pedagogy
	  • view themselves as part of a community
	  • help students to make connections between their racial, cultural, local, national, and global identities
	  • establish relationships with students that extend beyond the classroom environment
	  • establish a community of learners
	  • believe knowledge is constantly recycled, re-created, and shared by students and themselves – it is not static
	  • view teaching as students developing their knowledge
	  • use a variety of instructional strategies
	• Cultural competence
	  • knowledge about their own culture and the role it has in their lives
	  • knowledge of students' cultures
	  • understand the necessity to *study their students*
	  • ability to communicate (both verbal and nonverbal), understand, and interact with people from different cultures.
	• Critical consciousness
	  • knowledge of the larger sociopolitical context (i.e., knowledge of political and social realities in which they live beyond the television and newspaper articles)
	  • invest in students and understand students are important for who they are and who they can become
	  • plan and implement academic experiences that connect students to the larger social context
	  • believe the success of students has consequences for their quality of life

Results

Mrs. Thomas, an African American woman, began her teaching career 13 years ago at a school in the eastern section of the United States. After her husband received a job transfer, she accepted a position as an elementary teacher in a suburban district in the Midwest. During the past four years, Mrs. Thomas has taught third grade at the same school within this district. Currently, this school reports a student population that is 30.1% African American.

Mrs. Thomas' Knowledge of Equity Issues

Mrs. Thomas believes many teachers of African American students do not understand their culture and misinterpret their actions. As a result, they are essentially penalized for their African American identities. Mrs. Thomas explains:

> When we're teaching African American students, a lot of times we punish African American students for behaviors that they've been conditioned to. A student who goes to the principal's office for shouting out in class for disruptive behavior when the student was participating, the student was disrespectful or because the student was shouting out and the teacher had rules. And one of the things was we had to reconsider our rules based on the needs of the students. So, it's like you can't make students conform to a society when it's something that they're doing in their own home. And it's something that is considered normal, or when they go to a church it's considered normal. We're not saying that it's the way to function, but it's a way of understanding. (Stim-recall 1, 2009)

A statement like this suggests that Mrs. Thomas believes that teachers, white or black, are indoctrinated to filter students' actions through Eurocentric culture and beliefs. This culture dictates what is valued and how students need to behave and respond in a classroom setting. She works hard to try to help her white colleagues realize they need to understand African American culture and build relationships with students. Mrs. Thomas expresses,

> A [white] teacher thinks, well if I pay more attention to this student is because it's the

> white guilt. And so, I think they are missing the mark. Like I don't think they understand. They think well, I don't want to give special treatment to African American students. And it's not special treatment, it's just understanding differences. I mean I can't help it. There are differences. (Final Interview, 2010)

This quote illustrates that Mrs. Thomas believes white teachers feel they need to treat all of their students the same. However, she knows that equal treatment is not possible. Teachers need to acknowledge and understand there are differences and use those differences to support and build students' mathematical knowledge.

Mrs. Thomas' Beliefs and Knowledge of Equity Pedagogy

Mrs. Thomas believes effective mathematics teachers of African American students understand why individual African American students act in certain ways, and learns these aspects when he/she builds and establishes relationships with students. Mrs. Thomas explains:

> We don't want to stereotype them and say oh well since African American students are like this, I will teach like that because sometimes they don't fit a stereotype. There really isn't a stereotype. It's just that sometimes culturally in a household you know you just respond to things differently. Some students may talk in an aggressive way, but an aggressive way is joking. And if you don't understand that then you are always making that student have some kind of consequence. So, I think just understanding students culturally, where they're coming from, and when students do certain things it may not be an aggressive behavior, it's just who they are. If you pretty much have that relationship with that student and understand who those students are, then I think you'll be an effective teacher. (Final Interview, 2010)

Once Mrs. Thomas establishes relationships with her students, Mrs. Thomas contends her students cooperate because they trust her. They feel they can relate to her. She knows the students feel comfortable with her when they begin to call her "Mommy" (Final Interview, 2010). She believes her identity and cultural background allows her to make mathematics relevant. She clarifies, "If we go back to me being an African American, when I present something they think it's relevant because I'm doing it. They can trust me. I have that face that's familiar to them" (Final Interview, 2010).

Mrs. Thomas believes mathematics lessons must be relevant and African American students learn mathematics best when they solve mathematics problems that are related to their life experiences. As a result, Mrs. Thomas creates mathematics lessons using stories that her students share with her. For example, one of her students talked about a trip he made to the store to purchase an item. She had the class pretend they were going to the store to purchase a pack of gum and then posed problems about the amount of change the cashier returned. Mrs. Thomas finds that when she relates mathematics to real life experiences, her students are more successful. Further, Mrs. Thomas believes when she relates mathematics to something the students enjoy this connection enhances their mathematical knowledge. For example, one of her male students likes to dance. Mrs. Thomas used the context of the number of beats he used while dancing to help him build understanding for multiplication. Multiplication finally made sense.

Furthermore, Mrs. Thomas believes her African American students are visual and tactile learners. Mrs. Thomas believes her non-African American students also learn in a similar manner, yet she contends that her African American students have "unique learning styles" (Final Interview, 2010). She describes a student who constantly hears music when none is playing, and he likes to move. Mrs. Thomas realizes that if this boy is seated while she explains a mathematical concept, she loses him. To accommodate his learning culture, she incorporates movement during her mathematics lessons. She describes, “I try to incorporate having kids move around, do a lot of clapping, or do something with rhythms. I think that African American students, they participate more. They say, ‘ooh this is fun.’ They don’t say, ‘this is boring’” (Initial Interview, 2009). Mrs. Thomas believes all of her students benefit from movement, clapping, and rhythms, not just her African American students. Mrs. Thomas remarks, “I do pay attention to how they’re learning” (Initial Interview, 2009), and she believes it is imperative that she integrates ideas into her daily plans that will facilitate learning.

Mrs. Thomas uses a variety of instructional strategies to help her students develop their mathematical knowledge. She understands the African American culture and adapts her instruction to meet their needs. She explains:

> When you’re teaching students, especially students in the African American community, they like to shout out answers. Instead of penalizing them for shouting out answers, make it part of your lesson. So there are times when shouting out is okay. And I let them know that they don’t have to sit and wait on me to call you out or wait for a spinner. Sometime they just have to get it out…So I give them the opportunity to shout it out. (Stim-recall 1, 2009)

Mrs. Thomas recognizes this strategy is also effective with her non-African American students. She comments, “They love it” (Stim-recall 1, 2009). Further, Mrs. Thomas engages African American students by capitalizing on their cultural identity. She understands that part of the African American culture is participating in a call and response environment. Therefore, she creates informal dialogues during her mathematics lessons. These instructional strategies facilitate the participation of Mrs. Thomas’ African American students. Mrs. Thomas remarks, “Sometimes, unfortunately, you can’t get the students to learn the way that you want them to learn, but you have to find a way to make them successful (Final Interview, 2010). Mrs. Thomas understands that when she builds on her students’ strengths and makes the mathematical content relevant, her students are successful.

In Mrs. Thomas’ classroom, everyone must discuss his or her mathematical strategies. They cannot simply say, “Oh I did the problem, here it is.” They must explain their strategies to their peers and make sure their peers understand. In other words, students assume the role of the teacher. This instructional approach not only gives students ownership in developing their mathematical knowledge, it also deepens their mathematical understanding.

Mrs. Thomas believes that African American students must be active in the mathematics classroom. She thinks they should play mathematical games, discuss what they learn, and participate in hands-on activities and math workshops. Teachers need to instruct students by building conceptual understanding of mathematical concepts. Mrs. Thomas asserts:

> Kids become robotic with math. And they will say okay, you add the 6 plus 2 then you add this and then you put it together, but they don’t understand what it means. And so they have to talk while they’re doing it. So, they are talking about it, doing it to go with what they’re saying, and incorporating the vocabulary words. And they put all of that together they learn their best. (Initial Interview, 2009)

Mrs. Thomas does not want students to experience frustration or say that mathematics is hard because they have to solve a problem one particular way. She believes this is one of the reasons why there is an achievement gap in mathematics between African American students and other ethnicities. Consequently, Mrs. Thomas encourages students to use a variety of strategies to solve mathematics problems. She recalled a time when one of her African American students struggled to solve subtraction problems. He began to use a number line and counted up to solve subtraction problems. This approach worked for him and he was able to get the correct answer. Mrs. Thomas wants her students to understand there are many different ways to solve mathematics problems. When she gives her students a multiplication problem, some of them use their fact knowledge, some draw pictures, while others use repeated addition. They all end up with the same result although they use different strategies. Mrs. Thomas adds, “That’s what I love about it, the possibilities. It’s something I can always apply.” (Final Interview, 2010).

Moreover, Mrs. Thomas realizes all of her students are not *A* students, yet she understands that her students do not leave her class with the same knowledge they came in with. She wants her students to pass third grade so they can move to 4th grade, yet she desires more for her students. She explains:

> I think about where they’ll be 20 years from now. How will they compete in this type of economy? You can graduate from high school learning the basics, but not the higher level thinking, like matrices, calculus to get you into the 21st century…We have to get into what is beyond mathematics number sense. You know, why numbers are a certain way, why it makes sense, and always finding a way to incorporate it with real life experiences. That’s the big thing. So, I look at my kids, I want them to be able to do well in 10 years and give them all the skills that they need. (Initial Interview, 2009)

Above all, Mrs. Thomas aspires to equip her students with the mathematical knowledge they need to successfully compete in today’s society.

Discussion

Mrs. Thomas’ Knowledge of Equity in Teaching

Mrs. Thomas’ knowledge of equity in teaching focuses on knowledge that teachers, both black and white, are indoctrinated to structure the mathematics classroom and filter African American students actions through Eurocentric culture and beliefs. But, Mrs. Thomas uses the cultural capital of her African American students to help them become successful in the mathematics classroom. As a successful mathematics teacher of African American students, she believes she understands why African American students act in certain ways. Thus, in her pedagogy she positively incorporates those “actions,” which are manifested in her social processes. In this context, Figure 2 represents how Mrs. Thomas’ social processes are influenced by the interconnection among her knowledge of equity issues, beliefs and knowledge of equity pedagogy.

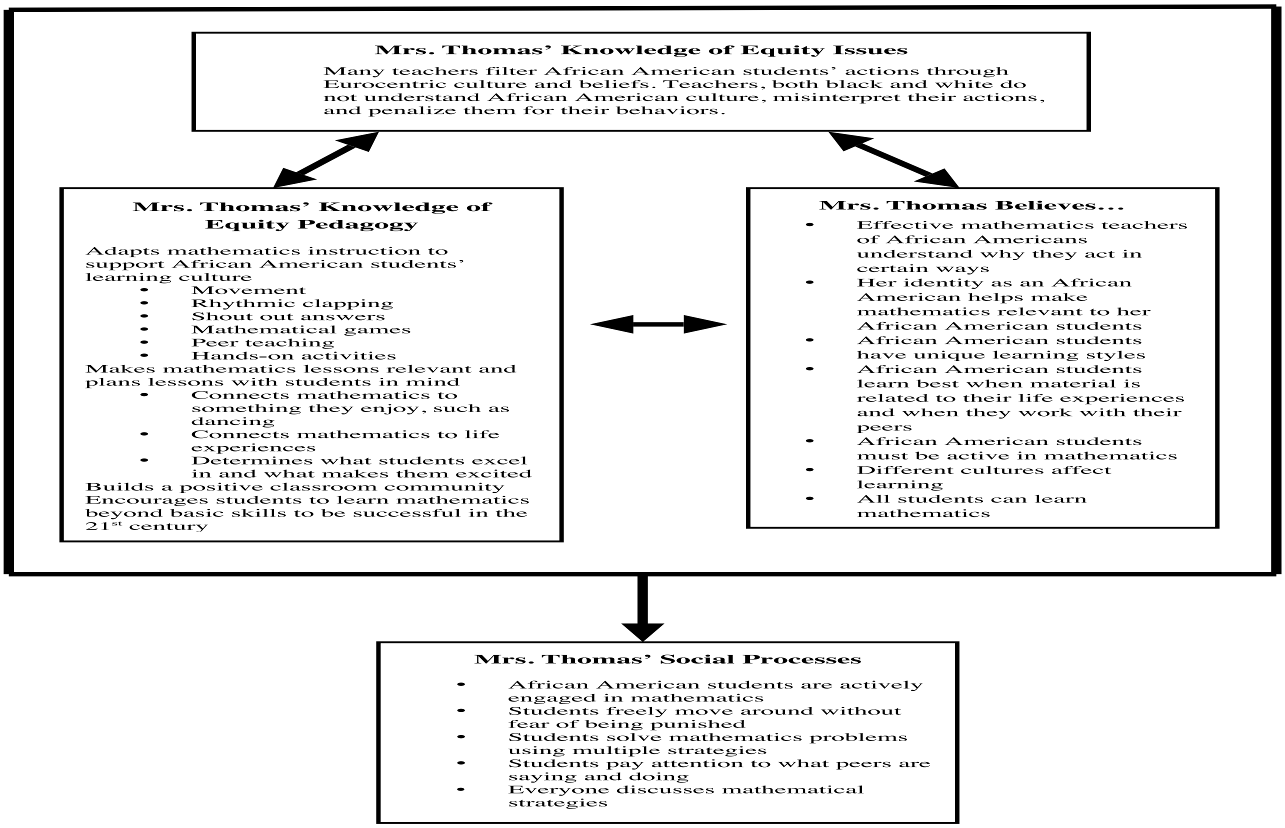

Figure 2. Mrs. Thomas' knowledge of equity in teaching.

Conclusion

The underlying theme of the Equity Principle in the *Principles and Standards for School Mathematics* (NCTM, 2000) is all students can learn mathematics. Unfortunately, many African American students are consistently demonstrating low mathematical achievement. Researchers in mathematics education have explored and studied a variety of methods to raise the mathematics achievement of African American students using what they term as equitable practices. In this study, I chose not to advocate yet another approach for teachers to use in the classroom. Instead, I qualitatively investigated an elementary mathematics teacher's knowledge of equity in teaching mathematics. I find that teachers who have been successful in teaching mathematics to African American students have productive beliefs about African American students, which influences and are influenced by their knowledge of equity issues and their knowledge of equity pedagogy.

If we want African American students to succeed in mathematics, we need to better understand the necessary knowledge base related to equity in teaching African American students. Currently, equity in mathematics education has not been fully theorized (Gutierrez, 2002). The findings from this study suggest this framework is a constructive approach to theorizing teachers' knowledge of equity in teaching mathematics to African American students.

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THE IMPACT OF FAMILY MATH NIGHTS ON PRESERVICE TEACHERS' PERSPECTIVES OF PARENTAL INVOLVEMENT

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This paper describes a Family Math Night designed to engage preservice teachers with parents in a high-poverty, minority setting and presents the results of a pre/post/follow-up survey examining the impact of the experience on preservice teachers' perceptions about low-income parents and their engagement in the education of their children.

Theoretical Framework and Context

There is little controversy about the benefits of parental involvement in children's education. Research indicates clear links between parental involvement and various indicators of school performance such as grades, test scores, graduation rates, and enrollment in institutions of higher education (Epstein & Dauber, 1991; Fan & Chen, 2001; Garcia & Hasson, 2004; Ingram, Wolfe, & Lieberman, 2007). Additionally, research indicates parental involvement impacts attitudes, school attendance, motivation, and behavioral outcomes such as suspensions, drug usage, and violent behavior (Ingram et al., 2007).

Perhaps because the connections between parental involvement and student performance are so strong, teachers often point to the lack of parental involvement as a key factor in poor student performance and in the achievement gap between poor and often minority students and their higher-income peers. Walker (2007) reports a common perception that the lower levels of support provided by low-income parents impedes their children's achievement. For example, educators surveyed about their beliefs regarding the reasons for African American male high school students' low achievement cited home factors as a primary factor (Lynn, Bacon, Totten, Bridges, & Jennings, 2010). Similarly, studies suggest that preservice (Gonzalez & Ayala-Alcantar, 2008) and in-service teachers (Patterson, Hale & Stessman, 2007/2008) attribute poor student performance and behavior to low-income Hispanic parents who, from their perspectives, do not value education and are not involved with the school.

Studies of teachers' perceptions of influences on student achievement in mathematics also point to the significance of parental involvement. Teachers report believing the lack of parental support for schoolwork and mathematics is a significant factor in the achievement gap (Bol & Berry, 2005). Sheldon and Epstein (2005) reviewed literature on mathematics achievement and concluded lack of home involvement is one of four explanations for low achievement. Yet, despite national policy calling for increases in parent involvement (US Department of Education, 2003) and evidence that increasing parental involvement improves students' attitudes toward school and their perceptions of their competence in math (Dearing, Kreider, & Weise, 2008), little attention has been devoted to improving home-school connections as a part of mathematics reform efforts (Sheldon & Epstein, 2005).

Increasing parent involvement requires that teachers believe parents will be involved and work actively to invite and encourage parents (Epstein & Dauber, 1991). Family Math Nights, first described by Stenmark, Thompson and Cossey (1986), are one such strategy for involving families. The goal of a Family Math Night is to help parents and children have fun and build

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positive connections with school and mathematics. Math games, a carnival-like atmosphere, and the provision of food are an integral part of the event (Hall & Acri, 1995; Kyle, McIntyre, Miller, & Moore, 2006; Taylor-Cox, 2005).

One wonders why Family Math Nights have not become common given the policy imperative to increase family engagement. The challenge, perhaps, is that many teachers believe poor and minority parents will not come, and therefore do use invitational strategies. Additionally, although universities have increased their focus on family involvement in recent years, the dominant focus is on topics like conducting conferences and using volunteers with little focus on skills related to conducting workshops like family nights (Epstein & Sanders, 2006).

Hindered by their assumptions that low-income parents do not value school and/or mathematics achievement and/or by the lack of skills, many teachers fail to invest time in strategies to involve parents. Lachance (2007) incorporated a Family Math Night field component into a preservice course to help teach parent-engagement skills. Participation in the event helped preservice teachers see families having fun and see students' various learning styles that must be accommodated. However, this event was not held in a low-income environment and the impact on preservice teachers' perceptions of parental involvement was not examined.

Practical field-based experience related to family involvement should be a component of preservice preparation programs (Epstein & Sanders, 2006). This paper reports the impact of a Family Math Night field component in a mathematics methods course on preservice teachers' perspectives about low-income African American parents and their engagement in the education of their children.

Methods and Data Sources

Participants

Preservice teachers (64 women, 3 men) in their fourth year of a five year elementary teacher preparation program participated in this study. The median age was 21. The participants came from three sections of an elementary mathematics methods course. Other than participation in the intervention, there were no differences in the content taught across the three sections. One of these was a treatment section (23 women, 1 man) and the other two were control sections (Control A: 20 women, 1 man; Control B: 21 women, 1 man). The majority of participants in each section identified themselves as Caucasian. Of the 24 participants in the treatment group, 18 (17 women, 1 man) responded to a follow-up survey one year later.

Intervention

The treatment group participated in a Family Math Night held at a local elementary school that serves a predominately African American (92%) community; 86% of the students receive free/reduced lunch. The school has been one of the lowest performing schools in the area according to the State Assessment System.

All students, their parents and other family members, were invited to attend. Food was provided at no charge. Students and their families were given free tickets, which they turned used to play a game. After completing a game, they were given raffle tickets for a door prize drawing. These characteristics of the event, as well as rigorous advertizing and responsiveness to parent needs, encouraged parents to attend, and created a fun atmosphere in which families could interact.

The Family Math Night was held approximately midway through the semester. Approximately 200 students and families attended. As a component of the course, members of the treatment section were required to participate. They were provided instructions for the games, assisted with preparation. Their role was to coordinate games on the evening of the event. Preparation of participants prior to the event focused on their role in facilitating the games. They were coached on how they should model the games, and how to explain the games to parents so they would be able to utilize similar games at home. The focus on the preparation activities was not on the games themselves, but rather on the idea that the preservice teachers should interact with parents in an engaging and non-threatening manner. At the event, attendees were provided with handouts that described the purpose of each game along with suggestions on how similar games could be played. The participants in the treatment group were encouraged to use this handout to facilitate parents' ability to transfer their experiences at the Family Math Night into mathematics activities at home.

Measures and Procedures

Participants in both the treatment and control groups completed a survey about their perceptions of parental involvement prior to and after the event (see Fig. 1). A follow-up online survey was administered to participants in the treatment group approximately one year later to determine if the changes observed were sustained after participants left the treatment and completed full time internship experiences.

ID Number :______________________________ Date:_____________________

Gender:_______________________________ Race: ____________________

In a POVERTY school, how many of the students' parents do believe will do each of the following?	**None**	**About 25%**	**About 50%**	**About 75%**	**Almost 100%**
1) Do an at home game or activity with their child					
2) Attend parent-teacher conferences					
3) Attend school activities such as PTA meetings and Family nights					
4) Communicate the importance of completing homework to their student					
5) Assist students with homework					
6) Examine completed work that the student brings home					
7) Ask you for ideas about how to help their child at home					

Figure 1. Pre/post survey used for data collection.

Results

Treatment and Control Groups Comparison

A Chi-Square test was used to determine whether there were differences between the treatment and control group before and after the intervention. There were no significant differences between the three groups on the pretest. However, there were four significant differences observed on the posttest. All differences were in the positive direction, demonstrating changes in perceptions of parents' willingness to: 1) to do an at home game or activity with their child (Chi-square(6)=13.09, p=0.04); 2) attend parent-teacher conferences (Chi-square(8)=15.35, p=0.05); 3) attend school activities such as PTA meetings and Family Nights (Chi-square(6)=26.89, p<0.001); and 5) assist students with homework (Chi-square(6)=11.81, p=0.07).

Treatment Pretest and Posttest Analysis

The Shapiro-Wilk test for normality was used to determine if a dependent t-test would be appropriate to explore differences between the pretest and posttest data within the groups. This test, as well as exploration of the graph of the normal probability plot and frequency distribution histogram, revealed that the data violated the normality assumption. Therefore the Wilcoxon Signed-Rank Test, the non-parametric equivalent to the t-test, was used to analyze the data. There were no significant differences between the pretest and posttest data for the control groups. In the treatment group, significant differences were observed on six of seven questions. All differences were in the positive direction, demonstrating changes in perceptions of parents' willingness to: 1) to do an at home game or activity with their child (Z=-2.92, p=0.003); 2) attend parent-teacher conferences (Z=-1.72, p=0.085); 3) attend school activities such as PTA meetings and Family Nights (Z=-2.62, p=0.009); 5) assist students with homework (Z=-2.43, p=0.015); 6) examine completed work that the student brings home (Z=-2.24, p=0.025); and 7) ask for ideas about how to help their child at home (Z=-2.55, p=0.011).

Treatment Pretest, Posttest, and Follow-Up Analysis

Due to similar conditions considering the normality of the distribution of the responses on the follow-up tests, the Wilcoxon Signed-Rank Test was used to compare the pretest, posttest, and follow-up test data. Three separate tests were performed to compare: 1) pretest to posttest data; 2) pretest to follow-up test data; and 3) posttest to follow-up test data for the 18 participants who completed the follow-up questionnaire.

On the pretest and posttest analysis, significant differences were observed on five questions. The question that did not have a significant difference on this analysis, but did in the analysis of the entire treatment group, was the one concerning parents' willingness to attend conferences. All significant differences were in the positive direction and related to parents' willingness to: 1) to do an at home game or activity with their child (Z=-2.23, p=0.026); 3) attend school activities such as PTA meetings and Family Nights (Z=-1.98, p=0.048); 5) assist students with homework (Z=-1.71, p=0.088); 6) examine completed work that the student brings home (Z=-1.73, p=0.083); and 7) ask for ideas about how to help their child at home (Z=-1.77, p=0.077).

On the pretest and follow-up analysis, significant differences were observed on only one question. This difference, again in the positive direction, was related to parents' willingness to: 7) ask for ideas about how to help their child at home (Z=-1.89, p=0.059).

There was also only one significant difference in the analysis of the posttest and follow-up test data. This difference occurred in the negative direction with the ratings dropping significantly from the posttest to the follow-up. This question related to parents' willingness to: 3) attend school activities such as PTA meetings and Family Nights (Z=-2.14, p=0.033).

Educational Significance

Parents play an important role in their children's academic success (Epstein & Dauber, 1991; Fan & Chen, 2001; Garcia & Hasson, 2004; Ingram et al., 2007), yet teachers often believe low-income, minority parents to blame for the achievement gap and don't reach out and encourage their participation (Bol & Berry, 2005; Lynn et al., 2010; Gonzalez & Ayala-Alcantar, 2008; Patterson et al., 2007/2008; Walker, 2007). These beliefs are neither productive nor professional, and must be remedied. The results of this study support previous research (e.g. Philipp et al., 2007) that coursework coupled with fieldwork can have a strong impact on preservice teachers. The participants who attended Family Math Night had more positive views toward low-income, minority parents and their involvement in their child's education than their peers. These effects, however, were not sustained after their internship experience, where they had very little control over interactions with parents and where events such as Family Math Night were not held. This suggests that sustained change in preservice teachers' beliefs about parental involvement may require multiple field experiences with opportunities for positive engagement with parents.

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ENGAGING IN MATHEMATICALLY-RELEVANT PLAY WITHOUT SIGHT

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This research examined whether mathematically-relevant play between a mother with a visual impairment and her sighted 15-month-old daughter could be facilitated by a five-week community-based early numeracy program. We aimed to identify the similarities and differences in mathematically-relevant input by comparing the 30-minute naturalistic free-play session conducted separately between the mother-daughter dyad and the sighted father-daughter dyad. Our findings reveal that the mother produced fewer mathematically-relevant utterances than the sighted father who did not participate in the early numeracy program. Implications for engaging in mathematically-relevant input by parents with a visual impairment with their sighted toddlers will be discussed.

Introduction

Empirical evidence suggests that the human brain is hardwired to engage in mathematical ideas and concepts (Butterworth, 1999; Dehaene, Molko, Cohen, & Wilson, 2004; Dehaene, Spelke, Stanescu, Pinel, & Tsivkin, 1999). However, empirical evidence also points to the critical importance of the environment on mathematical development. Parents and other significant adults have a pivotal role in supporting children's early mathematics development during the period between ages one to three years old. For example, parental or significant adults' teaching of mathematical concepts such as counting or providing mathematically-rich input has been found to relate to children's emergent numeracy skills (Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006; LeFevre, Clarke, & Stringer, 2002; Saxe, Guberman, & Gerhart, 1987). Mathematically-relevant input, as used in this context includes instances of joint attention between a caregiver and a child that mathematize elements of play, or engage in explicit mathematical play (e.g., counting, comparing, ordering, etc.) (van Oers, 1996). Additionally, mathematically-relevant input includes mathematical talk or language, as well as gestures (Klibanoff et al., 2006; Lee, Kotsopoulos, Tumber, & Dittmer, 2009; Nührenbörger & Steinbring, 2009; Singer, Radinsky, & Goldman, 2008).

Very few studies have examined adult mathematically-relevant input children receive at home on numeracy during their first few years of life. The few existing studies using interviews and checklists suggest that parental mathematical input is related to four- and five-year-olds' level of number knowledge, such as knowing differences in quantities and that numbers have magnitudes (e.g., Saxe et al., 1987; Blevins-Knabe & Musun-Miller, 1996; LeFevre, Clarke & Stringer, 2002). Given existing evidence on the importance of mathematically-relevant input in the environment for supporting the further development of mathematical understanding in young children, this research contributes to the dearth of studies by identifying (a) the nature of the mathematically-relevant input provided by parents with a visual impairment, and (b) how early educational programs such as LittleCounters™ (a community-based early numeracy program for parents with children between 12 and 36 months old, Kotsopoulos & Lee, 2009) could be tailored to overcome the challenges posed by visual impairment.

Our research questions for this case study were as follows: (1) Were there observational differences in the nature of mathematical talk during play between the parent-child dyads (mother-child versus dad-child)? (2) What are the potential learning implications of those differences (or not) for the sighted children of parents who are visually impaired?

Theoretical framework

Vision is important to the communicative and language development. It provides the ability of both the caregiver and child to be in tune with one another during conventional and normal cycles of interactions. Mutual gaze and visual exchange are important actions made to convey and understand messages during early communication, between parents and their sighted infants (Kekelis & Andersen, 1984; Rowland, 1984). Vision, is often seen as the primary integrative or mediating mode of communication. Visual information (i.e., eye gaze and gestures) aids the caregivers in understanding, interpretation and appropriately responding to their children's early speech or language (Kekelis & Andersen, 1984).

Maternal visual status has been found to influence children's early communicative development such as infant vocalizations (Rattray & Zeedyk, 2005). Infants of sighted mothers were found to be more vocal than infants with mothers with a visual impairment, particularly after 14 months of age (Rattray & Zeedyk, 2005). To date, most studies on visual impairment have focused on infants with the impairment and their language development. To the best of our knowledge, there is no known published research that examines how parental visual impairment would influence the provision of a mathematically rich environment for their sighted children, and might subsequently influence their children's mathematical development. The importance of the amount of mathematical talk children receive in the daily lives during the early years of life and its impact on their acquisition of mathematically language and concepts must be underscored. For example, developmental studies suggest that early mathematics representations such as numerosity (1 unit or 2 units of something) are linked to mathematics language, for example, the knowledge of count words (e.g, *one* and *two*) (e.g., Huttenlocker et al., 1994; Jeong & Levine, 2005). Thus, we currently do not have a good overview of what kind of mathematical talk young toddlers receive at home under the care of parents with a visual impairment.

The descriptive sentence 'parents leading from behind' becomes a very important reference in relation to caregivers doing more than just acknowledging or interpreting their child's communication attempt such as putting gesture into words, expanding on two-word utterances into complete sentences, and adding descriptive labels for him/her (Bruinsma et al., 2004; Kekelis & Andersen, 1984). Caregivers are teaching their children how to expand on ideas during speaker's turns, as well as how to respond contingently to their partner. Though existing research has been conducted on children with a visual impairment and their parents with sight, measurable research pertaining to visual impairment and parent-child dyads involving communication via gesture, speech, and touch does not examine how a child acquire linguistic and mathematical concepts with a parent who is visually impaired during the early stages of development (Bruinsma et al., 2004; Colombi et al., 2009; Kekelis & Andersen, 1984; MacDonald et al., 2006).

By examining the interactions between the two caregivers and their sighted daughter in a free play session with a standard set of toys picture books, differences in the quality of talk in terms of introducing mathematical-related words (e.g., '*one, two, three*', '*how many*') could be identified. The free play sessions were able to capture both the verbal and nonverbal ways in which mathematical words are presented to the child. Without the need for questionnaires or checklists that rely on parental memory, this approach allows us to examine the input that occurred in a more naturalistic manner and to obtain the frequency and nature of incidental mathematically relevant input that the child received at home with her parents.

Method

Participants

The current case study included a family of three individuals: a mother (age 32), a father (age 29), and a female child (15 months during the first home visit with mother and 19 months during the second home visit with father). The mother was born sighted. Due to a medical condition, she lost partial vision by eight years old before losing it completely at the age of ten. The mother is bilingual and speaks both English and Polish to her daughter. The father is a sighted Chinese-Canadian, currently working as a lawyer and only speaks English to the daughter. Both parents are highly educated and have completed four years of university education. The mother has completed three years of graduate work and the father has completed two years of professional work as a lawyer. The mother has reported informally that her sighted daughter is able to understand both English and Polish words, although she has yet to produce many words in either language.

Both parents engaged in a separate (mother-daughter dyad and father-daughter dyad) 30-minute naturalistic free play session in their home with the child. The first 30-minute naturalistic free play session occurred between the mother-daughter dyad in their home when the child was 15 months of age. The second 30-minute naturalistic free play session occurred approximately four months later between the father-daughter dyad in their home when the child was 19 months of age. The four-month interval between the two play sessions was due to the busy schedule of the father and the family's move to a new house. The family received a $10 gift card for their participation.

The mother and child participated in the LittleCounter™ program (Kotsopoulos & Lee, 2009) at a local library. As a result of participating in the program and the case of a caregiver with a visual impairment, the family was recruited for the current case study. In addition to participating in the LittleCounters™ program, the child was involved in other local community activities (e.g., educational programs) with the mother for approximately five to six hours a week, but does not attend daycare. The maternal grandparents, who speak Polish, also provided respite care to the child during the week.

Early Numeracy Program

The LittleCounters™ program consisted of five, 45 minute sessions (10:00 a.m. to 10:45 a.m.) over a span of five weeks at a local library. This community-based early numeracy program is offered free to families with young children between 12 and 36 months old. The goal of the program is to teach parents how to integrate mathematical relevant input into their child's play through the use of songs, games, stories, and movements. Supporting the development of the counting principles (i.e., stable order, one-to-one correspondence, order irrelevance, cardinality, and abstraction) was the basis of each of the five sessions. Additionally, a different mathematical concept is featured during each of the sessions: cardinality (i.e., last number represent counted represents total in the set), magnitude (i.e., more or less), ordinality (i.e., first, second, third, etc.), internal number line (small to large, left to right) and spatial effects (i.e., difference between bigger in size versus more in quantity), and parity (i.e., equal or the same).

For each of the sessions that the mother attended with her daughter, an instructor's aide (the third author) was by the mother's side to directly assist her with the activities of the program.

The instructor's aide assisted with the mathematical play and also ensured that the daughter was safe as she moved around the room independently.

Materials & Procedure

For the play session at home, in order to reduce variability between the activities in which the mother-daughter dyad and the father-daughter dyad engaged in, each dyad was provided with a standard set of toys. The toys included: four hand puppets, four oversized foam dice, one toy truck, three backpacks, a shape sorter bucket with plastic shapes (four green squares, four red triangles, and three blue circles), 40 small transportation toys (fire trucks, school buses, trains, cars, planes, and boats), 40 small animal toys (horse, cows, calves, pigs, piglets, ducks, rabbits, and sheep), 68 foam blocks in a variety of shapes (17 yellow, 17 blue, 17 red, 17 green), two children's pop-up books (one on color and one on numbers), three plastic squares, three plastic triangles, three plastic circles, three plastic rectangles (one blue, one yellow, and one red of each shape). The mother was told orally about the toys and invited to touch the toys prior to commencing.

Transcription and Coding

Each free play session of the parent-child dyads was videotaped, transcribed and coded using a portable Observer XT system (Noldus Information Technology, 2009) that allows the experimenters to work the two cameras using a remote control in a separate room from the dyad. The parent-child dyad was also asked to wear a wireless microphone each to ensure a high audio quality. Such an arrangement ensured that the free play session was as naturalistic as possible.

For the current research, mathematically-related talk referred strictly to utterances produced by parents and children to denote mathematical relations such as cardinality and quantity. All the adult speech were transcribed and coded by two trained research assistants for the mathematically-related input in the following categories: i. quantity or equivalence/non-equivalence words (e.g., how many, less than, more), ii. counting words (e.g., one, two), iii. counting objects in an array or in sets, iv. cardinality (stating or asking for the number of items in a set without counting), v. transformation of object arrays involving addition or subtraction (e.g., "If one cow went missing, how many do you have now?"), and vi. ordering numbers (e.g., 'Three, what comes after three?'). The coding scheme was adapted from Klibanoff, Levine, Huttenlocher, Vasilyeva, and Hedges (2006). Additionally, nouns and verbs used outside the context of mathematically-relevant input were also coded to account for the amount of non-mathematically-related talk.

Table 1. Coding scheme of mathematically-relevant input

Quantity or Equivalence/nonequivalence words	• How many, how much, so much, several, some, a few, altogether, most, more, same (only in the context of quantity, and not in the other context such as color), greater than, less than, count, lots, another, all, ones, all, some, little, little bit, a little, any, a couple, bunch… (Note: “match” or “anything else” are not quantity words)
Counting words	• Recite counting words such as ‘one’, ‘two’, ‘three’ (Similar to reciting the alphabets) • Count each word said as one occurrence. E.g., If the caregiver says “One, two, three”, count = 3, and not 1 because there are three count words in the utterance. • Number words such as ‘one’, ‘two’… Not associated with an object or an array of objects.
Counting objects	• Counting objects in an array or in sets (1,2,3,4..)
Cardinality words	• Stating or asking for the number of things in a set without counting them (e.g., Point to the child the number ‘2’ and ask him/her to point or show two items/objects or the above example “There are 3 books.”) • The number word (‘one’, ‘two’..) here is associated with an object or a set of objects (E.g., ‘SIX sides of a cube’, ‘THREE trucks’) • No counting involved
Transformation of object arrays	• Add, subtract, take-away (e.g., “If you take away from three, how many do you have?”) OR • One object item removed from an array of objects (e.g., 1 bird flew away, how many are left?)
Ordering numbers (different from reciting a list of number words)	• “Three, what comes after three?’

Results

Both parents produced more non-mathematically-relevant input than mathematically-relevant input during their 30-minute play session with their toddler. Since the daughter did not produce any words during both sessions, we did not include any of her data in Table 1. The coding of both visits by both coders individually was checked for inter-coder reliability by calculating the Cohen's Kappa and the population coefficient (Rho) for the entire 30-minute free-play session: the mother-daughter dyad (Kappa 0.83, Rho 0.99), the father-daughter dyad (Kappa 0.88, Rho 0.99). In terms of mathematically-related input, our results indicate that only two categories – counting and quantity words were produced by the parents. Though the mother with a visual impairment attended the 10 sessions of the LittleCounters TM program, she produced fewer instances of both counting and quantity words than the father with sight, who did not participate in the early numeracy program. A recent study on mathematically-relevant input produced by 19 sighted parents with 12 girls and 7 boys (with a mean age of 21.08 months, *S.D.* = 4.32) that have participated in the LittleCounters^{TM} program revealed that these parents produced an average of 9.84 counting words and 48.42 quantity words (Lee et al., 2010).

Table 2. Instances of word categories produced

Participants	Nouns	Verbs	Counting words	Quantity words	Total
Father	152	140	28	34	354
Mother	79	92	1	17	189
Total	231	232	29	51	543

Discussion

To begin our discussion, it is important to note that existing research on parents with a visual impairment who have children with sight has predominantly focused on practical considerations of parenthood such as caring for the infant. Our research examined the play environment between a caregiver (mother) with a visual impairment and her sighted 15 month-old daughter to explore the extent to which mathematically-relevant input was observed. Our findings did not suggest that the mother applied any skills taught within the two five-week sessions (10 sessions in total). Other related research on the LittleCounters^{TM} program has shown that participating parents produce more mathematically-relevant input than parents who have not participated in the program, over the same observation period (i.e., 30 minutes), using the same set of toys and books provided and the same research methodology (Lee et al., 2010).

Our findings also raise some questions in terms of the LittleCounters^{TM} program. Although an instructor's aide was provided to assist the mother during each of the sessions, perhaps the type of support was not consistent to what may have actually been necessary. For example, the support for the parent could have included building in strategies, with the parent's input, on ways in which the parent could communicate to the child on a particular object. Additionally, more hand-on-hand instruction for both the parent and the child could have been useful. Strategies for showing approval to the child could have also been useful. For example, when engaging in counting of fingers, we could have had all families stop to physically touch each other's fingers to ensure that the appropriate amount of digits was shown. It is possible that we may have relied too much on gesture and vocalization during our sessions which may not have been sufficient for a parent with a visual impairment.

Several studies have reported that mathematically-relevant input increases mathematical ability in young children (Klibanoff et al., 2006; Miller et al., 2000). Although this research is a case study, it provides us with a rare opportunity to examine the nature of the play environment between a mother with a visual impairment and her sighted child. For example, our results raise some concern over the level of cognitive engagement, particularly mathematically-relevant engagement that a child of a parent with a visual impairment may receive. Future policy and support directions could include provisions for providing additional resources to support parents during the crucial early developmental periods of childrearing. The early years of life have increasingly been found to be crucial in success or failure in formal schooling (Heckman, 2004).

Parents with a visual impairment could provide mathematically-relevant input by engaging in more "give me" instead of "can you show me" task during play. For example, a parent with a visual impairment, who is looking to determine whether a child understands one-to-one correspondence (one object is counted once), may need to hold the child's hand while doing the task to check the child's understanding. In sum, parents with a visual impairment may require additional coaching and support to learn how to assist young children during early childhood where shared understanding is much dependent on vision and gesture.

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HOW STEREOTYPE THREAT AFFECTS FEMALE MATHEMATICAL PERFORMANCE

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Stereotype threat has been shown to undermine the achievement of women within the domain of mathematics. This paper will review studies relevant to conditions that give rise to stereotype threat in women, types of mathematical tasks that are susceptible to stereotype threat, how stereotype threat interrupts performance, and interventions which lessen the effects of stereotype threat. Research on this phenomenon has far-reaching implications, not only for women, but also mathematics educators and university administrators as stereotype threat has been shown to account for most of the gender gap on the SAT, which in turn directly affects admissions and eligibility for scholarships.

Introduction

Performance differences between the sexes across mathematical content areas, problem types, and various instruments have been documented (Gallagher & Kaufman, 2005; Hyde, Fennema, & Lamon, 1990; Paek, 2002; Willingham & Cole, 1997), and in some situations these differences may be due to a lack of opportunity to learn about particular mathematical topics on the part of females. However, it has also been demonstrated that stereotype threat can contribute to performance gaps by causing equally qualified females to falter in the face of complex numerical reasoning tasks or word problems and opt to guess at an answer while their male counterparts are unaffected (Quinn & Spencer, 2001; Spencer, Steele, & Quinn, 1998). This paper brings together studies on stereotype threat, its affect on women's problem-solving strategies, the interaction between stereotype threat and cognitive capacity, and interventions which can limit the influence of stereotype threat on women.

Theoretical Perspectives

A landmark study (Steele & Aronson, 1995), that introduced the term *stereotype threat,* presented a possible explanation for some of the differences in achievement between males and females exhibited on mathematics assessments. Stereotype threat is defined in this study as a feeling of "being at risk of confirming, as self-characteristic, a negative stereotype about one's group." Further, it is stated that stereotype threat "may interfere with the intellectual functioning of these students [affected by stereotype threat], particularly during standardized tests" (Steele & Aronson, 1995, p. 797). This initial study focused on Black students who are stereotypically considered less able than White students in terms of general intellectual aptitude.

In the study, the researchers, working with a group of Black and White students, explicitly "primed" the stereotype for one subgroup of students (the experimental group) before administering the SAT verbal test. Specifically, before the experimental group took the exam, the researchers told them of historical diagnostic differences (that Blacks had generally scored lower on this test than Whites and that the test was diagnostic of ability). As a result, the Blacks in the "ability-diagnostic" group significantly underperformed (as measured by their previous SAT scores), while Blacks in the control or "nondiagnostic" group did not (Steele & Aronson, 1995).

The findings of Steele and Aronson's seminal study are relevant to research on performance gaps between the sexes on standardized mathematics tests since females must also deal with stereotype threat. In particular, females are susceptible to the stereotype that they are publicly perceived to be less able in mathematics than males. Steele and Aronson note the broad applicability of stereotype threat: "This threat can befall anyone with a group identity about which some negative stereotype exists, and for the person to be threatened in this way, he [or she] need not even believe this stereotype" (Steele & Aronson, 1995, p. 798).

The effects of stereotype threat on women's mathematics performance were later confirmed among college females who excelled at math and identified strongly with the subject (Spencer, Steele, & Quinn, 1998). In this study, on a relatively difficult test (composed from the advanced GRE [Graduate Record Examinations] in mathematics), females underperformed compared to males when informed before they took the test of historic sex differences in test performance, while they performed on par with males when told that the test was gender-insensitive. This study's findings confirmed what Claude Steele had concluded in an earlier article: That "stereotype threat may be a possible source of bias in standardized tests, a bias that arises not from item content but from group differences in the threat that societal stereotypes attach to test performance" (Steele, 1997, p. 622).

A test item can be linguistically biased if it uses terms that are unfamiliar to some subgroups, while the overall content of an item might be biased if it refers to situations that are less familiar for certain subgroups (Hambleton & Rodgers, 1995). While bias in item content has been confirmed and addressed by test developers over the years, the effects of stereotype threat are situational (dependent on the testing environment) and cannot be remedied by simply changing the language or premise of specific test items. One study (Shih, Pittinsky, & Ambady, 1999), that demonstrates the situational aspect of stereotype threat, examined the effects of racial and gender stereotype threat combined. While Asian-American women are stereotyped positively in mathematics due to their ethnicity, they are stereotyped negatively because of their sex. On a standardized mathematics test (SAT I quantitative section) given to Asian-American women, those whose ethnicity was primed performed better, and those whose gender was primed performed worse, than a control group for whom neither attribute was made salient (Shih et al., 1999).

A similar boost in achievement due to a positive stereotype was also observed in previous studies for White students when race was primed (Steele & Aronson, 1995) and for males when gender was primed (Spencer et al., 1998). This potential positive effect of certain stereotyping is supported by social cognitive theory: "Indeed, people who believe strongly in their problem-solving capabilities remain highly efficient in their analytic thinking in complex decision making situations, whereas those who are plagued by self-doubts are erratic in their analytic thinking," and "when faced with difficulties, people who are beset by self-doubts abort their attempts prematurely and quickly settle for mediocre solutions, whereas those who have a strong belief in their capabilities exert greater effort to master the challenge" (Bandura, 1989, p. 1176). This boost in performance for test-takers who are not part of the negatively stereotyped subgroup has been referred to as *stereotype lift*. A meta-analysis of 43 studies found that merely representing tests as diagnostic of ability was enough to induce stereotype lift in non-negatively stereotyped subgroups. Further, while the average effect size of stereotype lift (d =.24) was only half that of stereotype threat (d =.48) in these studies, it was still substantial (Walton & Cohen, 2002).

Stereotype threat can be primed in various ways. One is under the condition of *evaluative scrutiny*. In this situation test-takers know their results will be available to others, such as parents, teachers, administrators, or colleges. On standardized tests such as the SAT or ACT, some degree of evaluative scrutiny is always present. Another way to induce stereotype threat is via the condition of *identity salience*. This condition can be thought of as “the likelihood that the identity will be invoked in diverse situations” (Hogg, Terry, & White, 1995, p. 257). Participating in a mixed testing environment (e.g., males and females together) or having to identify one’s sex prior to a standardized test, as is often the norm, can cause one’s identity to become salient.

The effects of identity salience have been measured in studies conducted by the Educational Testing Service, commonly known as ETS, and the College Board. In one experiment, researchers administered the mathematics section of the GRE general test to males and females on an *individual* basis (GRE Board, 1999). In this study, the gap in scores between males and females was less than half that from the regular administration of the GRE general test (d =.40 versus d =.97) that same year, in which students generally tested in a mixed environment.

Another study measured the effect on performance from identifying one’s sex before a test (College Board, 1998). The experiment focused on students taking the Advanced Placement Calculus AB exam, and for those who indicated their sex on a standard background information sheet before the test, the performance gap effect size between males and females (d = .41) was more than triple that for those who identified their sex after the test (d = .12). These studies clearly demonstrate that while subtle in nature, the condition of identity salience can induce stereotype threat and thus cause women to underperform on standardized mathematics tests. A more recent study has actually measured the effects of stereotype threat on females’ SAT performance and found that it accounts for the majority of the gender gap on the quantitative section. In particular, the researchers (Walton & Spencer, 2009) found that stereotype threat is responsible for 19-21 points of the 34 point difference in scores between men and women.

Differences in Strategies Used by Males and Females under Conditions of Stereotype Threat

A study by Quinn and Spencer (2001) attempted to replicate a previous Gallagher and De Lisi (1994) experiment, which looked at the kinds of strategies employed by males and females when solving SAT I mathematics word problems. Gallagher and De Lisi found that females relied more on conventional strategies, while males resorted to using more unconventional strategies. Quinn and Spencer (2001) found further that a significant number of females were unable to come up with any problem-solving strategy at all.

In the second part of Quinn and Spencer’s study, when stereotype threat was primed by evaluative scrutiny and identity salience, females underperformed compared to males on a test of all mathematical word problems, but not on a test of the same problems reduced to their numerical equivalents (that is, no words, just equations). Quinn and Spencer concluded that stereotype threat lowered women’s mathematics achievement by hindering their capacity to generate and use problem-solving strategies (2001). They suggested that women had a diminished cognitive capacity as a result of having to deal with the effects of stereotype threat.

On the surface, the results of this part of Quinn and Spencer’s study might seem contradictory, since women, due to their documented edge in verbal skills, would seem to be favored in the domain of word problems. In this part of the study, however, the reason for women’s underperformance seemed to lie in difficulty with the processes and strategies of converting word problems into equations (Quinn & Spencer, 2001). Since then, another study (Bielock, Rydell, & McConnell, 2007) has confirmed that in particular, stereotype threat most greatly affects women’s performance on mathematical problems that rely heavily on verbal working memory resources. It was also demonstrated in this study that underperformance can *spillover* to tasks outside the domain of the stereotype that are reliant on verbal working memory resources (Bielock, Rydell, & McConnell, 2007). This may help to explain why women, who traditionally perform better than men on tests of verbal skills, score lower than men on the SAT critical reading exam, which is taken in conjunction with the SAT mathematics exam.

How Cognitive Capacity Interacts with Stereotype Threat

The human mind has been described as a "limited-capacity information processor" (Weinstein, 2005), and the attempt to control one's mental state can be broken down into the two processes of operating and monitoring (Wegner, 1994). Wegner notes that the operating process "promotes the intended change by searching for mental contents consistent with the intended state" of mind, while the monitoring process "tests whether the operating process is needed, by searching for mental contents inconsistent with the intended state" of mind. Both processes working together provide an individual's mental control and since these processes share the same mental space, at times (for example, when the mind is working on difficult or complex tasks) they must compete for limited mental capacity. Under conditions that reduce capacity, the monitoring process may override the operating process and thus lessen a person's focus on their intended task. Thus, the operating process is susceptible to distractions, either in the form of stress or from multitasking. "Anything that distracts the person's attention from the task of mental control will undermine the operating process and so enhance the effect of the monitoring process" (Wegner, 1994, p. 40).

Arousal plays a major role in many learning theories and is affected by levels of anxiety and stress. The Yerkes-Dodson Law, which describes the psychological effects of physiological arousal on performance, says that with appropriate levels of arousal, performance increases, but that when levels of arousal are too low or too high, performance decreases. The law (it is referred to as a law because the results of Yerkes and Dodson's experiments have been replicated numerous times since the original study) suggests that an inverted U-shape describes the relationship between arousal and performance (Yerkes & Dodson, 1908). At low levels of arousal, the subject (in our case, a student) is simply not motivated enough to perform well, while moderate levels of arousal are healthy and conducive to optimal functioning.

However, overarousal, caused, for example, by anxiety or stress, can negatively affect performance physiologically, cognitively, and psychologically. It can disrupt physiological states when the body reacts to stress by releasing hormones from the adrenal medulla (adrenaline) and the adrenal cortex (cortisol). This hormone release in turn increases heart rate, respiration, and blood pressure, which leads to hyperactivity (Selye, 1956). In addition to these physiological symptoms, cognitive activity can be disrupted via forgetfulness, confusion, and an inability to concentrate as a result of overarousal (Broadbent, Cooper, FitzGerald, & Parkes, 1982).

Overarousal can also disrupt emotional states through anxiety. It has been shown that overanxious persons entertain more task-irrelevant thoughts than nonanxious persons, and that these thoughts are frequently focused on negative personal characteristics, most notably during ability assessments such as achievement tests (Wigfield & Eccles, 1989). In particular, overarousal in the form of anxiety as a result of stereotype threat conditions has been shown to accompany the underperformance of females on standardized mathematics tests (Harder, 1999; Spencer et al., 1998). Further, it has been demonstrated that anxiety and evaluation apprehension are significantly related to females' performance on such tests (Spencer et al., 1998). Bandura (1989) states in an article on human agency that "threat is a relational property concerning the match between perceived coping capabilities and potentially aversive aspects of the environment," and "those who believe they cannot manage potential threats experience higher levels of stress and anxiety arousal" (p. 1177).

There is also evidence to support the notion of decreased performance for females under stereotype threat conditions (because available capacity for the operating process is reduced due to an increase in the monitoring process) in terms of working memory capacity. Working memory capacity can be thought of as the ability to maintain focus on a particular task while disregarding competing irrelevant thoughts (Engle, 2001). It is strongly related to performance on complex cognitive tasks such as problem solving. One study found that a measure of working memory capacity correlates significantly with scores on both the quantitative and the verbal sections of the SAT I (Turner & Engle, 1989). That is, those with more working memory capacity tended to perform better on these SAT I sections. Furthermore, there is strong evidence that "stereotype threat reduces an individual's performance on a complex cognitive test because it reduces the individual's working memory capacity" (Schmader & Johns, 2003, p. 449). A recent study suggests that, in particular, stereotype threat undermines performance through physiological stress, attempts to curb negative thinking, and an overactive monitoring process (Schmader, Johns, & Forbes, 2008).

The general notion that stereotype threat reduces cognitive capacity is supported by the results of earlier mentioned studies on sex differences in use of strategies on standardized mathematics test problems. In particular, these study results indicate that females rely more on standard classroom-presented algorithms, while males more often use unconventional strategies or short cuts (Gallagher, 1992; Paek, 2002). This difference might suggest that while females were limited to standard algorithms as a result of diminished cognitive capacity (caused by anxiety induced by stereotype threat), males had either more cognitive capacity available or a less distracted focus, which allowed them access to other, more direct methods to solve problems.

Females have also demonstrated less prior knowledge and fewer strategies than males who had similar mathematical backgrounds when working mathematics problems in mixed testing situations (e.g., males and females in the same group) (Byrnes & Takahira, 1993). This difference again could suggest that females in these situations may have been the victim of reduced working memory capacity. Another study of strategy use, based on mathematics items from the SAT I and the GRE general test, found that males had more strategy flexibility, and, as a result, outperformed females (Gallagher, De Lisi, Holst, McGillicuddy-De Lisi, Morely, & Cahalan, 2000). This greater flexibility demonstrated by males could also suggest that they had more cognitive capacity available than females burdened by stereotype threat resulting from mixed testing situations.

Along with stereotype threat, the rigor associated with a standardized test plays a major role in affecting the amount of cognitive capacity available. It has been documented that gender gaps in mathematics achievement widen as the difficulty level of problems increases (Feingold, 1988). The work by Spencer and colleagues (1998), described earlier, provides further support for this idea. The first part of their study focused on males and females who had performed well, both in a college calculus course (receiving a final grade of B or better) and on the SAT I or ACT quantitative sections (scoring above the 85th percentile). The control group was given a relatively easy (for their background) test (composed from the quantitative section of the GRE general exam), while the experimental group was given a difficult test (composed from the advanced GRE exam in mathematics). Although identity salience (from mixed gender testing) was a condition for females in both the “easy” and the “difficult” test groups, females underperformed compared to males only on the more rigorous test (Spencer et al., 1998). For these high-ability students, the general GRE quantitative section was not as challenging and made fewer demands on their cognitive capacity, which allowed the females to deal effectively with any intrusions by stereotype threat. The more rigorous test, however, required greater cognitive resources which due to stereotype threat were not fully available to the women.

Interventions that Mediate the Effects of Stereotype Threat

In this same group of studies (Spencer et al., 1998), the researchers sought to find out what would happen when they told an experimental mixed group of students in advance that the test they were about to take was insensitive to sex differences. As with the previous study, all the students were subject to identity salience from the mixed testing environment, but those who were *unprimed* (experimental group) exhibited no differences in achievement between males and females, while in the control group (students who were told nothing prior to the test) males outperformed females. McGlone and Aronson (2006) found that getting females in a mixed testing situation to consider positive personal attributes (such as their status as students at a select school) resulted in improved scores on the Vandenberg (Vandenberg & Kuse, 1978) mental rotation test. Seemingly, these types of interventions help to ease the condition of identity salience, thereby lessening the effects of stereotype threat. Further, it has been shown that practicing representative tasks prior to testing, thus moving the knowledge of procedures and concepts to long-term memory from the working memory, also moderates the effects of stereotype threat and improves female performance (Bielock et al., 2007)

Conclusion

Stereotype threat has been shown to have an effect on both the problem-solving strategies of females (Quinn & Spencer, 2001; Bielock et al., 2007) and their cognitive capacity (Turner & Engle, 1989; Schmader & Johns, 2003). However, various types of interventions have proven successful in mitigating the effects of stereotype threat (Spencer et al., 1998; McGlone & Aronson, 2006; Bielock et al., 2007). Even so, there has been little effort to date on the part of those who administer high stakes, standardized mathematics assessments or those who rely on the outcomes of such instruments to recognize or deal with existing inequities.

Approximately 92 % of four-year institutions require SAT/ACT scores from potential students and 75% routinely use these scores in making decisions on admissions (College Board, 2002). SAT scores also play a significant role in many institutions’ scholarship decisions. Likewise, AP and GRE scores play a role in students’ access to undergraduate and graduate education. The common thread between all these instruments is that they are high stakes assessments produced by the College Board and its psychometricians and they all exhibit significant gender gaps favoring males in mathematics that are not reflected in students’ future college performance (Bridgeman & Wendler, 1991; College Board, 2008; NACAC, 2008; Wainer & Steinberg, 1992). Studies conducted by the College Board affirm the effect of stereotype threat on females’ mathematical achievement (College Board, 1998; GRE Board, 1999) and the majority of the gender gap on the SAT quantitative section (boys outscore girls by approximately one third of a standard deviation) has been attributed to stereotype threat (Walton & Spencer, 2009).

Thus, the general underperformance by females on these instruments, resulting from stereotype threat, leads to inequities in terms of access to post-secondary opportunities. In particular, relatively low and unrepresentative scores on high stakes, standardized mathematics exams can serve as a roadblock to women interested in pursuing careers in the STEM fields, where females are currently and historically underrepresented. Mathematics educators at all levels and college officials need not only to be made aware of these inequities, but also what can be done to level the playing field.

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TRANSFORMING MATHEMATICS EDUCATION – APPLYING NEW IDEAS OR COMMODIFYING CULTURAL KNOWLEDGE?

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In Saskatchewan, the renewal of mathematics education has two major initiatives to contend with: the incorporation of transformative teaching and learning strategies, and the infusion of First Nations and Métis content, perspectives and ways of knowing. Playing leading roles in the development and implementation of Saskatchewan's renewed curricula, we have struggled with the enormity of these changes and their effective implementation. This has led us to reflect upon the traditional Western Paradigm as it relates to mathematics education, and to consider the importance and role of other paradigms, in particular, an Indigenous paradigm to our work. In doing so, however, we have come to question the actual origins of the foundations to the current trends in transforming mathematics education, and whether the acknowledgement of non-Western knowledges and perspectives is not due in this regard.

In Saskatchewan, as is happening in numerous places throughout North America and the world, mathematics curricula and resources are being rewritten, renewed, and transformed to address the most current findings in mathematics education research. This most recent renewal process for Saskatchewan, as a member jurisdiction of the Western and Northern Canadian Protocol (WNCP), began with the reflecting upon and revising of the original Western and Canadian Protocol (the predecessor of the WNCP) Common Curriculum Frameworks. Representatives of the WNCP jurisdictions met regularly in order to make "optimum use of the limited resources" (WNCP, 2000) of the individual jurisdictions, resulting in the completion of the *Common Curriculum Framework* (CCF) *for K-9 Mathematics* (WNCP, 2006) and *Common Curriculum Framework for 10-12 Mathematics* (WNCP, 2008). Among numerous other sources of research, the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (2000) was foundational to the renewal process.

There were, however, some issues related to mathematics education, such as assessment and time allotments for courses, were, through consensus, left out of the renewal process and frameworks. From the onset, the infusion of First Nations and Métis (FNM) content, perspectives, and ways of knowing was believed by at least one member WNCP jurisdiction to be too contentious to reach consensus, resulting in no attempt being made. The compromise, a brief description of the learning styles of Aboriginal students, was included in the introduction to the CCFs. In Saskatchewan, the number of self-declared FNM students in provincial schools has been steadily increasing (Saskatchewan Ministry of Education, 2010, p. 52, 53). As well, these numbers do not reflect the students within the Federally controlled Reservation schools in Saskatchewan, who, for the most part, also follow Saskatchewan's curricula. These numbers, in and of themselves are by no means worrisome, but when one considers that self-declared FNM students are consistently scoring below non-FNM students by as much as 18.6% in grade 10 mathematics (Saskatchewan Ministry of Education, 2010, p. 49), the seriousness of this situation becomes much clearer.

In the process of "Saskatchewanizing' the WNCP CCF's, it was decided that the outcomes and indicators of the CCF, and not just the upfront matter of the curricula documents, needed to be

written to reflect teaching strategies that are supported by the Constructivist Learning Theory and to support the infusion of FNM content, perspectives and ways of knowing. Throughout the writing process, and the accompanying research and deliberations (with a fair share of disputes), however, the realization of the complexity of the situation emerged. Not only did the curriculum writers need to find ways to address the two initiatives within outcomes and indicators, but they also began to worry about how such major changes were to be accepted and enacted the field.
But are these two sites of debate, research and action in mathematics education - the transformation of mathematics teaching and learning and the infusion of FNM content, perspectives, and ways of knowing - really distinct phenomena? In this research paper, we will present arguments and examples to support the notion that transformative educational strategies being proposed and implemented today are in fact the implementation of an Indigenous paradigm onto the realm of mathematics education.
To set this stage for this discussion, we need to consider not only Constructivist Learning Theory, but also the fields of Ethnomathematics and Multicultural Education. The following literature review provides a brief introduction to these areas.

Literature Review

This literature review is divided into two sections. The first, on Constructivist Learning Theory speaks to the current research that has informed both the WNCP CCFs (WNCP, 2006; WNCP, 2008) and Saskatchewan's Renewed curriculum documents (e.g., Saskatchewan Ministry of Education, 2009). The second part of the literature review considers the fields of Ethnomathematics and Multicultural Education that have helped to infuse Saskatchewan's mathematics curricula with FNM content, perspectives and ways of knowing.

Constructivist Learning Theory

Constructivist Learning Theory emerged from the work of two researchers, working separately and unknown to each other: Jean Piaget and Lev Vygotsky. Although the two researchers varied in how they believed children moved from one stage of learning to another, both based their work on the idea that, as humans we generate our own knowledge through the interaction of our experiences and ideas.
In Vygotsky's work, the role of the more knowledgeable other, someone other than the learner who helps in the scaffolding of the learning to move the learner through and beyond his or her current zone of proximal development, is key to knowledge development. Thus, social interactions, and especially play in the early years, are seen as important tools for the learner. It is through these interactions that the learner explores new contexts and ideas, being guided by the knowledge and experiences of not only themselves, but also of those around them. Vygotsky proposed a variety of stages of development within learners, according to different types of contexts such as the development of tools and communication, each of which the learner negotiated and moved through at a rate that was supported by their experiences and social interactions (Vygotsky, 1962; Vygotsky, 1978).
Piaget, proposed that learners go through developmental stages (sensory-motor, pre-operational, concrete operational, and formal operational), however; in his work, Piaget's focus was on the cognitive, and not the social. Piaget believed that all learners move through these stages at roughly the same ages, and in the case of the formal operational stage, many people would never reach it. Piaget argued that the learners construct their own knowledge through interactions between their experiences and their ideas, within the constraint of their developmental stages. As learners interact with their environment, Piaget asserted that they had to go through process

of accommodation and assimilation to make new experiences and new ideas “fit” into their knowledge schemata. (Piaget, 1970; Gallagher & Reid, 1980).

Out of these ideas of social and cognitive constructivism, teaching and learning strategies for supporting the learner as they construct new knowledge have emerged. These strategies vary from broad spectrum approaches to learning, such as inquiry and problem solving, to more specific methods, such as the use of children’s literature, and multiple representations, that support teachers and students in engaging in the more broad spectrum approaches.

Ethnomathematics and Multicultural Education

The notion of a field of study of mathematics in other cultures has existed since at least the late 1890s (for examples, see Taylor, 1874; Smith, 1923, or Struik, 1948), but in the 1980s, the field of Ethnomathematics, influenced by the groundbreaking work of Ubiratan D’Ambrosio, emerged. D’Ambrosio (1997) reflects “Not much has been done in ethnomathematics, perhaps because people believe in the universality of mathematics” (p. 14). Ethnomathematicians, however, continue to provide evidence of the cultural biases in the mathematics within North American schools. Despite Kline’s claim that the plant of mathematics remained “dormant for a thousand years” (as cited in Powell & Frankenstein, 1997, p. 52), mathematics was alive and expanding in those parts of the world that were beyond the grip of the Dark Ages that encompassed much of the European world. Ethnomathematics as a field has exposed the fact that there is not one mathematics, not one way to think mathematically, and not one way to do mathematics. It should be noted that Ethnomathematics does not, as a whole, question the role that Western mathematics (that which is currently taught in North American society) plays in the dispositions of modern society, but it does question the claim of singularity and absolute authority that mathematics has been afforded. By recognizing and valuing the mathematics of other cultures and societies, past and present, Ethnomathematics is expanding the world of possibilities for mathematics and mathematics education.

Banks and Banks (1995) define Multicultural Education as

a field of study and an emerging discipline whose major aim is to create equal educational opportunities for students from diverse racial, ethnic, social-class, and cultural groups. One of its important goals is to help all students to acquire the knowledge, attitudes, and skills needed to function effectively in a pluralistic democratic society and to interact, negotiate, and communicate with peoples from diverse groups in order to create a civic and moral community that works for the common good (p. xi).

In order to facilitate the creation of these equal educational opportunities for students, Banks (2002) identified four different approaches taken in curriculum writing and reform: contributions, additive, transformative, and decision-making and social action. In the contributions approach, ethic heroes, holidays, and celebrations are added into the curriculum on certain days, but set aside, or presented as distinct from, the rest of the curriculum. When curriculum writers include content, themes, or perspectives from certain ethnic groups, but do not change the actual structure or purposes of the overall curriculum, the approach being used is called additive and can often be identified by the addition of a book by an ethnic author or the inclusion of an additional, ethnicity-based, unit. When a curriculum’s basic assumptions of a singular point of view and perspective is changed to embrace students’ engagement with content and issues through multiple ethnic perspectives, the transformative approach is being applied. Finally, the decision-making and social action approach to curricula builds upon the transformative approach by requiring the students to make decisions and engage in social action.

Reflections from a Saskatchewan Curriculum Renewal Perspective

In the renewal of the Mathematics curricula in Saskatchewan, the Constructivist Learning Theory played a major role in the wording of outcomes and indicators, as well as in the writing of upfront matter regarding the teaching and learning of mathematics within which those outcomes and indicators are couched. With respect to the infusion of FNM content, perspectives, and ways of knowing, Ethnomathematics provided a sense of the breadth of possibility and its importance, while the theory of Multicultural Education encouraged and provided support for teachers to step away from the contributions and addition approaches used in past curricula to a more transformative approach. This approach is supported by the inclusion of a new goal for K-12 mathematics education in Saskatchewan, that of understanding mathematics as a human endeavor: "Through their learning of K-12 Mathematics, students should **develop an understanding of mathematics as a way of knowing the world that all humans are capable of with respect to their personal experiences and needs**" (Saskatchewan Ministry of Education, 2009), which opens up the classroom to students' sharing and valuing of different personal and cultural perspectives and ways of knowing mathematics. Through their learning of all the mathematics outcomes, it is a curricular expectation that students will achieve this goal.

Theoretical Framework

All of the research done for the renewal of both the CCF and the Saskatchewan curricula did not shed light, or even hope, on the ever-looming question of "how will Saskatchewan's teachers manage all these changes – effective incorporation of new teaching and learning strategies and the infusion of FNM content, perspectives, and ways of knowing. Throughout the renewal and implementation process, the notion that if one effectively incorporated the new teaching and learning strategies that the FNM component would or could happen was in the air, but justification for such a claim was nowhere to be found. The question that then arose was, perhaps it is because the two ideas (pedagogical change and FNM infusion) are being housed (unnecessarily?) in two different paradigms, or within a single, non-Western paradigm.

Consider first, the paradigm from which the new teaching and learning strategies have been emerging – that of the Western or Eurocentric world. This paradigm has its own "ontological, epistemological, sociological, and ideological way of thinking and being" (Kovach, 2009, p. 21) that can be distinguished from other paradigms, such as Eastern or Indigenous paradigms. Examples of the many characteristics attributed to the traditional Western paradigm include: claim to holding a monopoly on knowledge, the existence of one true or right answer, a society of specialists, scientific knowledge being based solely upon measurable attributes, reliance upon linearity, and the necessity of objectivity (Kovach, 2009; Little Bear, 2000; Meyer, 2003). This underlying paradigm makes the valuing and incorporation of FNM perspectives, content, and ways of knowing into education difficult as such action would call into question many of the fundamental principles assumed within society. How can there be multiple ways of knowing or perspectives if one believes there is only one true answer; and what happens to objectivity when one allows for multiple interpretations and understanding?

Although there is no one FNM or Indigenous paradigm, as each Indigenous group has fundamental differences from each other (Kovach, 2009; Little Bear, 2000), there are common characteristics across Indigenous groups that allow for the presentation of an overarching and broad Indigenous paradigm. An Indigenous paradigm is one that values wholeness with focus being on the group rather than the individual, on interdependence rather than independence; it values relationships, not just between people, but with all of creation; it values reciprocity; it values diversity within the group; and it values the place where knowledge is constructed and lives (Ermine, 1995; Hogan, 2000; Kovach, 2009; Little Bear, 2000; Meyer, 2003; Youngblood Henderson, 2000). Thus, an Indigenous paradigm invites varying perspectives and ways of knowing; it considers not only the impact of actions and beliefs on individuals, but on all people and on the world around them; and it seeks not only to gain, but to give back as well.

Are there benefits to considering the seemingly disparate sites of mathematics curriculum renewal – new teaching and learning strategies and the infusion of FNM content, perspectives and ways of knowing – through the non-Western lens of an Indigenous paradigm? The analyses from this perspective, as well as from the perspective of the traditional Western paradigm (which follow) will not only reveal the benefit of such a consideration, but also call into question the origins of the current transformations to mathematics education.

Analysis

In our analysis, we consider the question of which paradigm – traditional Western or Indigenous – best accommodates the changes to mathematics education that are supported by current Western research. To do so, we examine both initiatives (the new teaching and learning strategies and the infusion of FNM content, perspectives and ways of knowing) in light of the characteristics of each paradigm to determine whether one paradigm better aligns with where we are trying to go with mathematics education.

The Traditional Western Paradigm

Being from a world dominated by the traditional Western paradigm, we begin by considering how the initiatives of incorporating new teaching and learning strategies and the infusion of FNM content, perspectives and ways of knowing align themselves within the traditional Western paradigm. Within this paradigm, the notion of infusing content, perspectives and ways of knowing that are not inherently part of the traditional Western paradigm, which is the case for FNM knowledge, is in contradiction to the paradigmatic belief that the Western world holds a monopoly on knowledge. Acceptance and action upon this initiative within the traditional Western paradigm would be tantamount to admitting that there is some knowledge and some ways of knowing that do not originate within Western society. Moreover, FNM perspectives and ways of knowing value relationships and diversity, giving the resulting knowledge both subjective and objective attributes which also contradicts the traditional Western paradigm.

What of the new teaching and learning strategies that are emerging from ongoing Western research into mathematics education? How well does Constructivist Learning Theory fit into the traditional Western paradigm? At first glance and at the broadest level possible, the theory and the traditional Western paradigm would seem to fit. Both Vygotsky's and Piaget's theories assume a linear nature to learning, and the evidence for the theories is based upon scientific measurement and the application of objectivity. It is in the application of these theories, however, that a divergence from the traditional Western paradigm begins to occur. Inquiry and problem-based learning, although frequently taken to consensus, encourage students to consider alternative perspectives and representations, and the goal is no longer just the final 'truth', but also the processes and relationships developed along the way. Moreover, the use of literature in mathematics teaching and learning brings forward context and subjectivity, both of which are foreign to the Western paradigm of true mathematical knowledge. Likewise, strategies such as the use of multiple representations conflicts with the Western belief in one correct answer, and must ultimately give way to the penultimate symbolic mathematical form of representation. Thus, we contend that the traditional Western paradigm does not support the current transformative teaching and learning strategies proposed by mathematics education research.

An Indigenous Paradigm

We begin our analysis of recommendations for the transformation of mathematics education with respect to an Indigenous paradigm by considering the infusing of mathematics education with FNM content, perspectives and ways of knowing into mathematics curricula. In an Indigenous paradigm, the consideration of alternative perspectives and ways of knowing is a foundation rather than a contradiction to one's beliefs. An Indigenous paradigm not only recognizes, but welcomes new content and perspectives as a way of understanding the continuous flux of life (Kovach 2009; Little Bear, 2000; Meyer, 2003), making the infusion of FNM content, perspectives, and ways of knowing not only a possibility, but recognizing it as a necessity for the creation and application of knowledge. In an Indigenous paradigm, knowledge is grounded in place, and the infusion of FNM content into mathematics education is also about understanding the relationship between place and knowledge. As is probably not a surprise, the infusion of FNM content, perspectives and ways of knowing into mathematics education is one that is not only supported, but also endorsed by an Indigenous paradigm.

As noted previously, Constructivist Learning Theory, whether it be Piaget's cognitive development theory or Vygotsky's social development theory, does not, at the outset, require an Indigenous paradigm to fit into mathematics education; however, when one begins to consider the application of these theories to the actual acts of teaching and learning there becomes a much stronger alignment with an Indigenous paradigm over the traditional Western paradigm. Our earlier analysis noted how this realignment with paradigms begins to be apparent with the introduction of inquiry and problem-based learning. Now consider strategies such as the incorporation of children's literature, integration of subjects, and the use of multiple forms of representation in communicating and exploring mathematical ideas.

For Indigenous peoples around the world, language and oral communication are essential to the creation and maintenance of their cultural identities and knowledges (Absolon & Willet, 2005; Battiste, 2000; Kovach, 2009; Leavitt, 1995; Meyer, 2003). Little Bear (2000) notes that within Indigenous cultures, and thus within an Indigenous paradigm, "Storytelling is a very important part of the educational process. It is through stories that customs and values are taught and shared" (p. 81). The increasing use of children's literature books in mathematics instruction and learning in Western classrooms shares in these same paradigmatic beliefs about storytelling.

In an Indigenous paradigm, holistic views of the world and of knowledge are foundational - knowledge is not categorized into distinct disciplines; rather, it exists within the context of life experiences in a particular place (Ermine, 1995; Little Bear, 2000). Unlike the traditional Western paradigm, based in the Eurocentric belief that all knowledge falls into dualities and differences, an Indigenous paradigm does not try to separate out component parts of an experience, rather it encourages the valuing of the whole. In this way, the movement towards teaching and learning through the integration of different Western subject areas, is also in much closer alignment with an Indigenous paradigm than a Western one.

Likewise, the encouragement of students and teachers to learn, incorporate, and relate between multiple forms of representation, including oral communication, pictures, and concrete objects, and not just using abstract symbols, also pulls the teaching and learning of mathematics from the traditional Western paradigm and towards an Indigenous one. The openness of these teaching and learning strategies to alternative strategies and ways of knowing about mathematical ideas contradicts the Western need for objectivity, linearity, and a single, correct answer. These same features, however, bring such teaching and learning strategies in direct alignment with an

Indigenous paradigm by inviting students and teachers to openly consider and value alternative ways of knowing and understanding, and to celebrate diversity in thinking.

Discussion

Our analysis of the two initiatives of change facing mathematics education within the province of Saskatchewan, and undoubtedly elsewhere in the world, that of the infusion of FNM content, perspectives and ways of knowing and the introduction of new teaching and learning strategies, demonstrates how at least a significant portion of the transformations occurring in mathematics education are not supported by the traditional Western paradigm within which school mathematics currently sits. Instead, there is much evidence to suggest that an Indigenous paradigm might be a more appropriate choice for the enactment of these initiatives. One could argue that the new teaching and learning strategies aimed at mathematics education are as much a consequence of thinking within an Indigenous paradigm as they are aligning with the traditional Western paradigm. It is in this realization that the complexity of the changes being called for within mathematics education may find some resolution and simplification; that is, perhaps what we need to do in mathematics education is to focus on changing our Western paradigm to that of a more Indigenous paradigm, and thereby the educational changes we desire will become natural consequences of our beliefs rather than being at odds with them.

Perhaps, also, our analysis is an indication that the traditional Western paradigmatic view of mathematics education is already moving towards a more Indigenous paradigm. If so, perhaps we need to consider the words of Youngblood Henderson (2000) "In the Eurocentric construct of three-dimensional time, whoever masters the present moulds the past" (p. 65), and Smith's (2000) concern that the traditional Western paradigm seems to see the world as a "huge supermarket in which knowledge can be packaged up to be bought and sold…[without considering] the intellectual cultural property rights" (p. 218) of Indigenous peoples. If we are to let go of only some aspects of the traditional Western paradigm to let these transformations easily enter the world of Western mathematics education, then do we not need to be careful that credit is given where credit is due? Should we not be careful to avoid laying claim to the present findings by ignoring the past knowledge of others; should we not be taking care to not package someone else's wisdom under our own label? As much of the world struggles with the devastating impacts of colonization and other forms of intolerance and unsubstantiated assertion of power, do we not as mathematics educators and mathematics education researchers have an important role to play in this struggle? We believe that this convergence between an Indigenous paradigm and the transformation of mathematics education is not a mere coincidence, but the gradual evolution of parts of Western thinking into more Indigenous ways of knowing, and although the application of this thinking may be taking on forms different from those historically found within an Indigenous paradigm, the foundations for such thinking, which are also foundations of an Indigenous paradigm, should be recognized and honoured.

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"I SEE ZEROS": DEVELOPMENT OF PATTERN IDENTIFICATION SKILLS IN STUDENTS WITH LANGUAGE DELAYS

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Most research on teaching mathematics to students with learning disabilities has focused on explicit procedural instruction or rote learning. Instead, this study focused on what two first grade boys with language delays learned about patterns from Choral Counting, an instructional activity designed to encourage communication of mathematical ideas. This focus led to the research question: What do elementary students with language delays learn about identifying patterns as they engage in Choral Counting? Video interaction analysis taking a sociocultural perspective enabled me to investigate the resources that they use to express their mathematical ideas. Findings indicated that Choral Counting was a valuable instructional activity for these students: they grew in their abilities to count by various numbers, to identify patterns, to express their mathematical ideas, and to use nonverbal communication methods to clarify their meaning.

Introduction

Most research on teaching mathematics to students with learning disabilities has focused on explicit procedural instruction or rote learning (Miller, Butler, and Lee, 1998). The problems are broken down into their component parts, and the students are expected to master the components on their way to solving the problem. This type of instruction is time-efficient, makes overt the covert thinking strategies that support mathematical problem-solving, and does not depend on students' ability to intuit concepts from experience, or to work in a self-directed manner (Karp and Voltz, 2000, p. 208). Procedural instruction, however, also has its disadvantages. The students become dependent on the teacher's assistance, and there is limited opportunity for in-depth understanding, which leads to limited retention, generalization, and ability to apply the information (Karp and Voltz, 2000, p. 208). Because students may develop little conceptual understanding of the procedures, the procedures are also liable to be error prone (Woodward and Montague, 2002, p. 95).

Given the disadvantages of overemphasizing explicit procedural instruction and rote learning, research is needed on other instructional methods for students with learning disabilities. In particular, teaching through mathematical discussions has potential and is worth studying (Woodward and Montague, 2002, p. 97). In this study I focus on what students with language delays can learn by engaging in a particular instructional activity that encourages the communication of mathematical ideas. Specifically, I focus on what students with language delays learn about patterns when they engage in the instructional activity of Choral Counting.

Choral Counting is a 10-15 minute warm-up. Prior to the count, the teacher has thought about where to start and end the count, what to count by, whether to count forwards or backwards, and how to record the count. The class counts together out loud while the teacher records the count. At a strategic point, the teacher stops the count and encourages the class to discuss numerical patterns that students notice. During Choral Counting students practice counting, identify patterns, and engage in mathematical reasoning as they predict numbers that will occur in the count (http://sitemaker.umich.edu/ltp/home).

This study is an example of a self-study, as I was the resource teacher of the students being studied and was motivated by my own experiences. I tried to teach mathematics through direct instruction for a couple of years and was dissatisfied with the limited progress that my students were making in mathematics. While my students were able to do simple sums, they fared poorly on the state test that required them to solve novel problems and give explanations for how they had solved the problems. Consequently I wanted to experiment with another way to teach mathematics.

In my university, prospective teachers were being taught five routine instructional activities that encouraged students to interact with one another and communicate mathematical ideas and develop fluency with mathematical concepts and procedures (Lampert *et al*, 2010). Although I adopted most of these routines in my Resource Room class, in this paper I focus on one routine: Choral Counting. This decision led me to my research question: What do elementary students with language delays learn about identifying patterns as they engage in Choral Counting?

In the process of investigating this question I present the progression of numerical pattern identification of two first grade students with language delays over a three-month period. I present how they first thought of patterns and how they changed their conceptions of patterns as the months progressed. I present the differences between these two students in the types of patterns they tended to observe. I also present how they communicated their understandings of the mathematics. This paper expands current studies of mathematics learning because it examines a group of students who are rarely asked to communicate their mathematical ideas.

Conceptual Framework

A sociocultural perspective guides the view of learning in this study. From this perspective, learning is transformation of participation in a “skilled, valued sociocultural activity” (Rogoff, 1990, p. 39) and takes place within a social context with experts and novices engaging together in the activity. Participants have relationships with each other that mediate the individuals’ learning and this mediation occurs through discourse—communication that occurs through words, gestures, and actions (Cazden, 2001).

Moschkovich (2007) argued for using a sociocultural framework for understanding the resources that bilingual students use when communicating mathematically. These students struggled to find the words to describe the mathematical concepts that they had, and had Moschkovich looked at their words alone she would have concluded that they were not understanding the mathematical concepts. Instead she looked at the whole social context and realized that these students were using gestures, objects, artifacts, and their first language to express their mathematical ideas. When Moschkovich studied bilingual students’ participation in mathematics lessons from a sociocultural perspective, she could see all of the resources that they were using to construct meaning.

Like Moschkovich, I have chosen a sociocultural perspective to investigate the resources that students with language delays use when communicating mathematically, because this perspective allows me to examine their strengths. Students with language delays struggle to express their ideas verbally, but an interaction analysis taking a sociocultural perspective enables me to investigate other resources that they use to express their mathematical ideas. Students may use gestures and artifacts to communicate mathematical ideas that they are unable to express verbally. Such analyses afford a fuller estimation of the ability of students with language delays to do mathematics than a limited examination of students' verbal functioning.

Students with special needs do not often get the chance to talk about mathematics because "such children are assumed not to be linguistically prepared to participate in reform-based practices" (Hufferd-Ackles, Fuson, & Sherin, 2004, p. 82). However, students with language delays may actually need more opportunities to participate in mathematics discussions than typically developing students. They may need more practice communicating mathematically, just as they need more practice communicating in other modes. This means that mathematics lessons should be designed to support students' language goals as well as the mathematical content. These language goals will be more readily addressed with mathematical discussions than by direct instruction.

Methods

Subjects

The participants in this study were two first grade students receiving Special Education Resource Room services at Uhuru School (all names are pseudonyms) in the 2008-2009 school year. Their school was in an urban area in the Northwest U.S.A.

Martin (M) and Ali (A) were both members of my primary Resource Room mathematics group. They both had language delays (LD), with expressive language skills more typical of a 4 year-old than the 6 years-old they actually were. They tended to use 2-3 word sentences, often missing out grammatical features of the sentences such as verbs. Martin spoke more clearly than Ali, who tended to skip some of the consonants in words, which made his speech hard to understand. Despite these language delays, they both spent the majority of their school day in a general education first grade classroom and were making progress in the general education curriculum.

I chose to study Martin and Ali because at the beginning of the study they both had Individual Education Plan (IEP) goals related to counting, so the Choral Counting routine was directly related to their IEP goals. In September 2008, they both had mathematics goals on their IEPs that referred to rote-counting, one-to-one counting of objects, and recognizing numerals. Although these students had similar mathematics goals, there were differences in their goals. Martin's goals involved higher numbers than Ali's—reflecting their relative skill levels. In September, Ali still had difficulties counting above 12 by ones, whereas Martin could count to 100 by ones. Ali could reliably count ten objects, whereas Martin could count up to thirty objects. Ali recognized 5 numerals reliably--1, 2, 3, 8, 10—at a stage when Martin could recognize the majority of the numbers up to 50.

Martin and Ali had different home languages. Martin's family spoke English in the home. Ali's family spoke Somali at home, although his mother reported that he was more comfortable speaking English than Somali. Ali was receiving ELL services as well as Special Education services.

Procedures

In my primary mathematics class we engaged in Choral Counting approximately weekly from November 2008, until March 2009. We then continued to engage in Choral Counts once or twice a month from April 2009, until June 2009.

First, I chose a counting sequence for the students. These counting sequences were either by twos, fives, tens, ones, or backwards by ones. When counting by ones, the count started from a number in the low double digits because the students were very familiar with counting by ones from one. These counting sequences were selected because they are the ones that the state has identified as the essential counting sequences for first grade students. The *Washington k-12 Mathematics Standards* (2008) state that first grade students should be able to, “Count by ones forward and backward from 1 to 120, starting at any number, and count by twos, fives, and tens to 100” (p. 11).

Once I had introduced the counting sequence to the students, we would start counting together and I would write the numbers as we said them, having chosen ahead of time how to arrange the count on the board so that patterns would be discernable. After three or more rows or columns of numbers were written, I would stop the count and ask the students what patterns they were seeing. I would represent those patterns on the count and ask the students to predict the next number in the pattern (Kazemi and Hintz, pp. 5-9).

Data Collection & Analysis

The data for this study was collected by videotaping thirteen of my Choral Count lessons from November 2008 until June 2009. After watching all the videotapes, I chose to analyze three episodes that showed how Ali and Martin changed their understandings of what a pattern consisted of across the course of the study. I analyzed a count from mid-December, which was towards the beginning of my study when the students still believed that patterns consisted of shapes. The next count I analyzed came from mid-January, where both boys showed an understanding and recognition of patterns in the numbers. The final count that I analyzed came from early March, where Martin recognized a more sophisticated pattern than he had previously recognized. The December count was a count by 2s, the January count was a count by 10s, and the March count was a count by 5s.

I then transcribed the three episodes, just focusing on the parts of the counts where Martin and Ali were talking about the patterns that they had observed. I transcribed these episodes using conventions commonly used in interaction analysis (Schegloff, 1997). I wanted to pay close attention to both the words and actions of the discussion participants, because the words, hesitations, gestures, and the physical location of the participants were important clues to what the students understood about patterns and how confident they were about their answers.

I then repeatedly reviewed the video alongside the transcript and coded for instances of student learning and understandings, and students’ interactions with the artifacts. I annotated the transcript with notes about how the students were currently thinking about patterns, their positioning in relationship to the numbers on the board, and their interactions with the numbers.

Results

Growth in Pattern Identification

In this section I show a sequence for the growth in the types of patterns recognized: (1) shapes as patterns; (2) patterns where the digits stay the same; (3) digits that change systematically; and, 4) digits that show repetition while changing systematically. I also show how the boys established a norm for communicating about the numbers.

Shapes as Patterns

In the December episode, Martin thought that when I asked him to tell me about a pattern he saw, he was expected to respond with the name of a shape:

T: What’s a pattern you see Martin?

M: Umm. A square right there.

This was a reasonable answer if I had been looking for a shape answer. The count at that moment was a five by six grid plus two extra numbers (See Figure 1).

Figure 1: *Count by 2s, December*

2	4	6	8	10
12	14	16	18	20
22	24	26	28	30
32	34	36	38	40
42	44	46	48	50
52	54	56	58	60
62	64			

Martin's pattern
Ali's pattern

In Martin's world, patterns were shapes so even though there were no obvious other shapes to be seen in the count, he continued to look for shapes:

M: Ooh! A triangle

T: I don't see a triangle

M: Circle

A: I see it

M: Rectangle

These answers were less reasonable given the count, but emphasized his conviction that questions about patterns had a shape as an answer.

Patterns where Numbers Stay the Same

This was the second time that I had introduced these students to the idea of looking for patterns in the numbers, but Martin's responses made it clear that he was still focused on looking for shape patterns. So I reminded the students about the previous lesson where I had modeled finding a pattern in the numbers, "I'm seeing patterns in the numbers. Like last time we saw patterns in the numbers, Jose and I?"

Ali was the first to give me a number response, but it is unclear from his initial response whether he was just saying a number or he had seen a pattern,

A: Uh. I see fourteen?

T: Fourteen? Yeah? [Points to the number 14 on the board] What about fourteen?

A: Uh. Ecos e ba one an four {Because it has a one and a four.}

I decided to take his answer and emphasize a pattern that he could be seeing that involved 14,

T: What's the same about the one's place [Drops finger down along the 4s in the ones place] all down here? What's the same? [Moves finger up and down along the 4s.]

M: The four

A: four [Says number at the same time as M.]

At this point both boys were focused on looking at the numbers and they simultaneously responded that the 4s stayed the same in that column (See Figure 1).

Martin was able to take the idea of looking for digits that stayed the same and communicate a new pattern independently,

M: I see um zeroes

T: Where. Tell me where. [Martin gets up.] Tell me.

M: Uhhh. Right here [MF goes to the board and points to the zeroes in the final column. As he counts he points to each number.] Ten. Twenty. Thirty. Forty. Fifty.

He now saw that the final column always had a zero at the end (See Figure 1). In a minute and a half, Martin went from seeing patterns as always being about shapes to being able to communicate a pattern that he saw in the numbers.

In this interaction Martin initiated a norm for communicating about the numbers. I asked him to tell me which numbers he was talking about and instead he chose to go to the board and show me the numbers. This innovation was taken up by the other student and by January it had become a norm for the students to come to the board when they were trying to explain which numbers they were talking about.

In December it was not clear whether Ali really understood what it meant to look for a pattern in the numbers, but by the January episode he showed that he was looking for similarities between the numbers and he communicated this similarity. He saw that there was always a 10 in the numbers in the first row, 10, 110, 210 (See Figure 2).

A: Uh, a tens [comes to board, points to the first 10] got a number one zero, one one zero [points to 110.]

T: OK. So you've seen ten, ten, ten, ten. [T underlines 10 and the ten in 110, in 210, and 310. A walks back to his desk.]

In the January episode, Ali showed growth in pattern recognition from December. He now knew that he was meant to look for patterns in the numbers. He communicated that he saw a similarity between the numbers in the first row. The words that he said, "Uh, a tens…got a number one zero, one one zero," could have been easily misconstrued but because he came up to the board and pointed at the relevant numbers, I knew exactly which tens he was talking about. This growth in his communication skills allowed me to comprehend the pattern that he was indicating. The pattern that he saw in January was more sophisticated than the one he saw in December because this one involved two digits that stayed the same rather than just one digit.

Figure 2: *Count by 10s, January*

10	110	210	310	
20	120	220	320	
30	130	230	330	
40	140	240		
50	150	250		
60	160	260		
70	170	270		
80	180	280		
90	190	290		Martin's pattern
100	200	300		Ali's pattern

Patterns where Numbers Change Systematically

In the January episode, Martin also shows that he is now focusing on patterns in the numbers rather than shapes. He shows a pattern in the tens place where the tens place systematically changes by one (See Figure 2),

M: I see 2, 3, 4, 5, 6, 7, 8, 9, [points to each number in the tens column as he says it.] te...

A: Ten

M: Ten

As I did not ask him to continue his pattern in the second column, it is unclear whether he saw the pattern in the tens place as a repeating pattern: 123456789123456789123456789; or as a pattern that incorporates the hundreds place into the pattern he was seeing in the tens: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17. I suspect that this latter pattern is the way that he saw his pattern because of his use of the word "ten". In later counts, he consistently used this latter pattern, which strengthens my conviction that this is how he was seeing the pattern in this episode. However, his hesitation at "te…" suggests that he was unsure at this point and was not sure how his pattern continued.

In January, Martin shows that his understanding of patterns has evolved from seeing patterns as shapes, and even from looking for similarities between numbers. He is now looking for patterns in how numbers change systematically.

Patterns with Repetition within Systematic Change

In March, Martin identifies a new pattern. He sees a pattern of the form: 11223344, etc. (See Figure 3),

M: 1, 1, 2, 2, 3, 3, 4, 4, 5, 5 [M points to every number in the tens place as he comes to it.]
T: Ok. Let...let me. So in the tens place it goes 1, 1. Then it goes 2, 2 [Circles the numbers as she says them.]
This pattern shows that Martin has grown in his understanding of what a pattern can be. He can now see repetition within systematic growth.

Figure 3: *Count by 5s, March*.

5	10	15	20	25	30	35	40	45	50
55	60	65	70	75	80	85	90	95	100
105	110	115	120	125	130	135	140	145	150
155	160	165	170	175	180	185	190	195	200
205	210	215	220	225	230	235	240	245	250
255	260	265	270	275	280	285	290	295	300
305	310								

Martin's pattern

When Martin got to the number 95, he hesitated and so I asked him what came next:
T: Wha...What happens here?
M: Zero.
T: So does it go back to zero or does it go on to ten.
M: Goes on to ten right here. [Pointing to 10 in 100.]
T: OK. [Circles the number he indicates.]
M: Don't circle that one [Points to 0 in ones place.]
T: Ok. Ten
M: Ten [Points to number.]
T: Mm
M: Eleven [Points at the number.]
T: Eleven
M: Eleven [Points at the number.] Twelve...[Points towards number but does not touch it.] Twelve...Thirteen, thirteen, fourteen, fourteen, fifteen, fifteen, sixteen, sixteen, seventeen, seventeen, eighteen

Martin's hesitation after 95 showed that he was not sure how his pattern progressed once the numbers were in the hundreds, but he eventually decided that his pattern incorporated the hundreds place into the pattern he was seeing in the tens.

Discussion

Choral Counting was a valuable mathematical routine for these students with language delays. It helped them become more adept at counting by various numbers, including 3-digit numbers. Choral Counting was a useful routine to help these students with language delays grow in their ability to identify patterns—an important mathematical skill. It exposed them to the idea of using patterns as a way to help them skip count by various numbers. Choral Counting allowed the students to express their mathematical ideas and helped them become used to using nonverbal communication methods to clarify their verbal explanations. From a sociocultural perspective, these students learned these mathematical skills because they had the opportunity to engage in Choral Counting, so their ideas shaped and were shaped by their participation in the activity.

In three months, both students made growth in their ability to identify patterns. They moved from looking for shapes as patterns to looking for patterns in the numbers. Ali independently identified patterns where the digits stay the same. Martin could identify several different types of patterns in the numbers: digits that stay the same, digits that change systematically, and digits that show repetition while changing systematically. The routine of Choral Counting has encouraged these students to search for patterns in numbers, which is one of the major tasks of mathematicians.

These boys also showed growth in their communication skills over the course of the study. They both had difficulties communicating verbally due to their disabilities, but they found other ways to communicate what they were thinking. They established a norm that the student who had the floor would come up to the board to point at, and thus clarify, the numbers they were talking about.

One of the limitations of this study was that the two roles that I had in this study—teacher and researcher—were difficult to negotiate. Did I really see that evidence of understanding in the video? Or did I see evidence in some other part of the lesson that is not recorded and then read into the video evidence that is not there? Did I see some glimmer of understanding because I am their teacher and am invested in these children's lives, while an outside observer might not see the same evidence of understanding?

In future studies it would be interesting to compare the participation of students with language delays in activities that elicit their mathematical thinking, such as Choral Counting, with their participation in other activities that are typically found in mathematics lessons.

In future studies it would be interesting to work with groups of students with different disabilities. For which groups of students is Choral Counting a worthwhile activity?

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NEGOTIATING PRAXIS FOR EQUITY IN MATHEMATICS EDUCATION: HOW PD SUPPORTS TEACHERS' DEVELOPING CONCEPTIONS

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This study contributes to a growing body of research considering professional development focused on both mathematics and equity. The study examined how lived experiences provide a foundation for ways teachers take up equity in their mathematics teaching practice. Constructs of praxis and figured worlds frame the study. The figured worlds of teachers participating in professional development were explored with a focus on how they contributed to where the teachers located praxis for equity in mathematics. Contributions were noted from the figured worlds the teachers inhabited both in and out of the professional development.

Attention to issues of equity and diversity within mathematics education is an area of growing interest. For some, this interest grows out of an effort to understand the gap in achievement between historically underserved students and their White, middle-class peers (Lubienski, 2008). For others, a focus on the achievement gap masks circumstances at the local level (e.g., access to courses and experienced teachers, procedurally-focused instruction) that significantly impact school performance (DiME, 2007; Gutiérrez, 2008; Martin, 2006). While there is a rich body of research on professional development (PD) that focuses on attending to student thinking in mathematics (Franke, et al., 2009; Kazemi & Franke, 2004; van Es & Sherin, 2010) and a separate but equally fertile body of literature on PD that address general issues in attending to diverse learners (Cochran-Smith & Lytle, 2001; Sleeter, 1997), literature that attends to both is emerging. In this paper we report on a study that contributes to this growing body of literature that considers PD focused on both mathematics and equity (See for example Civil, 2002; Crockett & Buckley, 2009; Foote, 2010; Weissglass, 1994).

Contributing to a lack of equity in schools, is a socio-cultural distance that has been noted between the majority of teachers, over 85% of whom are White and middle class (Howard, 1999), and the ever more diverse groups of students whom they teach. PD provides a vehicle for researchers to examine this gap in lived experiences between teacher and student (Foote, 2010). Such a PD program was designed to provide elementary school teachers with an opportunity to examine their mathematics classrooms with an explicit focus on equity.

Theoretical Perspectives

Research suggests that high quality professional development: (a) is situated in the work of teaching, (b) is focused on student thinking and learning, (c) supports professional learning communities, (d) is sustainable and credible, and (e) is research based and supported (Whitcomb, Borko, & Liston, 2009). The PD we studied not only incorporated all of these elements, but also incorporated another that we think needs more attention in the PD research - how teachers' experiences in figured worlds shape how they take up the goals of PD and where they locate the praxis that the PD intends to support.

Praxis

In discussing actions teachers either took or envisioned in an effort to achieve greater equity in mathematics learning we are referring to praxis. The term praxis regularly appears in the discourse on PD without being specifically defined. In our work we draw on the definition used in critical race theory (CRT), "the connection between theory and practical work aimed at transforming concrete social institutions" (Delgado & Stefancic, 2000, p. 591). This definition goes beyond the Aristotelian view of praxis that considers human activity as directed at a particular end with a goal of realizing something worthwhile (Carr, 1995) and incorporates Freire's (2007) view of action for transformation. In our use of the CRT perspective, the "goal of realizing something worthwhile" is more specifically defined as working toward achieving equity. Praxis is not the end itself nor a product but action that leads to the goal, the process of putting theoretical knowledge into practice. Potential sites for praxis that a teacher might target include the classroom, school, district, and society. These sites exist in a nested configuration of classrooms within schools, within districts, within society, although we do not intend to suggest a linear interaction. Additionally, we view society as incorporating but not limited to such things as families and communities.

Experiences in the Contexts of Figured Worlds

As suggested by Gutiérrez (2002), teachers' practices are not just limited to their knowledge, beliefs, and experiences, but are influenced by the context in which they teach. In addition to the teaching context, the context of PD (and other contexts outside of the school setting) influences teachers' views on changing their practice (Borko, 2004; Kazemi & Franke, 2004). Identity is developed through participation in and interaction with this complex matrix of communities (contexts) with which we engage throughout our lives (Holland, Lachiotte, Skinner, & Cains, 1998). Holland and colleagues argue that identity develops within what they call figured worlds. For them, identity is situated (or contextual) and socially dependent (or relational) and therefore cannot be appreciated apart from its social context. This notion of figured worlds and the movement of teachers among the various worlds they inhabit is a useful one for considering the development of their identities as mathematics teachers and understanding why they identify particular places as needing change. To understand all the contributions to identity or to develop a complete explanation of the development of any individuals' identity is well beyond the scope of this paper. We can, however, consider how particular lived experiences in the figured worlds they have and continue to inhabit may contribute to the ways in which teachers take up certain issues, in this case equity in mathematics.

From among the figured worlds that participants inhabit and because of their experiences within these worlds, they identify the space in which they hope to take action or believe that action is necessary; in other words, where praxis should be located. This leads us to question how teachers' experiences in different contexts influence how and where they locate praxis regarding equity and the teaching and learning of mathematics?

Methods

Background of the PD and Participants

A yearlong PD seminar was developed to support and study teachers as they considered issues of equity in their teaching of elementary mathematics. In considering ways to weave together discussions of equity and mathematics, the researchers drew on scholarship in algebraic thinking (Carpenter, Franke, & Levi, 2003), PD linking equity and mathematics (Weissglass, 1994), multicultural PD design (Sleeter, 1997), and multicultural education (Nieto, 2004). During each seminar session, teachers discussed one chapter from *Affirming Diversity* (Nieto, 2004) and observed one video text of third graders working on algebraic thinking problems. Between seminar meetings each teacher considered a target student in her classroom from the perspective of two lenses: (a) algebraic thinking, and (b) how issues of equity and diversity might be impacting that child's mathematics performance in the classroom.

The PD met once per month during the school year for a total of nine, two-hour meetings. Participants were randomly assigned to one of four subgroups that were purposefully maintained throughout the year in hopes that participants could build community with a smaller group of teachers, something that we believed might be an important consideration during difficult discussions that can arise when equity issues such as race and poverty are raised.

The participants in the PD were 18 elementary school teachers from a single school district in a moderately sized Midwestern city. Participants included 10 classroom teachers, one substitute, two ESL teachers, two math resource teachers, and three district resource staff. From among these participants, we identified one case study teacher from each of the subgroups. Because of space constraints, in this paper we will consider one of our case study participants, using her as an exemplar of how past experiences, current experiences, and the PD itself contributed to where she located praxis

Data Generation

In order to support teachers in positioning themselves in terms of their experiences and contextual positions and to develop a deeper understanding of who the teachers were and how they might engage with PD (Connelly & Clandinin, 1988), teachers wrote a mathematical and diversity autobiography. These autobiographies revealed prior experiences and beliefs regarding the teaching and learning of mathematics on one hand and issues of equity on the other. In order to capture the nature of the conversations within each group, all group discussions were audio recorded and later transcribed for analysis. Each month, teachers wrote reflections about their target student and their own experiences in their school setting. Through this use of both written and verbal perspectives, we were able to triangulate the data.

Data Analysis

There were two phases of data analysis. First we examined the small group discussions to develop an understanding of the group contexts. Second we developed trajectories of individual teachers around their engagement in the figured worlds that they had and currently do inhabit. In analyzing the small group discussions, we drew on grounded theory (Charmaz, 2005; Glazer & Strauss, 1967) to identify and develop an understanding of the group contexts as they emerged and influenced our participants. Members of the research team met regularly to discuss their impressions of the sessions as a whole and the small group discussions in particular. This sharing of impressions helped inform a general view of the small groups which was then further refined based on analysis of the transcripts. After identifying the overall orientation of the different groups, the data sources were analyzed to develop an understanding of the initial positioning and trajectories of the case study teachers. Autobiographies were read to gain insight into the experiences and figured worlds contributing to teachers' identities as they entered the PD. The transcripts for the monthly meeting of each group were read and re-read; emergent themes were noted. Teacher reflections for each month were read and coded as to whether the teacher discussed the Nieto (2004) text, the video text, her target student, or her teaching practice more generally. Using these data, narratives for each of the case study participants were developed independently by each of the authors and then compared. These narratives provided both background on the initial positioning of the case study teacher and insight into her trajectory.

Results: Case Study of Reflection on Praxis

We present our results by focusing on our participant Deb. We first discuss the influence of experiences in her figured worlds on the where she started her thinking in the PD. We then turn to considerations of the individual PD group to which Deb belonged. Finally, we discuss the influence of identity and group dynamics on how she developed and where she located praxis.

Deb

Deb had a middle-class, suburban upbringing, but taught in two large urban districts for 16 years before moving to her present school. She completed a traditional teacher education program in a large city and became a classroom teacher in a school that was 100% African American and all of whose student body (with the exception of three students) lived in nearby public housing. She was horrified by the conditions at the school and the treatment of the students. This fueled an early interest in producing large scale change; Deb recognized early on that major changes were required well beyond her classroom. Deb brought some of this zeal with her to teaching in another urban district, but after the events of September 11, 2001, reconsidered life in a large city and eventually moved to the smaller city where the study takes place. At the time of the study, Deb was a school-based mathematics resource teacher in a school that was experiencing a rapid shift in student demographics. It was her first year teaching in the school and she was frustrated with many of the other teachers in the school who she saw as wanting the new, more diverse students to assimilate before they could be taught. She saw her target student as marginalized by his classroom teacher who dismissed the knowledge he brought to school. Deb was very conscious of what she perceived as racist attitudes in her school and in her target student's classroom. She also likened the problems she read about in a chapter in *Affirming Diversity* (Nieto, 2004) to those she saw with district sponsored PD on racism. Although she was aware she had more experience than most of her colleagues in teaching children from diverse backgrounds, Deb was also conscious of jumping to conclusions about her colleagues.

Despite the fact that teachers had been randomly assigned to subgroups, four of the five teachers in Deb's group were resource teachers either for mathematics or English as a Second Language (ESL) and all had extensive multicultural experience. All these teachers considered themselves and each other to be strong multicultural educators. Feeling that they had already explored their own selves and responsibilities, they looked instead at their schools as sites where improvement was necessary. Because of this orientation, there was limited space within the group for interrogating individual views; they adopted a taken-as-shared orientation in which they agreed that teaching mathematics for understanding was an important pedagogical stance and that valuing children's out-of-school experiences was important to student success. Together they were empowered to criticize other teachers at their schools who they felt (a) did not share their orientation toward mathematical teaching and learning and (b) did not appreciate the funds of knowledge (e.g., linguistic and cultural) that diverse students were bringing to the classroom.

Deb joined her colleagues in agreeing about what other teachers did wrong. She used her journal to consider ways she could help others change. The low expectations teachers at her school had for her target student and others like him worried her, and although she judged the teachers' views of the students, Deb was also aware of her own judgmental stance, "Maybe I am the judgmental one who needs an attitude adjustment. I will watch myself." (Oct. reflection). Toward the middle of the year, as she continued to experience resistance from the classroom teacher of her target student to reconceptualizing the student's abilities, her viewpoint shifted away from working with that teacher. Instead, she began to intervene directly with the target student to support him academically. Yet, in another instance, toward end of the year Deb began to question how, in her position as mathematics resource teacher, she could support teacher development when her estimation of what was necessary for teacher development went beyond mathematics and included understanding students.

> How can we move teachers toward understanding of the concept of "mutual accommodation" and the mindset of "academic success with cultural integrity?" How can we help her to understand what accommodation means, and show her that it is [currently] her students who are unlike her who have doing the accommodating? Can this understanding be taught to a veteran teacher who up until a few years ago worked only with students from similar backgrounds to her own? (May reflection)

Deb located her praxis within her school. Deb followed a patterned trajectory throughout the year from questioning and evaluation of a situation to developing a plan of action. She located the need for change in other teachers at her school and questioned how she could support change in the way teachers taught mathematics and in how they worked with diverse populations of students. Deb used Nieto (2004) to reify her beliefs about the teachers in her school. By April she wondered what her role could be if others had deeply held beliefs that must be changed first, both with regard to mathematics and with regard to equity. Yet, she defined places where she could take action instead of merely identifying others as in need of change. She developed plans for the following school year which included both (a) an immersion in mathematical problem solving for teachers at school-based PDs she would be planning, and (b) approaching the principal with ideas gleaned from the Nieto (2004) text and the PD more generally as to how to address teacher expectations of the new population of students entering the school.

Discussion

We return to our initial question about the ways in which teachers' experiences in different contexts influence how they locate praxis regarding equity and the teaching and learning of mathematics. In discussing the influence of the figured worlds, we look at Deb's experiences outside of the PD and how that shaped her initial positioning. We then draw attention to Deb's experiences within her small group where our discussion centers on what Horn (2007) refers to as the "conceptual resources" teachers bring in terms of their beliefs and how individual beliefs can influence the shared belief. Finally, we explore how these influences converged in the location of of her praxis.

Influence of Membership in Particular Figured Worlds

Deb brought a particular history to her participation in the PD. These varying experiences influenced the ways in which she took up the ideas of the seminar and brought them to bear on her practice. As a result of years of living and teaching in poor sections of major cities, Deb positioned herself as an expert on equity issues within her school setting and one who was in a position to support teachers with less multicultural experience. In addition she noticed and was offended by the teachers who she perceived were dismissive of what students (particularly her target student) brought to school and who looked at the home and community environments of their students from a deficit perspective.

Influence of Context

As mentioned earlier, there are multiple contexts that influence teachers' work. In looking specifically at the PD, this study revealed the impact of group context on engagement and participation. Deb, along with other members of her group, often engaged in a self-congratulatory discourse during the discussions in the PD, not using it as a place to examine closely their own practice, but as a place to criticize other teachers at their schools. Nonetheless, Deb used her journal as a vehicle to reflect on actions that she might take. This seems to have supported her in moving toward an action plan of her own within the context of her school.

Locating Praxis

Deb's focus was on the teachers at the school level. This is understandable given that her job was that of school-based mathematics resource teacher. She was explicitly interested in supporting teachers in reaching those students who were unlike the teacher in background and/or culture. She was further interested in supporting teachers in appreciating the knowledge and strengths that children had and brought with them to school and that the teachers often dismissed. We see specific examples of this in her dismay at the ways in which the classroom teacher of her target student marginalized him within the classroom and disregarded strengths in mathematical thinking and understanding he exhibited. Deb's discouragement at making headway with this teacher caused her to abandon her efforts to support the teacher. Instead she began to work directly with the student, supporting him in developing his mathematical understandings. Yet, ultimately Deb took action at the school level, the site she identified as in need of change.

Deb turned to working with her target student when efforts to support his classroom teacher in doing so failed. Yet Deb also continued with action plans at the school level on two fronts. On one hand she considered how during the next school year she would engage teachers in school-based PD that she was planning on children's mathematical thinking. On the other hand she thought of how she could interest the principal in exploring with the faculty in the subsequent year, some of the activities in *Affirming Diversity* (Nieto, 2004).

Conclusions and Implications

This study revealed how one teacher's experiences in her figured worlds contributed to the ways she engaged with PD and where she located praxis. This is a critical finding in that PD goals may not be attained unless attention is given to teachers' experiences both in and out of PD. Specifically in this study, the initial goal of the PD was to support teachers as they identified ways in which they could change their own practice to teach mathematics with an explicit attention to equity (i.e. locating praxis within themselves; taking action in their own teaching practice). Not all of the teachers responded to the PD this way. Rather, their experiences in their figured worlds positioned them to identify other sites for praxis (i.e. locating praxis in other teachers, schools, or society). Had we been more attuned to this, as PD facilitators we could have been more responsive to where teachers were locating the need for their praxis. For example, Deb saw the need for change in others yet she took action to support them in doing so; thus her own praxis supported the praxis of others. Had our initial PD design included a space for those teachers, such as Deb, whose experiences in their figured worlds support a more active role in working with others, we could have explicitly attended to that action. In the end, we conclude that it is critical to consider the role of teachers' figured worlds on teachers' interaction with PD.

Because of the structure of this PD we had the opportunity to notice how the context of the PD was different for each group (see Wager & Foote, 2011 for a more extensive discussion of this point than this paper allows). We were able to conclude that the context of any PD is critical in supporting teachers toward action. Our initial, more limited view of the PD assumed that the structure would support teachers in thinking about praxis in their classrooms. A consideration of the teachers' experiences and the constraints or supports offered by the differing dynamics of the small groups in the PD has revealed that a broader view of PD was called for. These two topics, experiences and PD context, shaped where teachers located praxis. We suggest that this study reveals an important consideration in future PD and studies of PD.

PD to support teaching mathematics with understanding is grounded in the premise that teachers first identify the strategies that their students bring and build on that understanding. In PD on equity and mathematics, this notion is further extended to consider the funds of knowledge students bring as well. Ironically, PD on equity and mathematics does not necessarily identify the strategies and funds of knowledge that the teachers bring. As discussed earlier, scholars have identified elements of PD that have proven successful (Whitcomb, Borko, & Liston, 2009). We argue that this vision for PD should be extended so that facilitators take into consideration the experiences within figured worlds of their teachers as well as the context of the PD itself. This consideration is necessary if the goals of PD are to be met. The questions facilitators should explore include: (a) how do the structures of the PD align with teachers' experiences and (b) what opportunities do teachers' experiences offer to extend the initial goals of the PD? In examining the structure of PD relative to teachers' experiences, facilitators might first assess teachers' understandings and then determine the best organization for small group and whole group discussions. Reflecting on our PD, had we redistributed the small groups in order to take advantage of varying expertise, there may have been more even distribution of participation. In closing, we suggest that linear views of PD, in which the facilitators' goals drive the process, underestimate the learning that might actually occur.

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A MULTI-STATE COMPARISON OF TEACHERS' BELIEFS ABOUT TEACHING ENGLISH LANGUAGE LEARNERS MATHEMATICS

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What mathematics teachers believe about teaching ELLs affects which mathematics teaching practices they employ (Sztajn, 2003). This study compares beliefs of mathematics teachers from Illinois, Texas, and Wisconsin. Surveys and quantitative analysis methods are used to compare beliefs of teachers by district demographics (high ELL population vs. low ELL population) and region (Midwestern school districts vs. Southwestern school districts). It was found that teachers' beliefs about teaching ELLs mathematics were mostly similar regardless of comparison made. One main difference was that districts with low ELL populations looked for outside help to address challenges in teaching ELLs mathematics while high population ELL school districts relied more on their own teachers' expertise to address challenges.

Since teachers provide ELL students access to mathematics, it is essential to understand how teachers implement curricula with these students. Research studies suggest the manner in which teachers interpret and implement curricula is influenced by their knowledge and beliefs (see Thompson, 1992). Thompson (1984, 1992) reports that teachers frequently treat their beliefs as knowledge. Sztajn (2003) noted that teachers' practices were based on their conception of students' needs. A teacher may implement problem solving with students from upper socioeconomic backgrounds, while using repetitive instruction and memorization with students from lower socioeconomic backgrounds. This variation in teaching strategies and activities based on teachers' perceptions is important, because teachers may perceive that ELL students would be more successful with memorization and lecture when the opposite is true (Winsor, 2007). Creating innovative, high quality lessons for students is most likely to occur when teachers are part of a collaborative learning community supporting such experiences (McLaughlin, 1993).

In order to be equitable, it is essential that ELLs receive a high quality mathematical education (NCTM, 2000) therefore, the present study focuses on mathematics teachers' beliefs about teaching ELLs in three states, Illinois, Wisconsin, and Texas (González, 2009). The comparison of different regions allowed the researchers to compare beliefs between teachers who teach in highly and sparsely Latino-populated areas. The hypothesis of the study was that teachers in regions with large Latino populations would hold similar beliefs. It was also hypothesized that teachers in regions with small Latino populations would hold beliefs that were quite different than the regions with relatively large Latino populations.

Literature Review

Teachers' practice affects ELL students' success in the classroom (Boaler, 2002). Two factors that affect teacher practice are their attitudes towards teaching ELLs and knowledge and beliefs that motivate attitudes towards teaching ELLs (e.g. Batt, 2008; Byrnes, Kiger, & Manning, 1997).

Boaler (2002) states that teachers do not hold malicious beliefs toward teaching ELL students; they just have misconceptions about how ELL students learn. One such misconception is that teachers believe ELL students' linguistic backgrounds keep them from participating in activities requiring a higher level of cognitive demand, such as problem solving (Hansen-Thomas & Cavagnetto, 2010; Sztajn, 2003). Hansen-Thomas and Cavagnetto (2010) found that teachers from all academic fields believe mathematics is the easiest course for ELLs to study because it uses only numbers and symbols. Reeves (2006) reports that teachers believe it takes two years or less to learn English, which leads them to believe ELL students' academic difficulties are due to ability and not fluency. Cummins (1999) proposed that in fact it takes ELLs four to seven years to gain sufficient fluency that allows them to negotiate academic situations.

There are two other teacher beliefs that seem to be systemic challenges. First, teachers feel unprepared to teach ELLs (Batt, 2008; Reeves, 2006), which often makes them reticent to teach such students; however, the vast majority want to learn effective teaching techniques to use with ELLs (Hansen-Thomas & Cavagnetto, 2010). Teachers also face added responsibilities that come with teaching ELLs (Batt, 2008), such as paper work and additional meetings outside of class.

Byrnes, Kiger, and Manning (1997) and Youngs and Youngs (2001) describe several factors that affect teachers' attitudes towards teaching ELL students. Both studies found that training specific to linguistic diversity had a positive impact on attitudes towards teaching ELLs. Training such as taking a foreign language class or a multicultural class (Youngs & Youngs, 2001) are examples of training that focused on linguistic diversity. Moreover, teachers that had ELL specific training in an in-service setting had positive attitudes towards teaching ELLs.

Another factor that had a positive effect on attitudes towards teaching ELLs was more direct experiences with linguistic diversity. Experiences such as living or teaching in a foreign country are examples of direct experiences with linguistic diversity (Youngs & Youngs, 2001). Byrnes, Kiger and Manning (1997) found that the region teachers live in affected their attitudes towards teaching ELLs. Youngs and Youngs (2001) found that teachers who had students from multiple language backgrounds were more positive towards teaching ELLs. Taken together, these studies suggest that if teachers have experiences that help them better understand and relate to ELL students, then they may be more likely to have positive attitudes towards teaching ELLs.

Methods

Research Hypotheses

1) Teachers from a district sparsely populated with Latino/ELL students will have different beliefs about teaching ELLs mathematics than teachers from districts densely populated with Latino/ELL students.

2) Teachers from Midwestern districts densely populated with Latino/ELL students will have similar beliefs about teaching ELLs mathematics to teachers from a southwestern district densely populated with Latino/ELL students.

Research Participants and Settings

Eighty-six secondary mathematics teachers from school districts in Illinois, Texas and Wisconsin participated in this study. Participants were recruited from school districts meeting criteria for (high vs. low) proportion of ELLs and that had established working relationships with the researchers. The four school districts will be referred to as follows: a high-ELL urban school district in the Southwestern United States (HUs); a high-ELL urban school district in the Midwest (HUm); a high-ELL rural school district in the Midwest (HRm); and a low-ELL suburban school district in the Midwest (LSm).

School District	Study participant demographics	District population that is Latino	District population classified as ELL
HUs	76% Hispanic	92%	24%
HUm	79% Caucasian (total participants from the Midwest)	20% total population, individual schools range from 40 – 97% Latino	5 % total, individual schools range from 20% to 47% ELL
HRm		71%	52.%
LSm		5%	3%

Table 1: Description of participant and school district demographics.

Forty-two secondary mathematics teachers taught at HUs. HUs school district was chosen because it was recognized as an urban district that is succeeding in educating their students (see Kitchen, DePree, Celedón-Pattichis, & Brinkerhoff, 2006). Twenty-three participants taught at HUm and HRm school districts, which were chosen because they had similar student demographics as HUs but were located in the Midwest. Twenty-one participants came from LSm school district, chosen for comparison purposes because of the scarcity of Latino students in the district.

Instrument

Researchers generated questions for a survey that focused on teachers' beliefs about teaching ELLs mathematics. The foundation for the questions came from research conducted on teaching ELLs mathematics (see González, 2009 for details). At an authors' university, the survey was piloted with a class of preservice middle school mathematics teachers. The results from the pilot helped researchers to refine questions based on preliminary responses. Data for this study was collected using a paper-pencil or an online version of the survey, depending on the preference of the school administrators and convenience for their teachers.

The data analyzed for this report came from two different sections of the survey. One section of the survey asks, "How effective are the following strategies in helping ELLs succeed in school?" Examples of these strategies are: grouping students by language proficiency level, hiring more Bilingual Education Assistants, hiring more ESL or Bilingual Ed certified teachers, etc. Rating categories were: "Strong Positive Impact", Weak Positive Impact", "No Impact", "Weak Negative Impact", "Strong Negative Impact" or "N/A".

The other section asks teachers to share their level of agreement (on a seven-point Likert scale) on statements about teaching ELLs mathematics. Two examples of statements are, "I use the same teaching methods with English Language Learners (ELLs) as I do with Native English Speakers (NESs)" and "Technology can help ELLs learn mathematics."

Data Analysis

For each survey item, one-way ANOVAs were conducted in Minitab to evaluate the relationship between a secondary mathematics teacher's beliefs and each of the two independent variables in the research hypotheses: (1) teachers' geographic region (either Midwest or Southwest) and (2) the ELL proportion of the district's student population. The latter was categorized into two groups -- a district predominantly Caucasian (70%), and less than 3% ELLs, and districts with at least 20% ELLs. A Bonferroni correction was applied to the simultaneous tests to guard against Type I error.

Findings

The first research hypotheses that LSm School district teachers would have different beliefs than those of teachers from the HUs, HUm, and HRm school districts might be seen as true. There are several responses where LSm teachers' beliefs are significantly different than those of teachers in HUs, HUm, and HRm (see table 2). LSm teachers also gave responses that were not significantly different than the teachers in HUs, HUm, and HRm. An example of two questions where there was no significant difference are: a) ELLs can be taught to problem solve as well as Native English Speakers (NESs) can and b) ELLs have the skills and content knowledge to contribute to my class that NESs do. All teachers in the study held positive beliefs towards both of the previous statements.

The second null hypotheses that HUm and HRm school district teachers would hold similar beliefs to those of teachers from HUs school district might be seen as false. Teachers from HUm and HRm held several beliefs that were significantly different than the beliefs of teachers from HUs (See table 1). There are multiple questions from the survey where there were no significant differences between the responses of the school districts.

	Strategy teachers were asked to rate effectiveness of	Comparison	
1.	Group students by same language proficiency.		(b)
2.	Hire more Bilingual Education Assistants.	(a)	(b)
3.	Hire more ESL or Bilingual Ed certified teachers.	(a)	(b)
4.	Create an ESL consulting teachers position to help teachers in math.	(a)	(b)
5.	Use a different education model.	(a)	(b)
6.	Change the ESL curriculum.		(b)
7.	Create a sheltered English academy within the school for ELLs.		(b)
8.	ELLs can be taught to problem solve as NES can.		(b)
9.	I use a variety of teaching methods with ELLs as I do with NES.	(a)	
10.	I collaborate with my colleagues to plan lessons for my ELLs	(a)	
11.	Technology can help ELLs learn math.	(a)	
12.	I feel adequately trained to teach ELLs math	(a)	

Table 2: Statements with significant differences by comparison (a) LSm vs. HUs, HUm, & HRm (b) HUm & HRm vs. HUs.

Comparison by Districts' ELL Population

The first section's statements related to strategies in helping ELLs succeed in schools. Statements that had significant differences, with $p < 0.05$, between LSm teachers' responses and HUs, HUm, and HRm teachers' responses were 2, 3, 4, and 5. The responses were in a Likert-type scale, in which 0 meant strong negative impact; 1, weak negative impact; 2, no impact; 3, weak positive impact; and 4, strong positive impact. (see table 3)

Statement	$F(df_w,df_b)=F$	M (SD) of LSm	M (SD) of HUs, HUm, HRm	p-value
2	$F(1,69) = 18.92$	3.36 (0.63)	1.53 (1.54)	0.000
3	$F(1,69) = 33.09$	3.71 (0.47)	1.30 (1.55)	0.000
4	$F(1,68) = 21.66$	3.31 (0.63)	1.25 (1.57)	0.000
5	$F(1,58) = 5.70$	2.55 (0.69)	1.68 (1.62)	0.020

Table 3: Means, standard deviations and ANOVA results of the statements' of the two groups of districts classified by ELL population.

The second section of the survey used a scale from 1 (strongly disagree) to 7 (strongly agree), with 4 being neutral. Statements 9-12 had significant ($p < 0.05$) results (see table 4).

Statement	$F(df_w,df_b)=F$	M (SD) of LSm	M (SD) of HUs, HUm, HRm	p-value
9	F(1,77) = 9.00	5.00 (1.73)	6.00 (1.0)	0.004
10	F(1,76) = 4.90	2.64 (1.90)	3.89 (1.91)	0.030
11	F(1,79) = 5.72	5.44 (1.03)	6.08 (0.94)	0.019
12	F(1,80) = 6.25	3.00 (1.97)	4.22 (1.74)	0.014

Table 4: Means, standard deviations and ANOVA results of the statements of the two groups of districts classified by ELL population.

Comparison by Region

In the comparison between regions, all statements in the first section had significant ($p < 0.05$) results (see table 5).

Statement	$F(df_w,df_b)=F$	M (SD) of HUm, HRm	M (SD) of HUs	p-value
1	F(1,58) = 23.54	3.35 (0.99)	1.55 (1.50)	0.000
2	F(1,55) = 123.95	3.77 (0.80)	0.70 (0.88)	0.000
3	F(1,55) = 177.68	3.35 (0.86)	0.43 (0.71)	0.000
4	F(1,55) = 135.27	3.17 (1.10)	0.36 (0.71)	0.000
5	F(1,47) = 18.87	2.60 (1.06)	1.27 (0.96)	0.000
6	F(1,47) = 16.67	2.57 (1.22)	1.20 (1.00)	0.000
7	F(1,52) = 5.90	2.19 (1.38)	1.21 (1.34)	0.019

Table 5: Means, standard deviations and ANOVA results for the statements of the two groups of the two regions.

Only the statement #8, "ELLs can be taught to problem solve as NES can," had a significant difference, with $p = 0.37 < 0.05$.

Discussion

LSm vs. HUs, HUm, and HRm School Districts

The researchers hypothesized that LSm teachers would have different beliefs than the other school districts because LSm teachers and students were predominantly Caucasian. There are certain factors that promote a positive attitude towards teaching ELLs (Byrnes, Kiger, & Manning, 1997; Youngs & Youngs, 2001) and few of these are present in LSm.

For the most part, there was not a significant difference between the responses that LSm teachers provided and responses from teachers in the other three school districts. Responses to two particular questions are especially intriguing. The first question where teachers in LSm gave similar responses as teachers in the other three school districts is "ELLs can be taught to problem solve as well as NESs can." In both the literature (e.g. Hansen-Thomas & Cavagnetto, 2010; Sztajn, 2003) and in the researchers' experience working with teachers in in-service settings, mathematics teachers tend to believe that ELLs cannot engage in mathematics that require higher levels of cognitive demand. However, both LSm and the other three school districts had a positive response to the question.

The second question where LSm teachers held the same beliefs as the other three school districts was, "ELLs have the skills and content knowledge to contribute to my class that NESs do." All of the school districts had positive beliefs towards the prompt. Once again, this conflicts with the literature (Vollmer, 2000) and the researcher's experience with other mathematics teachers, school administrators, and counselors. It may be that because teachers at LSm have not had many opportunities to work with ELLs, the teachers are just hypothesizing that ELLs could contribute to their class.

One area where LSm teacher responses were significantly different than teachers' in the other school districts was mostly with the questions that addressed effective strategies that help ELLs succeed in school. The four questions where LSm teachers significantly differed from teachers in the other schools are #2-5 in table 1. In each of the four questions, LSm teachers believed that each strategy had a weak positive impact (mean $\approx$ 3) on helping ELLs succeed in school. Teachers in the other school districts believed that each strategy would have a weak negative impact (mean $\approx$ 1) on helping ELLs succeed in school. It seems that the teachers in LSm are looking for outside help with teaching ELLs. The other school districts seem to believe that they do not need outside help. Perhaps, since the HUs, HUm, and HRm teach more ELLs, their teachers believe that they are already successful and do not need external help.

There are two other questions where LSm teachers differ significantly from teachers in HUs, HUm, and HRm school districts. The first is "I collaborate with my colleagues to plan lessons for my ELLs." It is not so much that the school districts' responses to the question differ significantly that is interesting; it is that teachers' responses had means that fell in the disagree range (2.6 and 3.8 respectively, where 4 is neutral and 1 is strongly disagree). It is unsettling that all teachers in this study generally choose not to collaborate in planning lessons for ELLs. In order to ensure students, including ELLs, receive innovative, high quality instruction, the literature indicates that collaboration is essential, but is less likely when working independent of a teacher community that values such work (McLaughlin, 1993). It may be that the prevailing culture of mathematics teachers is to work independently on all lesson planning, but working to change that culture appears essential to improving the instruction of ELL students.

The other question where LSm teacher responses significantly differed from teachers' responses in the other three school districts is "I feel adequately trained to teach ELLs math." Once again, the means of the two groups raises questions. The mean for LSm is 3, meaning that they feel unprepared to teach ELLs. The mean of the other three school districts is 4.2, which is a neutral response. It seems that this finding points to the necessity of training teachers how to teach ELLs mathematics. It is not surprising that teachers from LSm do not feel prepared to teach ELLs given ELLs' scarcity. In Batt, 2008 and Reeves, 2006, it was found that teachers from districts similar to LSm did not feel prepared to teach ELLs. What is surprising is that teachers in school districts with higher proportions of ELLs did not voice feeling at least somewhat prepared as a group to teach ELLs mathematics.

HUm and HRm vs. HUs

For the majority of the questions in the survey, the HUm and HRm teachers gave responses similar to the Hus teachers. There were eight questions where HUm and HRm teachers' responses differed significantly from the responses of teachers from HUs (See table 2). Seven of the eight questions fell under the category of "strategies that help ELLs succeed in school." What is interesting is that for questions #1-4 (see table 2) HUm and HRm had means of roughly 3, which indicates that the teachers felt that the strategies would have a weak positive impact on ELL's success in school. The means for the HUs for the same questions range from 1.5 (Weak negative impact) to 0.42 (Strong negative impact). It would seem that the HUm and HRm are looking for someone to help them teach their ELLs where HUs teachers feel that they do not need outside help. Perhaps many of the teachers at HUs had tried the different strategies and found them to be less effective. For questions 5-7, all three school districts responded on the negative end of the spectrum (Weak negative impact or Strong negative impact), with the HUs teachers answering more negatively on all there questions than the other two school districts' teachers.

Implications

Follow-up interviews with teachers would help researchers more fully understand participants' responses. Despite this limitation, the findings suggest several implications. First, teachers do not feel trained to teach ELLs mathematics. There is a need for specialized sustained training aimed at mathematics teachers in order for them to learn how to effectively teach mathematics to ELLs. Next, mathematics teachers need to cultivate a culture of collaboration within their community. Regardless of the students mathematics teachers are teaching, they should collaboration dramatically increases the likelihood of planning effective mathematics lessons that engage students (McLaughlin, 1997). Finally, further research needs to be conducted to understand teachers' beliefs with respect to teaching ELLs mathematics. Once researchers more completely understand the beliefs of teachers, training can be organized to effectively address teachers' misconceptions and reinforce positive beliefs and practices.

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AN EXPLORATION OF 'STUDENTING' IN HIGH SCHOOL GEOMETRY CLASSROOMS

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This paper examines the work that students do in instruction and the tacit knowledge that could guide this work. Classes of high school geometry students engaged with an animated scenario of geometry instruction and attempted to explain the actions of the animated students. Their responses to the animation were used to construct a model of the work of 'studenting': A set of instructional norms that partially accounted for the actions that students considered reasonable to do in high school geometry classrooms.

Objectives

As the actions that teachers take within instruction are often referred to as 'teaching,' in this study we look at the corresponding student actions, referred to as 'studenting' (Fenstermacher, 1986). We argue that studenting, while it may not be rational in the sense of correct, does reflect a rationality of sorts adapted to enacting the position of student. By the position of student we mean the expectations about how they should act in relation to the teacher, their peers, and the mathematical content they that they are expected to learn that individual adolescents come to be subject to when they enter high school. Even students who do not act according to these expectations are still exposed to them and are aware of them. It is these expectations of how a student should act that we are concerned with in this study.

This paper presents an empirical discussions of the question, 'what is the work of 'studenting' in high school geometry instruction.' The results of this study present a set of instructional norms that can be used to partially account for students' actions in classrooms. This report provides a brief example of the type of empirical analysis that was done and a brief overview of the results. For more details on this work see Aaron (2011).

The following are the primary research questions we explore in this paper:

- What are the actions that students consider reasonable to do in high school geometry instruction?
- How can hypothesized norms of instruction be used to justify student actions?

To study this question we begin with a list of hypothesized norms and examine the data to see whether and how students' comments in response to an animated classroom scenario reflect adherence to or departure from these hypothesized norms.

This paper begins with a discussion of a theoretical framework for viewing classroom interaction. We then describe the data and analytic methods used in this study. The results section consists of one in-depth example of the data supporting one instructional norm and the complete list of norms supported by the data in this study.

Theoretical Framework

To understand classroom interactions we use a model of classroom interaction developed by Herbst that is based on Bourdieu's notion of symbolic economy (1990, 1998) and Brousseau's notion of didactical contract (1997). According to this model, teachers and students act as if they are trading classroom work for claims that they have taught and learned a bit of the geometry curriculum. The foundational hypothesis is that inside educational institutions the teacher and her students enter into this economy because of their obligation to a didactical contract that brings students and teachers together to teach and learn geometry. A didactical contract specifies in rather general terms what it means to teach and learn geometry and what the geometry is that needs to be taught and learned.

The didactical contract can be modeled (by an observer) as a set of norms, or tacit rules that describe how instruction should proceed. Norms one set of cultural resources that actors use to construct their performances in particular settings. They are akin to Bourdieu's (1990) dispositions, "structured structures predisposed to function as structuring structures (p. 53)". That is, these structures for action are preexisting in the culture but when a particular actor enacts them in a particular moment they feel (to the actor and his companions) as if they are spontaneous improvisations in response to the current circumstances.

An important problem that a teacher and her students have to solve daily is to find out how the norms of the contract apply to specific chunks of work, or conversely, how their work on a specific task (Doyle, 1983; Doyle, 1988) contributes to meeting the demands of the contract. In "normal" instruction, this problem is handled through the existence of instructional situations (Herbst, 2006). Instructional situations are recurrent patterns of activity that organize the actions of the students and teacher so that they can engage in work that exchanges for claims on the contract. In general we hypothesize that these customary, recurrent patterns of activity make room for some canonical tasks saving people the need to negotiate how the contract applies for the task. In this study we focus on the instructional situations of 'making conjectures' and 'doing proofs' (Herbst & Brach, 2006). For a description of these situations see Aaron (2011).

This theoretical view of classroom interaction is useful in this current study because it highlights the role that norms play in shaping instructional activities. It also highlights the difference between instructional situations and the norms that support these situations. The results section contains three sets of norms; contractual norms that could be perceived to apply to any instructional situation in high school geometry classrooms, situational norms of 'making conjectures,' and situational norms of 'doing proofs.'

Data

The data used in this study was collected during one-time focus group sessions with classes of high school geometry students. These data come from eight classes in two schools. School 1 is a high-achieving public school serving 2,800 students in grades 9-12. School 2 is a low-achieving public school serving 1,200 students in grades 9-12. At School 1 we collected data in two honors level classes taught by Megan and two regular level class taught by Madison. At School 2 we collected data in two regular level classes taught by Jack and two remedial level classes taught by Sharleen. These schools, teachers, and classes were chosen so that the data corpus would represent a diverse group of students and experiences.

Near the end of the 2007-2008 school year the first author met with each class of students for one class period (about 50 minutes). Each class of approximately thirty students was shown the animated scenario, The Square (1), in five short clips. Each of the clips was between one and two minutes in length and highlighted the actions of a small group of animated students. The first clip was an exception because it featured only the animated teacher posing the angle bisectors problem to the class. After viewing each of the clips the moderator lead a discussion, in the form of a semistructured interview (De Groot, 2002), among participants with the aim of collecting general comments about what the participants had just viewed as well as collecting comments regarding the participants' views of the animated students they saw in the scenario.

While watching The Square, participants worked to make sense of the actions of the animated teacher and students: they reported on the actions that drew their attention, evaluated those actions, provided justifications for those actions, and suggested alternatives. These sessions were video and audio recorded and then transcribed and indexed for analysis. This indexing consisted of dividing the transcript based on the clip of the animated scenario being discussed. In the following section we describe the analytic methods that were used to examine the data and the methodology of collecting the data. We also include a description of the animated classroom scenario.

Methods

We consider the focus groups with students as modified breaching experiments (Garfinkel, 1964; see Aaron, 2011, for a full description of the use of animated scenarios as breaching experiments): The scenarios ushered students into a representation of their ordinary experiences where some of the usual features of such experience had been altered; participants were able to point to what they perceived as abnormal in familiar situations. We hypothesized that by showing participants the animated scenarios, which were designed to display actions that the participants could perceive as breaches in the normal running of a high school geometry class, we would be able to elicit from them the tacit perceptions they had of the norms of the work of being a geometry student. Because these perceptions are tacit, participants might not have been able to share them with an interviewer in a traditional interview. The immersive quality of the animated scenarios coupled with the provocation of the embedded hypothesized breaches of normal instruction prompted participants to share their perception of the geometry classroom.

Summary of the Square

The following is a brief summary of the actions of the animated teacher and animated students in The Square. This animation of high school geometry instruction was shown to the participants in this study as a data collection prompt. The animation can be seen in www.lessonsketch.org.

The Square begins with the teacher reminding her class that the angle bisectors of a triangle meet at a point. She then asks her class to make conjectures about the angle bisectors of a quadrilateral. The students work in groups on the task and then the teacher asks a student, Alpha, to share his conjecture with the class. He comes up to the board says that the diagonals of a square bisect each other. The teacher reminds Alpha that the problem is about angle bisectors, not diagonals. Alpha responds by saying that the diagonals cut the square in half and another student, Beta, interprets this to mean that the diagonals are the same as the angle bisectors. Another student, Gamma, points out that this is not true for a rectangle. Gamma comes to the board to present a counterexample. This leads to a restatement of Alpha's conjecture: The angle bisectors of a square meet at a point. The animated teacher asks the students how they might prove this and another student, Lambda, volunteers an argument using congruent triangles. After Lambda finishes, the teacher writes the 'given' and the 'prove' on the board and asks for a proof of the statement (2).

Coding for Norms

To analyze the data we look at participants' responses to the actions of the animated students. By examining these reactions one can see how appropriate the animated students' actions are from the students' perspective. The participants' comments are examined to see if they reflect a norm of instruction.

First, each comment from the participants was tagged as being in reference to a particular moment dealing with an animated student or the animated teacher. Second, within these moments, comments were summarized and similar comments were compiled. Third, the summarized and compiled comments were then coded as reflecting any situational norms. The full list of hypothesized instructional norms in not listed here but can be found in Aaron (2011).

Perceptions of animated students' actions, in relation to a norm, could be coded in one of four ways. The participants could see that the animated students complied with a norm and that this compliance was appropriate. The participants could also see that the animated students complied with a norm, however this compliance could be seen as inappropriate. That is, the participants could think that the animated student should have breached a norm. The participants could also report that a norm was breached by the animated students and that this breach was appropriate. Finally, the participants could report that a norm was breached by an animated student and that this breach was inappropriate, that is, the animated student should not have breached that norm.

Results

The follow section contains two types of results. The first is an example of the data that supports the claim that students perceived the instructional norm, *students' interventions should address the topic that the teacher has proposed*. The second is an entire list of the instructional norms, contractual and situational, that were examined in the current study.

Example of Data Supporting the Norm, Students' Interventions Should Address the Topic that the Teacher has Proposed

Below is an example of the results from this study. We describe one hypothesized norm, *students' interventions should address the topic that the teacher has proposed,* as well as animated student actions that activate this norm and evidence from the student focus groups that point to the existence of this norm. The norm *students' interventions should address the topic that the teacher has proposed* says that when students are responding to problems posed by the teacher, their responses should match that problem. This is, their responses should use the appropriate mathematical concepts, should connect their answer to the question, and students should not change the problem from how it is posed by the teacher. This norm was activated by two actions seen in The Square that are listed in Table 1. Also included is evidence from the data that supports the view that this norm was either appropriately breached or appropriately complied with by the animated students.

Table 1: Actions related to the norm, *students' interventions should address the topic that the teacher was proposing*

Action	Perceived relation to norm
A student presents an ambiguous solution	Appropriate Breach
A student provides a counterexample to another student's idea	Appropriate Compliance

The actions related to this norm are a student presenting an ambiguous solution, and a student providing a counterexample to another student's idea. These actions are preformed at different moments in The Square. In the beginning of The Square Alpha can be seen presenting an ambiguous solution, and later Gamma can be seen providing a counterexample to Alpha's idea.

A Student Presents an Ambiguous Solution (Appropriate Breach)

The participants' comments reflected the opinion that the first of these actions, a student presents an ambiguous solution on the board, was an appropriate breach of the norm *students' interventions should address the topic that the teacher was proposing*. The participants saw that Alpha's action in The Square was a breach of the norm because it was not clear how his conjecture connected to the problem that the teacher posed. However, the participants suggested reasons why it was appropriate for Alpha to provide the response that he did. Each of these reasons points to a cognitive difficultly that participants saw that Alpha could have had while working on the problem. One cognitive difficulty suggested by participants was that Alpha could have been thinking about perpendicular bisectors instead of angle bisectors. Art from Madison's third period class said that Alpha was "thinking of, like, perpendicular bisectors [instead of] angle bisectors." If this were the case, then the participants claimed that Alpha could have been making the conjecture that the diagonals of a square are perpendicular bisectors of each other.

Another cognitive difficulty suggested by participants was that Alpha did not have a fully formed conjecture, but simply had the idea to look at the case of the square. Denise from Madison's third period class said Alpha "doesn't know the answer, he just went up there and did a square." Gordon from Megan's fourth period class said, "Sometimes it's hard to put your thoughts into words especially if you are, um, shaky on the topic, so I think that's kinda what happened to Alpha." Gordon explains Alpha's breach of the norm by hypothesizing that the material was taxing for Alpha and therefore it was difficult for him to communicate his idea.

These comments from participants support the norm that *students' responses should address the problem* and provide an example of a cognitive difficulty that participants attribute to Alpha to explain why he was unable to comply with the norm. The participants see Alpha attempting to answer the animated teacher's questions but lacking the mathematical understanding needed to provide an acceptable conjecture.

A Student Provides a Counterexample to Another Student's Idea (Appropriate Compliance)

The participants' comments reflected the opinion that the second of these actions, a student provides a counterexample to another student's idea, is an appropriate compliance with the norm *students' interventions should address the topic that the teacher has proposed*. Participants saw that Gamma's contribution of thinking about the diagonals and angle bisectors of a rectangle was a counterexample to Alpha's conjecture and had an important mathematical component in terms of answering the problem posed by the teacher. That is, Gamma noticed that Alpha's conjecture was only true for squares and that Alpha's conflation of angle bisectors and diagonals is only non-problematic in certain quadrilaterals. The participants recognized Gamma's point, that in rectangles the angle bisectors are not the diagonals, as a mathematically valid point to make because the problem posed by the teacher is about quadrilaterals in general, not only squares. Bob from Madison's third period class said, "[Gamma] was trying to make it more general because [Alpha] was just talking about a square." Bob's comment reflects the opinion that it is appropriate for Gamma to provide a more general response than Alpha, which was too specific to answer the question posed by the teacher.

Also in light of Gamma's counterexample and this norm, some participants said that special cases, like Alpha's square, are not useful or are irrelevant when responding to general questions, like the angle bisectors problem. Muna from Megan's third period class said, "[Gamma] was just trying to point out that, um, what Alpha was saying was sort of irrelevant because they were talking about all quadrilaterals not just squares. So, I guess, answer the actual question." This comment from Muna highlighted the view that Alpha's conjecture did not address the teacher's problem, so it was appropriate for Gamma to provide a counterexample that was more clearly connected to the teacher's problem.

Other participants agreed that Gamma was making a distinction between squares and other quadrilaterals, but unlike Muna, these participants did not see the hierarchical relationship between squares, rectangles, and quadrilaterals so they did not see that Gamma's counterexample was a move towards answering the teacher's problem. Mary from Madison's third period class said, "Like, I don't know if [Gamma] realized it, but she was kind of saying that the rules for the squares are different than the rules for the rest of the quadrilaterals." Mary's comments showed recognition of Gamma's counterexample, but did not award mathematical value to Gamma's move from squares to rectangles.

The norm *students' interventions should address the topic that the teacher has proposed* was activated twice. The participants saw that it was appropriately complied with once and appropriately breached once. In the first case the participants saw that the norm was appropriately breached because a student did not have the cognitive capabilities to respond to the problem that the teacher posed. On the second case the participants saw that this norm was appropriately complied with since a student provided a counterexample to another student's idea. The participants saw that Alpha provided an answer to the teachers' question that only addressed a special case, so Gamma appropriately expanded on Alpha's response to make it a better answer to the teacher's original question.

List of Contractual and Situational Norms

The following is a list of the norms that were elicited in the conversations with the focus groups of high school geometry students. The first column labels the specificity of the norm as either applying to all instruction through the didactical contract, or only in the situations of 'making conjectures' or 'doing proofs.' The second column lists the norm. The third column describes the student action from the animation that prompted participants to discuss the norm. The final column relates the participants' stance toward the perceived norm.

	Norm	Action	Perceived relation to norm
Contract	*Students should share their ideas that are different than other students' ideas*	A student comes to the board to share her idea	Appropriate Compliance
	Students should complete incomplete arguments given in class	A student whispers a contribution to a solution	Appropriate Compliance; Inappropriate Breach
	Students' interventions should address the topic that the teacher has proposed	A student presents an ambiguous solution	Appropriate Breach
		A student provides a counterexample to another student's idea	Appropriate Compliance
	Students should be disposed to share ideas in public when so asked	A student is hesitant to present his solution at the board	Inappropriate Compliance
		A students shares his solution from his seat	Inappropriate Breach
	Students in the audience should support the presenter in public	A students whispers while another student is presenting a solution at the board	Appropriate Compliance
		A student clarifies another student's idea	Appropriate Compliance; Inappropriate Breach
	Students should be amenable to assessment	A student shares an ambiguous idea with the class	Appropriate Compliance
'Making Conjectures'	*Students should communicate their conjectures to the class*	A student is hesitant to share his idea at the board	Inappropriate Compliance
		A student whispers while another student presents a solution at the board	Appropriate Breach
		A student clarifies another student's conjecture	Appropriate Compliance
	Students can respond to other students' conjectures	A student provides a counterexample to another student's idea	Inappropriate Compliance
	Students should stop talking about a conjecture once it has been agreed upon or refuted	A student comments on a conjecture after it has been agreed up on or refuted	Inappropriate Breach

'Doing Proofs'	*Students should not make changes to the proof problem*	A student requests to modify a problem	Appropriate Breach

From these results one sees that the participants, while responding to the animated scenario, did provide evidence to support the hypothesized norms. This list of hypothesized norms is not a complete list, at least in regard to the norm 'doing proofs' in geometry instruction (Herbst & Brach, 2006; Herbst et al., 2009), but it does reflect the primary norms that can be seen to be active in The Square. Unlike previous research on instructional norms in geometry classrooms, this study focuses both on perceptions of norms during instruction (unlike Herbst & Brach, 2006 which focused on students' perceptions of norms while working on proof tasks in isolation) and on instructional norms from the students' point of view (unlike Herbst et al. 2009, which primarily looks at instructional norms from the perspective of the teacher).

Discussion

By understanding the student's position in geometry instruction we will be better able to understand geometry instruction as a whole. Understanding the complex system of geometry instruction is the first step to making informed improvements to the system. A change in curriculum could result in an unexpected student action that could be explained in terms of an instructional norm. That is, changes to any part of the system should take into account their effect on other parts of the system, and these possible interactions can only by predicted by first understanding the system as a whole. The current paper represents an attempt in that direction.

Unlike teachers, students go through no professional training for how to be a student, so their actions are guided solely by the rationality that they develop from doing the work of studenting. There is, therefore, a self-perpetuating relationship between studenting and the rationality of studenting. The actions that students engage in mold the rationality of studenting and the rationality of studenting shapes future actions. It is difficult, if not impossible, to change students' instructional actions directly, but if we can uncover the rationality of studenting we can use this as a lever for changing student action (see Lampert, 2001, chapter 10). The first step after understanding student rationality would be to design instructional activities where students' rationality could be use to enable instruction. That is, in cases where students' actions are undesirable due to students' interpretation of a task or situation, that task or situation could be reengineered in such a way that the students' rationality would result in more beneficial instructional actions. As students spend time engaged in mathematical tasks, performing actions that embody more desirable mathematical work, then their rationality of studenting could possibility change to naturally support more productive mathematical work.

Endnotes

1. The Square, along with all of ThEMaT's animated scenarios, can be viewed at www.lessonsketch.org

2. The authors would like to acknowledge the work of Ander Erickson in developing this summary of The Square.

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DEVELOPING AN UNDERSTANDING OF AREA MEASUREMENT CONCEPTS WITH TRIANGULAR UNITS

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This article focuses on six fifth grade students' responses to area measurement tasks involving square and right triangular units. Data were collected during individual interviews as part of a four-year teaching experiment. Although all six students were able resolve a square unit area measurement task, they struggled to resolve similar tasks with right triangular units. Results indicate that engaging in the triangular unit tasks helped our students realize the necessity of coordinating linear measurements, relating linear and area units, and constructing and iterating a row (or column) composed of identical units. Implications for instruction and future research are discussed.

The importance of area measurement is evident in both daily life and school mathematics. As students study area measurement, they "develop ways to mentally structure the spaces and objects around them," which provide a context for the development of mathematical reasoning (Cross, Woods, & Schweingruber, 2009, p. 35). However, most elementary and middle school students have developed an inadequate understanding of area measurement concepts (Outhred & Mitchelmore, 2000). In 1986, only 40% of 13-year old students correctly answered that the area of a 6 cm by 4 cm rectangle (with only these number labels demarcating the dimensions of length and width) was 24 square centimeters on the Fourth National Assessment of Educational Progress [NAEP] (Lindquist & Kouba, 1989). More recently, in 2000, only 14% of 13-year old students could determine the number of square tiles to cover a rectilinear region with number labels demarcating the dimensions on the Eighth NAEP (Sowder et al., 2004).

Kordaki and Potari (2002) and Outhred and Mitchelmore (2000) attributed the lack of understanding area measurement to the required coordination of multiple ideas. "The concept of area measurement is a rather complicated one for students to grasp as it consists of a network of concomitant related concepts: the conservation of area, the unit and its iteration and the counting of units" (Kordaki & Potari, 2002, p. 65). Other researchers have formally asserted that this inadequate understanding of area measurement is characterized by a rote application of the rectangular area formula (Simon & Blume, 1994). A relational understanding of the area formula depends on students' abilities to develop an understanding of *multiplication-as-repeated addition with the notion of an iterated row of identical area units* (Outhred & Mitchelmore, 2000; Sarama & Clements, 2009). Thus, research has demonstrated that students can structure a two-dimensional array and use the rectangular area formula by rote but without understanding crucial area concepts and principles, including identification of unit, coordination of linear and area units, tessellation of the plane, and relation of area measurement to multiplication (Battista, 2003; Clements & Sarama, 2007).

When area measurement is focused on counting units, students are sometimes unable to link area measurement to a structure that is two-dimensional and multiplicative (Outhred &

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Mitchelmore, 2000; Sarama & Clements, 2009). Research indicates that the use of physical tiles and square units has the potential to hide misunderstandings. Using concrete materials to tile a rectangular figure "may conceal the very relations they are intended to illustrate" (Outhred & Mitchelmore, 2000, p. 146). Because manipulatives pre-structure the array, students can correctly determine the area of a region without attending to the structure (Lehrer, 2003; Outhred & Mitchelmore, 2000). Thus, removing the concrete manipulatives has the possibility to reveal misconceptions.

Using square units to cover a rectangular figure – even if the concrete materials are not available – allows students to determine correctly area without attending to both dimensions. Because the rectangular array structure is not self-evident for children (Battista, Clements, Arnoff, Battista, & Borrow, 1998), they often imitate global organization without conceptually understanding it. In other words, some students can correctly structure an array with square units by relying on discrete counting of objects and using one-dimensional structuring techniques, or partial row structuring (Battista et al., 1998; Sarama & Clements, 2009). Students using one-dimensional structuring techniques struggle to *coordinate the linear measurements* because they do not use the width and the height of both the unit and the region. This coordination is especially non-trivial with non-square units when the orientation of the unit with respect to the region matters. These students also struggle to *relate linear and area units* because they do not understand the concept that the side length determines the number of area units that will fit along that length (Battista, 2003; Outhred & Mitchelmore, 2000; Sarama & Clements, 2009). However, missing from the literature is how students structure rectangular regions when provided with non-rectangular, non-concrete area units.

Theoretical Framework

The tasks presented here have been designed as part of our efforts to test and improve a hypothetical learning trajectory for area measurement (Sarama & Clements, 2009). A hypothetical learning trajectory consists of three components: a learning goal, a likely path for learning as students progress through levels of thinking, and the instruction that guides students along the path (Sarama & Clements, 2009). We used the hypothetical learning trajectory to focus our attention on students' internal representations, or mental images, and their external representations, or drawings, in the context of area tasks. In the construction of the hypothetical learning trajectory for area, Sarama and Clements (2009) reflected upon the interactions between external and internal representations, giving equal credit to how students construct both of them. This trajectory also guided our ideas concerning the underlying conceptual foundations of area measurement in terms of comparing, connecting the iteration of identical rows of area units with an understanding of multiplication-as-repeated addition, relating linear and area units, and coordinating linear measurements.

Purpose

It is our conjecture that area tasks involving only square units are not sufficient for revealing students' conceptions or misconceptions regarding area measurement. We claim that tasks involving right triangular units provide a rich context for teachers and researchers to investigate and also develop students' understandings of area structuring. We believe these tasks provide an opportunity for students to demonstrate their ability to adapt the standard area formula, which in these instances will not result in the number of triangular units needed to fill a region. The purpose of this report is to describe and analyze student responses as they engaged in area tasks involving right triangular units, specifically in terms of relating linear and area units, coordinating linear measurements, and understanding multiplication-as-repeated addition.

Research Question

1. How do area measurement tasks involving triangular units impact students' ability to relate linear and area units and coordinate linear measurements?
2. How do area measurement tasks involving triangular units impact students' application of the area formula?

Methodology

The sample consisted of six children (Abby, Anselm, Arielle, Danny, Jessie, and Owen) from two fifth grade classes at a Midwestern public school. Each student has been regarded as a case study within a four-year longitudinal study investigating children's thinking and learning across length, area, and volume for Grades 2-5.

The tasks described below were developed within the context of a teaching experiment (Steffe & Thompson, 2000). The teaching experiment consisted of a series of teaching episodes for which the research team generated a set of tasks and predictions for student responses and then later checked student responses against these predictions. Each teaching episode was videotaped, transcribed, and analyzed by a group of researchers, the authors. In this report, we consider data collected in a single teaching episode consisting of two 25 to 30 minute interviews, an initial interview and a follow up interview, for each student during the months of October and November of 2010.

We varied the tasks in terms of the area unit provided and followed the sequence of a square and two right, scalene triangular units. These task variations allowed us to assess and compare students' performance with the traditional square units as well as triangular units.

The development of each task we posed in this teaching episode was guided by a two-step framework (Cullen et al., 2010). We presented the students with two figural objects, a unit and a rectangular region, and asked them to compare the objects by area. We then asked, "How does the area of this tile compare to the area of the larger region?" If the student gave a qualitative comparison, we asked, "How much bigger?" in order to elicit a quantitative comparison. We conjectured that these two steps would draw students' attention to the underlying concepts of area measurement: comparing, connecting the iteration of identical rows of area units with an understanding of multiplication-as-repeated addition, relating linear and area units and coordinating linear measures.

Initially, we did not allow the students to use a writing utensil or their hands in order to prompt them to verbalize their internal representations of the area structuring. As a second part of each task, we asked the students to show how the unit would cover the region by drawing or creating an external representation. This allowed us to attend to both students' internal representations, or mental images, and their external representations, or drawings to be consistent with the learning trajectory in our analysis of students' understanding of area measurement.

The interviewer inferred the strategies children used to solve area tasks through both careful questioning as the students provided verbal descriptions of their internal representations and observations as students created external representations. The data analysis involved identifying and then classifying students' strategies for solving each of the four area tasks. Next, these classifications were examined for commonalities and variations among the six students.

Results

The results for this study are presented below and include descriptions of student responses to each of the three tasks. In each task, we asked students to compare an area unit of measure and a rectangular region to be measured, first verbally and then using their hands and a writing utensil to record their thinking.

Task 1: Compare area of a 5 by 7 rectangle to area of a 1 by 1 square-unit inside rectangle

The area unit for Task 1 was a square, inside of a rectangle (Figure 1). The linear units were represented as number labels on rulers along each side of the rectangular region.

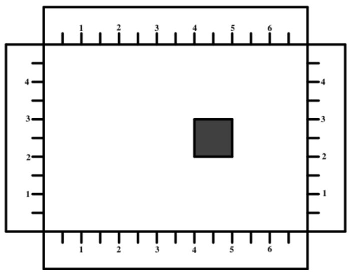

Figure 1. Task 1

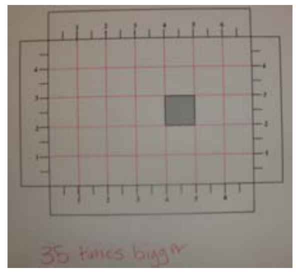

Figure 2. Abby's drawing for Task 1

Five students, Abby, Arielle, Danny, Jessie, and Owen, correctly used the rulers to determine the dimensions and correctly applied the rectangular area formula (i.e., $a=lw$) to calculate mentally that 35 squares would cover the rectangle without drawing. The sixth student, Anselm, used the rulers incorrectly, reporting the last numerals displayed on the rulers as the dimensions, thus missing the final linear unit for each. When asked to make a record of his thinking, Anselm drew one row of the units, realized that there were 7 units instead of 6, and changed his answer to 35. All six students could show how the tiles would cover the rectangle by either drawing the individual tiles or alternating between drawing individual units and parallel row and column line segments. Abby's drawing was the most sophisticated; she was the only student to draw a complete covering of parallel line segments from the linear unit markings on the four rulers to form a row and column structure of square area units (Figure 2).

Despite the variety in their drawings, all six students aligned square-units in rows and columns. All of them seemed to relate length and area units, coordinate linear measurements, and apply the rectangular area formula with understanding.

Task 2: Compare area of a 9cm by 8cm rectangle to area of a 3cm-4cm-5cm right triangle

The area unit for Task 2 was a right, scalene triangle (Figure 3). The linear units were represented as number labels along each segment in both the rectangle and the unit.

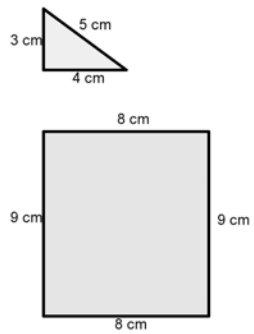

Figure 3. Task 2

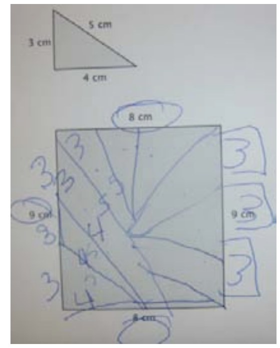

Figure 4. Owen's drawing for Task 2

Owen was the only student who could give a correct answer before drawing. All of the students except Jessie eventually resolved the task correctly by drawing. Owen discussed constructing columns of rectangles composed of triangles. Despite his correct explanation, he drew an entirely different representation without conserving the shape of the triangular area unit (Figure 4). After Owen completed his drawing, the interviewer asked him to reflect on his work in the preceding task in which he constructed a correct drawing of rows and columns of units, which is not mentioned here. This seemed to prompt him to draw row and column line segments; thus, he drew a second covering of triangles in rows and columns consistent with his initial verbal description. Jessie and Abby drew a complete covering of two columns and two rows of rectangles composed of triangles but did not attend to both dimensions. With prompting to think about what the number labels were referring to, Abby coordinated linear measurements and drew a correct covering.

Anselm and Danny gave incorrect initial estimates; however, while drawing, they correctly attended to linear measurements and operated on columns of rectangles composed of triangles. Arielle operated arithmetically on measures and thought about covering with columns of rectangles composed of triangles. She applied the area formula to both the rectangular region and the rectangular composite unit; she divided these quantities and obtained a quotient of 6. However, she incorrectly reasoned that she needed 5 more rectangular units or 10 more triangular units to cover the rectangle. When asked to make a record of her thinking, Arielle used both dimensions to correctly draw three rows and two columns of completed triangles inscribed in rectangles and stated that 11 more triangles would be needed.

Although all of the students successfully resolved Task 1, Task 2 was a challenge for them. During this task, we noticed that students struggled to resolve the task without drawing, to relate linear and area units, to connect linear measurements, and to connect the iteration of identical rows of area units with an understanding of multiplication as repeated addition.

Task 3: Compare area of a 12 by 8 rectangle to area of a scalene right triangle

The area unit for Task 3 was a right, scalene triangle (Figure 5). The linear units were represented as tick marks around the inside border of the rectangular region.

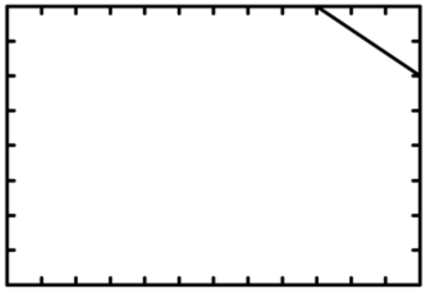

Figure 5. Task 3

Figure 6. Jessie's drawing for Task 3

Each student discussed the placement of units in rows or columns, but only three students, Arielle, Abby, and Owen, successfully resolved the task without drawing. Danny and Anselm used a non-adaptive area formula to resolve the task before drawing. Danny discussed placing eight triangles in a row and eight triangles in a column. He multiplied these quantities and said that the area of the rectangle would be 64 triangular units. Anselm discussed placing four columns and four rows but then multiplied 12 by 6 for an area of 72 triangular units, which he was unable to explain. While drawing, both Danny and Anselm realized their initial errors. Arielle, Abby, Owen, Danny, and Anselm attended to an organized row and column structure and related linear and area units in their drawings. The only student who did not resolve this task successfully was Jessie. She described a row and column structure of three rows of eight triangles for a covering of 24 triangular units. As she drew these three rows of eight units aligned in columns, she attended to only one dimension (Figure 6).

Discussion

Throughout the analysis of the students' responses and strategies when working with triangular units in area measurement, three main themes emerged. The first two themes concerning the underlying conceptual foundations of area measurement are already present in the existing area hypothetical learning trajectory (Sarama & Clements, 2009): 1) relating linear and area units and coordinating linear measurements and 2) connecting the iteration of identical rows of area units with an understanding of multiplication-as-repeated addition. The third theme emerged as the learning trajectory focused our attention toward students' internal and external representations; some mismatches appeared. The themes are explained in detail below.

1. Relating linear and area units and coordinating linear measurements

In Task 1, all six students correctly computed the area in terms of square units and justified their answers by showing how the tiles would cover the rectangle. In this task, Abby covered the array solely with row and column line segments. However, when covering with triangular units in Task 2, she demonstrated less sophisticated drawing strategies and only attended to one dimension. After Abby was prompted to use the linear measurements for both dimensions in Task 2, she not only resolved the task correctly, but also successfully drew an array in Task 3 with single row and column line segments. In other words, Abby's covering of a region with triangular units improved after she realized the necessity of coordinating linear measurements and relating linear and area units. Although Jessie successfully resolved Task 1, she was unable to coordinate linear measurements and relate linear and area units for the triangular unit tasks. While all six students seemed to demonstrate an understanding of the conceptual foundations of area measurement when working with square units, they were unable to do so when working with triangular units.

2. Connecting the iteration of identical rows of area units with an understanding of multiplication-as-repeated addition

By analyzing the results of Task 1 in isolation from the rest of the tasks, one could assume that the students were applying the area formula for rectangles meaningfully for square units. However, conflicting results occurred with the use of triangular units in the subsequent tasks. Therefore, we can conclude that they are unable to adapt the area formula, missing the underlying process. For example, Arielle's unit confusion in Task 2 made her end up with incorrect calculations as a result of inappropriate use of the formula. For Task 3, both Anselm and Danny applied a non-adaptive area formula; they did not connect using multiplication for area measurement with counting the number of area units needed to cover. In other words, their multiplication did not reflect the construction of a column as a composite unit of triangular units or the application of this composite unit repeatedly over the rectangle to cover (Sarama & Clements, 2009). The students did not have an understanding of why multiplication generates an array for square-units (e.g. Lehrer, Jaslow, & Curtis, 2003; Lehrer, 2003); as a result, they incorrectly applied the same formula for the triangular units.

3. Connecting internal and external representation

Asking students first to compare areas of regions without using their hands or a writing utensil and then to create a drawing of their thinking revealed some mismatches between students' internal and external representations. For example, Anselm gave incorrect initial answers for Task 1 when explaining his thinking. For this task, he counted incorrectly and realized his error while drawing, or translating his internal representation to an external representation. The drawing intervention also seemed to help Arielle and Abby. In contrast, Owen initially gave a clear description of his internal representation, which was correct. However, his external representation in the form of a drawing did not match his well-structured, internal representation as he described. These results indicate that some students, such as Anselm, Arielle, and Abby, may need to show their internal representations as external representations in order to correctly resolve tasks. However, for other students, such as Owen, showing their internal representations as external representations may be a challenge. The results of this study indicate that students' external representations may be mismatched from their internal representations.

Conclusion

Although all of the students successfully resolved the square unit task (Task 1), the results from the triangular unit tasks (Tasks 2 and 3) indicate that students may have a disconnect between the area formula and underlying area measurement concepts, including relating linear and area units, coordinating linear measurements (Battista, 2003; Outhred & Mitchelmore, 2000; Sarama & Clements, 2009), and connecting multiplication-as-repeated addition with the notion of an iterated row of identical area units (Outhred & Mitchelmore, 2000). By working with right triangular units, our students were confronted with novel situations, causing them to reflect upon their rote area measurement conceptions. We found that these tasks challenged them to adapt and extend the rectangular area formula to cope with this new situation. Engaging in these tasks helped our students realize the necessity of coordinating linear measurements, relating linear and area units, and constructing and iterating a row (or column) composed of triangles. These results indicate that students need more meaningful experiences with building and iterating identical rows (or columns) of area units to understand why multiplication works for determining the area of rectangular arrays using square units. We suggest that one such experience is to compare covering rectangular regions with square units to covering with triangular units.

By focusing separately on students' internal representations, or verbal descriptions of their mental images, and their external representations, or drawings (Sarama & Clements, 2009), we revealed mismatches between these representations. Therefore, in order to probe students' understanding in area measurement comprehensively, we suggest that researchers and classroom teachers ask students to describe their internal representations verbally and to display what they are thinking by creating external representations. One cannot assume that students' external representations are congruent to their internal representations.

Because of our small sample size, our conclusions concerning right triangular unit tasks can be regarded as researchable conjectures regarding fifth grade students' conceptions of area measurement. Further research is needed to investigate these conjectures across grade levels and in the context of classroom studies to establish the effectiveness of developing students' understanding of area measurement with non-square units, including but not limited to triangular units.

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USING DYNAMIC GEOMETRY TO DEVELOP STUDENTS' CONCEPTUAL UNDERSTANDING OF ANGLE

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This study expanded on research about incorporating technology into the geometry classroom by investigating specifically how the technology is used and if the different methods have any effect on students' conceptual understanding of angle. The primary purpose was to discover whether it would be beneficial for students to create their own constructions with Dynamic Geometry Software given definitions or discover definitions using teacher-constructed diagrams while learning the concept of angle. We provide evidence that students' understanding improved more when they were given teacher-constructed diagrams to learn the concept. Additionally, certain subgroups of students benefited from the DGS in more significant ways.

Introduction

Envision a class of average geometry students who had learned bits and pieces of geometry throughout several mathematics courses and enter into their only high school geometry class with previous knowledge from a variety of somewhat disconnected instructional experiences. As they discussed the description and properties of angle after looking at a geometric figure in a Dynamic Geometry System (DGS), one student defined angle as "basically a bend in a line, so it becomes two rays coming out of a single point." While two students analyzed a dynamic diagram of an angle they discussed how the degree of the angle "either increases or decreases and goes in a 360 degree" rotation. Finally when a student is asked to identify the angles in a quadrilateral you listened to a student express how a quadrilateral contains only four angles, "unless you count the over 180-degree ones," which would bring the total to eight angles. The question of how to best use DGS in a classroom is motivated by these kinds of experiences with students in classrooms.

Additionally, in recent years, more classroom emphasis has been placed on the multiple definitions of angle; research shows that the concept of angle is hard to comprehend (Keiser, 2004, p. 286). This can be a significant learning issue for students without which they cannot completely understand geometry or how an angle is represented every day in their lives. Therefore, we paid close attention to high school geometry students' understanding of angle and how to understand this essential concept with the use of technology. By connecting these two research areas, student understanding of angle, and DGS as a tool for instruction, we expanded upon the research on incorporating technology into the geometry classroom by concentrating on how the technology is incorporated and if the method has any affect on students' conceptual understanding. The primary purpose was to research whether it would be beneficial for students to create their own constructions with DGS or discover new information using teacher-constructed diagrams while learning the concept of angle. This new research will contribute to the mathematics education literature as well as provide ideas for using DGS in the classroom.

Literature Review

Angle

Close (1982) studied the assortment of angle definitions used in mathematics along with many authors who have noted the difference between the definitions as well as between dynamic (movement) and static (configurational) angles (Close, 1982; Kieran, 1986). In the traditional high school geometry class curriculum the definition of angle used is the definition described by Euclid. According to *The Elements* an angle is “the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line” (Health, 1956, p. 153). Euclid then continued to define angle introducing what we refer to in this research as the straight angle in which “the lines containing the angle are straight” this angle is called rectilinear (p. 153). “When Euclid specified the rectilinear angles contain straight lines, he was suggesting that other angles could be composed of lines that are not straight” instead they are curved (Keiser, 2004, p. 297). Mitchelmore and White (1998) suggested three different definitions in which the conceptual understanding of angle can occur. They proposed that an angle could be a rotation, a pair of half lines that extend from a common point, or as angle in a curve. These definitions were used in our study.

There are several articles that provide evidence that students have difficulty understanding the concept of angle (e.g. Mitchelmore & White, 1998; Munier & Merle, 2009; Kieran, 1986; Close, 1982; Clements & Burns, 2000). In elementary school, when students are introduced to the concept of angle, they are given a pictorial representation of an angle and taught how to classify it by measurements without really understanding the concept but instead memorizing a picture and a definition (Keiser, 2004). Keiser continues to explain that teachers consider the concept of angle to be a difficult concept to teach even at a preliminary level of understanding. Students do not understand the concept that the size of an angle has no dependency on the length of its rays (Munier & Merle, 2009, p. 1865). Close (1982) discussed the difficulty that students have the understanding that two rays with the same endpoint actually compose two angles. According to Close (1982) students have a hard time understanding that these two angles add up to 360˚.

Clements and Battista (1989) and Kieran (1986) discuss the fact that students often have many misconceptions about angle and experience difficulty learning the complex topic. In order to better prepare students for future mathematics courses teachers should use a more comprehensive approach, presenting angle using the variety of definitions given throughout history (Keiser, 2004). Teaching a diverse array of definitions may cause confusion about what is actually being measured when referring to an angle and therefore angle should be taught in great depth with concern for conceptual understanding as to reduce any misconceptions the students may gain with this multifaceted approach (Keiser, 2004).

Clements and Burns (2000) also mention a different misconception held by students about the turn of an angle and how they find it difficult to measure turn. In order to master this concept Clements and Burns (2000) use a piece of DGS called LOGO. “LOGO activities can be beneficial to students’ development of turn concepts and turn measurement” (Clements and Burns, 2000, p. 31). LOGO provides a situation of visualizing the turn of an angle (Clements and Burns, 2000). With the use of LOGO, students build active and flexible conceptual protractors unlike the traditional protractors used in everyday classrooms (Clements and Burns, 2000). The research on LOGO provides knowledge that supports the DGS research in angle.

Dynamic Geometry Software

Monoghan’s research supports the use of software can help the students' understanding of the concept “more easily than otherwise possible. Geometry software can provide dynamic diagrams that students can manipulate to understand concepts, which can't be done with traditional paper diagrams (Hollebrands, Laborde & StraBer, 2008). "When an element of such a diagram is dragged with the mouse, the diagram is modified while all the geometric relations used in its construction are preserved. … It is as if diagrams react to the manipulations of the user by following the laws of geometry, just like material objects react by following the laws of physics" (Hollebrands, Laborde & StraBer, 2008, p. 167).

A study conducted by Erez & Yerushalmy (2006) concluded that it is beneficial and "important to the study of learning basic concepts in geometry with the aid of the dragging tool" seen in a variety of DGS (p. 293). "Dragging allows changing a shape by direct translation of parts of its components on the screen… dragging the shape preserves the geometric relations according to which is was initially defined. Thus the critical attributes associated to this definition are preserved during dragging but the non-critical attributes are changed" (Erez & Yerushalmy, 2006, p. 274). Unlike constructing diagrams with pencil and paper, DGS can help students not only construct the diagrams but by being able to drag the points, lines, segments they can discover new information using previous knowledge (Skemp, 1976). Using previous knowledge with well-connected, conceptually grounded ideas provide a greater opportunity for students to gain a deeper understanding of the concept (Skemp, 1976).

Students may learn properties of the geometric representations with DGS, since the properties hold according to the constructions, and there could be fewer misconceptions due to the manipulation of the diagrams. There are numerous studies that have been conducted about the use of DGS in geometry classrooms. Mitchelmore and White (2000b) advocate a teaching method called 'teaching by abstraction', wherein 'students become familiar with several examples of the concept before teaching the concept itself." This kind of teaching works well with dynamic diagrams if it involves some exploration of the concept (familiarity). Then 'the concept is taught by finding and making explicit the similarities underlying familiar examples of that concept' (similarity). Lastly, 'as students explore the concept in more detail, it becomes increasingly mental object in its own right' (Munier & Merle, 2009, p. 1861).

Another study by Monaghan (2000) brought about similar results as the study by Erez & Yerushalmy (2006). "In this study, the cognitive conflict is brought about by asking students to describe in their own words the differences between pairs of quadrilaterals" (Monaghan, 2000, p. 180). Monaghan's (2000) underlying assumption is that students will gain a deeper understanding through conversation and hopefully reveal insights about quadrilaterals with the use of software and class activities creating an environment in which students can learn and share their discoveries (p. 180). The study concluded that these misconceptions about quadrilaterals would be reduced using software to aid the instructional activity, allowing students to explore their ideas about quadrilaterals.

The importance of technology in the mathematics classroom has been investigated through a variety of venues about a variety of topics throughout mathematics education research. This study is designed to use this research and expand upon it while investigating the importance of angle comprehension in the geometry classroom.

Methodology

The following research questions frame the overarching idea of researching whether using teacher-constructed sketches, called pre-constructed sketches by Sinclair (Sinclair, 2003, p. 289) or student-constructed sketches were more effective when teaching the concept of angle.

1. How do students learn the concept of angle supported by the use of dynamic geometry software and does the use of dynamic geometry software support diverse subgroups of students differently?
2. How is student learning of angle influenced by the use of student-constructed diagrams compared to the use of teacher-constructed diagrams using dynamic geometry software?

The study took place in the spring of 2010 in two academic geometry (non-honors) high school classrooms at the same urban school located in a large city in the Southeastern United States. There were a total of 27 subjects. In the SCD class, there were 7 females and 8 males, with more than half of the subjects being black or Hispanic. In the TCD class, there were 8 females and 4 males with the majority of the subjects being black. The student's ages ranged from 15 years of age to 17 years of age. Each of the subjects was first introduced to DGS in the current geometry classroom. The teacher and primary researcher had three years of previous teaching experience in two different schools. She has a bachelor's degree in mathematics education, has since completed her Master's Degree in Mathematics Education, and is pursuing a PhD in Mathematics Education.

The study consisted of a pre-test for students (45 minutes on Day 1), instruction using DGS in two different formats described below(270 minutes on Day 2, 3, and 4) and a post-test for all students (45 minutes on Day 5 which was 4 weeks after the completion of the instructional activity). The two classes were randomly assigned to be the student-constructed diagram (SCD) class and the teacher-constructed diagram (TCD) class. Each class was given the same pre-test and post-test but the two classes participated in a different set of instructional activities. The students in the SCD class were given definitions of introductory terms in geometry and asked to construct pictorial representations based on the definitions. The students in the TCD class were given teacher-constructed diagrams, figures created in DGS by the teacher prior to class, of the same terms as in the SCD instructional activity and asked to provide the definitions to go along with the pictorial representations. In both classes, the students worked as pairs in order to gain a deeper understanding through discussion and were required to expand upon prior knowledge from previous mathematics courses. Allowing this type of collaborative work helped the students gain a wide range of understandings (Keiser, 2004).

In both classes the students worked individually on their computer but discussed their findings with the DGS as a pair while they recorded their results on their investigation. After the students in both classes completed the three days of instruction using the DGS class continued as normal, going through the curriculum until a month had passed. One month after instruction, the students were given the post-test. There were also nine questions in the post-test similar to the questions in the pre-test, the questions were superficially different yet conceptually the same. The only question that differed conceptually was question 8; therefore, it was not used in the analysis of the study. The pre-test and post-test were used to compare the results in order to discover if the students showed a deeper understanding of angle due to the instructional activity, if any subgroups of students gained a better understanding with the instructional activity, and if one activity was more efficient than the other.

Data Analysis

Data analysis consisted of both qualitative and quantitative analysis. After the collection of data in the pre-test, instructional activity, and post-test all of the written responses were entered into a spreadsheet in order to make conjectures from patterns throughout the subjects and the questions. The written responses were coded and assigned a number to represent their level of understanding of the topic. After we identified the common misconceptions the students possessed before the study by analyzing the coding of the pre-test, we started considering categories or levels of some sort to demonstrate students' conceptual understanding of angle.

After consideration of the data collected and the instruction used, we decided to use a modification of the Mitchelmore & White (2002) categories (See Table 1 and 2).

Table 1
The categories for angle as defined in Euclidean geometry.

Category	Definition
A0	Subject shows no comprehension of angle.
A1	Subject understands angle as a single fixed measurement.
A2	Subject understands angle as multiple fixed measurement, such as right, acute, obtuse or straight.
A3	Subject understands angle as an intersection of two lines, or two lines that meet as a point.
A4	Subject understands angle as a bending line with an array of measurements
A5	Subject shows a complete conceptual understanding of angle as defined in Euclidean geometry

Table 2
The categories for angle as defined by a rotation about a single point.

Category	Definition
B0	Subject shows no comprehension of angle as a rotation about a point.
B1	Subject understands angle as a single rotation about a point.
B2	Subject understands angle as a variety of rotations about a single point .
B3	Subject shows a complete conceptual understanding of angle as a rotation about a single point.

The scoring described above was used to compare students gain scores from the pre-test to the post-test. The scores were analyzed using statistical software to perform a paired *t*-test. A paired *t*-test was used to see if the means on these two normally distributed interval variables differ from each other. A paired *t*-test is used when you have two related observations (for example, two observations per subject, in this case pre-test and post-test) (Rao, 2007, p. 143). In each case, the null hypothesis was that the results of the pre-test and post-test were equal and the alternative hypothesis was that they were different. The results of the *t*-tests were used to see if either, or both, methods of instruction were beneficial to student learning of angle and if the learning with SCDs or TCDs were more beneficial. Finally, we analyzed if particular subgroups of students were supported differently by the two styles of DGS use.

Results

All of the students had been introduced to angle in earlier mathematics courses, but this was their first exposure in their current geometry course. Every student showed some knowledge of angle, which is evidenced considering 27 out of 27 (100%) of the students could either identify an angle within a variety of shapes, draw their own angle, or both. After analyzing the instructional activity, we analyzed each student's specific responses during three phases of the study: Pre-test & Basic Terminology, Classifications and Relationships, and Postulates & Post-Test. The categories in which the responses were given in each of the three phases are described in the methodology section and shown in Table 1 and Table 2. There were two different sets of categories, one for the definition of angle according to Euclidean geometry (A) and as a rotation about a point (B). None of the students in the SCD class mentioned angle as a rotation so category B was not used in this section of the analysis.

Looking at the students within each phase of the study demonstrated at which category the student's responses were in throughout the study. When analyzing the categories for the SCD class, we found that 4 out of 15 (27%) students showed an improvement of conceptual understanding of angle throughout the study. 7 out of 15 (47%) were categorized the same in the 1st and 3rd phase of the study, yet each had different results during phase 2. 4 out of 15 (27%) students' conceptual understanding of angle decreased as a result of the study. After analyzing the SCD class we went on to analyze the TCD class. 6 out of 12 (50%) students showed an enhanced conceptual understanding of angle throughout the study. 4 out of 12 (33%) were categorized the same in both the 1st and 3rd phase of the study, yet each had different results during phase 2. 2 out of 12 (17%) students' conceptual understanding of angle seemed to decrease as a result of the study. Only 2 out of 12 (17%) of the students mention the rotation of an angle and the sum of a single rotation being 360 degrees. After comparing the categorizations of both classes, the SCD class did not gain the level of understanding of angle needed to be in category A4 or A5. This along with the number of students who improved their level of understanding throughout the study shows the value of the TCD class. The TCD class seems to be more successful and this was shown with the results of the quantitative analysis.

A paired *t*-test was used to compare the difference between the pre-test score and the post-test score of each individual student from each class to determine if one method was more successful than the other. The null hypothesis of the paired *t*-test's performed in this study are assuming that there is no difference between the students' score on the pre-test and the post-test, $H_0 : \mu_d \leq D_0$. The alternative hypothesis is assuming that the use of the software did in fact affect the students' scores on the post-test in comparison to the pre-test scores for each of the different classes, $H_a : \mu_d > D_0$. After analyzing the results from the paired *t*-test for the SCD class, we were not able to conclude that the instructional activity improved the students' level of understanding at a 90% confidence level (p-value of .1159). However, the result from the gain scores (+1.53) does show that the instructional activity did improve their understanding, just not at a significant level. After analyzing the results from the paired *t*-test for the TCD class (+2.92), it is evident that the instructional activity did improve the student's level of understanding with a 95% confidence level (p-value of 0.0087). Therefore, both instructional activities are successful yet the TCD classes' improvement from the pre-test to the post-test was significant.

To answer the other research question about which methods were best for different groups, more quantitative analysis was done. Table 3 shows the results of the categorized groups of students: male vs. female, minority vs. non-minority, and sophomores vs. juniors.

Table 3

Group	Number of students	Average Gain Score	Significant at 95% confidence level?	Significant at 90% confidence level?	p-value
Female (SCD)	7	2.857	No	Yes	0.0701
Female (TCD)	8	3.429	Yes	Yes	0.0219
Male (SCD)	8	0.750	No	No	0.6955
Male (TCD)	4	5.000	No	No	0.1152
Minority* (SCD)	10	2.600	No	Yes	0.0559
Minority *(TCD)	10	2.400	Yes	Yes	0.0145
Non-Minority (SCD)	5	-0.600	No	No	0.5291
Non-Minority (TCD)	2	5.500	N/A	N/A	N/A
Sophomores (SCD)	11	2.273	No	Yes	0.0683
Sophomores (TCD)	9	3.667	Yes	Yes	0.0100
Juniors (SCD)	4	-0.500	No	No	0.7027
Juniors (TCD)	3	4.000	No	No	0.5286

*African American and Hispanic.

Discussion and Conclusion

This study was conducted to investigate DGS use in developing students' conceptual understanding of angle. Results show that the use of DGS was a significant aid for the students to gain insight into the concept of angle as defined by Euclid (1956). Additionally, a few students showed a better understanding of angle as a rotation and no students demonstrated a better understanding of angle in a curve. The software was an important teaching tool yet the SCD class struggled whereas the TCD class did not report a high level of difficulty using the software. It was more difficult for the SCD class due to the fact that they had to learn how to use the software while learning what the terms looked like in order to be able to construct the terms, whereas the TCD class did not need to be extremely familiar with the software to explore their instructional activity. Being unfamiliar with the software created some frustrations throughout the activity. Although, the software used in the study did have the drag mode, the SCD class was not as apt to use the drag mode. Therefore they demonstrated less conceptual understanding than the TCD class, which was totally dependent upon the drag mode. The pictorial representations created by the SCD students with the help of the software was beneficial, especially since some of the terms were completely new information to the students, as they had not covered it in previous mathematical courses. The pre-constructed diagrams the TCD class used to define the introductory terms helped them visualize and define the terms with the use of the software and its many features. Although each instructional activity had its own instructional advantages, according to the mean gain scores and the paired *t*-test conducted on the data the TCD group improved significantly where as the SCD group did improve, but not significantly.

As far as the subgroups, for each group of female students, the software was beneficial and significant in helping their understanding of angle but had little influence on the male students conceptual understanding according to their mean gain score from the pre-test to the post-test. Looking at the results from the *t*-test for the minority students for each group, one can see that the instructional activity was obviously beneficial for their understanding unlike the results from the non-minority students who did not improve as much. After analyzing the results from the *t*-test for sophomores and juniors it is evident that the sophomores improved significantly where the juniors did not. Therefore, the use of the software was more beneficial for the younger students of the class.

According to these results the pre-constructed diagrams were more beneficial for entry-level geometry students, which might not be the case for upper level students or adult learners using the software. This issue of pre-constructed diagrams vs. constructed diagrams needs to be researched for all ages because constructed diagrams have been proven important to the study and learning of geometry, yet for this age the pre-constructed diagrams were more apparent. In summary, the software was more beneficial for the TCD class than the student constructed class no matter if the student was male or female, minority or non-minority, or if they were a sophomore or a junior. Therefore, dynamic geometry software would be best used in the classroom with teacher-constructed diagrams with similar conditions as the study conducted at hand. More research is needed to answer this question definitively, yet for this study, and for other classrooms with similar dynamics, teacher-constructed diagrams were shown to be more advantageous to promote student conceptual understanding of angle.

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TEACHERS' DEFINITION CONSTRUCTION: A STUDY BASED ON THE VAN HIELE THEORY

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This paper reports the results of analyses conducted to explore future and novice mathematics teachers' basic geometric knowledge of plane figures. Test scores suggest that teachers' responses to definition construction tasks often failed to include all necessary information to define a given figure. Other responses included redundant information and went beyond the information needed for a sufficient mathematical definition. Most teachers' responses corresponded to a van Hiele level of analysis, which aligns with national expectations for students in grades 3-5. Some teachers' responses were of the lowest van Hiele level, visualization, within which students should be operating in grades PreK-2.

van Hiele Theory and Geometry Instruction

The van Hieles' work to propose levels, as outlined in Table 1 below, for the acquisition of geometric knowledge is well known in the mathematics education research community (Fuys, Geddes, & Tischler 1988). As a framework, the van Hiele theory is useful when considering an individual's growth in geometric understanding (Malloy, 2002).

Level	Individual is able to…
Visualization	Focus on figures holistically; recognize and identify figures as general shapes
Analysis	Focus on the physical properties of figures; characterize figures by their attributes; describe the properties of figures.
Abstraction	Focus on the geometric properties of figures; order figures by their properties; interrelate properties and classes of figures using informal arguments.
Deduction	Focus on deductive reasoning; prove theorems deductively; establish relationships among theorems
Rigor	Focus on deductive geometry systems; establish theorems in different axiomatic systems; analyze and compare these systems

Table 1. van Hiele levels of thought

Researchers have shown that preservice and inservice teachers at both the elementary and secondary levels often are unable to operate at appropriate van Hiele levels for the material they are required to teach (Mayberry, 1983; Fuys, Geddes, & Tischler, 1988; Mason & Schell, 1988; Chappell, 2003; van der Sandt & Nieuwoud, 2003; Pandiscio and Knight, 2006). Concerns regarding preservice and inservice teachers' acquired van Hiele levels are not limited to the United States but rather seem to be a global problem (Gutiérrez, Jaime, & Fortuny, 1991; de Villiers, 1997; van der Sandt & Nieuwoud, 2003; Halat, 2008).

The National Council for Teachers of Mathematics (NCTM, 2000) in their *Principles and Standards for School Mathematics* have provided a cohesive vision for geometry instruction in the U.S. that involves expectations that students "analyze [the] characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships" (2000, p. 41). According to the NCTM (2000) students are expected to perform tasks which reflect the increased cognitive sophistication typical of higher van Hiele levels as they progress through the Pre-K-12 curriculum.

In grades PreK-2 students are expected to operate at the *visualization* level. Students are encouraged to use their own vocabulary to describe geometric objects. By Grade 2, tasks typical of *visualization* level cognitive activity have evolved to include identifying and sorting shapes by similarities and differences. Such tasks hint at *analysis* level cognitive activity. In Grades 3-5 students are expected to operate at the *analysis* level. Students should develop more precision when describing shapes. The NCTM (2000) indicates that during these grades appropriate terminology is of increased importance as is the use of the "specialized vocabulary associated with the shapes and properties" under consideration (p. 165). In Grades 6-8 students are expected to operate at the *abstraction* level. By this time the interplay of properties should become evident to students allowing them to classify shapes into a hierarchy. Students should be capable of synthesizing their knowledge of the interplay of figural properties and the mathematical hierarchy to generate proper mathematical definitions. Initial *abstraction* level activities include investigating the diagonals of geometric figures and using observations to create alternate definitions for the figures under consideration. In Grades 9-12 the NCTM expects students to operate at the highest van Hiele level, *rigor*. Expectations include analysis of properties, making conjectures and using mathematical proof.

Considering national expectations that students' van Hiele levels should be increasing with grade level, it is reasonable to wonder if teachers possess the necessary knowledge to facilitate students' acquisition of higher levels of geometric reasoning.

van Hiele Theory and Definition Construction

As key components of geometric understanding (Sfard 2000), definitions and their construction allow insight into the depth of teachers' conceptualization of geometric figures. Research focusing on reasoning processes has related definition construction to the van Hiele theory. Gutiérrez and Jaime (1998) have identified four processes of reasoning in geometry: 1) *recognition* of types, components, properties, and families of figures; 2) *definition* of geometric concepts; 3) *classification* of geometric figures (or concepts) into different families; and 4) *proof* and ways of convincing regarding geometric figures and statements. Of the four processes, they further parse *definition* into *use of definitions* and *formulation of definitions*. In their 1998 work, Gutiérrez and Jaime offered a matrix with distinctive attributes related to their processes of reasoning (*recognition*, *use of definitions*, *formulation of definitions*, *classification*, and *proof*) that provide evidence that an individual is operating at one of the first four van Hiele levels[1]. Table 2 reveals the attributes for the formulation of *definitions* by van Hiele level offered by Gutiérrez and Jaime.

	Visualization	*Analysis*	*Abstraction*	*Deduction*
Formulation of definitions	Able to list physical properties of figure	Able to list mathematical properties of figure	Able to provide necessary and sufficient properties of figure	Able to prove the equivalence of definitions

Table 2. Distinctive attributes of the process of reasoning in each van Hiele level

Individuals operating at the *visualization* level often classify figures based on basic prototypes of those figures. Analysis of the properties of figures is absent; rather the individual views geometric figures holistically. This can lead to identification and classification of figures based on visual properties that are irrelevant (Burger & Shaughnessy, 1986) including descriptions using words like “pointy,” “round,” “skinny,” and “fat” (Clements et al., 1999; Gutiérrez & Jaime, 1998). Moving from *visualization* to *analysis* involves more than just acquiring new vocabulary to describe properties; it involves recognizing existing and new relationships between properties (de Villiers, 2010). At the level of *analysis*, individuals are able to use definitions; however they are likely to reject alternate definitions counter to the definitions they hold to be true (Burger & Schaunessy, 1986). Definitions constructed at this level are not restricted to sufficient properties (Gutiérrez & Jaime, 1998, Burger & Schaunessy, 1986). When asked to construct a definition, individuals are likely to make a long list of properties with no regard for redundancies (Gutiérrez & Jaime, 1998). Whereas *analysis* involves recognizing properties within figures, *abstraction* involves recognizing the interplay of those properties via logical relationships (de Villiers, 2010). Individuals working at the *abstraction* level understand the hierarchy of geometric figures and are able to recognize relationships among geometric figures. Mathematical activity at the *abstraction* level consists of interrelating and deducing properties within and among geometric figures (Adele, 1996). Individuals at this level are capable of constructing economical mathematical definitions (Burger & Schaunessy, 1986; Gutiérrez & Jaime, 1998), although some redundancy may be present if the relationship between properties is not immediately recognized as being related (Gutiérrez & Jaime, 1998). Individuals operating at the level of *deduction* are aware that definitions are arbitrary. They accept equivalent definitions and are capable of proving their equivalency.

Because definitions are the foundations of geometric understanding, the process of definition construction and the associated van Hiele levels are of paramount importance to the teaching and learning of geometry. It is critical that teachers have knowledge of geometric figures needed to construct definitions as well as the ability to operate at the level of *deduction*. Research in the past has utilized the van Hiele theory as a framework to explore teachers’ and students’ geometric understanding. The van Hiele theory has been used to reorganize curriculums in Soviet Russia in the 1960s and more recently in the United States (Adele, 1996; NCTM, 2000). Definition construction too has been considered a crucial component of the mathematics classroom (Mariotti & Fischbein, 1997; Sfard, 2000; Fujita & Jones, 2007). The van Hiele theory has had a strong influence on the course and scope of geometry instruction at an international level, coupled with the fact that definitions are a fundamental component of the teaching and learning of geometry, it is surprising that no single work has used the van Hiele theory to consider teachers’ responses to definition construction tasks. The goal of this work is to build upon prior research by exploring future and novice mathematics teachers’ basic geometric knowledge of plane figures. Teachers’ responses to definition construction tasks will be analyzed to determine both the completeness of their responses as well as the van Hiele level associated with their responses.

This work addresses the gap in the literature. The research explored novice and future teachers’ responses to 12 definition construction tasks. More specifically the goal was to answer the following questions.

1. Do teachers have the knowledge of planar figures needed to construct concise mathematical definitions? How many key components do future and novice teachers include when asked to create definitions for basic geometric figures?
2. In which van Hiele levels do teachers operate when constructing definitions for basic planar figures?

Methodology

Participants

The 21 participants (16 females and 5 males) in this study were students in a graduate-level mathematics education course during the fall 2009 semester at a large research university in New York State. Eleven were future teachers; all of whom had Bachelor's degrees and were taking graduate courses in order to fulfill state certification requirements for mathematics teachers. While some of the future teachers had temporary teaching experience, none had experience as a full-time classroom instructor. The remaining 10 participants were novice teachers, full-time teachers with up to three years of experience. These 10 novice teachers had Bachelor's degrees and the first level of state teacher certification in mathematics. Four of the novice teachers had prior experience teaching a secondary school geometry course, while one novice teacher was teaching geometry for the first time while taking the course.

Data Collection and Scoring

The data collection instrument for the study was a paper and pencil test with four components. The first component collected demographic information including prior coursework and teaching experience. The second component included items that addressed basic geometric figures. The third component included items that addressed geometric calculations, theorems, and proofs. The fourth and final component included items related to pedagogical decisions related to geometry instruction. The four components of the instrument were administered on the first day of the course as a pretest. This study analyzes the second section of the pretest, focusing on the teachers' responses related to definition and drawings tasks of basic planar geometric figures.

The teachers were instructed to give a written definition for each of the following figures in the order listed: circle, triangle, quadrilateral, vertical angles, isosceles triangle, rectangle, parallelogram, chord, kite, trapezoid, sector, and rhombus. These geometric objects will be referred to as figures throughout this paper. This research considered the teachers' written definitions.

A four-point rubric with values ranging from zero to three was developed to score the teachers' pretest responses. In general there were two to three components for each of the written definitions and sketched diagrams making the four-point scale most appropriate. The scores for each level of the scale for the teachers' written definitions follow.

0 *Incorrect*: Blank, incorrect, or irrelevant response; response yields no real conceptualization of geometric figure; all key components are missing

1 *Mostly incorrect*: Incorrect, or limited correct information is given however two key components of the definition are missing

2 *Mostly correct*: Partially correct information is given however one key component of the definition is missing

3 *Correct*: Complete, correct response all key components of the definition are present

The second rubric used to analyze the data assessed the teachers' responses to the definition construction task and assigned a corresponding van Hiele level. The nature of a definition task involving geometric figures is such that it can reveal information only as high as *abstraction* level cognitive activity. In order to assess cognitive activity at levels higher than *abstraction* one would need to include tasks that involve deductive reasoning such as determining the equivalence of multiple definitions. Since such tasks are outside the scope of this investigation, the rubric has a maximum value corresponding to the level of *abstraction*. The rubric employed in this study was based on Gutierrez and Jaime's (1998) work related to definition construction processes (see Table 2). The rubric was scored using integer values from 0 to 3 as shown below. Examples for each level are provided in Table 3.

0 Provided no definition or a definition with irrelevant/incorrect information
1 *Visualization*: Provided a definition based on the physical properties of the figure
2 *Analysis*: Provided a definition that listed the mathematical properties of the figure
3 *Abstraction*: Provided a definition based on a set of necessary and sufficient properties of the figure

Teacher	Definition for Rectangle	Scoring Rationale	Score
A	A quadrilateral w[ith] 2 distinct sets of congruent sides	Definition is incorrect, exhibits signs of hierarchical classification error	0
B	A figure with 2 sets of opposite sides which are parallel. 1 pair of sides are equal. The other pair are also equal but not the same as the first pair	Definition is based on a prototypical image of the figure	1
C	A closed figure consisting of four line segments with the following conditions. • 4 rt [angles] • 2 pairs of opposite sides $\parallel$ • 2 pairs of opposite sides $\cong$	A list of mathematical properties is given	2
D	A parallelogram whose diagonals are $\cong$	Necessary and sufficient conditions are given with minimal to no redundancy	3

Table 3. van Hiele level sample responses for rectangle definition construction task

Results

The grand mean for the teachers' scores on definitions was 2.31. As mentioned above, a rubric scale value of 3 indicated that the definitions were complete whereas a mean score of 2 indicated that a teacher's response was missing a key component on the majority of the definitions. Mean scores of 1 indicated very little demonstrated understanding of the twelve figures present on the pretest.

Assuming interval level measurement, mean scores for the teachers' responses to the definition tasks ranged from 1.00 to 3.00 with a grand mean of 2.31. Nine of the 21 teachers' responses to the definition tasks had mean scores between 2.50 and 3.00, nine of the teachers' responses had mean scores between 2.00 and 2.50, and three teachers' responses had mean scores between 1.00 and 2.00. For measures of dispersion standard deviation values ranged from 0.00 to 1.35. Six of the 21 teachers' responses resulted in standard deviation values greater than 1.00. One teacher in particular received the highest possible scores on some items and lowest possible scores on others.[3]

The teachers' responses to the definition tasks for three figures were assigned mean and median rubric values of 2 or less. Two teachers did not respond to the chord definition task (M = 1.95, SD = 1.16; Mdn = 2, IQR = 2). One teacher referred to a personal concept image indicating

[3] Interval and ordinal descriptive statistics supported each other; however, more precision was evident when using mean scores. For the sake of brevity means scores are reported.

that a chord was "a combination of notes…" These three teachers' were assigned scores of 0. Teachers who were assigned scores of 1 or 2 for their responses most often referred to the chord as a line rather than a segment, or disqualified the diameter as a special case of the chord.

Two teachers did not respond to the kite definition task (*M* = 1.33, *SD* = 1.11; Mdn = 1, IQR = 1). Three teachers referred to a personal concept image. One teacher referred to a kite as "a recreational item used in high wind," another said "something you fly," and a third said "a diamond shaped polygon." Another teacher simply wrote the word "rhombus." These six teachers did not indicate and of the key components of the figure and were assigned scores of 0. Those teachers who were assigned scores of 1 or 2 for their responses to the kite definition task most often described the figure in terms of the parallelism of opposing sides or congruence of angle measurement. The key component of congruency of adjacent sides was absent in these responses.

Two teachers did not respond to the sector definition task (*M* = 1, *SD* = 0.95; Mdn = 1, IQR = 1). Two teachers referred to a personal concept image. One teacher referred to a sector as "a section of a vector," and another said "in business, it is used to explain that the office building is split into sections." These four teachers' responses were assigned scores of 0. Although a sector refers to the region enclosed by two radii and an arc of a circle, the remaining teachers whose responses were scored 0 most often believed that sector referred to bisection of a figure like a rectangle. Those teachers who were assigned a score of 1 indicated that the sector was the region bounded by a circle and a chord. Three teachers indicated that sector dealt with the perimeter of a circle rather than its area. One of these teachers referred to a sector as "a[sic] arc or line between two points," and another wrote "a piece of the circumference of a circle." These teachers were assigned scores of 2 as they included two key components the center point and radii.

The grand mean for the van Hiele level assigned to the teachers' responses for the definition tasks was 1.58 with no teacher's responses to the definition tasks resulting in mean values at a level of *abstraction*. A mean or median van Hiele level value of 3 indicated that a teacher' responses to the 12 definition tasks contained only those conditions that were necessary and sufficient to define the figures. In contrast values of 2 and 1 for a given figure indicated that some of the teacher's responses were *analysis* level or *visualization* level or both. *Analysis* level definitions and drawings included lists of the properties of the 12 figures while *visualization* level definitions were based on prototypical versions or the physical properties of the 12 figures.

For interval measures mean values for the teachers' responses to the definition task ranged from 0.33 to 2.67. Six of the 21 teachers' responses to the definition tasks had mean values between 2.00 and 2.67 corresponding to a level of *analysis*. Fourteen of the 21 teachers' responses received values between 1 and 2 corresponding to a level of *visualization* for the definition tasks. One of the 21 teachers' responses received a mean value of 0.33. For measures of dispersion standard deviation values ranged from 0.49 to 1.44. Twelve of the 21 teachers' responses resulted in standard deviation values greater than 1. For many teachers low values assigned to responses to the chord, circle, kite, and sector definition tasks resulted in larger spread.[2]

Discussion

Despite the fact that all of the teachers in the study had extensive mathematical coursework in their undergraduate degrees, their results on the pretest were unimpressive. It was hypothesized that the definition tasks would pose no difficulty for the teachers due to the basic nature of the 12 figures under consideration. The expectation was that the teachers would receive scores at or near the highest assessable score, a rubric score of 3; however, the grand mean for the teachers' responses to the definition tasks was $M = 2.31$. Results suggest that most teachers were missing one key component according to the scoring rubric. Mean scores by teachers for definitions provide evidence that many teachers' definitions were incomplete with ninety-five percent of the teachers scoring less than 3. Fourteen percent of the twenty-one teachers scored less than 2 indicating that their definitions were missing more than one key component. Many teachers demonstrated trouble recalling the components of the geometric figures in their definitions.

These low scores are attributable in large part to incomplete knowledge of the figures. These results suggest that the teachers in this study did not possess knowledge of geometric figures sufficient to teach the material indicated in the New York State Education Department's (2005) Learning Standards for Mathematics or the NCTM's (2000) PSSM document. Results support researchers' claims that there are deficits in teachers' content knowledge (Chappell, 2003).

Overall scores for definitions provided additional evidence that many of the teachers' definitions were not at a level appropriate for the coursework that they would be expected to teach. The teachers' responses to the definition task corresponded to low levels on the van Hiele model. Most teachers' definitions corresponded to *analysis* and were appropriate for grades 3-5 expectations according to national standards. Some teachers' responses aligned with the lowest van Hiele level, *visualization*, and were appropriate for grades Pre-K-2 according to national standards.

Recent curricular revisions in New York State, as well as suggested revisions in other states (e.g., Georgia), have brought a sharper focus on geometry content in the curriculum. This renewed focus on the geometry curriculum reminds us how important teachers' geometric knowledge is. The ability to define, and to do so with an awareness of what it means to define, is a critical component of effective instruction in geometry. Although the sample size in this study is limited and any attempt to extend these finding to a broader audience should be done with extreme caution, the data presented here are a platform for further exploration of teachers' knowledge of geometric figures, in general, and definition construction, in particular. Future work will address the use of the van Hiele theory to explore teachers' use of multiple representations when considering geometric figures.

Endnotes

1. Gutiérrez and Jaime (1998) purposely omit the *rigor* level indicating that they do not include it because its existence has not been confirmed. Level 4, rigor, is outside the scope of this research so its absence is of little import.
2. As with scores, the ordinal and interval statistics for van Hiele level values supported each other.

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AUTHORING MATH: MATHEMATICS LEARNERS AS BRICOLEURS

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This paper proposes that groups of middle school students who pose sub-problems as they solve open-ended mathematics problems might be considered to be "authoring" their mathematics. Using a definition of authoring as "the means through which a learner acquires facility in using community validated mathematical knowledge and skills"(Povey et al, 1999, p. 232) and of bricolage as a negotional form of reasoning, this paper argues that through problem posing students become bricoleurs, making discoveries by reassembling what they already know. In doing so, they create a storyline through their actions and ideas, in effect, becoming authors of their own mathematics.

Introduction

> *Mathematical work does not proceed along the narrow logical path of truth to truth to truth, but bravely or gropingly follows deviations through the surrounding marshland of propositions which are neither simply and wholly true nor simply and wholly false (Papert, 1980, p. 195).*

As a middle school teacher, I've found myself straddling the divide between mathematics and the humanities, teaching both math and English as my major subjects rather than the usual combination of math and science. One of the ways I've bridged this divide is through Problem of the Week (POW) (Tsuruda, 1994) assignments. Students are presented with a rich mathematics problem and given class time, and the opportunity to choose their own groups, to work on their ideas before composing a written account of their reasoning. They have quickly learned that the problems require thought, and can be solved in different ways. Despite the groaning that inevitably occurs whenever I announce it is time for another POW, at the end of each year when I survey students about what has gone well (or poorly) for them in math class, POWs are often highly rated. Although the questions are "hard," students often report that they like the challenge, the different kinds of math that they explore, but most of all, they are proud of their work. There is a sense of ownership similar to what I observe during English class when they share their creative writing with their peers. What is going on?

For my master's research, I had used my students' work on POW to consider the phenomenon of group flow. Looking at the transcripts again a few years later, it struck me that each of the groups that I considered to be functioning well (i.e. on task and working together) were asking questions – posing sub-problems as it were – about the POW as they worked towards an overall solution. This problem posing appeared to be a way of making the POW their own, of framing the mathematics in their own way.

Recalling my own experiences with English literature, it occurs to me part of the pleasure of being an author comes from the little conflicts you set for your characters and watching how their resolution serves to drive the main plot. Consider "classic" storylines that recur time and time again. Shakespeare's central problem of "star-cross'd lovers" in *Romeo and Juliet* is echoed in our contemporary *West Side Story* and even the more recent *High School Musical* – and Shakespeare's play itself is a descendant of Arthur Brookes' 1562 poem *The Tragicall Historye* *of Romeus and Juliet,* which is itself a translated interpretation of one of Bandello's Italian short stories *Novelle* (Drabble, 1985). Yet each author has made the storyline his/her own. While the overarching problem is the same (young couple from warring worlds come to a tragic end), it is how the smaller problems, or conflicts, are settled that makes each text unique. Might the same be said of middle school math? While the students are unlikely to develop ground-breaking theorems, there is much that is worthwhile to be drawn from playing around within the parameters of a larger problem and, in doing so, finding their own path through it. Could what the POW groups were doing in their problem posing be considered to be authoring in math? And might authoring be a way for students to be a part of the larger math community, to do math rather than to have math done to them?

The Art of the Bricoleur

The bricoleur "derives his poetry from the fact that he does not confine himself to accomplishment and execution: he speaks not only with things…but through the medium of things" (Lévi-Strauss, 1966, p. 21).

The central issue in this paper is: are we so conditioned to expect the act of mathematizing in school to proceed in a certain abstract formalized way that we are neglecting other ways in which mathematical understanding may emerge?

Papert is one thinker who has touched on this issue in his consideration of *bricolage* as a method students use when exploring geometrical concepts with computer tools (LOGO). Bricolage is a term that originates from Claude Lévi-Strauss's anthropological work *The Savage Mind* (1966). In it he contrasts what he considers to be the analytic method of scientific theorizing of the Western world, with the more concrete theorizing, or bricolage, employed in other more "primitive" cultures. He writes:

> The "bricoleur" is adept at performing a large number of diverse tasks; but, unlike the engineer, he does not subordinate each of them to the availability of raw materials and tools conceived and procured for the purpose of the project. His universe of instruments is closed and the rules of his game are always to make do with 'whatever is at hand', that is to say with a set of tools and materials which is always finite and is also heterogeneous because what it contains bears no relation to the current project, or indeed to any particular project, but is the contingent result of all the occasions there have been to renew or enrich the stock or to maintain it with the remains of previous constructions or destructions. (Lévi-Strauss, 1966, p. 17-18)

While the "engineer" carefully plans ahead and gathers the specific tools required before the project begins, the bricoleur does what he/she can with what is immediately available.

Some have argued that bricolage is an inferior form of reasoning, although not one limited to particular cultures as Lévi-Strauss appears to suggest (Berry & Irvine, 1986). In teaching, for instance, the bricoleur teacher borrows and uses old teaching methods and resources without improving them, resulting in pedagogical inadequacies (Hatton, 1989). In a slightly less negative view, bricolage is depicted as a method teachers are forced to fall back on due to working conditions – without enough time or resources to engage in proper planning, the bricoleur teacher ends up gathering lesson and unit plans from here and there in order to get by (Scribner, 2005).

These interpretations of bricolage as just wholesale "borrowing" are too simplistic. A teacher using another's lesson plan may be using what's on hand, but in doing so he/she is "reinventing based on evolving intentions" (Reilly, 2009, p. 383) and the results may be far different from the purpose of the sources. Although original meaning is sedimented in the artifact being used, because these artifacts are being recombined with other ones, the resulting connections develop into new codes of meaning (Barker, 2004). One can see this, for instance, with fashion (Harajuku, a Japanese street style that mixes, for instance, traditional Japanese attire with modern Western wear), or with writing (punk writer Kathy Acker's *Great Expectations* which "plagiarizes" heavily from Dickens's novel of the same name but to a much different effect). Bricolage, then, has the potential to be both creative and subversive.

Papert suggests that learning is itself bricolage:

> The process reminds one of tinkering: learning consists of building up a set of materials and tools that one can handle and manipulate. Perhaps most central of all, it is a process of working with what you've got. We're all familiar with this process on the conscious level, for example, when we attack a problem empirically, trying out all the things that we have ever known to have worked on similar problems before. But here I suggest that working with what you've got is a shorthand for deeper, even unconscious learning processes (1980, p. 173).

In a later study of computer programming students, Turkle and Papert define bricolage as "a style of organizing work that invites descriptions such as negotiational rather than planned in advance" (1990, p. 144). Working from Piaget's theory of intellectual development, although rejecting the hierarchy of his proposed stages, the authors characterize the engineer, who they call the "planner," as having a "formal" method as compared to the bricoleur who uses a "concrete" method. In comparing these learners, they write:

> The bricoleur resembles the painter who stands back between brushstrokes, looks at the canvas, and only after this contemplation, decides what to do next. For planners, mistakes are missteps; for bricoleurs they are the essence of a navigation by mid-course corrections. For planners, a program is an instrument for premeditated control; bricoleurs have goals, but set out to realize them in the spirit of a collaborative venture with the machine. For planners, getting a program to work is like "saying one's piece"; for bricoleurs it is more like a conversation than a monologue. In cooking, this would be the style of those who do not follow recipes and instead make a series of decisions according to taste. While hierarchy and abstraction are valued by the structured programmers' planner's aesthetic, bricoleur programmers prefer negotiation and rearrangement of their materials (1990, p. 136).

I quote this passage at length because it highlights important aspects which I will discuss further later, namely: the importance of mistakes as "mid-course corrections", the contemplation that takes place during the work process, and the characterization of it being "more like a conversation than a monologue" involving a negotiation and rearrangement of ideas. This, I believe, describes in part how a small group works. However, what tends to be privileged in mathematics classes is the more formal thinking that underlies the planner's process, the abstract reasoning that theorists like Piaget see as evidence of a later more mature stage of development. Turkle and Papert (1990) argue, however, that formal and concrete reasoning are not different stages of development, just different styles of organizing work.

Mathematics as Bricolage

That mathematics is largely perceived in North American schools as a timeless truth, external to human existence should be no surprise. There is a long tradition in Western thought stretching back to Plato and his ideal forms, of mathematics as an eternal absolute, and that it is only through thinking and theorizing by an elite group (i.e. mathematicians) that its laws and axioms can be uncovered. Lakoff and Nuñez call this "standard folk theory of what mathematics is for our culture" (2000, p. 340) the Romance of Mathematics, and they argue that its influence has had a number of negative effects:

It intimidates people, alienates them from math, maintains an elite and justifies it. It rewards incomprehensibility, and this inaccessibility perpetuates the romance. The alienation and inaccessibility contributes to the division in our society of people who can function in an increasingly technical economy and those that can't – social and economic stratification of society (2000, p. 341).

In the past century, however, an alternate view of mathematics has arisen – that of mathematics as a human invention that has been developed and refined by various societies throughout history. Lakatos, the philosopher who first set this idea out clearly (Ernest, 1998), argued for what he called "quasi-empiricism" in his *Proofs and Refutations* (1976). Here mathematics is not portrayed as static Platonic form that is discovered, but as a process, an evolving aspect of culture. The conversation between teacher and students as they discuss the Euler characteristic at first seems to be a Socratic dialogue where the teacher is apprenticing his students into traditional conventions of proper mathematical arguments. However, the alternative narrative provided by the footnotes undermines this interpretation, showing how "acceptable" mathematical strategies have varied during different eras of history, and pointing to an analogy between political ideologies and scientific theories (Lakatos, 1976, p. 49). Returning to the main storyline of the book, it becomes clear from the characters' arguments that the process of refining a mathematical proof is never-ending. There is always something else to consider. In the working, and reworking of Euler's axioms, Lakatos illustrates how mathematics is produced through a process of bricolage.

Authoring in Math

As math evolves, an invented tapestry of concepts, problems and invented rules is passed along in the community. Yet, even as an invention, mathematics is still largely presented to the general public as one unattached to any names or faces, ahistorical as it were. Unlike science, where Newton supposedly discovers gravity after being hit on the head with an apple, and Galileo defies the Church in his support of a heliocentric view of the universe, math has few founding stories. We can point to "the ancient Greeks," and maybe associate one particular ancient Greek (Pythagoras) with a hypotenuse, but there is little else to link math to the popular imagination. "The human meaning-making has been expunged from the accounts of mathematics that appear in standard texts; the contents are then portrayed in classrooms as authorless, as independent of time and place…. Mathematics becomes a cultural form suffused with mystery and power" (Povey *et al*, 1999, p. 235).

As a result, school mathematics is perceived by many students to be a series of rules imposed by an outside source, be it textbook or teacher, with little recognition that human thinking, even student thinking, generates math. "All the mathematical methods and relationships that are now known and taught to schoolchildren started as questions, yet students do not see the questions. Instead, they are taught content that often appears as a long list of answers to questions that nobody has ever asked" (Boaler, 2008, p. 27). And there are many teachers continue to present math as a set of rules and conventions, and to perceive math education as the indoctrination of their pupils into this long list of answers.

The concept of *authoring* math struggles against this sense of mystery. Brown (1996) suggests that, as the focus of math educators turns more to math activities rather than to the mathematics itself, interpretation plays far greater a role – for instance, the students' understanding of a mathematical situation, and how their interpretation changes as they notice new aspects of the situation and make new connections. This emphasis on interpretation, Brown argues, is similar to Gadamerian hermeneutics in that the meaning of the mathematics arises from the activity and the language used to frame it. And in that sense, it opens up the possibility of authorship to any of us who choose to engage in mathematics and communicate our interpretations to others. Mathematician Jonathan Borwein writes, "we respect *authority*, but value *authorship* deeply however much the two values are in conflict. For example, the more I recast someone else's ideas in my own words, the more I enhance my authorship while undermining the original authority of the notions" (2006, p. 3).

Povey *et al* (1999) define *authoring* as "the means through which a learner acquires facility in using community validated mathematical knowledge and skills" (p. 232). They use the term *author/ity* to play with the concept of *authority* – the traditional view of mathematical knowledge as external, fixed and absolute – splitting up the word to foreground the idea of there being an author (or authors) behind the scenes who negotiates and creates this knowledge. In this scenario, they write, "teachers and learners… work implicitly (and, perhaps, explicitly) with an understanding that they are members of a knowledge-making community.... As such, meaning is understood as negotiated. External sources are consulted and respected, but they are also evaluated critically by the knowledge makers, those making meaning of mathematics in the classroom with whom *author/ity* rests" (Povey *et al*, 1999, p. 234). In a knowledge-making community students are aware of the fact that someone did author the math in the books around them, and that in their work as bricoleurs they too are part of this authoring tradition. This helps to demystify math, and to alleviate students' fear of the subject. As well, not only does the notion of author/ity foster a more equal relationship between learners and teacher, but it also gives responsibility to the students to break away from the role of passive empty vessels waiting to be filled with facts, and instead make meaning of mathematics for themselves.

Problem Posing as Pushing the Story Along

In math, problem posing can be defined as both "the creation of questions in a mathematical context and to the reformulation, for solution, of ill structured existing problems" (Pirie, 2002, p. 929). One might say there are two kinds, depending on the purpose of the problem being posed (Silver, 1994), and where it occurs in relation to the problem solving process – i.e. when during the mathematical story process you happen to enter.

If you walk into the story outside of the problem solving process, either before it has begun, or when it is finished, the problem posing has the purpose of *problem formulation*, or "What new problems are suggested by this situation, problem or experience?" (Silver, 1994). Here the problem is generated from the situation itself, perhaps using techniques such as "what-if-not" (Brown & Walters, 2005). Silver (1994) suggests that it also occurs after a problem has been solved, akin to Polya's "look back and reflect" stage, to consider any new situations or problems that have arisen. Others disagree, arguing that "working from situations… is not the same as working from problems. Part of the activity is, in fact, the formulation of [local] problems that may arise out of definitions and rules that are developed in the discussion of the situation"

(Banwell, Saunders and Tahta, 1972). Perhaps the difference in these points of view is between creating a story based on your experiences of what's around you, and being inspired by the plot of another's story to develop your own.

If you walk into the middle of the story, while problem solving is occurring, problem posing takes the form of *problem reformulation,* or: "How can I (re)formulate this problem so that it can be solved?" (Silver, 1994). Here a related problem is generated in response to the original problem, as a way of making that original problem more accessible (Polya, 1957). This could take the form of modifications (Whitin, 2004), perhaps rewording the original problem, or setting it in a more accessible context, so that it can be directly solved. It could also take the form of purposely extending the original problem (Whitin, 2004) to see what else there is to investigate. Finally, problem reformulation might occur through the creation of sub-problems that are produced as one pursues the original problem, as a way of breaking the original problem up into more digestible pieces, or of dealing with situations that need to be resolved before the problem itself can be tackled. Despite what on the surface may seem a very pragmatic way to go about solving a problem, this reformulation is a creative process with the evolving problem taking on a life of its own (Kilpatrick, 1987):

Duncker (1945) offered the insight that each stage of the solution of a problem constitutes the problem's reformulation. Thus the mediating phases provide opportunity for problem-posing along the way. These re-formulations are products of creative thought. Finding and posing the problem is the critical outer layer of the problem solving process. Once that layer is peeled away, it reveals further layers within which new problems reside problems that must be addressed as steps in the finding of a grand solution (Lewis, Petrina & Hill, 1998).

Just as a literary story is propelled along through a series of unanswered questions, so too is a mathematical story. In describing how painters work, Sawyer notes that some artists have an improvisational style called *problem-finding* by creativity researchers, one which involves "constantly searching for her or his visual problem while painting" (2000, p. 153). He then argues that "[a]n improvisational theater performance is also, of necessity, a problem-finding process – albeit a collective one, akin to a brainstorming session" (2000, p. 154).

Improvisational theory is helpful in framing the interaction involved in these communications, "provid[ing] a means to characterize this process [the notion of self organization fundamental to the emergence of the collective] and offer[ing] a way of pointing to the conditions under which the collective might grow and develop" (Martin, Towers & Pirie, 2006, p. 159).

The nature of human behavior itself is improvisational:

> (T)o be thinking what he is here and now up against, [someone] must both be trying to adjust himself to just this present once-only situation ***and*** in doing this to be applying lessons already learned. There must be in his response a union of some Ad Hockery with some know-how. If he is not at once ***improvising*** and improvising ***warily,*** he is not engaging his somewhat trained wits in a partly fresh situation. It is the pitting of an acquired competence or skill against unprogrammed opportunity, obstacle or hazard. It is a bit like putting some ***new*** wine into ***old*** bottles (Ryle, 1979, p. 129).

Thanks to the popularity in recent years of improvisational groups such as Second City, and television shows like *Whose Line is It Anyways*, "improvisation" has become synonymous with spontaneity in North American popular culture. However as Ryle points out, with ad hockery comes know-how, and what is new is based on the old. It is, in this sense, a form of bricolage that recombines and reworks old ideas into new patterns. There is both structure and the freedom to work with/within that structure.

In an interview with musicologist Paul Berliner (1994, p. 66-71), jazz musician Lee Konitz suggests that there is a range of behaviours that can be considered improvisational, depending on the proportion of structure to spontaneity, a concept Weick terms “full spectrum improvisation” (1998). On the most rigid side of the spectrum (“Interpretation”), there is little change from the original structure. In jazz, this occurs when a song is performed and the musicians play all the notes in the original order, but they stress them differently in order to create their own interpretation. In theatre, the players follow the script, saying all of the words in the correct order, but they use their voices and make gestures to provide their own interpretation of what the written text means. An interpretation in a mathematics class might occur when students are first learning to work with example spaces (Watson & Mason, 2005). They are provided with an example (e.g. ½) and then asked to generate more on their own (e.g. 0.5, 4 ÷ 8, ¼ + ¼, etc). Interpretation, then, is like following a path through the woods – you can walk, you can skip, you can run, you can do a series of somersaults, but you keep to the constraints of the path.

At the other end of the spectrum (“improvision”) there is the potential for complete transformation. The song provides a starting point but parts of the melody are changed, or completely replaced, to such an extent that the resulting song does not resemble the original. In theatre, the actors may start with Cinderella, but then surround her with other characters (Superman, Jay Leno, etc.), strand them on a desert island, note that it’s Thanksgiving, and take it from there, throwing in other ideas and characters, and changing or removing others as the story moves along.

It is at this end of the spectrum that I propose mathematical problem posing belongs. “The learner becomes a *bricoleur* who assembles new things from old. In the process, an idea for an entirely new object may arise, leading to new mathematical questions, new exploration, and possibly revision of previous ideas” (Watson & Mason, 2005, p.157). In each of these cases, it is only when the performers/students look back that they can discern a pattern, or storyline, to their actions and ideas: “The improvisational process is one of laying a path down in walking” (Montuori, 2003, p. 246). It is through the laying down of this path that students become authors of mathematics.

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THE RELATIONSHIP BETWEEN TEACHER PEDAGOGICAL CONTENT KNOWLEDGE AND STUDENT UNDERSTANDING OF INTEGER OPERATIONS

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This study examined whether professional development (PD) related to integers improved pedagogical content knowledge (PCK) of teachers. The three week program involved 22 teachers and 341 grade 8 students from an urban district. The teachers and students explored a comprehensive representation for integers using vectors on a number line and engaged in argumentation about strategies for modeling integer operations. The results showed significant growth in teacher PCK following the PD and a significant association between teacher posttest PCK and student improvement in understanding even when controlling for years of teaching experience, teacher pretest knowledge, and student pretest knowledge.

Improving student understanding of integers and operations with integers is important for many educators; however, there is not one instructional model for integer operations that seems satisfactory in explaining all integer operations. Integers are the set of positive whole numbers, negative whole numbers and zero. Integers are a number and operations concept that is foundational in mathematics, and used in applications in other fields such as science, business, and statistics. However, students struggle with this foundational concept, particularly around negative numbers (Ryan & Williams, 2007).

Introduction

The mathematics education community has struggled to find an instructional model that effectively supports students in constructing a rich conceptual understanding of integers and of negative numbers, in particular. Such a model would need to address the purpose of the negative numbers and justify the arithmetical operations on them in order to be comprehensive. The following models are commonly used in classrooms K-8: Annihilation model with counters or chips, movement on the number line; transformations of an object's position and state (Schwarz, Kohn, & Resnick, 1993/1994); elevator or elevation model ; use of metaphors for example, people getting on and off a bus; technology, such as, calculators and virtual manipulatives; and other real world applications, to name a few, money, temperature, and yardage in football.

These models lack the comprehensiveness needed to address all operations with negative numbers (Fischbein, 1994). Due to the lack of a comprehensive instructional model, many students struggle with understanding integers and integer operations, developing misconceptions about these numbers (Ryan & Williams, 2007). This study is a step towards developing an effective and comprehensive instructional model using a number line vector representation along with professional development (PD) for teachers to build pedagogical content knowledge (PCK) focusing on engaging students in classroom discourse through argumentation.

Theoretical Framework

Teaching mathematics is a complex profession which requires more than just knowledge of the subject matter. According to Fennema and Franke (1992), the components of mathematics teachers' knowledge include the following: knowledge of mathematics, knowledge of mathematical representation, knowledge of students, knowledge of students' cognitions, and knowledge of teaching and decision making. Mathematical representations play an important role in helping students connect abstract mathematics into something they can understand. Some researchers have focused on developing what is referred to as pedagogical content knowledge (PCK). Grouws and Schultz (1996) include in their description of PCK the ways teachers provide students with useful unifying ideas, clarifying examples and counter examples, helpful analogies, important relationships, and connections among ideas. This interpretation includes more of the behaviors of teachers rather than just the types of knowledge they possess.

The theoretical framework for this study is informed by research on the relationship between teacher PCK and student learning. Turnuklu and Yesildere (2007) in their research of PCK in mathematics of 45 pre-service teachers in Turkey and their approaches to teaching fractions, decimals, and integers, found that many of the pre-service teachers had difficulty understanding student misconceptions related to integer operations, facilitating mathematical discussions that would engage students in mathematical thinking about integers, and asking questions to assess student understanding. Their work was informed by the research of An, Kulm, and Wu (2004) which they used to construct the following theory: (a) PCK is associated with teacher beliefs, content, knowledge and how one teaches, (b) teaching is associated with PCK and knowledge about students' thinking, (c) knowledge about students' thinking is associated with addressing students' misconceptions, engaging students in math learning, building on students' ideas, promoting students' thinking about mathematics, and (d) student learning is most closely associated with knowledge about students but due to the interrelationship of the model, student learning is associated with all of the four components.

This integrated approach of improving teacher PCK, developing the ability of teachers to facilitate conversations about mathematics among students in their classrooms, and the resulting student conversations, I propose, is vital to improving student learning. There is growing evidence that students in elementary, middle, and high school can reason, justify their thinking, make claims and warrants in a supportive classroom that is a mathematical community (e.g., Enyedy, 2003; Francisco & Maher, 2005; Goos, 2004; Mueller & Maher, 2009). However, developing this kind of environment requires expectations for behavior, norms for making claims and warrants, activities that promote different ideas, and facilitation of the development of reasoning through teacher questions to advance student thinking (McCrone, 2005).

Most mathematics teachers are familiar with mathematical reasoning, but formal reasoning is often reserved for students in high school when they are taught how to construct proofs as mathematical arguments using accepted statements considered facts such as definitions and theorems established by the mathematical community. They learn the acceptable form of justifying their thinking in the proof format expected by their teacher or the textbook. However, Stylianides (2007) and Francisco and Maher (2005), argue that this concept of mathematical proof can occur as early as elementary school where students are given opportunities to reason and justify their thinking. Students and teachers can collaborate to establish classroom norms for what is acceptable for a proof in their class community. These experiences can be important prerequisites for future work with more formal mathematical proofs used by participants in the larger mathematics community to communicate to other mathematicians.

Therefore, I determined that mathematical reasoning using argumentation would be an age appropriate activity for students in this study. My hypothesis was that teacher understanding of integer operations would improve by exposing them to activities that apply integers in real world contexts related to vector forces and temperature change and activities that explore integer operations using a number line. I also hypothesized that teachers would be enabled to facilitate student argumentation around misconceptions after experiencing their own use of argumentation around misconceptions and solution strategies and creating structures for their classroom to facilitate argumentation. Finally, I proposed that student understanding of integer operations would improve when they were provided with opportunities to engage in classroom discourse and argumentation about doing mathematics, which would be evident by improved performance on a posttest compared to the pretest. Specifically, the following research questions were investigated:

- What are the general patterns of teacher content and pedagogical knowledge of integers based on responses to a pretest and posttest?
- To what extent does PD impact teacher content and pedagogical knowledge as measured by growth between pretest and posttest?
- Is there a significant difference between the growth between students' pretest and posttest scores for 2010 compared to the growth made by students in 2009?
- Do differences in teacher content and pedagogical knowledge explain more of the variance in student performance than years of teaching experience of a teacher?

Methods

This research was conducted during a three-week summer program for grade 8 students in a large urban school district in Central Texas. This program serves students who had not passed the state mathematics assessment, Texas Assessment of Knowledge and Skills (TAKS) test, required for promotion to high school. Each year, program administrators are challenged to find teachers to teach in the program. Most of the teachers participate in summer school, so few remain that are available or interested in teaching in this program. Each year there have been several first year teachers teaching in the program. There is a need to ensure that the teachers understand the mathematics in the curriculum activities in the program as well as understand common misconceptions that students may have by including PCK features in the PD for the program.

Participants

There were 341 students and 22 teachers in the summer program. Twenty-one of the 22 teachers attended the PD and consented to participate. The age of the teachers ranged from 21 to 63 years with an average age of 34 years. Sixty-two percent of the teachers were female. The average years of teaching experience for these teachers was 7 years (ranged from 1 to 31 years). There were 24 percent of the teachers with a masters' degree and only 10 percent with an undergraduate major in mathematics. Thirty-eight percent had previously taught in this summer program which started the summer of 2008.

Intervention

This research study was considered an intervention that supplemented the current PD that had occurred in the past two years with additional emphasis on the development of teacher PCK related to integers, rather than just exposure to the student activities as was done previously. The six hours of PD that constituted the intervention for this study focused on improving teacher understanding of integers and integer operations. During the PD, the teachers read a chapter from *Children's Mathematics 4-15: Learning from Errors and Misconceptions* by Ryan and Williams (2007) entitled "Children's Mathematical Discussions." The discussion among the teachers resulted in the selection of three important features of a classroom environment that would enable this kind of safe argumentation: students are accepting of wrong answers, misconceptions are an opportunity to learn together, and students share their thinking and question the thinking of others.

The design of the PD for this study was also based on the guidance provided by Thompson and Zeuli (1999) for creating a transformative learning experience for teachers. To create cognitive dissonance, several problem were provided for teachers to engage in reasoning and sharing of their thinking with others that would bring up questions and uncertainty that would need to be resolved through inquiry and discussion, and to provide sufficient time for that process to occur. Through these experiences teachers were challenged to support their own claims about their understanding of integers which deepened their understanding and enabled them to experience the value of argumentation. An engaging discussion broke out when two groups were arguing over the claim that the number line was acceptable for addition, but it was not a good model for subtraction. The discussion that followed lasted almost an hour as teachers struggled with the meaning of representing subtraction on a number line, whether their method would work all the time or just sometimes, and the challenges they felt students would have with the model. In the end the teachers agreed that the number line model might make more sense to students then other models used previously, such as the cancellation model with colored chips.

For the purpose of this study, the curriculum for the integers unit of the program was revised based on a review of research literature on improving student understanding of integer operations to include more activities exploring integers with a number line. During the development of the unit, feedback was provided on improvements that could be made by university professors, graduate students, teachers in the district (including those who previously taught in the program), and the district Secondary Mathematics Supervisor. The goal was for this revised unit to develop the concept of operations on positive and negative numbers in a way that connects the concepts to a real world experience and a number line representation that is comprehensive in its application to all operations of integers. The number line revised unit for integers incorporated the use of vectors on a number line that integrated an application called NUMLIN on the TI-73 calculator with publicly available lessons called "Walking the Line" from the Texas Instruments education technology portal. Figure 1 provides an example of how multiplication of integers was explored with the calculator in these activities, where multiplication is represented as repeated addition or subtraction.

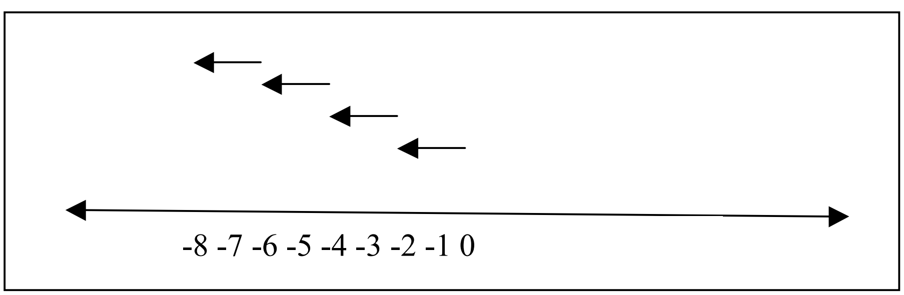

Figure 1. Number line Model for Multiplication of 4 x -2

The students used these number line explorations to discover the way operations on integers work by manipulating the vectors on the TI-73 calculator number line. The curriculum for the summer program also included activities from the *Navigator: Positive and Negative Numbers* curriculum module (America's Choice, 2008), which was developed to address student misconceptions and includes the vector number line representation. Daily journal activities were written for students to practice responding to misconceptions of other fictitious students

Data Collected

Teacher demographic data was collected with a short survey given to teachers during the PD. To measure teacher PCK, an assessment was designed based on three sources: a pre-service teacher's guide entitled *Fostering Children's Mathematical Power: An Investigative Approach to K-8 Mathematics Instruction* (Baroody & Coslick, 1998), research by Turnuklu and Yesildere (2007) with pre-service teachers in Turkey about their mathematical understanding of integer operations, and data collected in a pilot study from interviews of students in grades 7, 9 and 11 (Varma, Harris, Schwartz, & Martin, 2008). To measure student achievement the pretest and posttest from the America's Choice (2008) *Positive and Negative Numbers* curriculum were used, since they had been used for the two previous years, so that the student results of the 2010 summer program could be compared to the results of the 2009 summer program.

Data Analyses

The teacher PCK assessments were scored based on a rubric created from a review of the responses of a volunteer sample of 13 people (teachers and graduate students). The student assessment was scored according to the curriculum guidelines in the teachers' guide. A Hierarchical Linear Model (HLM) was used to examine the relationship between teacher content and pedagogical knowledge of integers and mathematics learning related to the fourth research question. A two-level Hierarchical Linear Model with student math scores as the level one outcome variable (a within-class model) and individual class variables, such as mean math score, as the level two outcome variables (a between-class model) was used to determine the variance in student test scores from pretest to posttest as a function of their class assignment. Classroom-level predictors (teacher content and pedagogical knowledge of integers and years of teaching experience) were added into the level two models to determine whether they explained the variance in mean class scores and pretest/posttest performance slopes. Because the relationships between variables measuring student learning are at least partially dependent on the larger context (the classroom) in which learning occurs, teacher PCK should partially explain the variance across classes, as should the years of teaching experience of a teacher. It was of interest to see whether PCK following PD would outweigh experience in explaining the variance across classes. In addition, an independent samples t-test was used to compare the performance of students in the 2010 program to the previous year's students.

Results

When the pretest and posttest responses of the 18 teachers were compared the change from pretest to posttest in teachers' PCK was positive and statistically significant ($t = 3.29$, $p < .01$). There was also a statistically significant increase ($t = 2.37$, $p < .05$) in teachers' ability to explain the sign of the answer to the problem -5 x (-8) as shown in table 1. Teachers also made statistically significant improvement ($t = 2.55$, $p < .05$) in providing real world and domain examples of applications for integers.

Question (N = 18 teachers)	Pretest Mean (*SD*)	Posttest Mean (*SD*)	Difference (SE)	*t*	*p*-value
Q1. 5 – (-8)	.67 (.77)	1.11 (.96)	.44 (.32)	1.41	.18
Q2. -5 x (-8)	.39 (.70)	1.00 (.97)	.61 (.26)	2.37	.03*
Q3. (-6) + (+7), 6 – (+7)	1.28	1.83	.56	3.34	$p < .01$

	(.83)	(.38)	(.17)		
Q4. 4 – 7 = 3	1.44 (.62)	1.61 (.61)	.17 (.12)	1.37	.19
Q5. 3 – 5, 3 + (-5)	.50 (.71)	.89 (.68)	.39 (.20)	1.94	.07
Q6. Prior Argumentation	1.06 (.87)	1.22 (.88)	.17 (.20)	.83	.42
Q7. Applications	1.06 (.54)	1.33 (.59)	.28 (.11)	2.55	.02*

Note. (a) These values represent the average raw score out of 2 points earned by teachers on the pretest and the posttest for each question. Teachers could earn 0, 1, or 2 points per question. The total questions were worth 14 points. (b) *Statistically significant difference at the $p < .05$ level.

Table 1. Improvement in Teacher PCK Following PD

When the pretest and posttest performance of students in the 2010 program was compared to the performance of students in the 2009 program there was statistically significantly difference between the improvements made for the students in the 2010 program compared to the 2009 program as shown in table 2.

Percent Correct out of 100	2010 Mean (n=177) (*SD*)	2009 Mean (n = 177) (*SD*)	Difference (SE)	*t*	*p*-value
Pre-test	37 (17)	43 (19)	-7 (4)	-1.77	.08
Posttest	51 (15)	49 (21)	2 (4)	.41	.68
Improvement	14 (17)	06 (19)	8 (3)	2.20	.03*

Table 2. Comparison of Student Knowledge of Integers 2010 vs. 2009

The HLM conditional model was as follows:

Level-1 Model

$$Y_{ij} = \beta_0 + \beta_1\text{*(Pretest)} + r_{ij}$$

Level-2 Model

$$\beta_0 = \gamma_{00} + \gamma_{01}\text{*(Teacher Experience)} + \gamma_{02}\text{*(Teacher PCK Pretest)} + \gamma_{03}\text{*(Teacher PCK Posttest)} + u_{0j}$$

$$\beta_1 = \gamma_{10}$$

The findings for this conditional model are summarized in table 3. The findings reveal that, after controlling for prior student and teacher knowledge as well as teacher experience, teacher's PCK significantly predicted student posttest performance (t (14) = 2.37, p = .033). Thus, as teacher's post-intervention PCK increased by one point, there was a 0.22 increase in student math posttest performance. When comparing the conditional model to the unconditional model, the addition of the teacher experience, teacher pretest knowledge and teacher posttest knowledge covariates explained 27 percent of the variance after adjusting for differences in student pretest. This positive relationship between teacher posttest PCK and posttest student math achievement suggests that when teachers have the necessary PCK related to integers and integer operations, students' overall learning of integer concepts and procedures for solving integer operation problems improve.

Parameter	Unconditional Model				Conditional Model			
	Estimate	*SE*	*t*	*p*	Estimate	*SE*	*t*	*p*
Fixed effects								
Intercept	50.79	1.73	29.28	.000	50.49	1.59	31.70	.001
Student Pretest	.44	.06	6.78	.000	.45	.06	7.03	.000
Mean Teacher Experience	—	—	—	—	.04	.23	.17	.866
Mean Teacher PCK Pretest	—	—	—	—	.00	.09	-.04	.968
Mean Teacher PCK Posttest	—	—	—	—	.22	.09	2.37	.033
Random effects	Variance	*df*	χ^2	*p*	Variance	*df*	χ^2	*p*
Intercept	33.80	17	53.74	.000	24.58	14	36.26	.001
Level 1 variance	135.02	—	—	—	135.31	—	—	—

Table 3. Comparison of Results of Conditional vs. Unconditional HLM Analysis

Discussion

The observations made during the PD were that most of the teachers were engaged in the activities where they experienced argumentation around representations of addition and subtraction of integers. There was evidence during the stations activities, that teachers were already making connections as to how they could use certain activities and certain questions to form the problem that would be the center of the argumentation for the students. On the teacher PCK posttest, several teachers shared experiences with students resisting participation in discussions due to low confidence in mathematics and a concern about being wrong. Others shared the challenges of facilitating small group discussions when one of the students was overly confident and verbal, causing the other students just to listen and not share their ideas for fear of being wrong. Twenty-four percent of the teachers admitted to not engaging students in argumentation. Close to 50 percent of teachers on the posttest continued to believe that the only way to understand why -5 x -8 was positive was because of a rule that when one multiplies two negatives (or an even number of negatives) one gets a positive, despite their exposure to other representations and strategies during professional development and through the curriculum they implemented with students.

In conclusion, the professional development and curriculum modifications resulted in statistically significantly positive gains in teacher PCK and student achievement. The change in the professional development, compared to the previous year resulted in less of a focus on student activities and more on developing teacher PCK. As a result, teachers who were originally unable to explain why multiplying two negative numbers results in a positive number were now able to explain their reasoning conceptually on the posttest. Other teachers who did not differentiate between an operation of subtraction and the negative sign of a number at pretesting made improvements in understanding the distinct roles of each of these components of a mathematical expression. The resulting positive change in teacher PCK was associated with positive gains in student understanding on the posttest. To have such a positive effect on teachers and students in just six hours of PD and three weeks of a curriculum and instruction intervention is encouraging for future PD development.

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STOCHASTICS KNOWLEDGE OF PRESERVICE TEACHERS

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This research examined preservice teachers' knowledge of probability and statistics. Clinical interviews were conducted with 2 preservice teachers in which they solved stochastics problems dealing with perceptions of randomness and strategies for a probabilistic game. Results highlight strengths and weaknesses of their stochastics understanding, as well as their perceptions of probability and statistics education in secondary schools. The implications for teacher education are discussed.

Purpose

Statistics and probability (stochastics) topics have been included as essential ideas in both National Council of Teachers of Mathematics (NCTM) standards documents (1989, 2000). However, there is a concern that mathematics teachers do not have the necessary background knowledge for teaching stochastics (Jones, Langrall, & Mooney, 2007; Shaughnessy, 2007; Stohl, 2005). This lack of knowledge is not surprising considering many elementary and secondary teachers have had few or no university courses or professional development dealing with stochastics (Stohl, 2005; Watson, 2001).

As noted in Jones et al. (2007), there is limited research on the stochastics knowledge of pre- and in-service teachers. This study aims to add to the body of research by examining the stochastics knowledge of two preservice teachers (PSTs). The purpose of my research is to discover what they know about probability and statistics and how that knowledge will affect their future teaching.

Literature Review

In this review of the literature, I will focus on several studies that examine preservice (or in-service) teachers' knowledge of stochastics and how teacher education programs prepare future teachers of probability and statistics. Specifically, I will highlight research concerning teachers' stochastic understandings and misconceptions.

Although there is little research on stochastics in teacher education, some studies have found areas of strength and weakness in PST knowledge (Batanero, Godino, & Roa, 2004; Baturo, Cooper, Doyle, & Grant, 2007; Canada, 2006; Koirala, 2003) and the knowledge of in-service teachers (Liu & Thompson, 2004; Watson, 2001).

In Canada's (2006) study of thirty elementary PSTs' notions of variability, subjects were enrolled in a mathematics content course in which they performed various stochastic activities. Canada found the participants had a good understanding of the difference between theoretical results and experimental results. However, a negative aspect of this understanding led participants to believe that if theoretical predictions did not match actual results, then all results are possible and equally likely. When Canada asked subjects to evaluate statistical graphs, most subjects referenced the idea of average. Some focused only on average, while many also looked at the other three components of distributional reasoning (Canada, 2006): range, shape, and spread. This study also commented on the link between proportional reasoning and probabilistic thinking. "An over-reliance on proportional reasoning can lead to a restricted expectation of variation, but an under-reliance on proportional reasoning can also lead to poor expectations" (p. 42).

Koirala (2003) focused on secondary PSTs' use of formal and informal reasoning when solving probabilistic problems in both individual and paired settings. On the individual task, PSTs used both formal (learned from university courses) and informal (based on intuitions and experiences) probabilistic reasoning. When in pairs, the PSTs wanted to use formal reasoning. When a formal thinker (from the individual task) was paired with an informal thinker, the formal thinker began the conversation and the informal thinkers agreed with their partners' solutions. In final individual interviews, all eight subjects initially offered formal reasoning. However, one informal thinker changed her mind and reverted to her informal (incorrect) answer in which she used a representative heuristic. In a similar but more difficult problem, only three of the PSTs used formal reasoning. The author attributed these inconsistencies to "conflict between their informal beliefs and their mathematical knowledge of probability" (Koirala, 2003, p. 153). Even students with a thorough math background did not have a complete understanding of probability. Many were likely to solve a problem incorrectly despite having the requisite formal reasoning required to solve the problem.

Koirala's (2003) results show the differences in reasoning when students are working alone or with a partner. However, she seems to imply that formal reasoning always led to correct solutions, while informal reasoning did not. It was not clear why Koirala seemed to think formal reasoning was always the preferred solution method.

In Liu and Thompson's (2004) work with eight high school teachers, they found their subjects held many viewpoints on probability, including using subjective judgment, not knowing a solution because the predicted event has not actually occurred, and a formal stochastic viewpoint. Positive findings included that teachers understood how variability decreases as sample size increases. Negative results showed that the teachers were not familiar with hypothesis testing, even though some of the teachers were required to teach that topic.

The study (Liu & Thompson, 2004) also found that some teachers thought that a probabilistic situation could have multiple interpretations, while others did not. Both groups, though, wanted to find the "correct answer". In trying to find those answers, the teachers trusted solutions by those they viewed as experts, but they were not confident in their own reasoning. Despite the discussions on formal probability theory, the teachers thought a correct answer could always be determined using a computer simulation. The teachers' reliance on computer models is interesting, because the teachers stated they believed students learned about probability by simply hearing the concepts explained correctly. "This way of teaching was also projected by the ways in which teachers negotiate meanings. We found that teachers often dismissed the opposing point of view as "mistakes", convincing others that they are wrong by telling them they are wrong, instead of trying to understand the other person's intention" (Liu & Thompson, 2004, p. 6).

Through the implementation of a self-designed survey, Watson (2001) sought to understand teachers' strengths, weaknesses, and opinions on the teaching of stochastic topics. Her subjects were 28 secondary and 15 primary level teachers in Australia.

When asked to identify important factors for teaching chance & data, teachers' responses were grouped into four themes: teacher, student, content, and school issues. Twelve teachers listed similar topics as being the most enjoyable for their students and their selves: graphing, normal distribution, surveys and data collection, analyzing and interpreting data, and various topics in probability. Many teachers said their students enjoyed hands-on activities such as probability games.

Teachers rated nine topics by their confidence in teaching each topic, 'odds' had the lowest mean, 'graphical representation' had the highest. All mean confidence ratings by secondary teachers were higher than the highest mean by primary teachers except 'odds'. Secondary teachers were significantly more confident about six topics: equally likely outcomes, average, basic probability calculations, median, graphical representation, and sampling. The difference in confidence levels was attributed to stronger mathematical backgrounds of the secondary teachers.

In one section of the survey, Watson (2001) asked participants to read stochastic problems, anticipate different ways students might respond, and explain how they might use this problem in teaching their classes. The author compared teacher responses with actual student responses to these same questions. In a problem dealing with odds, teachers successfully anticipated student answers.

Baturo et al.'s (2007) study highlighted some misconceptions held by elementary PSTs. The subjects were given a task in which they had to determine if a spinner was fair and confirm their answer by using three testing methods. After examining the spinner, only 41.9% of the participants correctly thought the spinner was fair. After using the first testing method (an overlay that divided the circle into 6 equal parts), many maintained the spinner was unfair, even though they could see that each color had an equal area. After performing & recording trials using the spinner, many teachers were still not convinced of the spinner's fairness. Baturo et al. attribute this to the PSTs' deterministic view of mathematics that would not validate a result that did not show exactly a third of the trials resulting in each color.

Methods

My initial goal for this study was to conduct a teaching experiment in which I would observe PSTs as they worked through a series of stochastics problems. I wanted to recreate Batanero et al.'s (2004) study, and compare their results with mine. Due to time limitations, my research was not a full-scale teaching experiment, but rather consisted of two separate clinical interviews in which I simulated the two learning experiences described by Batanero et al. (2004).

Batanero et al.'s study focused on preservice primary and secondary teachers enrolled in a mathematics methods course. All PSTs were statistics majors in their fourth or fifth years of study at the University of Granada in Spain. Throughout the course, the PSTs engaged in learning activities designed to engage them in learning and discussion of probability and statistics topics. The two activities presented in the study (Batanero et al., 2004) dealt with perceptions of randomness and developing a winning strategy in a probabilistic game. These experiences highlighted the PSTs understandings and misconceptions concerning stochastics topics.

The participants in my study were two PSTs enrolled in a master's of education program at a large southeastern university in the United States. Both students, Anna and Eric, majored in mathematics and took Statistics for Engineering as undergraduates. Neither PST had taken a teacher preparation course focused on probability and statistics.

Results

In this section, I will describe the learning activities and results from my clinical interviews, as well as report the findings from Batanero et al.'s (2004) research. The first task (Figure 1) concerns perceptions of randomness.

Perceptions of Randomness

Some children were each told to toss a coin 40 times. Some did it properly. Others just made it up. They put H for Heads and T for tails. These are Daniel and Diana's results:

Daniel: H T H T T H H T H T H H T T H T T H H T T H T H H T T H T H T H T H T H T T H T

Diana: H T T T H T T H T H T T T H T T T T H H T T T H T T H T T H T T T T T H T T T H T

Did either or both Daniel or Diana make it up? How can you tell?

Figure 1. Perception of randomness task. Batanero et al. took this task from Green (1991).

I told the PST this task had been used with secondary students to understand their ideas about randomness. The PST then gave their thoughts on this task. Both PSTs began by comparing the number of heads and tails in each sequence. They saw both patterns as possible, but each said that Daniel's was more probable because the outcome was closer to the expected number of heads and tails for forty coin tosses. In order to learn more about Eric and Anna's understandings, I asked them for their thoughts on the patterns of the sequences. Even with leading questions, they confused the probability of a certain sequence of heads and tails with the total outcomes of heads and tails. This finding is similar to Batanero et al. (2004); they also found that students recognized frequency properties more easily than run properties.

Next, I presented the PST with data from secondary students' work with the initial task as shown in Table 1. The results show that more students considered Daniel's sequence to be random, as well as differences between the 14- and 18-year-olds' responses. When asked to explain the students' thinking, Anna said that more students think Diana is cheating because they believed it was unlikely for her to obtain four tails in a row. This is the first mention of the pattern of the outcomes as opposed to the count of heads and tails. Anna did not see a noticeable difference between the younger and older students. Eric studied the table quietly for a few moments, and then noticed that a higher proportion of older students responded, "I don't know." He explained that more 18-year-olds might understand there is no way to actually tell whether Daniel or Diana cheated, because each sequence is a possible outcome. He went on to discuss how some students, possibly the 14-year-olds, would have a more rigid belief that when flipping coins, heads should occur close to (or exactly) half of the time. This idea is mentioned in the original study (Batanero et al., 2004), "Whilst a student can think he/she has 'proven' that a sequence is random, the statistician would just say that there is no reason for rejecting the sequence randomness" (Section 4.3, last para.). Batanero et al. (2004) did not report specific PST responses to the table, but they described how changes in the sequences or wording of the task might lead to different student results.

	14 year-old (n=147)		18 year-old (n=130)	
Response	Daniel	Diana	Daniel	Diana
He/she made it up	54 (37%)	83 (56%)	30 (23%)	63 (49%)
He/she did it correctly	86 (58%)	53 (36%)	82 (63%)	48 (37%)
I don't know	7 (5%)	11 (8%)	18 (14%)	19 (14%)

Note. This table shows the responses to the previous question obtained by Serrano (1996) from 277 secondary school students.

Table 1. Frequencies and percentages of secondary school students' answers.

The final discussion of the task occurred after I showed examples of student reasoning (Figure 2) to the PST. Both Anna and Eric explained what a student might be thinking if they gave each of the three responses. I then asked them to put themselves in the role of a teacher, and to consider what they would do with students to continue to explore the idea of randomness. Both of the PSTs suggested letting student perform the coin flipping activity in class. Anna would have her students compile their results and compare the aggregate data with the original data. This would help students see the likelihood of obtaining Daniel or Diana's results. Eric's reasoning behind performing additional trials was to show students that it is possible to obtain many heads or tails in a row. He was concerned that students would not believe that a long sequence of heads was possible unless they saw it happen. During this discussion, Eric brought up the idea of the gambler's fallacy, and mentioned that he would observe the students to see if they held that misconception.

Batanero et al.'s (2004) results focus mainly on how secondary students interacted with this problem. They explain how students consider irregular patterns and even distributions as indicators of randomness. Students considered Diana's outcome as too different from the expected outcome, so they labeled her as the cheater. For students who analyzed the runs, they either thought Diana's was too irregular to be random, or Daniel's was too regular to be random.

Some reasons given by students in Serrano's (1996) research to justify that Daniel or Diana were cheating were the following:

- The sequence pattern is too regular to be random, results almost alternate;
- The frequencies of heads and tails are too different;
- There are too long runs; heads and tails should alternate more frequently.

Figure 2. Explanations of randomness task responses.

Strategy for a Probabilistic Game

The second activity began with the description of a game shown in Figure 3.

We take three counters of the same shape and size. One is blue on both sides, the second is red on both sides and the third is blue on one side and red on the other.

We put the three counters into a box, and shake the box, before selecting a counter at random.

After selecting the counter, we show one of the sides. The aim of the game is to guess the color of the hidden side. We repeat the process, putting the counter again in the box after each new extraction. We make predictions about the hidden side color and win a point each time our prediction is right.

Figure 3. Description of probability game.

After making sure the PST understood the game, I explained how we would be using Fathom statistical software (Finzer & Swenson, 2001) to create a simulation of the game. Each game would consist of ten trials, so I gave the PST a sheet of paper to record their responses and outcomes (Figure 4). The purpose of the activity was to devise a winning strategy for the game.

Trial	1	2	3	4	5	6	7	8	9	10
Seen color										
Predicted color										
Hidden color										

Figure 4. Game score sheet.

Anna discovered the correct winning strategy immediately. She realized that if she predicted the same color as the seen color, then she would have a two-thirds chance of winning each trial. Before we played the game a second time, I asked her how many trials she expected to win. She said she would win about five or six of the ten trials. I asked her to elaborate, because she earlier she had mentioned a two-thirds probability. Anna explained that theoretically, she would win 66.6% of the time using her strategy. Since someone cannot win exactly two-thirds of ten games, she expected to win around six or seven games in each round. Anna pointed out that even with using the strategy, there could be a variety of outcomes, but on average, she expected to win two-thirds of the time.

It took Eric a few rounds to realize the winning strategy of the game. He was focused on the total reds and blues in the bag. Since there were three red sides and three blue sides, he correctly identified the probability of the counter being either color was one-half. I asked guiding questions to lead him to the correct strategy. Upon further review of the interview recording, I believe that I did not explain the game well and Eric did not realize I wanted him to develop a strategy. His thinking was at first centered on the final outcome of the counter, without regarding the ‘seen’ side. After Eric discovered the winning strategy, he was able to prove why the strategy worked. He used a tree diagram to help him understand the probability and justify his reasoning.

Batanero et al. (2004) found that less than half of the PSTs chose the correct strategy at beginning of game. They were given class time to discuss their strategies and try to convince each other of their reasoning. Even after most of the teachers came to the solution, not all could provide mathematical justification for why the strategy worked. This was surprising to the researchers because of the simplicity of the problem and the strong statistics training of the PSTs.

Finally, I showed Eric and Anna some of the strategies used by the PSTs in Batanero et al.'s study (2004) which are displayed in Figure 5. The PSTs gave examples of student conceptions that would lead them to each of the strategies. They also gave suggestions on how they might lead the student to discover the correct strategy. Using this problem with a group of students, both PSTs suggested playing the game multiple times and aggregating the win and loss data for various strategies employed by students.

Common strategies in this game are:
A: Predicting red and blue alternatively;
B: Choosing red (blue) in all the trials
C: Predicting the color at random;
D: Two blues, one red (or something similar)
E: Predicting the color shown on the visible side
F: Predicting the color contrary to that shown on the visible side.

Figure 5. Possible strategies for the game.

Conclusions

Although my study only considered two PSTs, I present some information that leads us to a better understanding of the stochastic knowledge of PSTs. Both PSTs in the study see the importance of teaching probability and statistics. Eric feels comfortable with teaching it, while Anna does not; however, both PSTs have strong content knowledge that will help them in the classroom. Even more encouraging is the way that each PST engaged with the mathematics. They seem confident that they can reason through a stochastics problem to find the solution, even if they do not initially have a strategy. I was also pleased that both PSTs stressed the importance of experimentation when learning probability and statistics. The idea of teaching rules or formulas was not suggested in either PST conversation.

Another outcome I noticed is that neither PST used much probability and statistics vocabulary to describe their thoughts. For example, although both PSTs discussed the idea of variability, neither one used the term. In addition, Anna and Eric were both not familiar with the term 'sample space'. They both seem to have a strong stochastics understanding, but I would not be surprised if they did not score high on a probability test, given their unfamiliarity with the vocabulary. The difference between test scores and actual problem solving ability might be an interesting area for further research.

Another possible topic for further study is to summarize the stochastics experience of mathematics teachers during their undergraduate, graduate, or in-service trainings. I would like to see if certain mathematics education courses made teachers more comfortable and competent at teaching probability and statistics topics.

In conclusion, the results of my study show promise for the future of stochastics education. More work must be done to ensure that all teachers are prepared to teach this important part of the mathematics curriculum.

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MIDDLE SCHOOL STUDENTS' GROWTH IN UNDERSTANDING OF PROBABILISTIC INFERENCE

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Reasoning informally about statistical inference is being able to make inferences about data without using any formal statistical method or procedure. In this paper we aim to share a probabilistic inference framework we developed to describe how students' informal probabilistic inference changes while being exposed to instruction. The study investigated changes in six middle school students' reasoning before, during, and after a 12-day instructional sequence using a probability simulation software tool. Findings indicate that students are more aware of the effect of sample size when drawing conclusions and making probabilistic inference at the end of the instructional program.

Introduction

Interpreting data and being able to use it to make predictions and decisions are essential tools in order to be informed, educated, and competent citizens in our modern society. Individuals must have the ability to critique, interpret, evaluate, and express their own opinion about the information they receive (Shaughnessy, 2007). Being able to evaluate evidence (data) and claims based on data is an important skill all students should learn as part of their educational programs since they will need to informally infer conclusions from data presented through different representations (e.g., tables, graphs) by the media in their everyday lives. According to Shaughnessy (2007) research has started to pay attention to students' understanding of statistics and students' statistical thinking. Building informal notions of inference within statistical and probabilistic contexts has been advocated for over the past decade within the U.S. (NCTM, 2000; CCSSO, 2010).

Theoretical Foundations

Many authors have described informal statistical inference in different ways. Rubin, Hammerman, and Konold (2006) described it as a reasoning that involves consideration of properties of aggregate including signal and noise and types of variability, the effect of sample size on the accuracy of population estimates, and controlling for bias in order to get a sample that is representative of the population. Zieffler, Garfield, delMas, and Reading (2008) defined it as the way students use argumentation and data-based evidence to support the connections they build between observed sample data and the unknown population. Pfannkuch (2005) stated that statistical inference is concerned with drawing conclusions about specific characteristics of a particular population based on evidence obtained by a sample. The authors have defined informal statistical inference as making generalizations beyond the data, using proper language, argumentation, and data-based evidence to support their findings. The inference is informal in the sense students have not had any formal instruction of methods of statistical inference.

A wide selection of studies have been conducted in order to study students' informal statistical inference (Ben-Zvi, 2006; Rubin et al., 2006; Shaughnessy, 2007). Research has found that even young children can collect data, display it in an adequate way, and make inferences about the underlying population based on the data collected. Most of these studies have had students using dynamic statistical software to analyze data and make informal statistical inference about a specific characteristic of a population based on a collected sample. For example, in Ben-Zvi's (2006) study, students were asked to make generalizations about height and arm span using data students collected through a questionnaire among peers.

Probabilistic inference is also based on data collected and the main objective is to predict the underlying probability distributions, obtained theoretical or empirically through experimentation or simulations. Probabilistic inference has been widely used with students in contexts where a theoretical probability distribution can be easily estimated (i.e., coins, dice, and spinners) (Jones, Langrall, Thornton, & Mogill, 1997); but how do students reason about situations where a theoretical probability distribution is not easily estimated? According to NCTM (2000), "through the grades, students should be able to move from situation for which the probability of an event can readily be determined to situation in which sampling and simulations help them quantify the likelihood of an uncertain event" (p. 51). Probabilistic inference can be use as a preamble to informal statistical inference, starting with contexts in which the theoretical probability is known, and then moving to situations in which a theoretical probability is unknown where sampling and simulations can be used to quantify the likelihood of an uncertain event.

From a wide selection of research on students' statistical informal inference a narrow selection have examined the growth of students' probabilistic reasoning while being exposed to instruction (Jones et al., 1997; Shaughnessy, 2007; Stohl & Tarr, 2002), and only a few have used technology as a tool for students to carry out simulations (e.g., Lee, Angotti, & Tarr, 2010; Pratt, Johnston, Ainley, & Mason, 2008; Stohl & Tarr, 2002). Various technology tools have been used with positive results to study students' probabilistic reasoning. Of importance to our work, the results of these studies have shown that using simulation tools help students develop a better understanding of empirical probability, particularly the importance of sample size in being able to make generalization beyond data.

This study intends to bring some insights on how students develop important concepts in probabilistic inference, and how they may be influenced by instruction. In particular, we seek to answer the question: *How do middle school students' ability to make probabilistic inferences change over the course of instruction?*

Methods

The data in this study was part of a larger study that took place in a public middle school in the southern United States. A 12-day instructional program engaged an average-level 6th grade mathematics class (n=23) in a probability unit using *Probability Explorer* (Stohl, 1999-2002) as a simulation tool during the first month of classes (see Stohl & Tarr, 2002, for details on the instructional design). Prior to instruction, six students were carefully selected to be used as case studies. The selection was based on students' scores from two tests: a standardized mathematics achievement test, and a pretest on probability concepts. The students were chosen by their relatively consistent ranking on both tests (high, middle, low). The six students included four boys (two Caucasians, one Hispanic, and one African-American) and two Caucasian girls. Six sources of data collected in the larger study were used for the current focused study: a pre-interview and a post-interview of six case study students, students' worksheets, a pre-test, post-test, and a retention test. The pre-test was used to obtain evidence about students' knowledge of probability and students' informal statistical inference before instruction. After the pre-test, but before instruction started, semi-structured interviews were conducted with six case study students. During instruction, students worked in pairs and completed a series of tasks and recorded their work on handouts. Immediately after the instructional sequence, a post-test was given that was parallel in construction to the pre-test. Within a week after the last day of instruction, a post-interview was conducted with the six case study students. Ten weeks later a retention-test was administered. After examining all sources of data,13 tasks, some from each data source, were selected to be used to more closely analyze for evidence of students' probabilistic inference (see Table 1). Tasks were selected based on whether or not students were explicitly asked to make inferences and whether or not responses from all six students were gathered.

Task and Sources	**Context**	**Concepts/Constructs**
Task1PreT *Task2PreT* (Pre-Test)	Fairness of an old buffalo nickel found on the street.	• Importance of sample size. • Variability within and across samples. • Propose data collection (sample size larger than 50) and how data would be analyze (looking at the distribution).
Task3PreI (Pre-Interview)	A bag of marbles	• Given the total number in the bag, predict its content. • Importance of sample size.
Task4PreI (Pre-Interview)	Estimate the likelihood a plastic cup would land upside down.	• Estimate proportion of unknown population. • Importance of sample size in order to make generalizations beyond the data collected.
Task5Inst (Instruction)	Bag of Marbles with known total, but unknown content	• Given the total number in the bag, predict its content using PE • Importance of sample size.
Task6Inst (Instruction)	Fish in Lake with 2 types of fish, but unknown total	• Estimate proportion of unknown population. • Importance of sample size.
Task7Inst (Instruction)	Decide whether or not the dice produced by certain companies were fair or not	• Estimate proportion of unknown population • Importance of sample size. • Variability within and across samples
Task8PostT *Task9PostT* (Post-Test)	Bottle cap found in the sidewalk. Is it fair?	• Importance of sample size. • Variability within and across samples. • Propose data collection (sample size larger than 50) and how data would be analyze (looking at the distribution).
Task10PostI (Post-Interview)	Two flavors in the jar. Proportion of each flavor?	• Given the total number in the bag, predict its content using PE • Importance of sample size.
Task11PostI (Post-Interview)	Estimate likelihood toothpaste cap would land upside down	• Estimate proportion of unknown population. • Importance of sample size.
Task12RetT *Task13RetT* (Retention Test)	Fairness of unusually shaped button	• Importance of sample size. • Variability within and across samples. • Propose data collection (sample size larger than 50) and how data would be analyze (looking at the distribution).

Table 1. Tasks and items selected from the sources of data

Development of a Probabilistic Inference Framework

In order to characterize the complexity of students' ability to make informal statistical inference within a probabilistic context, it was necessary to develop a coherent framework. Although several frameworks were found in the literature (Jones et al., 1997; Makar & Rubin, 2009) there was not a framework to study students' inferential reasoning in a probabilistic context. Similar to the frameworks developed by others, we based our framework on the SOLO

Taxonomy (Biggs & Collis, 1982). The SOLO (Structure of the Learned Outcomes) Taxonomy postulates that all learning occur in five modes of functioning: *sensorimotor*, *ikonic*, *concrete symbolic*, *formal*, and *post-formal*. The growth within each mode is characterized through a learning cycle. The sequence of levels refers to a hierarchical increase in the structural complexity of the responses in a particular mode and can be used to classify the outcomes of learning within any given mode. The five levels of response within each mode are: (1) *pre-structural* – the learner is frequently distracted or mislead by irrelevant aspects of the situation; (2) *uni-structural* – the learner focuses on the problem, but uses only one piece of relevant information; (3) *multi-structural* – the learner uses two or more pieces of data without perceiving any relationship between them; and (4) *relational* – the learner can now use all data available, integrating each piece of information.

Comparing the literature on students' informal statistical inference there are similarities and differences in the key principles researchers believe are important when investigating how students reason about informal statistical inference. We identified three constructs as important key principles to be included in the framework: *generalization, variability, and investigation* (see Gonzalez, 2010, for details about the development of the framework). Once an initial draft of the framework was developed based on the research literature, successive refinements were made through sampling students' work on a task and considering how well the categories characterized their work. A sample of tasks was separately coded by both authors and when disagreement existed, the authors discussed the coding in order to reach agreement. Several back and forth coding and refinements were made to result in the final framework shown in Table 2. For each construct, a brief description is included for what was expected for each of the four levels.

	Pre-structural	**Uni-structural**	**Multi-structural**	**Relational**
Generalization (G)	• Makes generalizations based on their previous experiences and their intuitions. Ignores sample size.	• Makes generalizations based on a single part of the information given ignoring the others.	• Makes generalizations based on most of the given information or data collected but still ignoring others.	• Make generalizations based on all the information given or the data collected. Recognizes the importance of large samples to make generalizations.
Variability (V)	• Attributes variability to luck or chance in all contexts. No recognition that variability can be described/controlled.	• Recognizes variability among individual results within a single sample (e.g., I got more 5's than 3's). • Has low expectation of variability from an expected distribution based on some theoretical probability distribution.	• Recognizes variability among individual results within a single sample and recognizes variability across different samples (e.g., last time I got more 5's, this time I got more 1's). • Has a better sense of expecting some variability from theoretical probability distributions, but may expect too much variability.	• Recognizes variability within and across samples and relates variability to characteristics of the underlying probability distribution.

Investigation (I)	• Does not propose a data collection process • Believes sampling does not tell you anything about the underlying population.	• Proposes a data collection process, either appropriate or inappropriate (e.g., uses a small sample size), and does not describe how to assess the results.	• Proposes an appropriate or inappropriate data collection process and describes how the results could be analyzed, although not using the most appropriate approach.	• Proposes an appropriate data collection process (e.g., a large sample size) and describes how the results will be analyzed (e.g., from the distribution of the outcomes)

Table 2. The Probabilistic Inference Framework

Once a final coding framework was developed, each of the 13 tasks for each of the six students was coded by the first author. A sample of 2-3 tasks for each student was selected for the second author to code in order to provide reliability checks. At this point in the coding, it was rare for there to be disagreements between coders. When a few disagreements arose, the researchers discussed reasoning behind the codes until a final agreed-upon code was reached.

For the purpose of this paper, we are focusing on students' informal probabilistic inference within the generalization construct. Generalization builds upon the ability students have to move from looking at individual cases to look at the data as an aggregate and beyond. Generalization is making a claim about the aggregate that goes beyond the data (Makar & Rubin, 2009). Statistical generalizations are abstractions from particular cases (sample) that are applied to a broader set of cases (population). Generalizations should not be based on intuitions or previous experience; it has to have data as evidence. This evidence should provide information about the plausibility of students' claims or predictions. Students could provide graphs, tables, or any other representation they believe would help them support the generalizations they are making. In making generalizations beyond the data students should take into account the size of the samples they are using to draw their conclusion. At higher levels, it is expected that they change their perceptions about the role sample size plays when making inferences as well as how to use the information obtained through the simulations to draw their conclusions and generalizations.

Results

The generalization construct was used to observe the way students were making generalizations beyond a data collected. Some students used their intuitions and ignored the fact that sampling could provide a good estimation about the underlying population distribution. Others reasoned and made generalizations based on small samples using the "total weigh approach" strategy (Lee, 2005). And others, although not reasoning appropriately about how to use the result to make generalizations about the underlying population, did propose a large sample size. Table 3 illustrates the assigned levels used to characterize each student's responses. The levels are represented with the letters P (pre-structural), U (uni-structural), M (multi-structural), and R (relational). All names are pseudonyms; Lara and Dannie composed the high-scoring group, Manuel and Brandon the average-scoring group, and Greg and Jasyn the low-scoring group.

During all the *pre* tasks (Task1PreT through Task4PreI) Lara reasoned at a pre-structural level, she was not paying attention to sample size and was not using the distribution of the outcomes to make generalizations. For example on Task1PreT, when given data that a coin was tossed 1000 times resulting in 643 tails and 357 heads and asked if the coin was fair, she concluded a coin "can always be a fair coin unless it always lands on one side." This represents a pre-structural level response because she ignores the large sample size and seems to base her reasoning on an intuition that if both sides of the coin occur it is a fair coin. During instruction (Task5Inst through Task7Inst) her reasoning increased to a multi-structural level, as she demonstrated attention to sample size and using the distribution of outcomes to make generalizations, although not in the most appropriate way. She continued on a constant level of reasoning toward the end of the unit.

A similar pattern is observed in Dannie's generalization reasoning. She started either making no generalizations or using small samples to infer characteristics from an underlying population, and during and after instruction she reasoned at a multi-structural level, using a larger sample to make a generalization and using the distribution of outcomes in some way to assess results. As an example, during Task1PreT she concluded a coin was unfair because the outcomes "should be closer to even." This is evidence of her reasoning at a uni-structural level because although she used the distribution to make a generalization, she did not attend to sample size. During Taks8PostT, after a bottle cap was tossed 10 times she did not make a generalization stating that "it's a small amount of times you flipped it," but after 1000 tosses she said the bottle cap was unfair because the outcomes "should have been closer to be fair" reasoning at a multi-structural level because she is paying attention to the distribution. Dannie's reasoning was observed to be in the highest level (relational) during Task12Ret where she used several aspects of the situation, making connections about the different samples. In general, Lara and Dannie seemed to be able to maintain at least a multi-structural level of reasoning through instruction and post instruction assessments.

	Individual Pre-Test		Individual Pre-Interview		Paired Work with Software During Instruction			Individual Post-Test		Individual Post-Interview		Individual Retention Test	
Task	Task1PreT	Task2PreT	Task3PreI	Task4PreI	Task5Inst	Task6Inst	Task7Inst	Task8PostT	Task9PostT	Task10PostI	Task11PostI	Task12RetT	Task13RetT
Lara	P	P	P	P	U	M	M	M	M	M	M	M	M
Dannie	U	P	U	P	M	M	M	M	M	M	M	R	M
Manuel	U	U	P	U	M	M	R	P	U	M	U	P	M
Brandon	P	P	U	P	M	M	R	P	P	M	U	R	U
Greg	U	P	M	U	M	M	M	U	M	R	M	M	M
Jasyn	U	P	U	P	M	M	M	U	U	R	U	U	P

Table 3. SOLO Levels for Generalization construct before, during and after instruction.

Manuel and Brandon, the average-scoring pair, reasoned at either a pre-structural and uni-structural level during the *pre* tasks and showed an increase in their level of reasoning moving to the multi-structural and relational levels during the instructional tasks. For example on Task1PreT, when given data that a coin was tossed 10, 100, and 1000 times, Manuel generalized the coin was unfair paying no attention to sample size. This is evidence he was reasoning at a uni-structural level. During Task12RetT, when asked how to determine if an odd-shaped button was fair Manuel specified he would have "flipped [the button] 50 times and saw the outcomes." Since he is considering sample size and using the outcomes he was coded as reasoning at a multi-structural level. With Manuel and Brandon the mode of engagement in the task enhanced their reasoning, having a steady performance during the instructional tasks where they worked together as a pair and had access to the software. Their inconsistencies in levels of reasoning on the post tasks indicate that they were capable of reasoning at higher levels (M and R) on some tasks, but that perhaps they did not see the need to apply their understanding of generalization in all tasks.

Jasyn's level of reasoning was often coded as uni-structural, with only two pre-structural instances (Task2PreT and Task13RetT) and one relational instance (Task10PostI). The questioning by the interviewer within this particular task in the post-interview seemed to encourage him to reason at a higher level. His reasoning demonstrated to be stable and higher during the instructional tasks when he was working with Greg. Greg's level of reasoning increased from the *pre* tasks to the instructional tasks. His level of reasoning was mostly uni-structural during the *pre* tasks (with the exception of Task3PreI) and multi-structural during and after instruction (with the exception of Task8PostT and Task10PostI).

Overall, students' reasoning in the generalization construct increased through experience with instruction focused on informal inference. The students were more aware about the importance of sample size and were able to use the distribution of outcomes to make generalizations. They seemed to be recognizing that sampling could be used to gather information that could help them make inferences about an underlying population. Within this construct students reasoned at a higher level during the instructional tasks and the interviews than on individual written assessments.

Discussion

Growth in the ability to make probabilistic inference means for the student to be able to make generalizations beyond the data using an empirical distribution to make predictions about a theoretical probability distribution, recognizing the importance of sample size and types of variability, and, when possible, their abilities to propose an investigation process. All tasks included in the analysis asked students to make generalizations beyond the data; from predicting the distribution of a bag of marbles (knowing the total of marbles as well as how many different colors were in a bag) to estimating the probability distribution of a die sold by a certain company. Some students used their intuitions and ignored the fact that sampling could provide a good estimation about the underlying population distribution (Rubin et al., 2006). Others reasoned and made generalizations based on small samples using the strategies mentioned by Lee (2005) such as the "total weight approach." And others did propose a large sample size, although not always reasoning appropriately about how to use the results to make generalizations about the underlying population,.

Recognizing the importance of sampling and sample size is one of the biggest problems with informal inference (Pfannkuch, 2005). For most of the time, these six students believed that either small or large samples were representative of the underlying population. Even during the instructional tasks, they often used several samples of small sizes to predict the distribution of the colors in the bag of marbles, and to estimate the proportions of Blue Bass fish in the Mystery Fish on a Lake task. Through the study, students used small samples (n=4) and very large samples (n=5000). During the *pre* tasks (Task1PreT through Task4PreI) most of the students were not aware of the importance of sample size in order to draw inferences about the underlying population. Only one was able to recognize it was not appropriate to use a sample of size 10 to make inferences about the underlying population. Five out of six students only used the "total weight approach" discussed by Lee (2005) during the *pre* tasks.

The use of the Probabilistic Inference framework provided insights about how middle school students reason about informal probabilistic inference. The framework could also be used in future research and curriculum development in statistics education that intends to facilitate students' reasoning about important aspects of probabilistic inference, such as, generalization beyond data, variability, and the investigation process. By attending to and coordinating different constructs within informal inference, researchers and teachers may be able to design better tasks, assessment items that facilitate informal probabilistic inference and lead to a better understanding of such reasoning may develop across ages or grade bands.

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PROBABILITY KNOWLEDGE AND A PRESERVICE TEACHER

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Introduction

The essence of probability is uncertainty, which lies in stark contrast with the typical problems encountered in mathematics, which are deterministic. Probability can be defined as or framed along a continuum where some things will occur with certainty to the other end of the continuum where some things will never occur. The ability to confront and make decisions under uncertainty is a useful skill in our data rich world in addition to addressing uncertainty within our personal lives. The importance of probability and data analysis as a part of the K-12 curriculum is apparent by its inclusion in the National Council of Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000; also see NCTM,1989). Before the NCTM embraced data analysis and probability, early research on probabilistic learning in children, went for the most part, unnoticed by educators, but was embraced by psychologists (e.g., Piaget & Inhelder, 1975).

Earlier research addressed heuristics as a strategy that people used to reduce a complex task into a simpler task in order to assess probabilities or predict values when making decisions under uncertainty (Tversky and Kahneman, 1982a). Later research noted the complexity of reasoning, when people make decisions under uncertainty that extended beyond the heuristics people use as identified by Tversky and Kahneman (Shaughnessy, 2003; Konold, 1988). More recent research addressing the curriculum and the teaching of probability examines the initiatives taking place in the U.S., Australia, and the United Kingdom (see Jones, Langrall, and Mooney, 2007). However, research addressing the probabilistic reasoning of preservice teachers, which curriculum and instructional initiatives depend, is minimal. This study explores the strategies, knowledge, and reasoning of one preservice teacher when engaged in situations of uncertainty.

Theoretical Framework

The dependence on heuristic principles are not limited to lay people, experienced people are also inclined to rely on heuristics when thinking intuitively (Tversky & Kahneman, 1982a). When people struggle to make a decision that consists of lists of possible outcomes, Tversky and Kahneman (1973) found most people try to reduce the complexity of the situation by embracing judgmental heuristics, "a strategy … that relies on a natural assessment to produce an estimation or a prediction" (p. 294), where people "categorize objects and events, which are related to mental models such as prototypes or schemata" (p. 295) that simplifies the situation. They identified and named three types of simplifying heuristics or strategies people use to solve problems, representativeness, availability, and adjustment and anchoring, which may lead people astray.

The representativeness heuristic is used when a person assesses the probability of an event or situation based on how closely it resembles the population. This heuristic comes into play when people are confronted with the following types of questions: "what is the probability that object A belongs to class B? What is the probability that event A originates from process B? What is the probability that process B will generate event A," (Tversky & Kahneman, 1982a, p. 6).

Fundamentally, people make judgments based on the degree that A is representative or similar to B. However, Kahneman and Tversky (1982a) indicate that to be "representative, it is not sufficient that an uncertain event be similar to its parent population. The event should also reflect the properties of the certain process by which it is generated, that is, it should appear random" (p. 35). People view randomness as unpredictable but fair, which means they expect samples to maintain the same or close to the same proportions found in the population.

In addition, Kahneman and Tversky (1982b) indicate that the representativeness heuristic is directional. That is, a person "evaluates the input [or]…predicts the outcome" (p. 57). An example of representativeness given by Tversky and Kahneman (1982b) is as follows; a person is described as a shy, meek, and timid, what occupation would they most likely hold when given the list librarian, doctor, sales person or farmer from which to choose. Most people from their research choose the librarian as the most probable, based on the stereotype of a librarian in our society. Another example, provided by Kahneman and Tversky (1972), involves a maternity ward with five newborn baby boys and two newborn baby girls. The nurses believe the next baby to enter their ward will be a girl so that the number of baby boys and the number of baby girls will even out. The mental model prescribes the probability that a new born baby is a boy is one-half and the probability that a new born baby is a girl is also one-half, which is another way to stereotype. That is, the proportions in samples must be equal at all times or at least remain close to the proportions in the population so that the sample represents the population from which the sample is drawn. Tversky and Kahneman (1982a) warn us that this type of reasoning leads to serious errors because people ignore other factors that impact judgments on probability, such as sample size, prior probability outcomes, illusion of validity, misunderstandings of regression, and misunderstandings of chance.

There are other categories of misconceptions that may arise from heuristics, which include *availability, adjustment and anchoring* (Tversky & Kahneman, 1982a) and the *time axis effect* (Fischbein & Schnarch, 1997) that are beyond the scope of this paper. In essence, the foundational issue revolves around the ability to choose an appropriate strategy when there are tensions that make the decision more difficult. That is, students may wrestle with probability activities due to "(1) fragmented understanding of proportional reasoning, (2) probability ideas [that are] counterintuitive and conflict with students experiences, and (3) … [a student at the collegiate level] experiences probability in an abstract and formal context" (Garfield and Ahlgren, 1988, p. 47). This paper focused on one secondary mathematics preservice teacher as she engaged in several problem solving tasks. This raised our awareness of the complexities associated with her strategies, her mathematical knowledge, and her conceptions about probability, where her reliance on the representativeness heuristic emerged.

Methods

This study used an extended clinical interview setting to explore one preservice teacher's experiences with probability in order to better understand her reasoning when solving probability problems. The interviews consisted of two sessions occurring over a three week period. The interviews involved both an interviewer and an observer. The first author filled the role of observer and took field notes during the interview. The two interview sessions were captured, in their entirety, with the video-audio recorder on the interviewer's computer. The recordings were later transcribed by the observer and later analyzed with her field notes. The transcription from the video-audio was coded first according to the behaviors displayed by the participant when engaged in drawing diagrams, formula calculations, and calculations using diagrams. The next iteration of the analysis involved coding of the data according to the participant's mental processes, i.e., thinking and reasoning. We were interested in the thinking strategies associated with calculations or the ability to recall formulas as well as the reasoning associated with the participant's explanations. The focus of the analysis was then narrowed to the sections of the data where misunderstandings emerged.

Participants

The study drew participants from one mathematics education class consisting of 18 senior mathematics preservice teachers, who intend to teach secondary mathematics. The mathematics preservice teachers attend a large university in one of the Mid-Atlantic States. The demographics of the target population, this class, is diverse in culture and gender, however, the one mathematics preservice teachers who was willing to participate in the extended interview sessions was white and female.

Instruments

There were three instruments used in the study. The first instrument was a pre-test instrument used to identify students who have misunderstandings with probability. The second and third instruments were probability activities. Each of these instruments is described next.

The first instrument was a pre-test that was taken from Fischbein and Schnarch (1997) and it was administered to all 18 preservice teachers in the course. The pre-test instrument consisted of seven probability problems, where the last question had two parts and was entirely open response. All questions except one had three multiple choices answers from which to choose and the remaining item had two multiple choices options. Our instrument was modified from Fischbein and Schnarch (1997) by adding ample space below each question for the mathematics preservice teacher to explain the reason for their multiple choice selection. The questions were designed to identify common misconceptions that emerge when employing the representativeness, negative and positive regency effect, simple and compound events, conjunction fallacy, availability, and the time axis heuristics. The goal of the questionnaire was to evaluate the target population because there was a concern that the mathematics preservice teachers from this target population may not display any of the common misconceptions. Based on the pre-test assessment all but one student demonstrated at least one misconception related to probability. As a result, 17 mathematics preservice teachers were invited to participate in the study. Two mathematics preservice teachers volunteered and this paper examines one of these preservice teachers.

The second and third instruments were tasks used by Shaughnessy (2005). It is the third instrument that is the focus for this study. It is a probability experiment adapted from tasks used by Shaughnessy (2005) called *Coin Flipping – Keep Your Head Up*. The participant explored the probabilities for the various numbers of heads appearing when tossing six coins. The activity is divided into four parts. The first part of the activity has the participant estimating the probability, when flipping six coins that results in six heads, five heads, four heads, and three heads. Following these estimates, the participant performs the experiment recording the results on a tally chart. In the second part of the activity, the participant is asked to make a list of all possible outcomes for flipping six coins. This is followed by the development of a theoretical model for the outcomes of flipping six coins. The third section of the activity has the participant compare the results of the experimental probabilities with the theoretical probabilities, followed by drawing a graph of the experimental and theoretical probabilities.

Procedures

The first interview session began with the preservice teacher sharing prior school experiences with mathematics, attitudes toward mathematics, and attitudes toward teaching mathematics. Following the conversation, the preservice teacher is presented with a probability activity adapted from tasks used by Shaughnessy (2005). As the mathematics preservice teacher solved the problems, she was encouraged to verbalize her thinking and reasoning in detail. The second interview session, which is the focus of this study, had the preservice teacher engaged in a probability experiment based on an activity from Shaughnessy (2005). The activity had three parts.

Results

This study explored the prior experiences with probability and the reasoning of one preservice teacher, Cassie, when engaged in a probability experiment. In the first interview session Cassie explains that she learned probability and statistics by memorizing formulas and definitions.

In the second interview session Cassie was asked to perform an experiment called Coin Flipping – Keep Your Head Up. The activity has three sections that focused on the probabilities of getting various numbers of heads when flipping six coins. The first part of the activity involves estimating the probability, when flipping six coins, of observing six heads, five heads, four heads, and three heads. Cassie wrote down her probability estimates for each of these events and the interviewer noticed Cassie had written $½* ½ * ½ * ½ *½ * ½ = ½^6 = 1/64$ next to the six heads estimate and asked Cassie to explain the method she used to reach that result. Cassie responded:

> Because every time you flip it you have half of a chance of getting it [heads]. So, if you did the tree diagram, you would get heads or tails and then you'd get heads or tails... [and] …there would be 64 choices…and you would have one path with all heads.

The interviewer noticed her response for getting 5 heads was 1/32 and she says to Cassie,

> For 5 heads you [did] the same thing, 1 over 2 to the 5th [or] 1 over 32 … you would say that is one over 32 and another way of thinking about it you'd think [of] all of the paths [so] it would be two over 64, is that what you mean?

Cassie agrees and then explains. "Yeah, because you could do all heads and one tail or all tails, (she stops herself) I'm sorry, or one tail and all heads afterwards". She continues, "I think ….maybe [it's] wrong…hold on" she makes false starts and then begins to draw a tree diagram. The diagram is unclear and incomplete because most of the branches stop short with the fifth coin position. Cassie continues to work and then admits, "Actually I'm confused". After further attempts by the interviewer to help her grapple with these estimates, the interviewer stops this line of questioning and starts Cassie with the coin tossing experiment. She performs the experiment flipping six coins 50 times and tallied the results on the chart. The experiment results are as follows: she tossed six heads once, five heads six times, four heads twelve times, three heads ten times, two heads eleven times, one head nine times, and zero heads once for all 50 tosses. There were no strategies, reasoning's, or calculations involved in this section of the activity, but the interviewer asked her, "What is the probability of getting at least one head"? Cassie replies, "49", which the interviewer responds, "Why would you say that?" She explains, "Because you add up all the probabilities of getting one or more…" The interviewer asks her about the probability of getting at least two heads, which Cassie answers, “You add up 2, 3, 4, 5, and 6 so its 40 out of 50” and follows with the explanation, “because if you get one head it’s not at least two or zero you didn’t get two so you exclude those so you count up all the others that are two or higher”.

The second part of the activity had Cassie generate a sample space of all possible outcomes followed by developing a mathematical model to find the theoretical probabilities for the outcomes of flipping six coins. This time, she neatly draws a tree diagram correctly with all possible outcomes represented. During the conversation, the interviewer randomly selects an end point on Cassie’s tree diagram that showed a branch with three heads and three tails, at which point the interviewer asks Cassie, how many other branches would show three heads and three tails”? Cassie responded, “so half of…so 32”. When the interviewer asked Cassie to explain her answer she replied:

> Every level there’s one head or one tail so if you started with tails, you could (she is thinking and looks confused). Well, I am trying to figure out… if it’s for all 64 possible outcomes … the first one could be either heads or tails, the second one could be heads or tails, so if you just pick …three of them to be tails and three of them to be heads it’s, just a moment let me explain this... or six choose two”?

The interviewer asked her to explain her last comment, “What do six choose two represent”? Cassie is trying to recall the formula for combinations, choosing two out of six, as she incorrectly writes “${}^{6}{}_{2}C_{6}$”. She says, “I haven’t done combinations in a while”. She begins drawing a second diagram showing six short horizontal line segments that represent each position of the six coins laid next to each other. This diagram she drew will be referred as her *slot diagram* in the remainder of the paper. On top of each line segment she writes the numeral two. Cassie continues:

> Ok, so you have six [pennies] and you have two choices, two choices, two choices two choices all the way across for all six of them [she says as she points to each slot]. Then there are…64 different ways you could do it. I think its 32 ways you can have three tails and three heads but, I can’t explain why.

This section of the activity we see her struggle with diagrams and with her partial recall of the combinations formula.

Next, the activity in this section has Cassie develop a theoretical model of the probabilities for the outcomes of flipping six coins, which Cassie struggles at first to understand the task as she looks at her experimental results while considering a theoretical result. She returns to her slot diagram, which supports her and allows her to accurately find the theoretical probability for the event of tossing six heads. Prompted by the observer to next consider the case with zero heads, which is equivalent to observing six tails, she finds the probability of tossing six coins where no heads appears correctly. Prompted by the interviewer she continues this process of addressing the counterpart outcomes. That is, she next finds the probability of one head appearing and then finds the probability of five tails appearing, because it is the counterpart of five heads and one tail. Once she found all of the possible outcomes for flipping six coins, Cassie explains her system using her slot diagram. “If you get a tail in the first one, you have five other places where you can get the second tail”. She continues with three tails, “there are three other places, fourth there are two other places and in the fifth there is one other place. So it would be 15 ways of doing it [referring to four tails and two tails]”. The prompt to consider the complement of zero heads, which is six tails, shifted the conversation to addressing tails, although she understood the same reasoning applied to heads. When asked to find the theoretical probability of getting three heads, she correctly responded with, “I added up 1/64, 1/64, 6/64, 6/64, and 15/64 plus 15/64 and I got 44/64 so that means the rest of the ways for it to have to be 3 heads… and that is 20 over 64”. She used the process of grouping probabilities and using the counterparts of probabilities to fill in all of the probabilities for the various numbers of heads that appear when tossing six coins.

The third part of the activity has Cassie compare the experimental probabilities with the theoretical probabilities. The interviewer asks her to determine “how well do these probabilities agree?” Cassie says, as she is pointing to the theoretical calculations and the experimental results,

> The six heads and the zero heads are pretty close, one out of 50 and one out of 64 for both of those. The same with five heads and one head that’s pretty close, too. Except here, in the theoretical model three heads should have occur[ed] more...but in the experimental, four heads and two heads both occurred more than three heads.

Cassie sees that the experimental frequencies lack the strong symmetry found in the theoretical probabilities. The interviewer closes the session with a final question, “Were there any assumptions that you made when you were doing your experiment”? Cassie responds, “When I did my experimental probability [it] was based on the experiment so, I’m assuming this would always happen in every 50 coin tosses”.

Discussion and Conclusions

This study explores the mathematical reasoning, strategies, and calculations of one mathematics preservice teacher when participating in two probability activities. The activity highlighted her strategies, which include heuristics, diagrams, and incomplete formal knowledge about probability. Cassie’s prior experiences with probability align with other collegiate students who experience probability in a traditional abstract context (Garfield & Ahlgren, 1988).

The second interview session, Cassie performed an experiment called: Coin Flipping – Keep Your Head Up. She was to find the probabilities of various numbers of heads appearing when tossing six coins, which was designed to address the representativeness heuristic. Her attempt to estimate the probabilities associated with six heads, five heads, four heads, and then three heads appearing was problematic for her. She found the probability of getting all six heads by multiplying ½ six times to get 1/64. Her reasoning for the probability of getting five heads with the six coins was 1/32. It appears she focused on the number of heads showing and multiplied ½ that many times ($1/2^5$). This seems to suggest she is finding the probability for five heads using the same process she used for six heads because she knew her model for six heads ($1/2^6$) was correct. This may imply she is using the representativeness heuristic because she used the model for tossing six heads as the model for the tossing five heads, unaware she was changing the problem.

In the second part of the activity Cassie made a list of all possible outcomes for flipping six coins, where she neatly draws a completed tree diagram. The interviewer randomly chooses a branch on her tree diagram, which showed three heads and three tails, and asked her the probability of obtaining three heads and three tails. Cassie responded with 32 but admits she cannot explain why this is the case. It appears she is taking 64 and dividing it in half because out of the six coins, half of the coins would show heads and the other half of the coins would show tails. Another explanation may be that she chose to divide the total number of possible 64 outcomes in half because of the probability of tossing one coin once is one-half. In either case, it suggests she is relying on the representativeness heuristic. We see the tensions between her partial recall of the combinations formula, "${}^{6}{}_{2}C_{6}$", the tree diagram, and the slot diagram, where all of these strategies fail to help her make appropriate estimates of the probability of getting three heads and three tails. Therefore, it appears she relies on the representativeness heuristic to make sense of this problem.

The next activity in part two has Cassie finding the theoretical probabilities for tossing six coins. She struggles at first to understand the question as she looks at her experimental outcomes while considering a theoretical result. She returns to her slot diagram, which supports her and allows her to accurately find the theoretical probabilities for the event of tossing six heads and then, prompted by the observer, to explore zero heads, which is the same as six tails. Cassie explained her system using her slot diagram, where each position of a coin is placed next to each other forming a horizontal row of coins. She explained that a head in the first position allows a second head to appear in one of the remaining five positions, which shows there are 15 ways that two heads and four tails can appear. She continued with this line of reasoning for three heads, where there are three positions possible for the three heads and there are three positions possible for the three tails. The prompt to consider the complement of zero heads with six tails was provided as an aid to reveal the symmetry among the theoretical probabilities, which Cassie was able to grasp. This aid, along with using the complement property of probability, helps her to find the theoretical probability for all of the possible outcomes. When asked of the theoretical probability of getting three heads, she correctly responds with, 20/64.

The third part of the activity has Cassie comparing her experimental and theoretical results, where she sees the experimental results lacked the strong symmetry found in the theoretical model and she notes that the occurrence of three heads is greater in the theoretical model than in the experimental model. The interviewer closes the session with a final question addressing Cassie's assumptions that she may have held prior to performing the experiment. Cassie responds that anytime six coins are tossed 50 times the result would look exactly like hers. This reveals a misconception associated with variability in experiments, which is a common result when someone relies on the representativeness heuristic. That is, people tend to ignore the role that variability plays in sampling.

This study explored one secondary mathematics preservice teacher as she engaged in a probability experiment. The research identified several misconceptions which appear to be associated with the representativeness heuristic. For example, the participant used the same method for finding the probability of tossing six heads, which is ½ multiplied six times, as to finding the probability of tossing five heads by multiplying ½ five times, which is an incorrect assumption. The representativeness heuristic seemed to emerge when she mapped the probability of tossing three heads and three tails to the theoretical probability of tossing one coin once. Additionally, it seems the heuristic emerged when she said that tossing six coins 50 times would result in the same numbers she found in her experiment. Her fragmented formal knowledge, when she tried to recall the combination formula and her struggles to find a useful strategy may have prompted her to rely on the representativeness heuristic. This study proposes that preservice teacher's struggles with probabilities may be a concoction of fragmented formal knowledge, ineffective strategies, and intuitive heuristics, such as the representativeness heuristic. Therefore, future research should focus on the complexity of the problem solving process by teasing out the relationships between fragmented formal knowledge, strategies, and intuitions, which may promote a reliance on the representativeness heuristic, when solving probability problems.

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QUANTITATIVE REASONING AND MATHEMATICAL MODELING IN ENVIRONMENTAL SCIENCE

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The National Science Foundation MSP Pathways in Environmental Literacy Project is focused on creating learning progressions for environmental science across grade levels 6-12, which includes collecting and analyzing data on students' development of conceptual understanding about the carbon cycle, water cycle, and biodiversity. The Quantitative Reasoning(QR) Theme of the project is researching the impact of quantitative reasoning on students' ability to make informed decisions about the environmental grand challenges facing their generation. This paper will report on findings based on clinical interviews and written assessments of students' QR ability in the context of environmental science.

Pathways in Environmental Literacy is a National Science Foundation Mathematics and Science Partnership project which is establishing learning progressions for environmental science. The learning progressions span 6^{th} to 12^{th} grade, studying the trajectory of learning that leads to the development of environmentally literate citizens capable of making informed decisions. The strands (areas of enduring understanding) selected for study were the water cycle, carbon cycle, and biodiversity. It was hypothesized that comprehension of these three concepts was essential to development of environmental literacy. In addition, three themes were identified as being integral to developing informed environmental decision makers: quantitative reasoning, citizenship, and cultural relevance. The project researchers believe that these three themes must be integrated across the three strands if students are to become informed decision makers. This report will focus on the quantitative reasoning theme.

The Pathways project is a collaboration of six universities, four Long Term Ecological Research Sites (26 LTER sites are funded by NSF to support ecological research on established field sites), and LTER partner school districts. Research teams were established at partner universities to lead efforts on study of the three strands (carbon, water, and biodiversity) and three themes (quantitative reasoning, citizenship, and cultural relevance). Research on quantitative reasoning for the Pathways project was assigned to the University of Wyoming, which had established a Quantitative Reasoning and Mathematical Modeling (QRaMM) research group. QRaMM is one of three research strands initiated by the Wyoming Institute for the Study and Development of Mathematical Education (WISDOM^{e}). QRaMM brings together researchers from multiple universities across the country to share ideas and conduct research on quantitative reasoning and modeling within an interdisciplinary context. As part of the work of QRaMM and in collaboration with the National Science Foundation MSP Pathways in Environmental Literacy Project, a national virtual seminar QRaMM in Science was initiated in Spring 2011. The seminar engaged 15 speakers from across the country in the online video conference, including experts in quantitative reasoning and modeling from the fields of science, computational science, mathematics, science education, and mathematics education. The context for the discussion of quantitative reasoning and modeling was environmental literacy. The issues in psychology of mathematics education for the seminar included students' development of quantitative reasoning and mathematical modeling from 6^{th} to 12^{th} grade, creation of parallel QRaMM learning progressions for those being created by the MSP Pathways project for environmental literacy, and study of the impact and interplay of QR and modeling on students' development of environmental literacy. The national experts informed a QRaMM Science research team consisting of faculty and graduate students who developed Environmental Science QR interview protocols based on MSP Pathways learning progressions. Data from these protocols lead to the creation of Environmental Science QR assessments and research-based professional development that supports learning environments that lead to environmentally literate citizens capable of making informed decisions about the grand challenges facing the next generation.

Purpose of Study

The primary research question for the Pathways project is: What are the characteristics of a 6^{th} to 12^{th} grade learning progression aimed at developing environmentally literate citizens? The purpose of the QR Theme is to determine the quantitative reasoning aspects of the learning progression leading to the primary QR research question: What are the essential quantitative reasoning abilities that are required for the development of environmental literacy? Specific research questions include the following. How does quantitative reasoning integrate and interact with environmental literacy? What is the relationship between learning progressions for quantitative reasoning and environmental literacy – should they be integrated or is a separate learning progression for quantitative reasoning required? What is the current state of quantitative reasoning supporting environmental literacy at the 6^{th} grade level (defined as the lower anchor for the learning progression)? What is the required level of quantitative reasoning for an environmentally literate citizen at the 12^{th} grade level (defined as the upper anchor)? How does the learning progression inform professional development supporting the development of environmental literacy?

Theoretical Framework

The Pathways project is based on the theoretical framework of learning progressions. The Consortium for Policy Research in Education defines learning progressions as follows:

> Learning progressions are hypothesized descriptions of the successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time (Corcoran, Mosher, & Rogat, 2009, pg 37).

The panel identified essential elements of learning progressions to be:

- Upper Anchor: target performance or leaning goals which are the end points of learning progression and are defined by societal expectations, analysis of the discipline, and requirements for entry into the next level of education.
- Progress Variables: dimensions of understanding, application, and practice that are being developed and tracked over time.
- Levels of Achievement: intermediate steps in the developmental pathway(s) traced by a learning progression.

- Learning performances: tasks students at a particular level of achievement would be capable of performing.
- Assessments: specific measures used to track student development along the hypothesized progression.

The Pathways learning progressions for environmental literacy are based on research in science education and cognitive psychology, foundational and generative disciplinary knowledge and practices, and strive for internal conceptual coherence. The QR in environmental literacy frameworks build on these characteristics, incorporating mathematical and statistical frameworks.

The Pathways learning progressions have a lower anchor which is the typical accounts of environmental issues given by students at the upper elementary and middle school level (Anderson, 2009). These accounts are empirically tested through a cyclic research process of clinical interviews informing the learning progression and leading to the development of written assessments given on a large scale. The Pathways learning progressions upper anchor is based on experts views of what a scientifically literate citizen should know and be able to do by the 12th grade. The upper anchor is much like a NSTA or NCTM standard, but learning progressions differ from standards in that the lower anchor and intermediate achievement levels are research-based, reflecting the actual trajectory of student learning. A limited number of achievement levels (4 or 5) are identified as plateaus in students' development of more sophisticated ways of thinking about enduring understandings, concepts, and processes. The progress variables for the Pathways project at the meta-level are the carbon cycle, water cycle, and biodiversity, which are considered areas in which students must develop conceptual understanding if they are to become environmentally literate citizens. Within each of these areas progress variables are identified. For example, in the carbon strand the progress variables are generation-photosynthesis, transformation-food chain/web/biosynthesis, oxidation-cellular respiration, and oxidation-combustion. Learning performances are exemplars drawn from the clinical interviews and written assessments which demonstrate student responses at different achievement levels. Learning progression matrices are created by cross tabulating achievement levels (rows in matrix) with progress variables (columns in matrix). A number of learning progressions in science are currently under development including: tracing carbon in ecological systems (Mohan, Chen, & Anderson, 2009), particle model of matter (Merrit, Krajcik, & Swartz, 2008), modeling in science (Schwarz, Reiser, et. al., 2009), genetics (Duncan, Rogat, & Yarden, 2009), chemical reactions (Roseman, et. al., 2006), data modeling and evolution (Lehrer & Schauble, 2002), explanations and ecology (Songer, Kelcey, & Gotwals, 2009), buoyancy (Kennedy & Wilson, 2006), atomic molecular theory (Smith, Wisner, Anderson, & Krajcik, 2006), and evolution (Cately, Lehrer, & Reiser, 2005). Examples of three of these learning progressions are provided in (Corcoran, Mosher, & Rogat, 2009).

The QR Theme of the Pathways project is using the learning progressions theoretical framework to research the impact of QR on students' development of environmental literacy. The overarching goal is to study the capacity of students to understand and participate in evidence-based discussions of socio-ecological systems. Three overarching progress variables are hypothesized: quantitative literacy which consists of arithmetic understandings supporting science, quantitative interpretation which is the process of interpreting scientific models to determine trends and make predictions, and quantitative modeling which is the creation of models by the student. Within each of these areas four progress variables are identified that are hypothesized to be critical QR for science (Table 1).

The development of learning progressions is an iterative process typical of design-based research. The Pathways environmental literacy learning progressions began with hypothetical frameworks based on theories about reasoning about processes in socio-economic systems, including discourse, practices, and knowledge, as well as linking processes between lower and

Progress Variables	Quantitative Literacy	Quantitative Interpretation	Quantitative Modeling

Components	Numeracy
	• Number Sense
	• Small/large Numbers
	• Scientific Notation
	Measurement
	• Accuracy-precision
	• Estimation
	• Dimensional Analysis
	• Units
	Proportional Reasoning
	• Fraction
	• Ratio
	• Percents
	• Rates/Change
	• Proportions
	Basic Prob/Stats
	• Empirical Prob.
	• Counting
	• Central Tendency
	• Variation
		Multiple Representations (rule of 4)
		• Written/oral
		• Tables
		• Graphs/diagrams
		• Equations
		    • Linear
		    • Quadratic
		    • Power
		    • Exponential
		Science models
		Statistics
		• Normal Distribution Model
		• Linear Regression Model
		• Statistical plots
		• Correlation
		• Causality
		• Z-scores
		• Confidence Intervals
		Logarithmic Scales
			Logic
			Problem Solving
			Modeling
			• Normal Distribution Model
			• Regression Model
			    • linear
			    • polynomial
			    • power
			    • exponential
			    • logarithmic
			• Logistic Growth
			• conceptual science models
			• table and graph models
			Inference
			• Inference
			• Hypothesis testing

Table 1. QR Progress Variables

upper anchors. Challenges for the QR Theme researchers are to determine which of these theories carry over to QR aspects of environmental literacy and to discover potential theories that are more QR-centric. For example, students achieving the upper anchor in the environmental literacy learning progression should function as informed decision makers within the socio-economic system in Figure 1 at three levels: discourse, practices, and knowledge. “Knowledge is embedded in practices, which in turn are embedded in discourses” (Anderson, 2009). We all participate in multiple discourses that associate us with communities of practice (Gee & Green, 1998) and as we gain understanding of a phenomena as our discourse around it matures. The lower anchor discourse for the environmental literacy progression is force dynamic, relying on students' theory of the world with a focus on actors, enablers, actor's purposes, conflicts between actors, and settings for the actions (Pinker, 2007; Talmy, 2003). At the upper anchor scientific discourse is essential, with students moving away from actors in settings to laws that govern the work of systems. Certainly there is a parallel development of mathematical and statistical discourse that is essential for quantitative reasoning. There are four practices that are essential

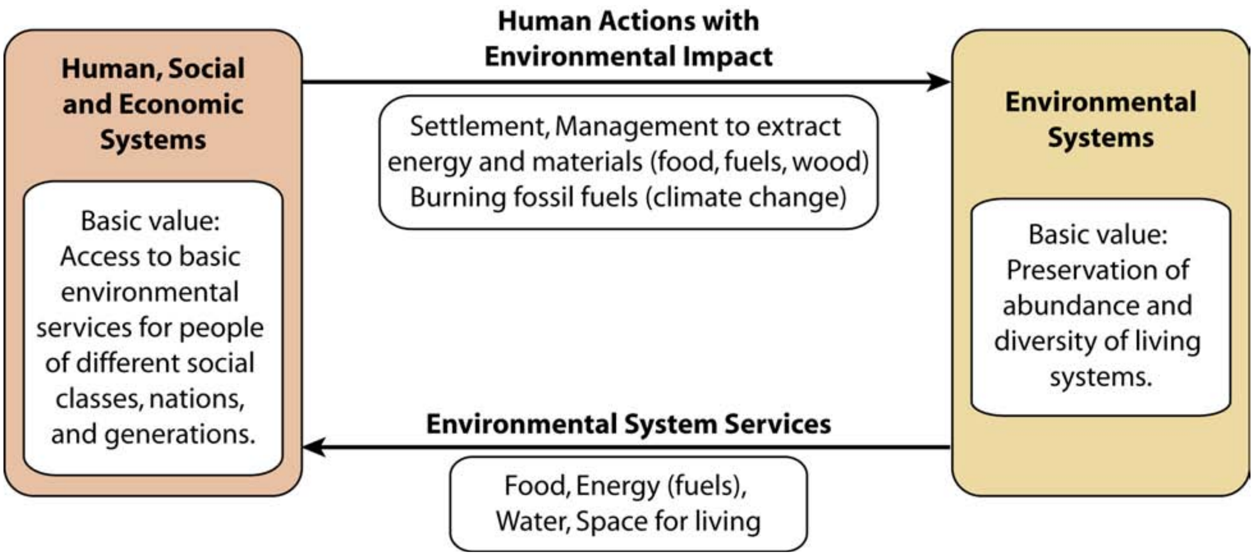

Figure 1: Socio-Economic System

for environmentally responsible citizenship: inquiry (What is the problem and who do I trust?), accounts-explaining (What is happening in the system?), accounts-predicting (What are the consequences of my course of action?), and deciding (What will I do?). At the lower anchor explaining and predicting practices are expressed in force dynamic discourse, while at the upper anchor they are expressed using the language and theory of scientific discourse. These practices reflect the public roles (voter, advocate, volunteer) and private (consumer, owner, worker, learner) roles in which decisions are made. Informed explaining and predicting practices often require QR, which allows for connecting observations to patterns and models, as well as analyzing data from a scientific perspective. Finally knowledge is embedded within discourses and practice. At the lower anchor students focus on knowing facts, while at the upper anchor scientific knowledge is applied to create coherent systems of observations, patterns, and models which serve as the basis for scientific inquiry and accounts. QR parallels this trajectory from knowing mathematical or statistical algorithms to an understanding of enduring concepts that allow one to analyze problems quantitatively.

A framework for the Pathways learning progression consists of progress variables (matrix columns), levels of achievement (matrix rows), learning performances (content of matrix cells),

Carbon-transforming processes	Generating organic carbon	Transforming organic carbon			Oxidizing organic carbon	
Scientific accounts	Photosynthesis	Biosynthesis	Digestion	Biosynthesis	Cellular respiration	Combus-tion

Linking processes	Plant growth	Animal growth	Breathing, exercise	Decay	Burning
Informal accounts	Plants and animals as actors, accomplishing their purposes in life, using their abilities, if their needs (food, water, sunlight, and/or air) are met			Natural process in dead things	Flame as actor consuming fuel

Figure 2: Carbon Linking Processes

and linking processes which are common processes that are recognized by students at all levels of achievement (Anderson, 2009). The carbon framework in Figure 2 provides an example of linking processes within the carbon strand. What are the linking processes for QR? How do they interact with the environmental literacy linking processes? Another aspect of the Pathways learning progressions which is highly quantitative in nature is scaling. Students at the lower anchor often function at a macroscopic or "individual" scale in which they view the world from their sensory purview. Hence, their accounts of environmental issues are based within their daily perceptions at this scale. Moving students to the upper anchor requires that they scale up to a global view, as well as scale down to a microscopic view and even an atomic view. Regardless of the scale, moving up or down the scale becomes intensively quantitative.

Methods

The validation of learning progressions requires conceptual coherence, compatibility with current research, and empirical validation (Anderson, 2009). Conceptual coherence means the learning progression tells a comprehensible and reasonable story of how naïve students develop mastery. Compatibility with current research requires learning progressions to adhere to research on science and mathematics content, pedagogy, and cognition. All learning progressions for QR will be treated as hypotheses which are to be empirically tested. There are five general approaches for hypothesizing progressions: (1) extrapolation from current and conventional teacher and curriculum practice (i.e. standards); (2) cross-sectional sampling of student performance using assessments, observations, or interviews; (3) longitudinal samples of student work over time; (4) closely observed classroom interventions; and (5) disciplinary understanding of the structure of the key concepts in the discipline (Corcoran, Mosher, & Rogat, 2009). The original hypothetical learning progression for QR arose from the first and last approaches, with approach two being the primary method for iterative testing and revision of the learning progression.

Clinical interviews are the initial method of gathering data to validate and revise the hypothetical learning progression. Clinical interviews allow for a deeper insight into student reasoning with faster turnaround for revising the progression. Guiding principles for developing the clinical interviews (Anderson, 2009) are that they are based on the progression framework, built around practices, linking processes, and standard representations, with branching probes to explore discourses, principles, and themes. The interviews are transcribed and coded using the Grounded Theory method. Analysis of the interviews informs the revision of the framework. The revised framework is used to revise the assessment and develop a written summative assessment used to track progress in teaching and learning QR aspects of environmental literacy. Development of the written assessment requires administering them at multiple sites and grade levels across the Pathways project, developing databases of student responses, developing coding rubrics that tie responses to achievement levels and progress variables in the framework, and conducting validation and calibration analysis. Teaching experiments and professional development on QR aspects of environmental reasoning are then conducted, guided by the learning progressions and assessments. The cycle is then repeated.

Results

The QR Theme Team is early in the research cycle. In Fall 2009 through Summer 2010 the QR Theme Team developed QR assessment items to be included in the carbon cycle, water cycle, and biodiversity strands assessments. In Spring 2010 a small sample of students were interviewed using QR items developed for the strands. Based upon initial assessment data piloted in grades 6-12, we hypothesize students function on four different levels: 1) students avoid quantifying the problem even when having a basic understanding of science; 2) students fail to quantify the problem correctly; 3) students quantify the problem correctly but fail to relate it to the science concept; and 4) students quantify the problem correctly and relate it to the science concept. This has led to the development of questions that differentiate on which of these levels students are functioning.

Another observation in the development of assessment items has been the implementation of QR into science concepts where it is not necessary. QR must be incorporated only when it is essential to understanding the science concept. Questions must also be written carefully to illicit informative responses to correctly assess student's QR abilities in the context of science. Specifically, a key element of learning progression assessment items is that they must be open-ended (i.e., must be based on a linking process) allowing for a response from any student. Development of open-ended QR items is particularly difficult, as the QR linking processes must be identified. Simply asking for a numerical response would likely eliminate many student responses.

The integration of QR into learning progressions for ecology has been a challenge. At this stage, we are unsure if the best approach is to integrate learning progressions for mathematical and statistical components of QR into the science learning progressions, to have parallel learning progressions, or if the integration of a learning progression for QR is necessary for the science learning progression. An additional complication is that formal learning progressions for the QR components do not currently exist. Importantly, the data provides an attempt to determine whether the science concepts, the mathematics or statistic concepts or integration of both is the barrier that results in the inability to quantitatively reason about environmental issues.

Discussion

Currently the QR team is working on a number of barriers to the QR aspects of developing environmentally literate citizens. The team believes that QR is not often expressed when students are functioning at the local or individual level, however, as students move across scales up to landscape or down to microscopic and atomic, the need for QR becomes paramount. If students are to become environmentally literate citizens they must address the grand challenges facing their generation, which will require them to scale up to global and down to atomic levels. To address these barriers the QR team is currently working on development of interview protocols, assessments, and professional development rooted in QR in environmental science.

The QR team will pilot global environmental challenges QR clinical interview items in spring 2011. Analysis of data from these interviews will be used to revise the QR learning progressions and explore interrelationships between QR and environmental science learning progressions. Development of a written assessment based on this analysis will begin late in Spring 2011 with the goal of developing a pilot written assessment for use at Pathway LTER sites in Summer 2011. By the time of the PME-NA Conference in October 2011 the Pathways QR Theme Team will have significant data analysis to report.

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SEEING COMPLEXITY IN THE PROCESS OF CHILDREN'S BLOCK PLAY

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Nearly a century of research has established a relationship between the complexity of children's block play and mathematical thinking and reasoning in other contexts and in later years. However, most of this research has evaluated complexity by examining children's completed structures. This video-ethnographic study looks at mathematical thinking and reasoning during *block play as a way of contributing to learning trajectory research in early childhood mathematics that values diverse ways of thinking. The paper highlights opportunities for mathematical reasoning that emerged during the construction of "less" complex structures.*

The importance of children's block play for developing spatial thinking and geometric reasoning has been established through multiple studies (e.g., Casey et al, 2008; Guanella, 1934) These studies, dating back almost a century, have looked at ways to increase the complexity of children's block structures as well as ways that this complexity can support more complicated mathematical thinking, particularly in reference to spatial visualization skills, such as imagining flips and rotations (e.g., NRC, 2009). In addition, this body of work has documented relationships between block play and mathematical thinking more broadly (e.g., Caldera et at, 1999; van Nes & van Eerde, 2010). For example, recently Wolfgang, Stannard and Jones (2003) demonstrated a relationship between complex block play at age 4 and mathematics achievement in the 7th grade.

One emerging line of research, developed in response to increased interest in young children's mathematical learning trajectories (NRC, 2009), is the role that teachers can play in scaffolding children's spatial and geometric learning through block play (Clements & Samara, 2007; Kersh, Casey & Young, 2008; NRC, 2009). Gregory, Kim and Whiren (2003) demonstrated that trained educators could support students in creating more complicated block structures by talking with children as they played with blocks. Their study, like most others, evaluated the complexity of the block building through the examination of the qualities of children's completed structures.

The purpose of the current study is to add to this emerging line of research on children's learning trajectories in block play by evaluating children's thinking and reasoning as they go about the process of building with blocks. In doing this work, we hope to contribute to diverse ways of evaluting and scaffolding children's mathematical thinking during block play. Nearly all previous researchers have based their analysis of complexity on the completed block structures. For example, in 1934 Guanella described block structures as ranging in complexity from "pre-organized," where no structure is built, to piles and rows, to solid forms that include closed spaces, to three-dimensional structures (Guanella as cited in Casey et al, 2008). In their recent study, Gregory, Kim and Whiren (2003) expanded on the categories for analysis developed by several previous blocks researchers to evaluate student structures based on three broad categories: the complexity of the building, the complexity of arches, and number of dimensions (use of points, lines and planes) in the finished structures.

We would like to consider the possibility that complex mathematical thinking might not be entirely represented by increased complexity of completed block structures. In other words, a child might set a task for herself in work with blocks on a single plane that is more mathematically complex than one involved in creating a structure on multiple planes. We believe that the documentation of children's learning paths in their work with blocks ought to include an examination of the various cognitive demands and opportunities made throughout the building process as well as documentation of progression in creating more and more complex structures.

Guiding our study were the following research questions:

- What kinds of mathematical thinking and reasoning emerge during children's play with various kinds of blocks in both formal and informal contexts?
- How do interactions with others shape the mathematical thinking and reasoning that occurs?

Theoretical Framework

This study draws on Vygtoskian (1978) traditions that see play as central to children's development and see learning as an outcome of engagement in social activity. In his work, Vygotsky emphasized the tools that children draw on in their learning. These tools include the physical – such as Lego or wooden blocks – and also the social – such as ways of thinking or playing that are learned from other humans in social settings. Although Vygotsky's work around scaffolding is most often taken up to analyze interactions between children and adults, in this study we would like to acknowledge the role that children can play as the knowledgeable others in their peers' play. Particularly, during free time where some children may spend far more time than others with certain materials and thus develop expertise, children themselves may be able to scaffold the block play of their peers toward more complexity. Vygotsky also described the role that a cultural tool like "self-talk" can play in scaffolding a difficult task. He describes a child standing on a chair to reach a cookie, vocalizing each step in the process as a way of managing the difficult task. In a similar way, many children talk to themselves as they play with blocks. Following Vygotsky, we believe it is important to capture and analyze this talk as a way of gaining further access to children's mathematical thinking and reasoning.

Methods

This project is embedded in a larger ethnographic study of children in a rural community as they move from kindergarten to first grade, which seeks to explore their mathematical learning during this time span. As a small, rural school, Taylor County Public School is an ideal site for a longitudinal study because there is only one classroom for each grade. Nearly all of the students who attend the school are eligible for free lunch and about 90 percent are African American. These characteristics make this an important site for broadening the research base on learning trajectories around block-building because much of the reseach on blocks has occurred in settings where a majority of the children are European American and come from middle-income families. Data collection for the larger project includes weekly classroom observations supported by video- and audio-taping, individual video-taped assessment interviews with each child, audio-recorded parent interviews and focus groups, and video- and audio-taped observations of Parent Math Nights. Data collection and analysis for this paper has been informed by the multi-year relationship the first author has with the school.

As an ethnographic study, the research team seeks to understand children's experiences through both observation and participation. For this aspect of the project, this means that rather than comparing students' play and their block structures to pre-existing criteria, we attempted to document the mathematical interactions students engaged in during the process of play. In the preschool classroom, we took fieldnotes, audio-recorded conversations, and documented interactions with video cameras during both formal mathematics lessons and free-choice time, when children were allowed to choose among the many materials in the room, including Lego blocks, wooden blocks, and unfix cubes.

Data analysis was supported by the qualtiative data analysis program NVivo9. Using this program, the research team was able to code excerpts of field notes as well as segments of video clips for analysis. For example, a 30-second video segment could be identified and coded separately from a longer video clip. The observer who recorded the video also used the Nvivo9 program to describe the action going on in the video and transcribe some speech. Initially, data were coded based on the setting, the type of interaction, the materials used, and the mathematical concepts. The current codes are listed in the text box to the right; however, coding, particularly around describing the block play and mathematics, is being continually refined. At the time of writing this proposal, we still have another three months of scheduled observations in the prekindergarten. As a result, the analysis reported in the following section is tentative and still evolving. Over time, we hope to create a map of the opportunities for thinking and reasoning possible in the midst of block play.

Results

In this section, we describe several representative episodes of block play captured by videotape or fieldnotes over the course of the study. These episodes were chosen to demonstrate the mathematical thinking and reasoning that occurred as children interacted with different kinds of blocks, in different settings, and with different people. While all are unique, the kinds of thinking and interactions that occurred in each episode can be found across our data record.

Episode 1: Carter and Wooden Blocks

One afternoon, Carter, who frequently played with all kinds of blocks during free choice time, chose to build with wooden blocks. The block set available in the classroom was relatively traditional, with a variety of rectangular and triangular prisms, cylinders, and arches. On this occasion, Carter decided to build a road out of blocks, announcing his intention to no one in particular. Carter began by laying down a 6-by-3-inch block. He then put together four 3-by-1.5-inch blocks, choosing from dozens of blocks spilling from the tub around him. For the next road segment, he chose two 6-by-1.5-inch blocks, putting them side by side. He then used another 6-by-1.5-inch block and then two 3-by-1.5-inch pieces. Finally, he chose two 3-by-3-inch blocks for the last segment in the road.

The wooden blocks, which were designed to make many equivalent shapes possible, allowed Carter to experiment and practice with composing and decomposing shapes. Carter seemed to embrace this activity as part of his play. After all, several of the large 6-by-3-inch blocks were available and Carter could have chosen to create a road out of identical blocks. Instead, he seemed to go out of his way to make as many equivalent combinations as possible in the building of his road. This ability to compose and decompose shapes has been highlighted as an important mathematical concept for young children because children "who can compose shapes develop better understanding of composing and decomposing numbers" (NCTM, 2010, p. 56) as well as a foundation for geometric concepts. In addition, play like this afforded Carter the opportunity to practice visualization because he needed to orient blocks in the proper direction to make equivalent shapes and to choose the desired block from a pile in the tub.

It is also important to note Carter's intention to build a "road." Traditional rubrics for scoring the complexity of block structures would categorize the road as less complex than other structures because of the lack of vertical building, arches and enclosures. However, this analysis does not take into account Carter's intentional construction of an object that he likely visualized in his head before he created it as well as his efforts to create equivalent segments. In considering complexity, it is important to consider not just the finished product (in this case, a straight line), but also the particular tools used to construct the line (blocks of various sizes and shapes rather than all identical blocks and equivalent block units rather than varyingly-sized units). In this episode, Carter played by himself with little scaffolding from others; however, the available tools contributed to his mathematical thinking. This includes both the opportunities for visualization, identification, composing, and decomposing provided by the blocks, but also the story scaffolding provided by other toys. Casey and colleagues (2008) found that telling stories contributed to children building more complex block structures, and in many ways available toys served this purpose for children in the classroom. For example, in this case, it is unlikely that Carter would have chosen to build a road if cars and trucks had not been available to drive on it.

Episode 2: Markus and Unifix Cubes

One morning toward the end of free-choice time, Markus came upon a bucket of unifix cubes scattered across the carpet. Initially, he bent and began to stack as many as he could into a tower nearly as tall as himself. After holding his tall tower against himself, he sat and began breaking the tower into smaller chunks. He then picked up one of these smaller towers and counted each cube, touching it as he said the number. After counting seven cubes, he reached out grabbed a loose cube and added it on. He then repeated the process, making four other towers of eight cubes each. He then laid these towers against each other on the floor. When he made his fifth tower, he laid it down by the others to check its length and then took off two blocks when he realized it was too tall. At this point, Dahlia and Trevor approached and began to stack the blocks into tall towers. Markus corrected them, saying "We need to make eight! For Bingo." Trevor and Dahila accepted this and began to make towers of eight blocks, each of them counting the blocks in their finished towers one at a time, touching each block and laying their completed towers down next to the pile Markus had made. Other children started to approach, making various lengths with the cubes. Markus reiterated that the towers had to be eight long. At this time the teacher approached and asked Markus, "How many are in your tower?" He picked one up and counted for her, touching each block, "1,2,3,4,5,6,7,8." She replied, "Very good."

Unifix cubes are not often considered in block research; however, as we watched children play with them during the informal moments in the classroom, it seemed odd to exclude them. Because these cubes could only be used as isolated ones or as towers, the mathematical thinking they were most likely to elicit involved counting and comparison. Students compared towers to each other (as Markus did when he was making units of eight), compared them to themselves (as Markus did when he made his first tall tower), and compared them to objects in the room. Students also often counted these blocks, practicing one-to-one correspondence as they did so. Sometimes, this counting seemed to serve a particular purpose, as it did for Markus, who knew that the blocks were passed out in groups of eight for Bingo because the boards had eight spaces. However, more often students seemed to count simply because they had the blocks. This may have been because the teacher frequently used these blocks for counting in formal lessons and also tended to ask children to count the blocks whenever she saw them being used during free-choice time. The unifix cubes seemed to offer fewer opportunities for spatial reasoning than other blocks, although opporutnities to compare lengths did encourage students to talk about more, less, longer and shorter, and to make adjustments to their towers as a result of these analyses. Markus's use of his already constructed towers of eight to measure his new towers demonstrated an ability to reason about length and quantity as he was able to recognize that towers of the same length would have the same amount of cubes without having to count again and was also able to adjust towers by taking off or adding on cubes to get the correct lengths. The task Markus set for himself and the way he set about it reveal aspects of his thinking and reasoning that would not be apparent in the completed pile of towers of eight blocks.

In this episode, the blocks themselves and the potential of a Bingo game later provided scaffolding for Markus to explore length and comparison and to practice counting. Markus, himself, provided this scaffolding for other students, by repeatedly naming the task at hand. Although the other students did not talk about the Bingo game, it is likely that this real-world purpose, served to persuade them to engage in the counting activity Markus set up. On the other hand, the teacher's intervention in this case did little to scaffold mathematical thinking. Markus had already counted the blocks in multiple towers by the time the teacher asked him to count. Indeed, he already knew there were eight blocks in the tower he was holding when she asked him to count the blocks. He performed the task anyway and her response did little to either acknowledge the thinking he had already done or to push him toward new mathematical engagements.

Episode 3: Xavier, Carter and Legos

At the beginning of free-choice time, Carter pulled out the Dulpo Lego blocks. He placed a large blue mat (25 by 25 nodes) on the floor and began to attach blocks along the perimeter of two sides. Nearby, Xavier began to build a tower of 2-by-4 blocks. Once the tower was 10 blocks tall, Xavier attached it to the mat on which Carter was working. Carter rejected this effort, saying "No, I'm making something right here." He reached out and removed Xavier's tower. Xavier accepted this correction and moved away from Carter, who followed him asking, "You want to help me?" Xavier tried one more time to make a tower and attach it, but Carter again removed it from the board. He gestured to the blocks he had placed along the perimeter and said: "I'm doing this." Xavier began to join Carter in filling up the board with a single layer of blocks. This went smoothly until the entire board was nearly full. The boys then began to search for pieces that would exactly fit the few remaining holes. Carter dug through the box to find a 2-by-2 square, which he placed in a matching open space in the center of the board. Xavier turned a rectangular block to fit a remaining hole. The last space in board required a 2-by-8 block; however, one of the surrounding blocks had plastic googly eyes projecting from the side, which made it impossible to fit an appropriately sized block in the adjacent space. After both Carter and Xavier tried a number of correctly sized blocks with little success, they appealed to the researcher observing. She asked them to feel the side of the block with the raised eyes. Xavier did so and then took that block out, replaced it with an identically sized one, and then filled in the last hole before beaming up at the camera. In all, this process took Xavier and Carter 25 minutes.

The Dulpo Legos, with their defined working spaces on mats, primarily rectangular shapes, and ease of fitting together, scaffolded different kinds of mathematical thinking than wooden blocks or unifix cubes. Children were able to build much taller structures with the Dulpo Legos than with the wooden blocks. In addition, the concepts of area, perimeter, and rotations were frequently explored during Lego block play. As Carter did in this example, students frequently created a perimeter first and then filled it in. This required choosing blocks that would be exactly the same length as the mat and negotiating the tricky business of turning corners. Carter, who spent a lot of time in the block area, performed these tasks with certainty, rarely needing to remove a block and replace it with one that had a different length or width, while other children frequently needed to make such adjustments.

These differences in the ways children went about building similar structures reveal differences in development of visualization skills and understandings of length and width and would be unlikely to be captured in the evaluation of a final product. Similarly, in this case, many of the scoring systems for complexity would rank a structure as more complex if it had the tower that Xavier proposed adding; however, Carter's sustained commitment to filling the board demonstrated perseverence with a task and also created an opportunity for both him and Xavier to think about how to fit geometric pieces together exactly, which required anlysis of shapes, lengths and widths and work with rotations. Typically, when children added towers to constructions the play became focused on creating cities or sometimes parking garages, rather than on a task like filling all of the available space on the mat. Thus, Carter scaffolded a more mathematically complex task for Xavier through his commitment to working only in one layer.

Episode 4: Dahlia, Her Mother & Legos

At the Parent Math Night, one of the stations consisted of several boxes of Legos, including special pieces, such as wheels and trees, and written directions for making specific projects. Dahlia came to the event with her mother and her six-year-old sister, Tamara. Her mother took the written directions for a house, held them in front of Tamara, and quietly talked her through the process of building. Dahlia looked on for a moment and then began to build a tower with 2-by-8 blocks. After watching Tamara for a moment, she searched through the box to find window and door pieces the same size as the blocks. She pulled on her mother's sleeve and said "See my house." Her mother looked over, saying, "That's pretty, Dahlia. What you going to do next?" Dahlia pressed her tower onto a Lego board and began to add blocks to her board. After a moment her mother smiled told her to look at Tamara's house, which had a roof, doors, and other accessories, as well as four walls creating the structure. Dahlia looked on for a moment and then began to add additional blocks to her construction.

In this episode, Dahlia, with her mother's help, used her older sister's construction as a model for her own work so that while Tamara constructed a house that mirrored the one pictured in the Lego directions, Dahlia turned her tower into a "house." In someways, viewing the tower as a house was merely cosmetic, such as in Dahlia's decision to add the clear bricks to stand in for windows. However, Dahlia's building of a tower was her independent translation of Tamara's four-walled enclosure. In her attempt to emulate her older sister, she needed to watch what Tamara was doing and decide how to replicate it with the blocks available to her, a process that encouraged the development of visualization skills and manipulation of geometric figures. Her mother encouraged her to add complexity to her structure by prompting her to think about what she should do next and presenting Tamara's house as a model. Unlike the teacher and parapro, both in the classroom and at the Math Night, the parents we observed rarely asked their children to count or to name shapes. Rather, parents scaffolded their children's work and thinking through engagement in the same or similar tasks. For example, the father and son at the table with Dahlia both made their own airplanes. This sort of scaffolding seemed to allow more opportunities for spatial thinking and reasoning to unfold than the teachers' efforts, which continually redirected children back toward traditional preschool mathematics, such as counting.

Discussion

Given the concerns expressed in many recent reports (e.g., NAEYC & NCTM, 2002; NRC, 2009) about the quality of mathematics experiences many young children receive, particularly poor and minority children, we believe it is important to consider ways that opportunities for complex thinking and problem solving can be incorporated into preschool and kindergarten school days in both formal and informal spaces. In the current climate of standardized testing, many early childhood teachers have become focused on basic skills, which has led to a reduction in free play. However new research has shown that these play opportunities are critical for complex mathematical thinking later (NRC, 2009).

This ethnographic analysis of block play raises a number of issues for mathematics educators to attend to as they develop early childhood mathematics curricula, engage in professional development with early childhood teachers, and do further research on young children's learning trajectories. First, it is important to recognize that the complexity of mathematical thinking young children engage in during block play may not be entirely captured in the finished structures. The research community would benefit from increased attention to creation processes, and children and teachers would benefit from increased opportunities to talk about their thinking during and after their work. Second, children's play with blocks needs to be scaffolded to maximize opportunities for mathematical thinking; however, this scaffolding does not always need to come from a teacher in either a formal or informal setting. Well-chosen materials can scaffold particular kinds of thinking, and other children and parents can also support children in problem solving. More research needs to be done on how to build on the strengths parents and children bring to this process. In addition, teachers need to learn both how to watch children during block play to figure out the mathematics that may be present and to recognize the more complex process skills that young children need to develop along with counting and shape recognition. Finally, we would be remiss not to note that in our study, as in much previous work (e.g., Kersh, Casey & Young, 2008), gender mattered in block play. Nearly all of our classroom clips with Legos, wooden blocks, and unifix cubes show the work of boys. However, during Parent Night and in the assessment interviews, girls engaged enthusiastically with the blocks, as Dahlia did in the described episode. Creating spaces where girls were expected to build with blocks and where they did not have to struggle to maintain access to the materials seemed important in giving girls access to these learning opportunities. Guided play as part of formal lessons in classrooms could potentially provide another welcoming space for girls to engage in block play.

Developing learning trajectories around blocks that include both images of what finished structures may look like and descriptions of processes that children go through as they engage with these materials is important to the study of children's early learning broadly and, more particularly, to the study of the relationship between formal and informal learning. Teachers will be able to use these trajectories to learn about the deeper mathematics embedded in block play and to make connections between formal and informal lessons. It is important to both theory and practice to develop these trajectories by looking at the thinking that goes on during block building in a variety of social settings not simply at a progression of completed structures. This theoretical view is likely to be more inclusive of diverse mathematics and of diverse children.

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CONCEPTUAL METAPHORS WITHIN MATHEMATICAL PROBLEM SOLVING

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Current research in mathematical problem solving suggests that language, and specifically metaphors, are influential in a student's ability to perceive, solve, and learn from a mathematics problem. This paper summarizes the results of a pilot study in which high school students attempted to solve and justify three mathematics problems. The student was video recorded and then allowed to watch himself/herself solving the problems. The student commented on the process used while solving the problems. Three analyses focused on the student's use of conceptual metaphors, the word like (myopic like), and how such factors influenced the student's performance.

Metaphors are a means to relate experiences through language, thought, and action. The relationship between the experiences of the teacher and the student are vital to mathematics education. Specifically, teachers and students share an experiential set: solving mathematics problems. However, the student's and teacher's perspective of what constitutes mathematical problems and/or solutions are complex in structure. (Lakatos, 1976; Lesh & Zawojewski, 2007; Pólya, 1945; Schoenfeld, 1992). Metaphors are culturally designed to make these implicit perspectives explicit. Moreover, they have been found to encourage and incite cognition (Lakoff & Núñez, 2000; Sfard, 1997).

This paper summarizes the results of a pilot study into how metaphors are used in mathematical problem solving, specifically:

Q1. How do metaphors help in understanding a problem?
Q2. How do metaphors guide the process of solving a problem?
Q3. What metaphors are used to describe mathematical problem solving?
Q4. How do metaphors connect problem-solving and proof?
Q5. How can knowledge of conceptual metaphors help teachers improve student learning?

THEORETICAL FRAMEWORK

The theoretical framework for this study has evolved from the firm belief that mathematics is embodied (Lakoff & Núñez, 2000). Fundamentally, embodied mathematics states that mathematics is not mind-free and dependent upon human perception. Concomitantly, this research perceives of mathematics education under the philosophical axiom of cognitive science; one can understand and interpret how people think, and more specifically, learn (Gardner, 1987). Within this realm, this study focuses on the influence of metaphors to one's ability to learn from and solve mathematical problems.

George Polya (1945) emphasized two types of problems in mathematics: problems to find, and problems to prove. This distinction is significant, yet this study will show how these two concepts are not mutually exclusive via linguistics. Another significant area of focus of Polya (1954) was the use of analogies; how analogous problems can be identified and used to solve foreign mathematics problems. This is the springboard from which this article begins because it was through this branch of thought that Polya began to see the influence of language in solving problems.

Polya's linguistic inspiration was paralleled in the 1970's by cognitive scientists, whose emerging field was a culmination of artificial intelligence, cybernetics, anthropology, psychology, philosophy, linguistics (Gardner, 1987), and now education. This interdisciplinarity attracted mathematics educators such as Alan Schoenfeld (1985) to model students' approaches to problem solving. Schoenfeld's attempts to interpret how students solve mathematical problems demonstrated a high level of complexity which suggested a categorization of problem-solving characteristics rather than a sequential how-to model. Schoenfeld's initial categories included heuristics (as reinvented by Polya, 1945), resources, controls, and beliefs. He found that student's expression of their problem-solving process emphasized aspects of control, beliefs, and explicit knowledge of their own cognition. He clarifies this as metacognition, a valuable skill students possess (Schoenfeld, 1992). Combining Schoenfeld's concept of metacognition with Polya's concept of analogy suggests that the language through which students express their cognitive process may demonstrate how students discern between isomorphic (analogous) and non-isomorphic mathematics problems.

The language that the students use to express their metacognition is insulated by experience. In this manner, mathematics is embodied (Lakoff & Núñez, 2000). One's knowledge of mathematics is dependent upon one's perspective and experiential learning of that knowledge. For example, Lakoff and Nunez (2000) demonstrate how the understanding of limit is complex and embodied within one of two conceptual metaphors. One can think of limits graphically; claiming that as x *approaches* (motion metaphor) a number, the function will also *approach* a number. Weierstass viewed limits by saying; if x is within the *proximity* (proximity metaphor) of a number, then the function will also be within the *proximity* of a number. Both metaphorical perspectives are valuable, and distinct in their techniques of proof. However, the logic necessary to solve limit-based problems is analogous in both metaphors. Thus an elusive bond exists between problem solving and proofs in which metaphors are squarely centered (Q4).

Polya's concept of analogy is distant in linguistics from metaphors, yet close in cognition. Analogies reference two concepts already firmly defined in the learners' mind for purposes of mapping aspects of one concept onto the other (Sfard, 1997). However, metaphors frequently model a new conceptual structure with a pre-existing structure. The accommodation of known structures into new concepts can define the new concept, and is considered an aspect of conceptualization (Kövecses & Benczes, 2010; Sfard, 1997). Justifiably, Sfard (1997) refers to such conceptual metaphors as implicit analogies. Hence for mathematical problem solving, application of analogies follows the learner's understanding of conceptual metaphors.

Linguists classify these conceptual metaphors into three hierarchal categories: structural, ontological, and orientational (Kövecses & Benczes, 2010; Lakoff & Johnson, 1980). In all three conceptual metaphors, there is a source domain and a target domain. The source domain is the experientially-known domain and the related concept is the target domain. Thus in the metaphorical linguistic expression "The solution escapes me", the target domain is solutions while the source domain is prey. Hence the conceptual metaphor is "SOLUTIONS ARE PREY". It is important to note that despite the use of the being verb "are", the phrase is unidirectional (Target→Source). Structural metaphors strive to describe a complex concept, such as time, in terms of a concrete experiential object, such as a limited resource, i.e. "Don't waste my time". Ontological metaphors provide target domains with less structure and a new reality in which they may be defined. Personification is regularly ontological; as is the phrase "the solution escapes me". Orientational metaphors are the most difficult to relate experientially according to linguists. They are a broad concept with a specific direction inherent in our development as humans. The metaphorical linguistic expressions "Things are looking up" and "He fell ill" are examples of the conceptual metaphor "HEALTHY IS UP". How these metaphors directly influence a student solving of mathematics problems is the focal point of Q1, Q2, and Q3. Applying this metaphorical influence pedagogically is the purpose of Q5.

RESEARCH DESIGN

This pilot study used a naturalistic paradigm (Donmoyer, 2001) to study how metaphors influenced student's problem solving because the lack of current research within mathematics education (Sfard, 1997) mandates trustworthiness. In tandem, phenomenological inquiry (Short, 1991) was used to search for the essence of mathematical problem solving. Thus, students were chosen according to a list of criteria that indicated the student had a propensity towards mathematics and expressing their thoughts. The population of interest is high school students who are planning on going to college. Characteristics of the sample varied but were limited to participants with a genuine interest in mathematics for learning and recreation beyond grades, participants who could clearly communicate their thoughts without fear of the researcher's judgment, and participants who were willing to make mistakes and continue despite disappointment. Participants were neither included nor excluded based on ethnicity, sex, age, health or association with a special class. Twenty students were asked to participate in the study and nine students volunteered.

Nine students at a suburban high school in Ohio participated: three freshman, two sophomores, and four juniors. There were three females and six males. Each student met with the researcher individually after school for an hour. The students were given the three mathematical problems shown below:

P1. Imagine you had a piece of string. How would you bend this string to make a triangle bounded by the string with the greatest area?

P2. Humans have classified numbers on the number line into two categories, rational and irrational. Rational numbers are those that can be written as fractions, irrational numbers cannot be written as a fraction. Suppose I have an irrational number. If I add one to that number will it be rational or irrational?

P3. How could you cut a cylindrical birthday cake so that you have 8 slices using only 3 straight cuts with a knife?

The techniques and justification for each problem varied mathematically to identify differences or similarities in problem-solving techniques and metaphorical conceptualization. Moreover, the problems were specifically designed to be metaphorically sterile so as to evoke conceptual metaphors from the students without bias. The problems could be done in any order and manipulatives were available to help the students, including cork board, dry erase markers/board, string, pencils, paper, calculators, thumb tacks, pipe cleaners, and straight edges.

As Steffe (1983) poignantly noted in studies involving children solving problems, there are multiple interpretive mediums involved. The experience of the student is expressed by the student, interpreted by the researcher, related via the researcher's experience, and then expressed by the researcher. To minimize the amount of interpretation of the researcher and to maximize the metacognitive expressions of the student, Reynolds' (1993) design was applied where the student would attempt to solve the above problems (primary video) and then immediately watch themselves solving the problems with explicit instructions to explain their thought process (secondary video). Thus students worked with the researcher on the above problems for 30 minutes and then watched the video of their problem-solving process with the researcher for 30 minutes. The names of the students were coded according to the first nine Greek letter of the alphabet.

RESULTS

Using mixed methods, two analyses are complete, while two are still in progress. The first analysis was qualitative and interpreted the student's conceptual metaphors of problem solving. The second analysis was quantitative and ascertained two results; linguistic hierarchy of conceptual metaphors is viable within mathematical problem solving, and the student's use of the word *like* has a negative correlation with their performance on Q1, Q2, and Q3. The third analysis is a continuation of the first two analyses, currently ongoing, and discussed below.

First Analysis: Qualitative

The initial analysis involved multiple observations of all the videos using a phenomenological design and recording significant metaphors, problem-solving techniques, and all justifications. The first analysis revealed that the students evoked metaphors rich in context and culture. For example, when working with P2, Beta stated "Adding rational and irrational numbers is kind of like mixing oil and water". Initially, this analogy helped conceive of the question, but then raised complicated issues when deductive reasoning was needed. A more surprising result of the first analysis was the abundant use of the word *like*. Students used the word *like* frequently demonstrating examples or counterexamples to guide their intuition. When trying to understand the problem, students used the word *like* for inductive reasoning rather than deductive reasoning. The second qualitative analysis delves deeper into these results.

There was significant evidence that a classification of metaphors (structural, ontological, orientational) was applicable to problem solving. Students were consistently able to discuss their problem-solving techniques as if their brain was a separate entity. In the secondary video, students evoked ontological metaphors personifying their mind as an entity from which they were analyzing. For example, Epsilon changed from one question to another because she had to let her "subconscious work on it for a while". Zeta stated "my mind plays games on me." The following is a list of conceptual metaphors (mainly structural) that the students related to their problem-solving strategies through metaphorical linguistic expression in the first analysis:

Table 1: Conceptual metaphors of problem solving.

Target Domain		Source Domain
PROBLEM SOLVING	IS	A JOURNEY, STRATEGIES, A HUNT, A BATTLE, A PRODUCT, A DESTINATION, A BUILDING, A GOAL, TRICKS, DISCOVERY
THE PROBLEM-SOLVING PROCESS	IS	CONSTRUCTING,EXPERIMENTING, ILLUMINATING, TRAVELING, PLAYING, SEARCHING

Notice how problem solving is never personified as a creature with its own intelligence. Yet, the problem-solving process always involves an action-based orientation. For example, no one suggested PROBLEM-SOLVING PROCESS IS THINKING. These results demonstrate a metaphorical system (Kovecses, 2010) with certain coherent (Lakoff, 1980) traits.

Second Analysis: Quantitative

The second analysis (quantitative) had two parts. The first attempted to verify Kovecses's (2010) research in linguistics, that there is a hierarchy between structural, ontological, and orientational metaphors in mathematical problem solving. The second attempted to verify that the frequency of the word *like* was related to the student's performance. It is important to note this voluntary pilot study included only nine participants and thus nine degrees of freedom which limited the study (N<30).

The first part of the quantitative analysis was calculated by counting the number of times each conceptual metaphor was used during the primary and secondary videos. The descriptive statistics are shown below:

Table 2: Descriptive statistics for conceptual metaphors.

Descriptive Statistics	Mean	Std. Deviation	N
Structural	32.22	10.462	9
Ontological	19.78	6.438	9
Orientational	14.78	7.870	9

MANOVA was performed on the data, and the Wilk's Lambda showed a strong significant variability between the conceptual metaphors ($F(3,9)=14.292$, $p=.003$ with $alpha=.05$). Additionally, the structural metaphors were most frequent ($\mu=32.22$) followed by ontological metaphors ($\mu=19.78$) and then orientational metaphors ($\mu=14.78$) as was expected according to cognitive linguists.

The second part of the second analysis demonstrated that the total number of times a student used the word *like* was related to their overall score. For each problem the student solved, but could not justify, the student was given a score of ½. For each problem the student solved and could justify, the student was given a score of "1". Students could receive an overall score between 0 and 3. There was a nearly-significant moderate negative correlation between the overall score and the number of times a student used the word *like* ($r=-.634$, $p=.067$). There was a strong negative correlation to their score on P3 to the number of times a student used the word *like* in P3 ($r=-.937$, $p<.001$). Both of these results demonstrate that the more the student used the word *like*, the worse their solution to the problem.

Third Analysis: Mixed Methods

The third analysis is both quantitative and qualitative and currently still underway. The third qualitative analysis has two parts. First, the analysis develops a metaphorical system (Kovecses, 2010) for problem solving of teachers and students and determines which perceptions of problem solving converge and what perceptions diverge. Conclusions from this study are still being resolved. The second aspect of the third qualitative analysis has yielded results to how the word *like* is used in mathematics by high school students. My study has found that despite the word *like* taking nearly every form of speech, with respect to comparison theory of metaphors, all of the study's participants used the word *like* in only four frameworks.

These four distinct structures were true to mathematics education. Yet, linguistic theories shed light on their interaction within mathematics. Thus before designing a categorization, prototypical metaphors were examined to understand connotations:

Beta: "Adding rational and irrational numbers is kind of like mixing oil and water"
Gamma: "I will treat one side of the triangle like it was fixed"
Delta: "This triangle problem is like extending the rectangle proof"
Iota: "Consider a vector space like three dimensions"

The word *like* was exclusively used within these four instances to clarify the significance of discovery. Other uses of the word *like* did not belong in this classification. Consider the phrase "I was like, 'no way'!" This comparison does not align with the above exemplars because it relates an expression in response to a situation without directly stating the situation. This phrase compares the emotional response which must be assumed by the context. Another use of the word *like* that was never used was "equilateral triangles and the like", making specific reference to a generalizable group of triangles comparable by the equilateral property. Many other uses of the word *like* exist beyond these two, making the results of this study encouraging to mathematics education.

Consider the four exemplars for categorical purposes. Beta used *like* to compare multiple objects and similar operations between the objects. The linguistic structure comparable to Beta's use of the word *like* is an *analogy*; rational numbers are added to irrational numbers as oil is mixed with water. Gamma's conjunctive *like* can be replaced by *as if* which is a common use of *like* to refer to something happening in a literally comparable manner. I will treat one side of the triangle *as if* it was fixed (as literally being fixed). Delta compares the current problem to that of a different problem involving a different shape in hopes of searching for a common process. As these two shapes are unlike in structure, this is best referenced as a *simile*. These three uses of *like* closely resemble comparisons designed by linguist and cognitive psychologist, George Miller (1993). For purposes of interdisciplinary uniformity, Miller's comparison terms will be used here. Specifically, we will refer to the three comparison statements used by students within this study as analogies (Beta), literal comparisons (Gamma), and similes (Delta).

Notice, one can replace Iota's like with the phrase *for example*. This use is referenced in the Merriam-Webster Dictionary (2007) as a possible adjective definition, yet research in linguistics and mathematics education does not mention Iota's use of *like*, despite its frequent use. Iota's use of *like* differs because Iota is emphasizing a specific example (three dimensions) in reference to the broad concept (vector space). Thus Iota narrows, or focuses, to a specific example so as to concretely work with the mathematical problem within a familiar context. Iota's *like* transfers from global to local understanding. This use of the word *like* resembles nearsightedness. This is not to attach a negative connotation to this use of the word *like*, but rather clarify that one focuses on closer (more tangible) objects rather than farther (less tangible) objects. For this reason, I will use the metaphor of *myopia* to reference this use of the word *like*.

Table 3: Comparison Statements of the word *like* in mathematics education.

Like	Structure	Prototypical Example
Analogy	A is to B like C is to D.	Beta: "Even and odd numbers are like rational and irrational numbers."
Literal Comparison	A as if B, or A is literally like B.	Eta: "I was hoping like something would come to my mind".
Simile	A is like B despite A and B being distinctly different.	Gamma: "Get the base of the triangle like the base being the chord of a circle"
Myopia	B is an example of A.	Epsilon: "A quantity can't be irrational like a quantity in real life isn't really irrational because it is a fraction of something."

The purpose of the myopic like was significant and consistent with all students. If a student used the word *like* in solving any of the problems, the myopic like was in attendance. The quantitative analysis of myopias is still under investigation and is not included in these results. Additionally, a larger study (*N*>9) would give a stronger quantitative argument for the significance of myopias.

The lack of research into the myopic model illuminates a crevice between theory and practice. The myopic like has not been studied because mathematics educators perceive this use of the word *like* as misused. The myopic like is improper in formal writing, and thus not studied in how students speak. Yet, its existence mandates linguistic study if we are to rectify student misconceptions. While improper on paper, it is cognitively significant because every participant who used the word *like*, generated a myopia. Moreover, the negative correlation between score and frequency of the word *like* suggests an aid to the practical educator. Students who overuse the word *like* could be focusing on examples and inductive reasoning rather than deductive reasoning. Such a student continues to revolve around tangible examples rather than looking to aspects of those examples that are generalizable (Polya, 1945). Thus educators should be aware that a student's use of the word *like* may indeed cause myopic problem solving without the teacher's help.

SIGNIFICANCE TO MATHEMATICS EDUCATION

This study's primary purpose was to develop and improve upon the understanding of conceptual metaphors within mathematical problem solving. Three conclusions can be drawn from the results of this study. First, students are able to model and describe their problem-solving process metaphorically. Students used personifying ontological metaphors to describe their mind as an external object and structural metaphors to describe how their minds interpreted problem solving and the problem solving process (Table 1). These conclusions offer insight into Q1, Q2, and Q3.

Secondly, the quantitative analysis showed that the linguist's cognitive hierarchy of conceptual metaphors is tenable within mathematical problem solving as the student's metaphorical frequency was greatest with structural metaphors and least with orientational metaphors. This analysis also demonstrated that the more students frequent the word *like*, the lower their performance on mathematics problems that require justification (problems to prove). This negative correlation between the use of the word *like* and the student's performance suggests that if students are unable to move beyond inductive reasoning, they will be unable to deductively reason its logical truth or falsity (Q4). Hence, if a teacher listens for the student's over use of the word *like*, it may be an early indicator that the child is struggling with the concept at hand. The understanding of the word *like* may improve student learning (Q5).

The third mixed methods analysis is still underway, but the results show promise towards two features discovered in the first and second analysis. The third analysis interprets the model of conceptual metaphors for problem solving via a cohesive system for the purposes of helping educators better interpret and teach high school students. Additionally, the third analysis identified four (and only four) uses of the word *like* while solving mathematical problems: analogy, literal comparison, simile, and myopia. The frequent use and discovery of the myopic like suggest more quantitative analysis is needed. Both aspects of this mixed methods study encourage interdisciplinary research between mathematics education and linguistics with the philosophy shared by cognitive science that mathematics is embodied (Lakoff & Núñez, 2000).

This theoretical framework of embodied mathematics demands that pedagogy is influential in mathematics. The metaphors used by teachers and students influence how both parties think and more specifically, learn (Sfard, 1997). This study can aid educators and researchers because it offers specific and tangible observations of cognitive theories, such as conceptual metaphors, the myopic like, and embodied mathematics. The language chosen by students to express and construct mathematical thoughts is not arbitrary. Their ability to use metaphors develops incrementally. The metaphorical hierarchy of conceptual metaphors suggests one such increment, while the myopic like supports another. This analysis has suggested ways in which conceptual metaphors and comparison theory can be used to improve teaching. Hopefully, this pilot study will lead to future studies encouraging and confirming that metaphorical conceptualization can aid mathematics educators and students.

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CHALLENGING FACT FAMILY REASONING WITH INSTRUCTION IN NEGATIVE NUMBERS

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Through "fact family" instruction, students learn the commutative property of addition and inversion principle. However, problems like 3-9, challenge students' understanding of these principles due to their novel arrangement. Sixty-one first graders were randomly assigned to one of three instructional groups. Pre- and post-test interviews indicate that students who practiced operating through zero had greater gains in principle understanding.

Introduction

One of the fundamental goals of elementary mathematics instruction is to help students develop number sense; for children in grades K-2, in particular, number sense involves understanding the composition and decomposition of numbers as well as the relationships among numbers in addition and subtraction problems (National Council of Teachers of Mathematics, 2000). An important part of this process includes learning three principles: the commutative property of addition (e.g., 4+5 = 5+4), the inversion principle (e.g., 4+5=9, so 9-5=4), and the subtraction complement principle (e.g., 9-4=5 so 9-5=4). Knowledge of additive commutativity tends to appear before the other two principles, with inversion presenting more difficulty for young children (Canobi, 2005).

Instead of only addressing these part-whole relationships individually, US textbooks frequently incorporate a series of lessons on "fact families" which aim to help students see these connections all at once (Lovin, 2006). In the California version of the first grade *enVision* mathematics curriculum, there is one lesson dedicated to the commutative property of addition and seven lessons on relating addition to subtraction and using "fact families" (Pearson Education, Inc., 2009). Although the use of "fact families" or "turn-around facts" as an instructional focus seems widespread, there is little research on their effectiveness. Because studies show students understand that 4+5=5+4 and teachers see students correctly filling in the "fact families", we might believe that students *do* understand all of these principles; however, other studies show that students solve problems like 62–48 by subtracting the smaller number from the larger number in each column, regardless of their placement (Fuson, 2003). These results suggest that students may not have full understanding of these principles. This paper takes a small step at exploring how early instruction in negative numbers might facilitate first grade students' judgment of when and how to use the commutative property and inversion principle.

Theoretical Framework

As children learn addition and subtraction and develop more efficient strategies for solving arithmetic problems, they move through three "conception of quantities" levels (and sometimes a transition level) (Fuson, 1992; Murata, 2004). At Level 1, students count all quantities; for 3+4 a child would count out three objects, count out four objects, and count the combined collection to find the total. Students at Level 2 shorten this process by counting on (or counting back or up for subtraction); for 3+4, this student would say "three", knowing that it is not necessary to count out the three, and then count on "four, five, six, seven." Finally, at Level 3, students use composition and decomposition methods, especially involving groups of ten, to solve problems. Learning and using these part-whole relationships, such as those emphasized with fact families, is one of the most important goals in elementary arithmetic (National Research Council, 2001).

Aside from guiding students towards part-whole strategies, effective mathematics instruction also needs to help students develop mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (National Research Council, 2001). According to the National Research Council, "Examining the relationships between addition and subtraction and seeing subtraction as involving a known and an unknown addend are examples of adaptive reasoning" (p. 191). Along with fostering adaptive reasoning, presenting all forms of a fact family together (e.g. 4+5=9, 5+4=9, 9-5=4, and 9-4=5) might also help students deepen their *conceptual understanding* of the commutative property, subtraction complement principle, and inverse principle as students make connections among these relationships. However, by spending limited to no time learning the principles separately, it is also possible that rather than understanding these complex relationships, students learn that order does not matter for addition and likewise assume that the numbers in subtraction problems can be moved around in multiple ways.

A more *procedural understanding* of these principles could lead students to incorrectly think they can solve 2–7 as they would 7–2 by misapplying the commutative property or by thinking, "What plus 2 equals 7?" due to a misapplication of the inversion principle. Alternatively, students may use these misapplied methods because they do not know about negative numbers but feel the need to provide an answer anyway. Students who know about negative numbers and their relationship to positive numbers may be willing to think about obtaining answers other than positive numbers. Furthermore, learning that the commutative property does not hold for subtraction and having experiences correctly solving subtraction problems with smaller minuends might help these students better understand both the commutative property and the inversion principle.

Research Questions

The following questions guided this analysis:

1) How do students reason about the commutative and inversion properties before and after instruction in negative numbers?

2) How is instruction on the order and value of negatives and/or addition and subtraction with negatives related to students' understanding of commutativity and inversion?

Methods

Subjects and Site

This data comes from a study conducted at an elementary school in northern California, in which 47% of its students are English language learners (California Department of Education, 2010). Out of a possible 79 first graders at the school, 61 first graders (30 male, 31 female) agreed to participate and complete the interviews in English. The study took place in the spring, so the first graders had already learned about addition and subtraction.

Materials and Data Collection

The study employed a pre-test, post-test design with an instructional intervention. Both pre- and post-tests were conducted as individual interviews and involved similar questions; some post-test problems were identical to those on the pre-test, while others just had different numbers. During the interviews, students were asked to explain how they solved the problems.

While the questions covered a wide-range of integer concepts, including addition and subtraction problems, this paper focuses on a small-subset of question categories: counting backward, integer value comparisons, commutative property, and subtraction problems. Table 1 lists the questions from both the pre- and post-tests.

Table 2. Items students completed during the pre-test and post-test interviews.

Question Category	Pre-Test	Post-Test
Counting backward	"Start at five and count backwards as far as you can. Is there anything before <last number child says>?"	"Start at five and count backwards as far as you can. Is there anything before <last number child says>?"
Integer value comparison *"What are these two numbers? Circle the one that is greater."*	8 vs. 6*	6 vs. 4*
	3 vs. -9	5 vs. -7
	-2 vs. -7	-3 vs. -1
	-5 vs. 3	-8 vs. 4
	-8 vs. -2	-6 vs. -2
"Two children are playing a game and trying to get the highest score. Circle who's winning."	4 vs. -7	5 vs. -9
	-7 vs. -3	-8 vs. -6
Commutative property *"Just look at these two problems. Do you think they will give you the same or different answers?"*	4 + 5 vs. 5 + 4	2 + 5 vs. 5 + 2
	3 – 1 vs. 1 – 3	4 – 1 vs. 1 – 4
	6 – 4 vs. 7 – 4 (distracter)	7 – 3 vs. 7 – 4 (distracter)
	5 – 8 vs. 8 – 5	2 – 9 vs. 9 – 2
	3 + 2 vs. 3 + 3 (distracter)	6 + 3 vs. 2 + 6 (distracter)
	9 – 6 vs. 6 – 9	0 – 8 vs. 8 – 0
Subtraction *"Solve this problem."*	1 – 4 =	1 – 4 =
	3 – 9 =	3 – 9 =
	6 – 8 =	6 – 8 =

*All students solved these problems correctly, so they were excluded from further analysis.

After the pre-test, students (regardless of classroom) were randomly assigned to one of three instructional groups, so that each group had an even mix of students (in terms of initial understanding of negative numbers, teacher ratings of their math performance, and gender). Each group participated in 8, 45-minute lessons. During their group's instructional time, students met in a separate room, and the author provided instruction. Group 1 (N=20) received instruction on the order and value of negative numbers along with how to add and subtract with them; this included learning that subtraction is not commutative. Group 2 (N=21) only received the "adding and subtracting" lessons from Group 1's instruction, without learning about the order and value of negatives, while Group 3 (N=20) only received the "order and value" lessons from Group 1's instruction. Both Group 2 and Group 3 had additional practice games similar to their original lessons so that they received the same amount of lesson time as Group 1. Table 2 provides an outline of the lessons for the three groups.

Table 3. Lesson topics for the three instructional groups.

	Day 1	Day 2	Day 3
Group 1 (N=20) Integer Value & Order, Add, Subtract	Discuss and explore symbols	Explore minus sign versus negative sign	Vertical number line with integers; Game: Which is greater?
Group 2 (N=21) Add, Subtract	Explore lack of commutativity for subtraction compared to addition, no specific mention of negatives		
Group 3 (N=20) Integer Value & Order	Discuss and explore symbols	Explore minus sign versus negative sign	Match negative numbers vs. positive numbers

	Day 4	Day 5	Day 6	Day 7	Day 8
Group 1 (cont.)	Explore commutative property and lack of commutativity for subtraction compared to addition		More (move right) vs. Less (move left)	More positive = larger, more negative = smaller	Less positive = smaller, less negative = larger
Group 2 (cont.)	More (go right on NL) versus Less (go left on NL)	More positive= larger, more negative= smaller	Less positive= smaller, less negative= larger	More positive, less positive, more negative, less negative	
Group 3 (cont.)	Negatives on a vertical number line; Game: Which is greater?		War: Which integer is Greater?	Game: Get three or four consecutive integers in a row	

Analysis

Students' solutions were coded for number correct in each category, and their verbal reports were transcribed for each. When counting backward, students only had to count to "negative one" in order for their count to be considered correct. Using the terminology "minus one" or "penalty one" also counted. Regarding the commutative property questions, all distracter questions were removed from this analysis (only three students gave an incorrect answer for one or both of these). Students' explanations were coded based on their reasoning for why they thought the answers would be the same or different (e.g., Commute = commenting on the numbers being switched around, Inverse = justifying a subtraction problem based on an addition problem). Finally, for the subtraction problems, answering "0" or a negative number was counted as correct since these answers meant students were not reversing the order of the numerals, which was all the study explored. The explanations of students' solutions were coded as to whether they reversed the numbers or used inversion reasoning in solving the problems.

Results

Table 3 lists each group's percentage correct scores on the pre- and post-tests for each question category, along with their overall percentage gains. On the counting backward and integer value comparison tasks, both groups who had instruction on integer order and values (Groups 1 and 3) made pre- to post-test gains that are almost double the gain of Group 2, who did not receive this instruction. Unsurprisingly, on the pre-test, all groups demonstrated understanding of the commutative property of addition, judging that 4+5 would give them the same answer as 5+4; because of their high initial performance on this item, there were small to no gains on this item across groups.

When asked if subtraction problems and their reversals would give them the same answer, Group 3—who learned about the order and value of negatives—made a large gain in understanding that the answers would be different. However, when they were asked to solve the problems, they made no gain in providing negative or zero answers. On the flip side, Group 1—who received instruction in all of the topics—made no gain in identifying that the subtraction problems with smaller minuends would have different answers but made a modest gain in actually providing negative or zero answers when they had to solve them. Group 2—who practiced adding and subtracting on both sides of zero—improved on both of these question types.

Table 3. *Percentages correct (and gains) for each group on pre- and post-tests by question type.*

Counts Backward into the Negatives	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	15%	60%	**45%**
Group 2 (N=21)	14%	38%	24%
Group 3 (N=20)	20%	65%	**45%**
Correctly Compares Integer Values	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	18%	73%	**55%**
Group 2 (N=21)	17%	32%	15%
Group 3 (N=20)	20%	90%	**70%**
Commutative Property of Addition	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	85%	95%	10%
Group 2 (N=21)	90%	95%	5%
Group 3 (N=20)	90%	90%	0%
No Commutative Property of Subtraction	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	37%	32%	-3%
Group 2 (N=21)	11%	37%	**25%**
Group 3 (N=20)	15%	48%	**33%**
Solving S – L, where L>S, S, L > 0	**PreTest (%)**	**PostTest (%)**	**% Gain**
Group 1 (N=20)	33%	45%	12%
Group 2 (N=21)	38%	63%	**25%**
Group 3 (N=20)	47%	45%	-2%

Students' Responses

While Group 2 was the only group to show growth for both the commutative questions and the subtraction problems, there were similar trends and student variation in each of the groups. Over all groups, students reversed numbers to solve 3-9, 6-8, and 1-4. An additional nine students specifically misapplied inversion reasoning to justify their positive answers; this is likely an underestimate since several students provided positive answers without justifying why they did so. Each group had at least one student who originally claimed problems such as 3-9 and

9-3 would have different answers but then on the post-test stated they would have the same answers. Furthermore, each group had students who demonstrated no gains from pre- to post-test (although some of these students changed their reasoning about their answers). Finally, all groups contained students who improved in realizing that the subtraction problems would give them different answers, and more specifically, that subtraction problems with smaller minuends would not have positive answers. See Table 4 for examples of how students in these subsets reasoned from pre- to post-test. The examples are from students in Group 1, but their reasoning is reflective of the other groups' reasoning.

Table 4. Examples of students from Group 1 whose performance decreased, stayed the same with different reasoning, and improved from pre- to post-test.

ID	Test	Answer	Explanation	Code
209	Pre-Test	3 – 1 ≠ 1 – 3	Three and taking one is (writes 2). There's one and you took away three and there's no more, and it's zero (writes 0).	Zero / Positive
	Post-Test	4 – 1 = 1 – 4	Four. Four. One. One.	Same Numbers
213	Pre-Test	3 – 1 ≠ 1 – 3	When you minus, it can't equal the same number.	Not Equal
	Post-Test	4 – 1 ≠ 1 – 4	This one [1 – 4] would equal a negative and this one [4 – 1] would equal a positive number.	Negative/ Positive
101	Pre-Test	3 – 9 = 6	Six plus three is nine.	Inversion
	Post-Test	3 – 9 = -6	(counted back on fingers) Two, one, zero, negative one, negative two, negative three, negative four, negative five, negative six.	Count Through Zero

Discussion and Conclusions

While all of the groups improved in a couple areas, some of the findings are clearer than others. The differential instruction influenced students' improvement on certain questions due to their emerging understanding. The two instructional groups who learned about the order and value of negative numbers (Groups 1 and 3) had greater gains on the counting backward and integer comparison tasks. This is unsurprising since students in these groups practiced counting backward through zero and played games which focused on the value of integers. Group 3, on the other hand, only saw negatives written on a number line in one lesson, and the numbers were not named or pointed out to them.

As found in previous studies, these first graders also showed consistent knowledge of the commutative property of addition. The patterns for the subtraction problems with smaller minuends are less clear. Group 2 showed gains in identifying that subtraction problems, when reversed, will result in different answers; they also made gains applying this knowledge in order to solve the subtraction problems. They transitioned from giving positive answers to answering mostly negative numbers or zero. During instruction, this group spent several lessons playing games where they moved back and forth across zero as they acted out addition and subtraction problems. These motions may have facilitated their developing conceptual understanding of why the answers would be different and helped them avoid the perceptual inclination to think that commuting in subtraction is okay because the problems contain the same numbers.

Group 1, however, also received this movement instruction, but their gain was much smaller than Group 2's. A possible reason for this is that Group 1 spent less time on the addition and subtraction activities because they also had lessons on integer values. It is reasonable that the length of time students practice moving beyond zero is related to their ability to reason about why subtraction problems with smaller minuends would have different answers than their reversals. The results of this study only hint at a possible connection, but further study is needed.

Although Group 1 did slightly improve in concluding that the subtraction problems with smaller minuends would not be positive, they did not improve in judging that subtraction is not commutative. How can we account for these contradictory results? On the one hand, more students in this group started with knowledge that subtraction is not commutative compared to the other groups, so they had less room for growth. On the other hand, four of the students had lower performance on the post-test, so it is possible that the combination of both types of instruction was too much to keep straight in such a short period of time. Again, investigating instruction over a longer time period or with slightly older students who may be better able to integrate the many aspects of the lessons may provide insight into this issue.

Finally, while Group 3 showed gains in identifying that subtraction is not commutative, they did not make any gains in applying this knowledge. Over half of the students could count into the negatives, yet they did not use this strategy to solve the subtraction problems. Furthermore, two students stated that subtracting a larger number from a smaller one would be a negative number, yet they still wrote positive answers to the subtraction problems. One possible explanation for this result is that students are used to inverting problems from working with fact families, and they continue to invert even in contexts where this application is incorrect—even if they know it!—since they did not have the experience (as Group 2 did) breaking out of this habit.

A second possible explanation is that students may be able to operate at Level 3 of the conception of quantities either on a procedural/perceptual level *or* on a conceptual level. One hypothesis was that students in Group 3 should have been more willing to get negative answers because they had instruction in their existence. However, they also may have had a surface understanding of the commutative and inverse properties—believing that one can change the order of the numbers in any way. Since they did not have instruction in solving negative problems, students in this group may have felt it easier to use one of the properties they "knew"—such as "switching" the numbers—rather than attempting to solve the problems in a new way.

Clearly, the results of this study raise more questions than provide answers, and we need to investigate children's learning of the commutative and inverse properties further. However, one noteworthy finding is that if children have knowledge of the commutative property of addition this does not mean that they understand its limits. A stronger argument could be made for the inversion principle. By using 2+3=5 to solve 3-5, children demonstrate that the numbers, rather than their order and relationship to each other are more salient features to them. These results suggest that students will develop deeper conceptual understanding as well as the flexibility to know when they can apply learned rules and reasoning by having experiences with those situations which require them to expand or challenge their original understanding. If we can help students develop deeper schemas for concepts and procedures earlier, they will have a stronger mathematical base on which to build future concepts.

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SUPPORTING FRACTION-BASED ALGORITHMIC THINKING: PAVING THE WAY VIA ESTIMATION AND NUMBER SENSE

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This report shares early findings from research on teacher practice associated with fraction-based algorithmic thinking. It describes key practices used by a "skillful" experienced sixth-grade teacher while implementing a curricular unit that leads students to develop algorithms for fraction operations using a guided-reinvention approach. These practices emerge from the initial days in the unit where estimation and number sense are the focus of lessons that set up the work to come where students will be developing algorithms for operating with fractions. These practices highlight mathematical thinking patterns, representations and language development that this teacher emphasized.

Purpose

Many have discussed the work of teachers and how the practices a teacher draws upon directly influence what students learn. In their review of the collective literature on teaching and classroom practice, Franke, Kazemi and Battey (2007) offer that effective teaching involves more than having a rich task or eliciting students' thinking. How one orchestrates mathematical discourse in a particular mathematical domain needs more attention. For example, what is it that a teacher should attend to when they elicit students' thinking in a particular domain? What are the core activities that should occur regularly when teaching in a domain? The research findings reported here are from a larger study focused on examining teacher practices that support students to engage in algorithmic thinking (Gravemeijer & van Galen, 2003) associated with fraction operations. In this research report I draw from the work of one of four experienced "skillful" teachers whose practice is being studied. The focus is on the initial days of an instructional unit where students will be developing algorithms for adding and subtracting fractions. This teacher's local instructional theory (Gravemeijer, 2004) for using estimation in preparation for developing approaches for finding actual sums will be presented and examined in terms of mathematical thinking, representation and language development.

Theoretical Framework

In their discussion of an *emergent modeling-reinvention* or *guided-reinvention* approach to algorithm development, Gravemeijer and van Galen (2003) emphasize that when instruction emphasizes algorithmic thinking based on student-generated approaches, that students engage in generating and developing their own mathematical knowledge. A guided-reinvention approach to algorithm development relies on teachers who have topic-specific instructional theories to organize the emergent ideas of their students around.

> A local instructional theory describes, with arguments, how enacting a series of instructional activities can support a specific long-term learning process. It would for example, describe how children first learn informal ways of addition and subtraction and how they may thereby develop more standard procedures for addition and subtraction. Or it would describe how children develop calculation procedures that may or may not be similar to the standard algorithms for multiplication or division. (Gravemeijer & van

Galen, 2003, p. 117)

This research seeks to make explicit such theories and the supporting arguments that guide the instructional choices that teachers make when positioning students to developing meaning of and algorithms for each fraction operation.

Gravemeijer (2004) argues that “if justice is to be given to the input of the students and their ideas built upon, a well-founded plan is needed” (p. 126). He distinguishes between Simon’s (1995) *hypothetical learning trajectories* used to plan instructional activities and *local instructional theories* which provide a rationale for an “envisioned learning route as it relates to a set of instructional activities for a specific topic” (p. 107). Drawing from Simon’s (1995) travel metaphor Gravemeijer suggests that a local instructional theory offers a travel plan with a destination that is steered toward as teachers choose instructional activities and design hypothetical learning trajectories for their own students. An important point made by Gravemeijer is that local instructional theories can never free teachers from having to design hypothetical learning trajectories for each class of students they work with. In the work presented here, project teachers were using a particular curriculum unit, Connected Mathematics Project II (CMP II) *Bits and Pieces II: Using Fraction Operations* (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006) which focuses on developing meaning for operations, allows algorithms to arise through student engagement with contextual situations, and explicitly asks students develop algorithms for each of the four fraction operation. The curriculum is viewed as an initial hypothetical learning trajectory or travel plan. It provides a place to start. However, the exact path of travel involves how the teacher uses the provided curriculum, including alterations that they make to the curriculum, and the students’ ideas that emerge.

Reasoning and proof is an important part of the guided-reinvention approach. Regarding reasoning and proof, Ball and Bass (2003) offer two foundational ideas related to practices of teaching mathematics: developing a base of public knowledge and the development of mathematical language. A practice that supports student to engage in reasoning and proof should take into account how ideas are developed visually, what ideas are made public, how ideas are identified and preserved, and how to use records of practice to support the development of new ideas. In the data presented, the use of representational records and how they are used to support the development of mathematical language is highlighted.

Kazemi and Stipek (2001) suggest that implementing reform-minded mathematics instruction is challenging. In their research (summarized in Figure 1) they found that teachers whose practice regularly incorporated sociomathematical norms, rather than the general social norms associated with mathematics reform, involved a higher press for conceptual thinking on the part of the students. They offer that teachers need to “understand what a sociomathematical norm is and construct pedagogical strategies for can be applied in a variety of contexts” (p. 79). Closely related to the work on sociomathematical norms is the ability of a teacher to sustain the level of cognitive demand (Stien, Smith, Henningsen & Silver, 2000) when implementing instructional tasks.

Figure 1. Related Social Norms and Sociomathematical Norms

Social Norms	Sociomathematical Norms
Students describe their thinking	An explanation consists of a mathematical argument, not just procedural description
Students find multiple ways to solve a problem and share their strategies with classmates and teacher	Mathematical thinking involves understanding connections across multiple strategies
Students can make mistakes—this is a normal part of the learning process	Errors provide opportunities to reconceptualize a problem
Students collaborate to find solutions to problems	Collaborative work involves individual accountability and reaching consensus through mathematical argumentation

The contributions of those like Ball and Bass (2003) or Kazemi and Stipek (2001) are acknowledged as important. However, building upon the call by Franke, Kazemi and Battey (2007) to identify key routines of practice associated with a particular mathematical domain, this work seeks to extend such contributions. Teachers need to develop local instructional theories about fraction algorithm development as part of determining what is involved when deciding what ideas to develop visually, what ideas to make public, how to address misconceptions that emerge, what records of practice would be useful, etc., as one navigates. One might envision a third column added to the chart in Figure 1 that fleshes out the intersection between socio-mathematical norms and supporting students to engage in algorithmic thinking associated with fraction operations.

Methodology

The findings reported here draw from the work of one of the four teachers during the initial lessons in a unit on fraction operations. This teacher is using the Connected Mathematics Project II (CMP II) curricular unit *Bits and Pieces II: Using Fraction Operations* (Lappan, Fey, Fitzgerald, Friel & Phillips, 2006) as the primary curriculum. Having taught mathematics using a reform-based approach for approximately 15 years, he adjusts or modifies the curriculum as needed for his students. He draws from his past experience with learners as well as what is happening with the current group of students he is teaching. He is true to the mathematical intent of the curriculum but as is the case with the lessons examined for the findings reported here, he has redesigned some of the lessons, or parts of lessons, for very specific reasons. This is an important characteristic of the four teachers who were studied—they each have a personal perspective on what experiences students need to participate in if they are to move toward articulating algorithms. For this teacher and the others, the adopted curriculum is a tool used in conjunction with what they know about how students make sense of and come to articulate fraction operations.

This study uses a qualitative design. The primary data sources were video of daily lessons, audiotaped reflections by the teacher on the daily lesson, weekly debriefings with teacher and responses to structured interview questions. The analysis was guided by Erickson's (1986) interpretive method and participant observational fieldwork, which addresses the need to understand the social actions that take place in a particular setting. The analysis of this teacher's practice, along with the articulated reasoning that emerges from reflections and interviews is an important part of understanding his local instructional theory. Identifying and classifying the teacher's declarative statements and questions posed during lessons, served as the initial starting point for articulating themes. The findings reported here focus on the prevalent mathematical reasoning patterns, representational development (diagrams as well as written symbolic) and the mathematical language being supported in the initial 3½ days of a unit where estimation and number sense are the focus of lessons that set up work to come developing algorithms for adding and subtracting fractions.

Findings

A majority of the coded classroom dialogue involved the teacher asking questions of students as they worked on problems in small groups and when they presented their work in whole class discussion. When the teacher asked a question a response was expected. While the teacher did ask some generic questions that positioned students to work together (i.e., "Did someone try this a different way?" or "Does somebody in their group think they can help?"), most of his questions served specific mathematical purposes. His declarative statements were designed to draw out particular mathematical ideas and reasoning patterns. Statements often supported students to develop a written record or a representation that captured their thinking. Figure 2 summarizes the major themes found in the class transcript data.

Figure 2. Major Themes and Number of Occurrences across 3½ Days of Instruction

Questions that…	Declarative Statements that…
Explicitly or implicitly assessed or developed meaning for numerator and denominator (17)	Purposefully drew out students' thinking to make it evident for all (4)
Assessed or pushed for development of how far a fractional quantity is from 0, ½, or 1 whole (29)	Prompted for or pointed out that reasoning needs to be provided—an answer without reasoning is not valid (12)
Prompted students to articulate their understanding of how to estimate (15)	Positioned students to engage in certain types of fraction-based reasoning (17)
Focused on distinguishing between an estimated sum and/or an exact sum (9)	Acknowledged how students found exact sums, but then redirected them to estimate (8)
Focused on using estimation as a strategy for determining reasonableness of an exact sum that emerged from a proposed algorithm (6)	Supported students to show their thinking using a written record or representation (10)

Making a Reasonable Estimate

While reflecting on his teaching, the teacher shared that students will need to draw upon their benchmark strategies later in the unit when they are developing algorithms. "I want them to be able to refer back to their estimation strategies so they don't get lost in the algorithms they start inventing". The first task the teacher had the students work on involved determining where a fraction is placed on a number line, and determining how far it is from 0, from ½, and from 1 whole. The second part of the task involved determining if the sum of a pair of numbers is closest to 0, 1, 2, or 3. For each part, students were asked to show work to support their answers. The teacher read the first problem to the class, "On this number line, where does 4/9 belong? Where to you think 4/9 would be if you were estimating? Where would you put it and then why are you doing that?" Student pairs begin to work as the teacher moves from group to group asking questions such as "How far is that from 0?", "What would be half of 9?", "Is that closer to 0, ½, or 1 whole?", "How can you show that?", "How far is 4/9 from 1 whole?" and "What is the 9 [in 4/9] telling you?"

During both small group and whole class discussion there is a strong emphasis on understanding what the numerator of a fraction represents, what the denominator indicates in terms of the numerator and the use of counting, or addition and subtraction, to determine how far a fraction is from 0, ½, or 1 whole. For example, when discussing if 4/9 is closer to 0, ½ or 1 whole, a student came to the front to share where she thought 4/9 was on a number line pre-marked with 0, ½ and 1. When this student placed her mark to the right of the half mark on the number line the teacher asked "Is everyone okay with that?"

Teacher: Is everyone okay with that? Caleb.

Caleb: It would have to be before the half because half of 4/9 would be about point five away from the half mark and so it would be a little way but on the other side [of half].

Teacher: Stephanie do you understand any of that?

Stephanie: Yes.

Teacher: What is half of 9?

Stephanie: 4½.

Teacher: So why don't you help yourself and above the ½ write 4.5 over 9. This would be a good skill to use on all of these problems. So then, the 4/9, would it be before that or after that?

Stephanie: Before.

When the next question in the set is posed (Is 4/9 closer to 0, ½ or 1?) a second student brings up his work. He tried to partition the number line into 9ths but struggled to integrate the marks for ninths with the one-half mark provided on the number line. With some suggestions the class and the student work together to adjust the markings and partition the number line into ninths. One-half is marked as 4.5/9 and 4/9 is marked with a star. They return to the question of how far 4/9 is from 0 and from 1. While some students can easily calculate the distance, counting is encouraged as away to determine how far.

Teacher: How far is 4/9s from 1? Abby?

Abby: 6

Teacher: 6 what?

Abby: 6 ninths?

Teacher: Okay, let's see. Tanner can you start at the star and count over and see if is it 6 ninths?

Tanner counts and gets 5 ninths. This thinking pattern is then prompted for as students work on other problems like this. Some students can do this thinking mentally. Others need to partition a zero-to-one number line or use grid paper to make and mark fraction strips/bars that they can count on. Like Abby, students were prompted to name the unit being quantified—6-ninths rather than 6. Many of the questions and declarative statements posed by the teacher focused on assessing and developing reasoning patterns for determining "how far".

This focus on developing strategies for *determining how far* is critical if students are going to be able to use benchmark reasoning as a tool for later determining if their algorithms are leading to reasonable results. If students do not have a way to determine if a quantity is closer to 0, ½, or 1 whole they will then struggle to determine if the sum of two fractional quantities is closer to 0, 1, 2, or 3. The teacher used numerous problems like this to draw out this type of thinking. The teacher elicited three important estimation ideas: using the denominator to determine how many of a particular fractional unit makes one-half and one whole, representing the fractional names visually on a number line (with full partitioning or renaming 0, ½ and 1 for a particular fractional unit), and counting fractional units to show how far away from 0, ½ or 1 whole a fraction is.

Development of Written Record and Representation

For students with well-developed number sense and computational ability, they were able to efficiently mentally calculate which fractions represented 0, ½ and 1 whole. For example, with 16ths the benchmarks 0/16, 8/16 and 16/16 were easily identified and used. Others struggled. The teacher provided grid paper for students who needed to draw and visualize a fraction bar representing 16ths. Students could mark and count as needed to determine if 9/16 was closer to 0, ½ or 1 whole. Throughout the 3½ days of instruction there were a few instances where students would randomly a guess how far a quantity was from 0, ½ or 1 whole. The teacher would support them to use one of the above strategies without taking over and doing the problem for them. For example, "I don't want you to guess. I want you to [make] a whole fraction strip here. You can cut it up into 8ths." Once the student made the strip, he then assisted him or her to label the representation. Some of his statements included: "Up at the top label it as one [whole]. Under here write how many 8ths that is. Now find where the halfway mark is and label it as ½ above with how many 8ths below." (Etc…). With the fraction strip labeled, he then asked the student, "How far is 5/8 from 0? From ½? From 1 whole?" If the student would guess or struggle to use mental computation, he would then ask them to use the fraction strip representation and count. As larger denominators such as 100ths were introduced, the counting strategy shifted to using subtraction. For example, 26/100 is 24 hundredths (50 - 26 = 24) from ½ or 50/100. So, 26 hundredths is closer to ½. In this teacher's practice there is a clear emphasis on using representations as a tool to give meaning to the quantities that students were being asked to work with. When a student indicated that they thought a fraction was closer to 0, ½ or 1 whole, the teacher would follow by asking why. To support developing and using visual and symbolic representations as reasoning tools, he would say, "So, let's use our strip and see where it does end up." Or "Let's write 50 - 26 = 24 hundredths to show why you think 26/100 is closer to ½".

Students needed similar support when trying to show the reasoning they used when finding an estimated sum for two fractions. On Day 2, after the teacher had observed students working on estimated sum problems, he gave the class a common problem (7/8 + 5/9) to find an estimate for. The students turned in anonymous work that was then displayed and rated. "You guys are going to look at this and rate it on a scale of 3, 2, or 1. Do we think that math is right and do we think that the reasoning they have shown supports their opinion?" The written notation may have been quite simple—"7/8 is about 1 and 5/9 is about ½ so the answer is about 1½". Some students rated this particular response as a 2 because it was important to say for example, that 7/8 is 1/8 from a whole making it is closer to 1 and that 5/9 is close to 4.5/9 or ½. This exercise provided students the opportunity to consider and discuss various expressions of benchmark reasoning.

Benchmark Reasoning, Estimation and Algorithmic Thinking

When students were working to determine if the sum of two fractional quantities was closer to 0, 1, 2 or 3 etc., they were asked to consider if an estimated sum was an overestimate or an underestimate. Using the context of decorating a dollhouse, students knew they had 5 ½ feet of molding to trim two rooms. Needing 3¼ feet for one room and 2 3/8 feet for the other, they were asked to use estimation to consider if 5½ feet of molding would be enough? Some students began to move away from benchmark reasoning toward exact reasoning. Using the *determining* *how far* reasoning, a group of students offered the following, "You know that you have 5 5/8 altogether." When asked how they knew this they explained that "3 plus 2 is 5 and 2 eighths plus 3 eighths is 5 eighths" and "4/8 would be exactly ½."

While this is legitimate reasoning on their part, and will clearly support them when they begin to work on finding exact sums and differences, the focus of the lesson was on estimating. The teacher did not wanted to lose the focus on the thinking patterns for estimating and did not want to discouraging students who could argue using exact equivalent quantities. He used the opportunity to have students consider if their reasoning led to an estimated sum or the exact sum. Students finding exact sums did not always realize they were not estimating. After listening to the reasoning of the student above, the teacher asked,

Teacher: Were you estimating or did you get the exact answer?
Student: We got the exact.
Teacher: Yes, and what were we suppose to be doing in this problem?
Student: Estimating.
Teacher: Yes, we wanted to estimate but I understand that these numbers are pretty compatible…that is good math but we are still gonna keep working on finding an estimate.

This distinction is an important one and will become important later when students are asked to find an estimate to show that their proposed algorithm is leading to a reasonable solution. The use of estimation as a strategy for determining the reasonableness of a sum, as well as what a reasonable sum would be close to, was noted in the discussions the teacher had with students.

In the next problem, a potential exact-sum approach was put on the table for discussion. This problem asked students to use estimation to determine if the solution 12/20 is reasonable for 7/12 plus 5/8. A student offered that 7/12 is 1/12 more than 6/12, or ½, and 5/8 is 1/8 more than ½, so the exact sum has to be more than ½ plus ½ or more than one whole. Here the *how far* reasoning is apparent. Another student, Abby offered that you cannot add the numerators and add the denominators. This is what is implied by the solution 12/20 presented in the problem. However, Abby is not able to say why you cannot do this. The teacher concluded by saying, "We are hoping that your estimate will show that. When you did your estimate, you should have seen that 12/20 is not close to what they real answer should be." Here the teacher positions students to engage in a certain type of reasoning. While Abby struggles to say why, the important point is that estimation can be used as a tool. In this case, the estimate shows that the algorithmic approach of adding numerators and adding denominators does not lead to a reasonable estimate. This reasoning will be important when students start to develop and assess their own algorithms.

Conclusion

These findings build upon the research on teacher practice while developing a content-specific view as suggested by Franke, Kazemi and Battey (2007). The findings highlight aspects of this teacher's local instructional theory for engaging students in fraction operation algorithm development. His practice asks students to engage in reasoning and proof (Ball & Bass, 2003 while developing specific reasoning tools for supporting algorithm development. Representations such as partitioned fraction strips and number lines are tools for *determining how far*. The *determining how far* reasoning is also a tool. Gravemeijer (2004) argues that in a guided-reinvention approach the focus is not teach a set of strategies but to develop a framework of number relations that build a foundation for flexible mental computation. This teacher used questioning and dialogue to help students distinguish between an estimate and an exact answer.

When redirecting students to estimate they were pushed to consider how an estimate can be a tool for determining the credibility of a computational result. This will support them when they design algorithms and work to justify and prove they are mathematically reasonable.

In these results, generic practices such as designing a task or making ideas public (Ball & Bass, 2003) are evident. There is a clear focus on Kazemi and Stipek's (2001) sociomathematical norm that an explanation consists of a mathematical argument, not just a procedural description. By considering these generic practices within the context of fraction algorithmic thinking, specific thinking patterns are made visible for inspection. In this teacher's practice there are purposeful ways in which students were supported so they could make an argument for why they were choosing a particular benchmark. They were encouraged to use models and to develop mathematical symbolism so they could make their benchmark reasoning public and open for inspection. Through this process students were developing language and symbolism to further support their ability to engage in fraction-based algorithmic thinking.

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Exploring Preservice Teachers' Abilities to Pose Division Scenarios

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The study examined preservice teachers' (PTs) abilities to pose scenarios that match a division computation with a remainder. The results showed that PTs were able to pose various scenarios but most of them focused on partitive division interpretation. They preferred to leave the remainder as a left over and discarded rather than partitioned as a fraction/percent/decimal, forced to the next whole number, and rounded to an approximate result. More emphasis on providing rich scenario problems during teacher preparation is necessary so PTs will be able to understand the division concept in diverse contexts and deal with remainders accordingly.

Over the past few decades problem posing has gained considerable attention within the mathematics education community gradually familiarizing educators with making aware the benefits of the strategy for improving the teaching and learning of mathematics (Silver, 1994). The National Council of Teachers of Mathematics [NCTM] outlined the importance of problem posing in the *Professional Standards* (1991) document, "students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem" (p. 95). When students formulate problems, they become curious about mathematical ideas and develop deeper understandings of particular concepts (English, 1997, 1998; Silver, 1994). The creation and reformulation of problems within new contexts engages students' thinking and their inquisitiveness in learning mathematics (e.g., Crespo, 2003; Dickerson, 1999; Grundmeier, 2003; Knuth & Peterson, 2003; Leung & Silver; 1997).

Problem posing refers to both the generation or formulation of new problems and the reformulation of given ones (Silver, 1994). Skinner (1991) defined a *problem* as a question engaging someone in searching for a solution. Whereas, Krulik and Rudnick (1982) stated that a *question* required recall, an *exercise* provided routine drills and practices, and a *problem* required careful thought and synthesis of knowledge. These researchers argued that a problem at one time may be an exercise or even a question at some later stage of that individual's mathematical development.

The *Principles and Standards of School Mathematics* (NCTM, 2000) highlights the responsibility of a teacher in choosing and posing worthwhile mathematical tasks to develop and nurture students' understanding of certain concepts. In reality, most mathematical problems come from textbooks as teachers assign them to students. Crespo and Sinclair (2008) argued students of all ages, including those who subsequently become teachers, have limited experience posing their own mathematical problems. Research studies document that many prospective and inservice teachers lack the skills and confidence to go beyond solving a problem (Gonzales, 1994; Silver, Mamona-Downs, Leung, & Kenney, 1996). As problem posing is part of mathematics education reform, teachers must develop skills in formulating mathematical tasks that include appropriate situations to engage students in the classroom (Gonzales, 1994, 1996).

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Problem posing, especially with division can be a challenge (Silver & Burkett, 1994) as research has shown that many school teachers have difficulties dealing with division of whole numbers involving remainders (e.g., Simon, 1993; Sowder, Philipp, Armstrong, & Schappelle, 1998). For instance, in a two-year long project of the Teaching and Learning Rational Numbers and Quantities Working Group, Sowder et al. noted the confusion of teacher participants when they were presented $37 \div 5 = 7\ R\frac{2}{5}$. The teacher participants believed that 7 remainder 2 and 7 remainder $\frac{2}{5}$ (i.e., $7\ R\frac{2}{5}$) were the same as $7\frac{2}{5}$. During the continued discussion, a facilitator in the project asked them to pose a story problem for which the answer was 7, 8, 2, $7\frac{2}{5}$, 7.4, and 7 remainder 2. When the teachers debated their word problems, they gradually were able to distinguish clearly the contexts when a remainder would be appropriate to consider in each case. Through problem posing, the facilitator helped the teacher participants to realize that 7 remainder $\frac{2}{5}$ was not the correct answer for $37 \div 5$.

In this present study, we explored PTs' capabilities to generate a scenario problem involving division with a remainder. Specifically, we focused on Silver's (1994) formulation of problems emphasizing the creation of a scenario based on a given long division computation with a remainder.

In this study, we explored PTs' capabilities to formulate a scenario problem involving division with a remainder. Also, problem posing was used to assess PTs' understanding of this concept based on the solution of the scenarios they posed. Specifically, we focused on Silver's (1994) formulation of problems emphasizing the creation of a scenario based on a given long division computation with a remainder.

METHOD

The analysis reported in this paper was part of a larger research study examining the problem solving and problem posing skills of PTs. This present study replicated research done by Silver and Burkett (1994) on PTs' conceptual knowledge and contextual connections to division with remainders. Nevertheless, with different perspectives, we attempted to determine the structure of scenario problems individual PTs posed to match a given division computation. The research questions guiding this study were:

a) What kind of scenario problems associated with a division computation and a remainder did PTs pose?
b) What strategies did PTs use to pose scenario problems associated with a division computation and a remainder?

Initially, a pilot study was conducted in Fall 2010 using a group of elementary PTs who volunteered to be part of the project. They completed an open-ended problem posing task as illustrated in Figure 1, adapted from Silver and Burkett (1994). The researchers evaluated response scripts based on a priori coding schemes that emerged from the literature. The results from the pilot study showed that these PTs posed scenario problems that did not match the division computation. The researchers suspected the PTs were not able to understand the written instructions during the administration of the task. Therefore, the instructions for the task were modified to make the task clearer to the participants.

$$\begin{array}{r} 13 \\ 40\overline{)540} \end{array}$$

400
140
120
20

Figure 1. The task adapted with permission from Silver and Burkett (1994).

Subsequently, in Spring 2011, a convenience sampling technique was utilized for data collection. The problem posing task was administered to 33 PTs during the fourth meeting of a 15-week problem solving course. Although the course emphasized Poyla's four-step problem solving process, the professor regularly discussed the importance of problem posing with the PTs during class instructional periods. The participants were working towards their middle grade teaching certification at a southwestern public university in the United States. The majority of the PTs were female (96%), juniors ranging in age from 18 to 22 years old. Each participant completed the problem posing task by writing his/her responses on the answer sheet. They were given 20 minutes to pose multiple scenarios that matched the long division computation and provide different answers to each of the problems posed.

The responses were analyzed by using the classical content analysis that focused on the frequency of each category occurring (Leech & Onwuegbuzie, 2007). The coding was based on Van de Walle, Karp, and Bay-Williams' (2010) description of division of whole numbers. The coding procedure was divided into two phases in order to make the data easier to analyze. In the first phase, a posed-scenario was identified as to whether it matched the long division computation and was categorized into *partition* or *measurement*. Sowder et al. (1998) argued that knowing these two interpretations of division was important when dealing with a context or situation. For example in partition, 540 cookies shared among 40 people, each person has 13 cookies with 20 cookies left over. On the other hand, the situation would be different in measurement when giving 40 cookies from 540 cookies to each one and asked, "*How many people can get 40 cookies each?*" In this case, the units of measure would not be the same -it would be 13 people and 20 cookies left over. Sowder et al. believed that the referent units were the important idea when presenting division within a context. As the task required a context that matched the division computation, the study assessed whether the kind of scenarios PTs posed made sense or were meaningless to the solution they provided. Then, if any posed-scenarios did not satisfy the classification for partition or measurement, they were coded as other including all the responses that used different arithmetic operations such as addition, subtraction, multiplication or combination of operations.

The second phase of the coding procedure dealt with implications of division remainders. According to Carpenter, Fennema, Franke, Levi, and Empson (1999), the context of a problem posed can influence the way the remainder is treated. These remainder effects were categorized as *left over*, *partitioned as a fraction/percent/decimal*, *discarded*, *forced to next whole number*, and *rounded to approximate result* (Van de Walle et al., 2010). Below are examples of responses that were categorized in each type of division remainders:

> *Allison is counting out beads for an art project, in which she needs 540 beads. If she counts the beads by 40s, how many groups will she have?* (Answer: 13 groups and 20 left over, a left over).
>
> *Molly owns a car, which the maximum speed is 40 mph. How much time should she allow to drive 540 miles? (*Answer: $13.50, partitioned as a fraction/decimal).

Jim has 540 ft. of chain. He needs to cut the chain into 40 ft. segments. How many segments of 40 ft. can he make? (Answer: 13 segments, discarded).

There are 540 people waiting to get on a boat. If each boat can carry 40 people, how many boats do you need to carry all of those people? (Answer: 14 boats, forced to next whole number).

Mrs. Jones has 40 students in her class. This year for Christmas, they are making popcorn necklaces to decorate their class tree. If each student gets an equal number of popcorn pieces and Mrs. Jones has 540 to divide up, how many will each student get? (Answer: 13 pieces, rounded to approximate result).

Two researchers coded ten scripts based on the coding procedure (i.e., Phase 1, Phase 2) and achieved 77.1% agreement. Any disagreements were resolved before the first researcher coded the remaining scripts. Then, data were analyzed descriptively (e.g., frequency, proportion) using *Statistical Package for Social Science* (SPSS) version 16.0 (SPSS Inc., 2007) to answer the research questions and the results were presented using tables and charts.

RESULTS

The results were used to portray the structure of scenario problems posed that matched the division computation and to illustrate the strategies PTs used to deal with division remainders in the context they posed. In general, the descriptive statistics from the SPSS results showed PTs were able to generate 170 scenario problems and successfully showed the answers for each scenario they posed. Each of the PTs posed at least three scenarios and 15 PTs generated at most six problems. The analysis of the scenarios is presented in further detail to answer the research questions.

Research Question 1: Kind of Scenario Problems Posed

Most of the PTs (82%) posed a combination of division scenarios associated with measurement and partition. The proportion of the unexpected responses (1.8%) was small as compared to Silver and Burkett (1993). This might be because the PTs were given clear instructions during the administration of the task that they were expected to pose only scenarios associated with division and a remainder. As noted earlier, 170 scenarios were posed in total; however, two of them did not meet the requirements of the task in which the scenarios PTs posed were supposed to match the division computation. In addition, one problem had missing information making it meaningless even though the solution was stated. These are the examples of the unexpected responses:

For every group of 54 oranges in a grocery store there are 40 apples. If this pattern is always consistent, how many oranges would there be if there are only 10 apples? (Proportion problem)

If there are 50 students riding the bus to school and all of them receive a ride, how many buses are needed? (Missing information)

Of the 167 posed problems, 42 were considered redundant because they had similar contexts and repeated the same solution. Specifically, the study demonstrated ten PTs posed at least two similar scenarios without changing the structure of the problem, using the same interpretation of division and remainders. For example, initially one PT posed six scenario problems with correct proposed solutions, but after closely examining the structure of each division problem only two scenarios were considered valid for further analysis. Here are two similar scenarios posed by this participant:

A group of 40 students gathered all of their candy from Halloween. They totaled 540 pieces of candy. If they share equally, how many pieces of candy will each student get? (Answer: 13 pieces of candy)

At a 4th grade party, 540 pieces of pizza were delivered. If there are 40 students at the party, how many pieces will each student get? (Answer: 13 pieces of pizza)

The above examples clearly demonstrate the way this participant interpreted the meaning of division using partition. Even though this PT posed six problems, she/he was not able to generate any scenario associated with division that related to measurement showing this PT had difficulty seeing division in this dimension. In fact, only 89 (47%) of the scenarios that PTs posed were focused on measurement as illustrated in the following:

A convention center must prepare their showroom for the TMTA Piano Convention. To maximize the number of vendor booths, 40 tables can fit in one row across the room. If there are 540 tables, how many rows of 40 tables can you create and how many tables are left over? (Answer: 13 rows and 20 tables)

Timmy has 540 chickens. He needs to determine how many chicken coops he should buy. Each chicken coop can hold 40 chickens. How many will he buy? (Answer: 14 coops).

On the other hand, the majority (53%) of the problems posed were based on partition interpretations in which the results were similar to Silver and Burkett (1994). In fact, five of 33 PTs posed only scenarios associated with partitive and one participant focused solely on measurement. This study showed that PTs were able to pose reasonable scenarios and correct answers with an appropriate referent unit (e.g., 13 rows and 20 tables, 13 pieces of candy, 14 coops) in each case. It seemed that most of PTs were able to understand the relationship among the dividend, divisor, quotient, and remainder. In addition, the study also found two PTs posed contexts by interchanging the divisor and quotient of the original computation. One participant consistently posed two scenarios that used 540/13 to illustrate the meaning of division using measurement and partition.

The World Eating Championships are being held in Los Angles this year and is sponsored by McDonalds. McDonalds decides to give 540 Big Macs for the competition. If the average competitive eater can eat 13 Big Macs, how many people should they hosted to ensure they do not run out of Big Macs? (Answer: 41)

EA Sports is hosting an Online FIFA "Goalathon". A "Goalathon" is a rare for all FIFA players to reach 540 soccer goals in 13 weeks. How many goals must a player average a week to reach 540 goals in 13 weeks?Round number to two decimal points (Answer: 41.65).

Based on the responses, this participant was aware that the proposed answers would be different as he/she interchanged the values that showed his/her understanding of division problems. Also, similar to Silver and Burkett's study, the results showed that PTs posed a number of 'Yes/No' scenario problems (6.5%), for example as following:

There are 40 people at a conference that need to be fed equally. The chef makes 540 cookies that can be eaten. Each of the 40 people must eat the same amount of cookies and cannot eat half of a cookie. Will there be any leftover?

Even though this problem was associated with the partition interpretation, the researchers believed it was not challenging and did not engage students' thinking as it could be answered simply by stating yes or no. It did not fit with the definition of a *problem* as defined by Krulik and Rudnick (1982). Also, most of the PTs were able to provide an appropriate solution for the scenario they posed and considered the effect of remainders on the answers. More detailed discussion on remainders and answers for the posed problems that were provided by PTs are presented in the following section.

Research Question 2: Strategies with Division Remainders

Part of the requirement of the problem posing task was to provide different answers, so the researchers examined each of them in depth as they were related to how remainders were handled. The results provided a source of information about PTs' understanding of division involving remainders that can be categorized as a left over, partitioned as a fraction/decimal/percent, discarded, forced to next whole number, and rounded to approximate result (Van de Walle et al., 2010). Of the 167 responses, 42 were eliminated because they had similar contexts and repeated the same solution. In addition, six were dropped because of incorrect proposed solution. Therefore, the analysis of division remainders was based on 119 valid responses. As displayed in Figure 2, the researchers found PTs preferred to leave the division remainders in form of a left over in which 35% of the scenarios were in this category. The category of division remainders used least by the participants was the forced to next whole number in which only 14 (12%) scenarios were coded for this category.

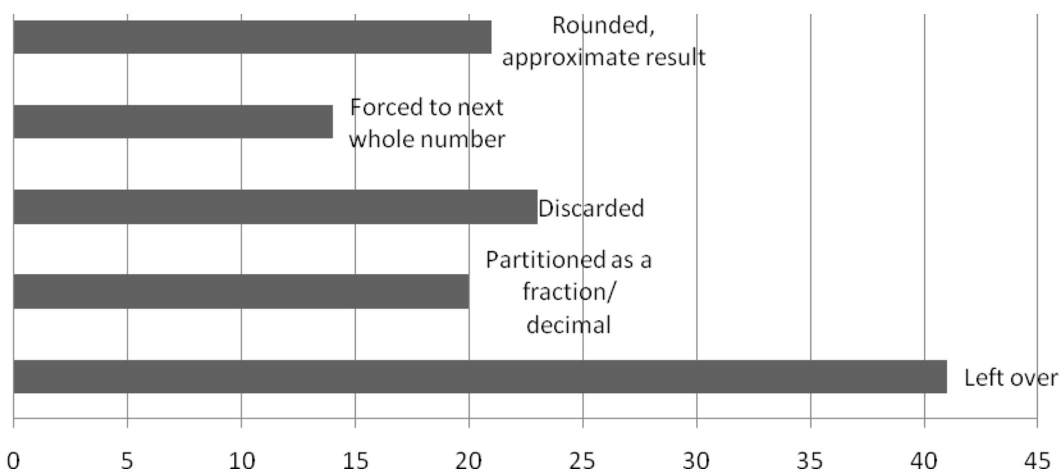

Figure 2. The analysis of division remainders

In addition, 11% of the scenarios (13 of 119 responses) were noted as errors because they provided unreasonable answers that did not match the remaining context of the posed problems. For instance, '*There are 540 children going to camp and 40 buses to get them there. How many children will be on each bus?*' The answer given was 13 children, which was not realistic that showing the remaining 20 children were ignored and making the problem meaningless and not practical. The scenario clearly stated '*There are 540 children going to camp*', which meant all children were supposed to go to the camp. Based on this context, these 20 children should be together with other children on the buses so the appropriate answer would be about 14 children as it forced the figure to next the whole number. Two PTs made a 'silly' error by mistakenly writing 13.2 instead of 13.5 as a solution and three PTs gave incorrect answers to the posed scenarios.

Discussion

The study reveals that the selected PTs were able to show their understanding of division with remainders. In regard to the first research question, the researchers found from the structure of the scenarios posed that PTs were more comfortable with sharing scenarios (partition) compared to the equal size groups (measurement). The selected PTs were able to pose reasonable scenarios and to provide correct answers with an appropriate referent unit (e.g., 13 rows and 20 tables, 13 pieces of candy, 14 coops) in each case. It seemed that most of PTs were able to understand the relationship among the dividend, divisor, quotient and remainder.

In regard to the second research question, even though they were able to put the division computation in a context, they could not deal with the remainder accordingly. They were not able to think of the remainders as fractions or as rounding-up, rounding-down the results depending on the context they posed. Specifically, most of PTs preferred to deal with division remainders in form of a left over and discarded. When PTs discarded the remainders, most of the problems were thus meaningless and not practical.

Teacher educators should place more emphasis on providing rich scenario problems during instruction so that PTs can see the different meanings of division clearly and be able to understand these concepts profoundly. In addition, teacher educators should encourage PTs to deal with a remainder instead of just leaving it as a left over or discarding the remainder. Much more remains to be understood about PTs' abilities with problem posing and their understanding of division; therefore future investigation is necessary to explore the benefit of using of problem posing in discovering PTs knowledge and understanding of a particular mathematical concept.

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THE POWER OF INVENTING PERSONAL REPRESENTATIONS: A CASE STUDY OF DAVID

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This study describes how nine-year old David, in exploring the ordering of fractions on a number line, illustrates his understanding of fraction ideas by offering multiple representations of his solutions. David begins his problem solving by building a physical model with Cuisenaire rods. He then justifies his solution to the class using verbal explanations accompanied by gestures, drawings and symbols.

Introduction

Fractions are too often introduced in schools using standard symbols and notations that do not carry meaning for students. For decades, researchers have documented the shortcomings of traditional instruction that focus on rules and algorithms for the learning of fractions and their operations (Erlwanger, 1973; Freudenthal, 1983; Maher, Davis and Alston, 1993; Steencken and Maher, 2002; Streefland, 1991). Studies have shown that learning mathematics, when using and linking various forms of representations, can promote inquiry and higher levels of thinking (Ball and Bass, 2003; Even, 1998; Richardson, Berenson, Staley, 2009). The process by which students build their own understanding is complex process and involves building personal, meaningful mental representations that can take on many forms (Davis, 1984; Goldin, 1998, 2003; Goldin and Janvier 1998). Students can then use their personal representations to express their understanding. In this paper, we explore the variety of personal representations used by David to express his understanding of number concepts. In particular, we investigate: "What representations does David use to express his understanding of fraction as number?"

Theoretical Framework

The view that learning should be meaningful is not new. Decades ago, Dewey (1938) stressed the importance of learning by doing. Today, mathematics educators emphasize the importance of building meaning in the learning of mathematics, citing the importance of student engagement in thoughtful problem solving s context for new learning (Maher, 2010; Maher, Powell, & Uptegrove, 2010). According to Davis (1984) and Davis and Maher (1997), the process of making sense of a problematic situation calls for the restructuring of knowledge. Previous experiences that learners bring to a problem-solving exploration can serve as assimilation paradigms for new learning. As learners examine and compare earlier experiences, they either solve the new problem or restructure their existing knowledge to accommodate the new conditions of the new problem. Making use of personal representations is key in the process. The variety of personal representations that a learner can bring to a problem-solving activity makes the understanding richer and, ultimately, more durable, in that the ideas can be related and restructured as needed. Recent studies have documented the durability of learning when it is built in a meaningful way (Aboelnaga, 2011; Ahluwalia, 2011; Sran, 2010; Steffero, 2010, Schmeelk, 2010). These studies show that students build on and extend their knowledge when supported by conditions that support collaboration with appropriately designed tasks. Also, they demonstrate the durability of learning over time. Representations are key in that learning. According to Goldin and Janvier (1998), representations can be interpreted in the context of external physical situations, linguistic embodiments, formal mathematical constructs and individual's internal cognitive configuration. The PME working group on representations has developed extensive discussions on various representation forms, and from these discussions, there has arisen an understanding that in analyzing the various representations, there is a need to understand mathematical notations, branches, recordings, individual productions, external situations, as well as individual mathematical learning and understanding (Goldin 1998). Translation between and among multiple representations is individual, varied and independent (Superfine, Canty and Marshall, 2009; Kaput, 1998). As a result, individual, personal representations that student's use to invent are more powerful than those that are imposed (Vergnaud, 1998). It is our view that students need to develop their understanding of mathematical ideas using the schemes and representations that are personal. Access to manipulatives, useful as tools to illustrate external representations, can assist the development of student internal representation schemas. According to Gattegno (1961, 1963), using Cuisenaire rods can foster student understanding of fractions. He suggests a multi-stage introduction to Cuisenaire rods, including (1) free play and (2) free play that is accompanied with directed activities with the rods, in which discussions and observations for the rod relationships are informally discussed. Then, what might follow is more free play accompanied with directed activities with the rods, in which formal notation is introduced without directly assigning number values to rods. Finally, free play accompanied with directed activities, in which the formal notation is extended with directly assigning number values to the rods, might follow. Components of the framework proposed by Gategno were integrated into this research in students' investigation of fraction ideas with Cuisenaire rods available as a tool.

Study Context

This research took place in a fourth-grade rural/suburban classroom consisting of 25 heterogeneously grouped nine and ten year old students. In this school's mathematics curriculum, fractions and operations were not introduced until the following year. It was important for the researchers to study how the students made sense of the fraction/rational number ideas though their problem solving and sharing of ideas before fractions were formally introduced as part of their fifth-grade mathematics curriculum.

Data and Methodology

The data were collected from multiple sources. All sessions were videotaped with multiple cameras during extended classroom sessions that lasted from fifty to seventy-five minutes. The video data and their full transcripts, along with student work and researcher observation notes, were used to triangulate data. Student work consisted of personal written inscriptions and collaborative overhead inscriptions. The video data have been digitized and are stored, along with transcripts, in a repository as a component of the Video Mosaic Collaborative (see http://videomosaic.org).

Background

Twelve sessions transpired before the focus sessions that are described in this report. During the earlier sessions, students explored fraction as number, fraction as operator, equivalence, ordering and fraction comparison ideas. The student reasoning in the form of conjectures and argument strategies is described by Reynolds (2005). Detailed reasoning forms used by the students, including upper and lower bounds, cases, contradiction and recursive were analyzed by

Yankelewitz (2009) and Yankelewitz, Muller and Maher (2010). The first seven sessions were analyzed for students transitioning from an understanding of fraction as operator to fraction as number (Steencken, 2001; Steencken and Maher, 1998; 2003). In these studies, the students' verbal and inscription representations were examined in detail for how they built their understandings. All of the studies reported that the students' verbal expiations became increasingly more precise over time.

Results

For this report, we describe three episodes that provide as examples of the variety of representations that David used to express his growing understanding of rational number ideas. For more detail and full transcripts, see Schmeelk, 2010. Video clips can be viewed on the Video Mosaic website: http://videomosaic.org/.

Episode 1: 11-01-1993

In this episode, David was working on the problem: "Which numbers are bigger and smaller—1/2, 1/3, 1/4, 1/5?" After building rod moles with his partner, he offered a solution to the class and used gestures to represent an imaginary number line. He demonstrated (see Figure 1) four fractions—one half, one third, one fourth and one fifth—are larger. The clip exemplifies his awareness of the partitioning of a unit interval into regions representing the fraction parts.

Line	Speaker	Transcription
78	Researcher	[Writes one half, one third, one fourth and one fifth on over head] So, if I were to say things, like one half, one third, one fourth, one fifth, right? If I were talking about these numbers then would you know which are bigger and which are smaller? How many think you know which are bigger and which are smaller? Who could explain why?
79	Class	[Many students raise their hands.]
80	Researcher	David, what do you think?
81	David	If you have one half cuts right there. [motions ½ on an imaginary unit] If you have one third it cuts right there [motions to where one third would cut on a unit] You would have three pieces.

Figure 1. David gestures as he orders fractions on a number line

The researcher then asked David to use the overhead projector to explain his ideas. David, made drawings of rod models and, again, justified his ordering of fractions (see Figure 2). At the overhead projector, he drew rod pictures to show the magnitude of the five fractions with respect to a unit rod that he labeled whole.

Line	Speaker	Transcription
82	Researcher	You want to come draw that for me? You all hear what David is saying?
83	David	[Walks up to OHP in front of room.]
84	Researcher	Want to draw your one. Call something one and zero?
85	David	Maybe the orange
86	Researcher	Just sketch it …
87	David	If this is the one here
88	David	… then one half would be there …
89	Researcher	Can you mark one half right where you put it, like put it right underneath so we can see it?
90	David	What do you mean?
91	Researcher	Just write the number one half where you want to show one half.
92	David	Then, one third
93	David	Then, one fourth
94	Researcher	Then, one fifth. Thank you very much.

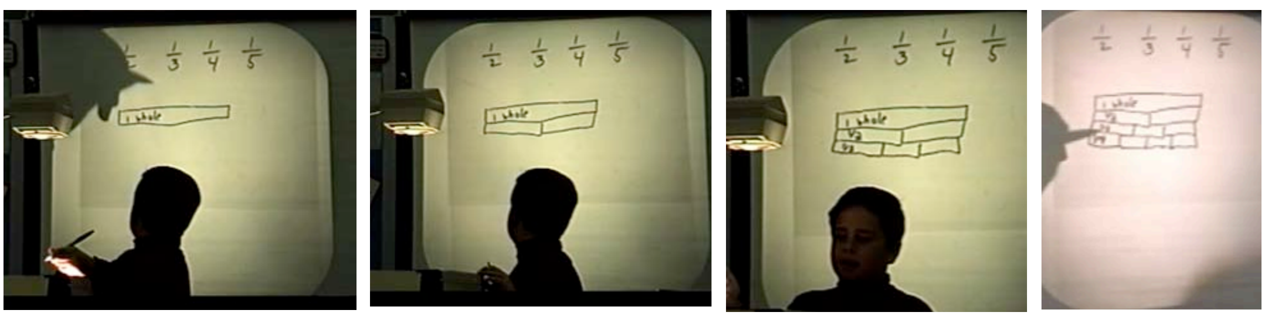

Figure 2. Images of David ordering fractions on the overhead projector

Episode 2: 11-03-1993

In this episode, the researcher summarized an earlier discussion about how the numbers on the number line continue. The researcher then asked the students to review their assignment, which was to think about the numbers between two whole numbers. David responded with a metaphor of a microscope, suggesting how a microscope could show the space between numbers, indicating that an observer would see multiple numbers through magnification.

Line	Speaker	Transcription
208	David	I think that you really can't see it to well. But, if you use a microscope, then you are seeing closer and it looks like you are seeing more; but,

		you're really not—you're just looking closer than before.
235	David	I think that you can take the little smallest thing and, then, put it under a microscope; and, you will have a lot more space. But, you really don't. It looks like a lot more space; but, it really isn't. You are just magnifying it.

Later in the episode, the class was asked to investigate on where the number one one-hundredth might be placed on a number line. David used a ruler to place the number one one-hundredth on his number line model at his desk (see Figure 3).

Line	Speaker	Transcription
251	David	On my paper, I had a ruler, that I was using, put up to [the number line]. It think it was a millimeter or something. I had a ten inch number line, so I put [the number one one-hundredth,
252	Researcher	So that is how you placed one one-hundredth
253	David	Yeah

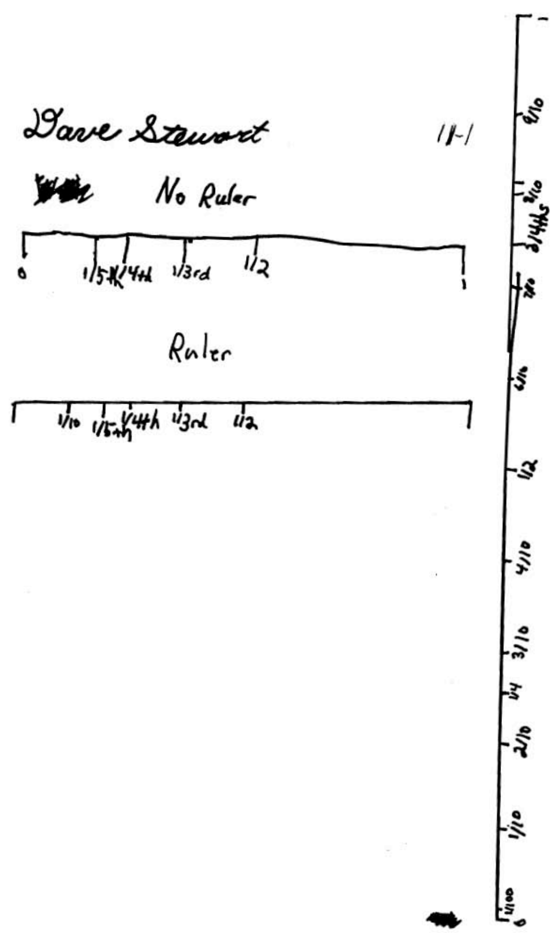

Figure 3. David's sketches for the placement of one one-hundredth on a ruler

Episode 3: 11-12-1993

In this episode the researcher extended the bottom of the model to represent a number line and asked David to place the fractions. David responded by correctly labeling the number zero, one fourth, one half, three fourths and one. Under the number named one half, David also placed the number two fourths. The clip (see Figure 4) showed that David used Cuisenaire rods as an assimilation paradigm to assimilate he transitions from fraction as operator using rod models to fraction as number in placing the numbers on the number line, noting the equivalence of one half and two fourths.

Line	Speaker	Transcription
111	David	Four fourths equals one whole. Two halves equals one whole. And, two fourths equals one half because one half is half of one whole. So, two fourths is one half.
169-173	David	I just drew it like that because that's the one whole [gestures to top level]. These are the one fourths [gestured to the second level]. And, that's the half [gestures to the bottom level]. This would be zero [gestures to the left most vertical line] and that would be one [gestures to the right most vertical line].
174	Researcher	Let me ask you to something here that might help me. I want theses to be here [draws in lines to turn the rods into a number line]. Can you place your numbers, here, now?
175	David	You mean like one half?
176	Researcher	Where's zero?
177	David	[Draws in zero, one fourth, two fourths, one half, three fourths and one].

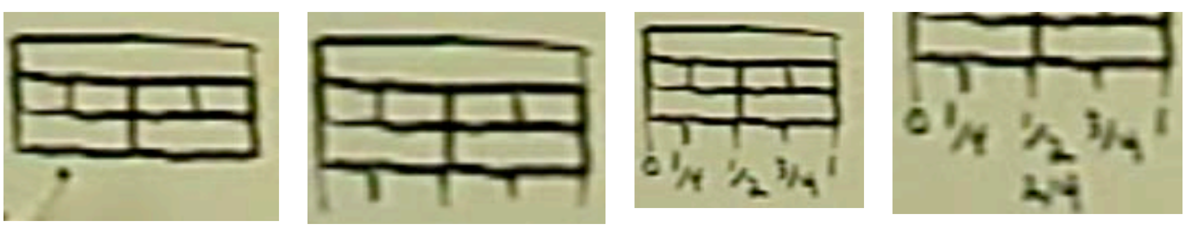

Figure 4. David's diagram showing the placement of two fourths

Conclusions and Implications

David's rod model became a useful representation for transitioning to the placement of fractions on a number line. In episode one, David used gestures in explaining his rod model and drew diagrams of rod models justify his fraction placements. In the second episode, David used external microscope representations to convey his understanding of density. Later, in the same episode, he accurately places the number one one-hundredth on a number line using a ruler. In the final episode, he accurately conjectures that one half is equivalent to two fourths, indicating some understanding of fraction equivalence, using model drawings to justify his conjecture. David was developing and expressing his awareness of the idea of the density of fractions, using a metaphor of a microscope to indicate that there is more space between two numbers on a line for the placement of other numbers. David's exploration of fraction ideas shows that he is able to coordinate the ideas of fraction as operator and fraction as number, an important foundation for growth in fraction knowledge. Exploration and free play, accompanied by opportunities to be engaged in thoughtful problem solving are important components for building meaning of mathematical ideas. David's problem solving activity was thoughtful and productive and shows successful transitioning of a variety of representations.

Endnotes

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DEVELOPING FRACTION NUMBER SENSE WITH PRIMARY LEVEL PUPILS

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This article describes the experience resulting from applying the Rational Number Project (RNP) sequence of activities with a group of 6th grade pupils which consists of working with concrete material at all times and making constant reference to the other representations. The reasoning expressed by the students gives a clear idea of the benefits they obtained with this approach, since they were able to transfer their knowledge to new situations. Additionally, it was shown that the use of concrete materials is not a waste of time, contrary to what many primary teachers argue, since there was enough time to cover the contents with encouraging results.

Some studies (Cramer, K., Behr, M., Post, T., Lesh, R., 1997; Cramer, K., Henry, A., 2002; Huinker, 2002) suggest that in order to learn and understand fractions young children require: to experience handling a variety of concrete materials; to communicate their mathematical ideas through oral expressions, pictures and symbolic representations; to solve real life problems; to be given the opportunity to express and share their mathematical ideas with teachers and peers; as well as to be involved in a teaching environment that emphasizes the understanding of concepts before formal symbolic and algorithmic work is introduced. This article analyzes the experiences resulting from a Fractions Workshop, in which the lessons suggested by the Rational Number Project (Cramer, K., Behr, M., Post, T., Lesh, R., 1997) were applied. The general aim of the Workshop was to help the participants develop a deep understanding of the concept of a fraction.

Some of the questions guiding the study were: How can we help students acquire correct ideas about fractions? How can we help pupils feel confident about their mathematical abilities? Which teaching sequences bring about a better understanding of the concept of fractions, of fraction equivalence, of the relationships between the different modes of representation, and their relevance in real life situations?

Conceptual Framework

There's a general concern amongst mathematics teachers of different levels with respect to the way students learn fractions. Most educators agree that there has to be a change in the way fractions are taught, but there is no consensus as to how to carry out such change.

Several authors have made interesting attempts at broaching this subject; for example, Behr (1992) suggests that in the initial stages of introducing fractions the teacher should present situations such that pupils are faced with the possibility of constructing their own qualitative concept of order, equivalence and size of fractions. Likewise, due to the complexity of the concept of fraction and the variety of situations to which fractions apply, it's important to focus on the development of a quantitative understanding which can be achieved by using concrete models and emphasizing meaning instead of procedures (Bezuk, 1989). It is also necessary to present children with relevant situations which will allow them to make connections between their academic work and real life contexts, and between the different representations (Scaptura, 2007). In this sense, Lesh (Cramer, K., Behr, M., Post, T., Lesh, R., 1997) suggests that children learn fractions when they have the opportunity to explore ideas and make connections between

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the different modes of representation. Lesh put forward the translation model shown in Figure 1 as a way to represent how fraction mathematical ideas are related. This diagram was used as a model for the development of the Rational Number Project (RNP) curriculum.

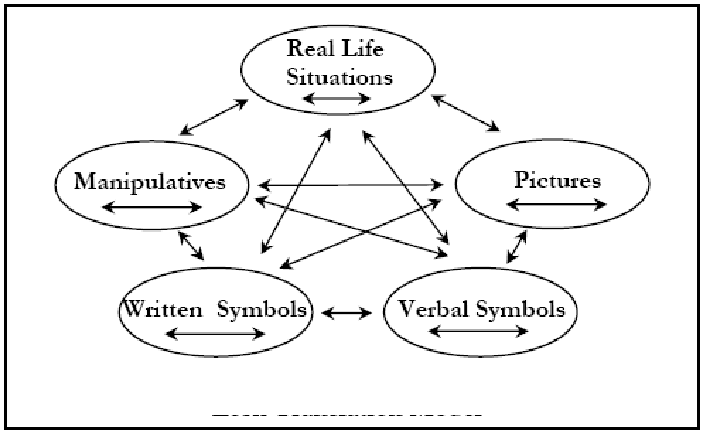

Figure 1. Lesh Translation Model

RNP Curriculum

The RNP curriculum suggests a sequence, alternative to that which is traditionally presented in primary text books. It bases its philosophy on the principle that it is necessary for the child to use concrete materials in order to: be able to understand and feel confident with the different fraction modes of representation; perceive them as part of a whole, as part of a collection of objects, or as a division; acquire a solid understanding of fraction order and equivalence; all this as a foundation necessary for meaningful work with fraction operations. With respect to the teaching of fractions, the National Council of Teachers of Mathematics (NCTM) establishes as aims: the development of number sense, the use of a variety of concrete models for representing fractions, the application of the teaching of fractions in context and to develop the ability to make connections between the different modes of representation (NCTM, 2000). The RNP curriculum strives to achieve such aims (Behr, 1992).

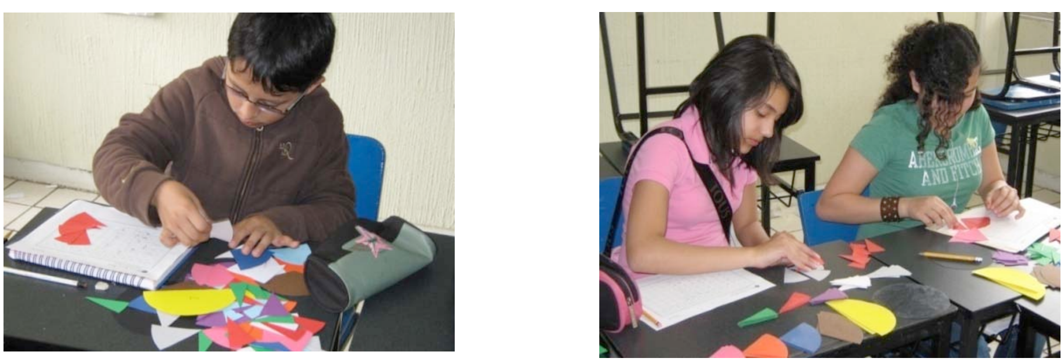

Figure 2. Working with concrete materials.

Participants, Method of Investigation, Procedures

The aim of this study is to carry out a qualitative analysis of the results obtained through working with some of the activities suggested by the RNP Level 1, during a workshop organized in a private school in the State of Michoacan, Mexico, with a group of 6th grade students (12

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year-olds). In order to select the 15 pupils who would receive most benefit from such work, a diagnostic test was applied to a group of 60 students; those who appeared to have greater difficulties regarding the topic of fractions were invited to participate. Whilst the Workshop was offered as a voluntary activity, the pupils made a commitment to participate throughout the sessions. The Workshop consisted of thirty hourly sessions, which took place three times a week during the afternoon. The material used was that which the RNP suggests: coloured circles divided in sections of different sizes, paper strips for folding, and chips to be used as discrete material. The aim of the Workshop was to help students develop number sense of fractions by: understanding the concept of flexibility of the unit, understanding and using fraction equivalence, ordering and comparing fractions using different strategies, adding and subtracting fractions.

Results

Diagnostic Exam

This exam consisted of twenty questions, designed to discover the performance level of the pupils with respect to: shading part of a whole -in discrete as well as continuous quantities-, locating fractions on a number line, reducing fractions, finding equivalences between common and decimal fractions, adding and subtracting fractions with the same and different denominator. Some aspects worthy of mentioning regarding the results of the diagnostic exam were that: most pupils were able to divide a figure in equal parts and shade a given part, but they found it difficult to shade a set of objects (see Figure 3). This could indicate that during their school years they have had more experiences using continuous rather than discrete quantities. In relation to the operations, most of the group managed addition and subtraction of fractions with like denominators, but were not able to do the operations with unlike denominators. This could lead us to think that the pupils had a poor number sense of fractions.

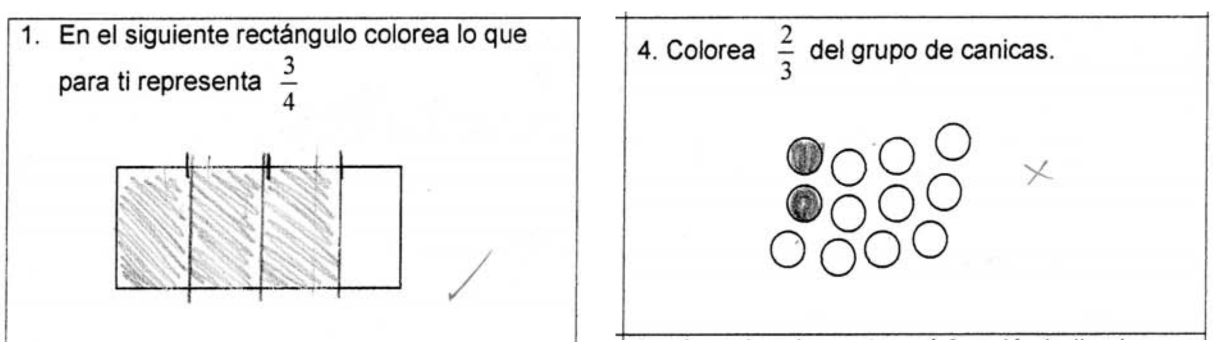

Figure 3. Daniela's work in the Diagnostic Exam.

Experiences during the Workshop

The whole teaching sequence was based on the RNP, making adjustments in time only: the material for 20 lessons was covered during 30 Workshop lessons. During the first session of the Workshop the pupils were given their circles for them to cut out, in order for them to become familiar with the colors. The pieces were stored in an envelope and were used throughout the sessions. The methodology used consisted of reviewing the topics of the previous session, covering new material and closing the session with reflections and comments from the pupils. Basically the sessions were teacher-guided, with brainstorming and individual participations and, sporadically, pair work. The participants had a notebook where they registered their work as well as the ideas they came up with. Some worksheets, included in the RNP, were also used.

The last two sessions of the Workshop were devoted to evaluation: a written, individual summative evaluation, to determine whether the aims had been reached; and an oral, group, formative evaluation designed to discover the pupils' opinion regarding their perception of the Workshop as well as to give them the opportunity to suggest ideas of how to improve it.

From the beginning, the students were very enthusiastic in participating; their informal comments reflected that the topic of fractions caused them anguish and were very interested in improving their understanding of the same. Whilst they cut out their circles they shared some of their discoveries: "the greater the denominator, the smaller is the piece"; also, when they put some pieces on top of other pieces they discovered that not all of them have equivalences (for example that eights and fifteenths only coincide in the unit).

During the sessions devoted to working the concept of flexibility of unit, i.e. using different pieces as the unit, the pupils reached the conclusion that the whole doesn't have to be a complete circle (see Figure 4); also that the number of pieces that fit in a larger piece constitute the denominator. In words of a participant:

Miri: So, if the red piece fits 5 times in the brown piece then you can say that red is 1/5 of the brown.

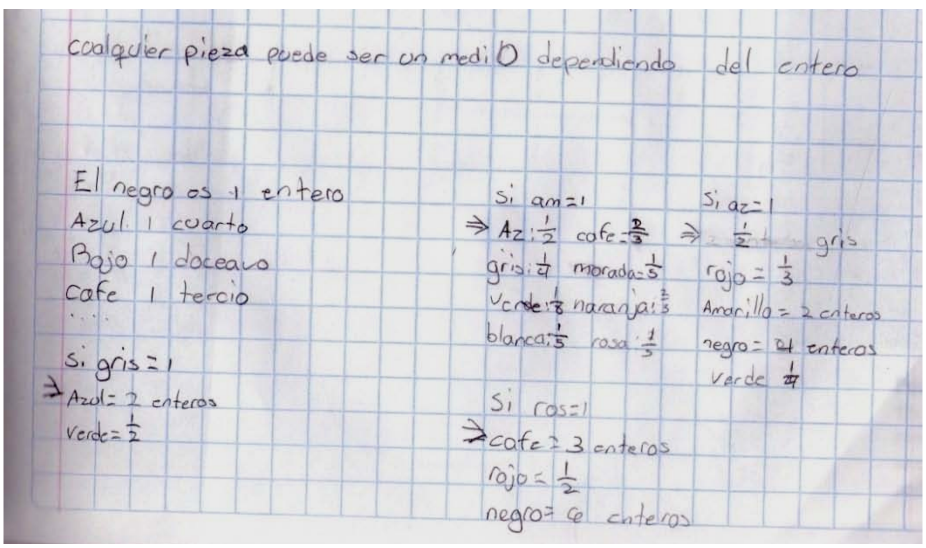

Figure 4. Aitana's work on the flexibility of the unit.

Equivalences came to them naturally. In previous explorations with their material they had come to the conclusion that there are pieces that take up the same space as others. The work started with fractions representing ½ , and afterwards moved on to other equivalences. When they compared the use of concrete materials with the written symbols they quickly realized that if one multiplies numerator and denominator by two, one obtains a number which represents the same fraction but is divided into a different amount of parts. They soon made a generalization: given a fraction, if you multiply numerator and denominator by the same amount you obtain an equivalent fraction:

Ricky: It is by 4 above and below. That is, the one above and the one below will always be the same. [If you multiply above and below by the same amount you get an equivalent fraction.]

Thus, the children were able to infer an idea that is generally used as a formula in traditional instruction: to obtain an equivalent fraction you have to multiply numerator and denominator by the same number.

In the sessions that addressed fraction comparison, the pupils used different strategies. Some made drawings; others made use of the transitive property of numbers:

Monse: ¾ > ½ because since 2/4 = ½ then 3/4 has to be larger.

Others used the idea of the unit:

S. Paz: you look at the number on top and check how much it needs to complete a unit. If there's more than one half missing then the fraction is smaller than 1/2.

These inferences show that they were starting to apply their knowledge of equivalent fractions (1/2 = 2/4), and that they had a clear idea that the denominator indicates the number of parts into which the unit is divided.

Afterwards, the topic of fraction addition was broached, starting by making estimations and deciding whether the answer was smaller or larger than ½ or of the unit. Some pupils felt safer with the concrete material and gave the answers using their circles. Others showed very interesting strategies, for example:

Sagar: I'm going to add 1/8 + 9/10, since 9/10 needs 1/10 to become a unit and 1/8 > 1/10 then the sum will be greater than 1.

Also:

Aitana: If I want to add1/3 + 3/4, 3/4 needs 1/4 to become a unit, since 1/3 > 1/4 then the sum will be > 1. As for 1/3 + 2/6, I can see that they are equivalent, then the sum will be 4/6; since 1/2 = 3/6 then 4/6 > 1/2.

Still other participants used the notion of equivalent fractions and gave exact answers, ignoring the instruction of trying to find a close estimate.

This is the extent of the material covered during the Workshop because, even though the initial objective was to reinforce addition and subtraction of fractions, a time limit of 30 sessions had been established, as was mentioned afore.

Once the Workshop concluded, three pupils were interviewed; these interviews were recorded and transcribed. It's important to point out that the problems broached during the Workshop were different to the ones described herein. The following situations were among some of the most outstanding contributions. (I = Interviewer).

I: [Solve the following problem]. Marco and Memo each have a packet of M&M's, with each packet having the same amount of candies. Marco ate 2/3 of his packet and Memo ate 3/4 of his. Can you tell me whether they ate the same amount or one of them ate more candies than the other? Explain why.

Aitana: Memo ate more because let's say he divided his M&M's into 4 groups and ate 3 of those groups then he was left with a small group, whereas Marco divided his into three groups and ate two of those and was left with a larger group, therefore he has more M&M's left than Marco.

[…]

I: OK, now we are going to use these cubes. Thomas made a tower of cubes. He used 6 cubes to make 1/5 of the tower. Can you tell me how tall the tower was?

Aitana: How tall was the tower?
I: That's exactly what I want you to tell me.
Aitana: Right, so how tall was the tower? Let's see, it must have been 30 cubes tall because if I divide 30 into 5 equal parts I get 6 and if I choose one of those [groups] then I chose 1/5 of the tower.
I: OK, now how about this problem: Thomas made another tower using 8 cubes, which represent 2/3 of the whole tower. Can you tell me how tall the tower was?
Aitana: The tower measured 12 [cubes], because if you are telling me that 8 cubes represent 2/3 [of the tower] then one third has to be 4 [cubes] because I understand there are 3 groups.

It is evident that this pupil has a clear idea of the concept of a fraction since she was able to give logical explanations when solving situations that were unknown to her.

During the last session, where the participants were asked to evaluate the Workshop, they commented that they considered too much time was wasted in forming their circles at the beginning of each session, thus diminishing the amount of time of significant work. They also suggested that they would have liked more games related to real life situations. As a reply to this suggestion it could be interesting to add to the Workshop some baking sessions which would give the pupils some practice with using fractions in real life situations.

After carrying out a general analysis of the sessions, it is deemed important to promote teamwork (Hagelgans, 1995), as well as to encourage written communication of their mathematical ideas through the use of a mathematics journal (NCTM, 2000).

Final Comments

At the time of implementing the activities it became evident that the pupils found it difficult to move away from traditional learning, which privileges procedural learning of mathematics as opposed to trying to make sense of the conceptual ideas. Though the diagnostic exam made it clear that they had a poor comprehension of fractions, their procedural knowledge gave them a sense of false security.

Throughout the 30 sessions of the Workshop most of the pupils showed progress in their comprehension of the meaning of fractions, equivalences between fractions, ordering strategies, and translations between the different modes of representation. It was interesting to notice that in different situations: the pupils used their own explanations, they "invented" their own method of solving the different activities instead of depending on the algorithms they had learned, they were capable of explaining their reasoning to their peers, and they had the ability to solve problems which were new to them. All in all, they trusted their reasoning instead of asking for explanations or solutions. Nevertheless, even though some pupils achieved good results and felt safe using their concrete models, not all of the participants were able to make the translation to the symbolic representation. The issue of whether increasing the number of sessions would help these pupils achieve this matter still remains an uncertainty. Another question that remains unresolved is if teachers were to implement these strategies from the moment the pupils first come in formal contact with fractions (in the third grade of primary school), would their pupils be able to understand them better.

It is not easy to quantify the learning that took place in this group of pupils. Only time will evidence if the experience they lived was significant in their construction of the concept of fractions. An undeniable fact is that the experience started with a group of pupils with many doubts about their mathematical abilities, who felt uneasy about participating, and unable to answer most of the exercises of a fractions exam. The end result of the 30 sessions was to have a group of more confident pupils who were happy to have participated in the experience, and possessing the knowledge that fractions are a topic far easier than they had imagined and with the certainty that if they don't know how to solve a problem they can ask without feeling afraid of being judged. These facts are well worth the time invested in the Workshop.

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PROSPECTIVE ELEMENTARY TEACHERS' DEVELOPMENT OF FRACTION NUMBER SENSE

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Can elementary teacher candidates "unlearn" harmful algorithms used with fractions as they are invited to develop fraction number sense? This study examines the development of prospective elementary teachers' fraction number sense during an intermediate (grades 5-8) mathematics methods course. During this course, participants were involved in a variety of activities and tasks used for the development of fraction number sense.

Number sense "represent[s] a certain way of thinking rather than a body of knowledge that can be transmitted to others" (Sowder, 1992, p.3). In order for teachers to create opportunities and experiences for students to develop meaningful understandings of fractions, teachers not only need to have a well developed and meaningful understanding of fractions themselves but also pedagogic practices that provide experiences for students to explore and construct ideas about fractions. This research is based on the premise that prospective teachers bring with them to their methods courses a somewhat limited understanding of fractions and that a methods course taught from a constructivist orientation may change and extend that understanding. Thus, the purpose of this study was to explore changes in prospective elementary teachers' fraction sense. The specific research question was: Can carefully selected experiences in a mathematics method course designed for elementary education majors affect their fraction sense?

Theoretical Framework

According to Schneider & Thompson (2000), number sense includes a solid understanding of the meaning of numbers and numerical relationships; flexible thinking about numbers (e.g. can see ten in a variety of ways). Based on these descriptions of number sense, researchers in this study define fraction sense as an intuitive feel for fractions and fraction relationships; flexible thinking about fractions (e.g. can use benchmark fractions to determine reasonableness of fraction operations).

There have been many calls for change in fraction instruction to move from procedural instruction to helping students develop conceptual understanding (Lamon, 2005; National Council of Teachers of Mathematics, 1989, 2000; Van de Walle, 2007). Deborah Ball posed the following question for mathematics educators in a paper presented at the eighth annual meeting of PMENA posed the following important questions for mathematics educators:

- What is taught in different mathematics methods courses and what do prospective teachers learn?
- What goes on in the mathematics courses that teachers take at the college level and how do these experiences fit with students prior experiences in mathematics?
- How can teacher educators productively challenge, change, extend what teacher education students bring?

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Methodology

Participants were 42 female (2 non-traditional) prospective elementary teachers enrolled in an intermediate mathematics methods course. This course is a part of their last year of coursework and focuses on teaching mathematics in grades 5-8. All participants had completed twelve credit hours in mathematics including a course designed specifically for elementary education majors that addresses the foundations of numbers (set theory, numeration, and the real number system), number theory, algebraic systems, functions and applications, and probability. Additionally, all participants had successfully completed a three credit hour primary mathematics methods course prior to enrolling in this semester.

As a part of this second mathematics methods course, participants were engaged in a variety of experiences that were designed to help them think about the mathematical content and pedagogy of teaching of fractions. Fraction instruction started informally at the end of a class session with the question:

Consider the following three fractions: $\frac{99}{100}$ $\frac{6}{7}$ $\frac{15}{16}$

Which fraction is the largest? ___________

Explain the reasoning behind your choice.

The next class session started with a discussion of their responses and followed with activities and discussions involving comparing and ordering fractions.

Participants created the Marilyn Burns Fraction Kit and explored the cover-up and un-cover activities using the fraction kit (Burns, 2001) and additionally worked with a variety of tools (e.g. fraction bars and circles) used to aide students in fraction sense development. Students used these tools to demonstrate and solve a variety of fraction problems including writing and modeling (in at least three ways) fraction addition, subtraction, multiplication, division contextual problems.

One of the texts used for this course was *Developing Mathematical Fluency* (Wheatley & Abshire, 2002) which includes a variety of activities (e.g. fraction sequences, fraction two-ways, fraction balances) that aide in development of fraction sense and fluency with fraction operations. Each of these activities had a 1-2 page reading that supplied instructions and the pedagogical rationale for the activity. In addition, course readings included chapter sixteen from John A. Van de Walle‘s (2007) *Elementary and Middle School Mathematics: Teaching Developmentally* and a variety of articles from the NCTM journal *Mathematics Teaching in the Middle School* including Reys, Kim & Bay’s (1999) Establishing Fraction Benchmarks; Warrington’s (1998) Multiplication with Fractions: A Piagetian Constructivist Approach; and Nowlin’s (1996) Division with Fractions. As a part of the course, all of the prospective teachers were required to tutor an elementary student in grades 3 through 7 one hour per week for 10 weeks. During this experience, most prospective teachers worked with their tutee on fraction concepts.

A pre-test was administered during the class period prior to any instruction on fraction concepts and a post-test was administered at the end of the 15-week semester. The pre/post test contained 12 questions dealing with basic fraction concepts. Questions on the pre/post-test included

- 2 questions dealing with using benchmarks to determine size of fractions;
- 5 part-whole tasks (2 using an area model, 2 using a set model, and one contextual) (eg. If this rectangle is one whole, find one-fourth.);

$\frac{1}{5}$

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- 1 fraction sequence (Complete the sequence ___, ___, 4, ___, ___, 5 , ___);
- 1 pedagogy question that involved comparison of two fractions (each missing one piece); and
- 3 ordering fractions (1 set of unit fractions, 1 set with a common numerator, and 1 set with each fraction missing one piece).

Prior to data analysis the researchers determined how data would be coded (see Table 1). Then each question of the pre/post tests was analyzed independently by each researcher. Explanations were further analyzed for emergent themes using a constant-comparative method (Strauss & Corbin, 1998). The alpha level of significance of .05 was used for in inferential analyses.

Score	Scoring Explanation
0	Wrong Answer
1	Correct Answer, Incorrect Explanation
2	Correct Answer, Incomplete Explanation
3	Correct Answer, Complete Explanation

Table 1. Scoring rubric for questions with an explanation required

Results

The first two questions on the pre-post test involved students consideration of a pair of fractions (A. $\frac{1}{3}$ or $\frac{3}{4}$ and B. $\frac{5}{8}$ or $\frac{3}{4}$) and determining which of the two fractions were closer to ½ and to explain the reasoning for their choice. A paired t-test analysis revealed that students use of benchmark fractions to compare fractions revealed a significant increase in students understanding ($t(41)=-2.866$, $p=.006$) after fraction instruction. Further analysis (see Table 2) revealed that on their comparison of $\frac{5}{8}$ and $\frac{3}{4}$ an additional nine (21.4%) students scored had a correct answer with a complete explanation on the post-test while on their comparison of $\frac{1}{3}$ and $\frac{3}{4}$ an additional five (11.9%) students had a correct answer with a complete explanation on the post-test.

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Score	$\frac{1}{3}$ or $\frac{3}{4}$ Pre-Test n(%)	$\frac{1}{3}$ or $\frac{3}{4}$ Post-Test n(%)	$\frac{5}{8}$ and $\frac{3}{4}$ Pre-Test n(%)	$\frac{5}{8}$ and $\frac{3}{4}$ Post-Test n(%)
0	10(23.8)	9(21.4)	16(38.1)	3(7.1)
1	9(21.4)	7(16.7)	4(9.5)	7(16.7)
2	12(28.6)	10(23.8)	9(21.4)	10(23.8)
3	11(26.2)	16(38.1)	13(31.0)	22(52.4)

Table 2. Prospective teachers' scores on benchmark questions

Students' incorrect or incomplete explanations tended to focus the missing portion of the whole or how close they perceived the pieces were to ½ or 1 (See Figure 1).

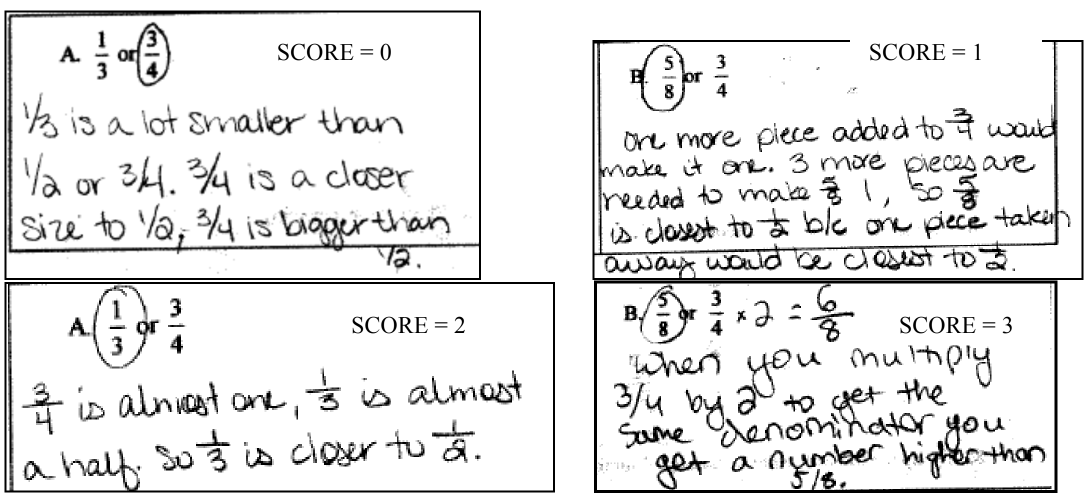

Figure 1. Sample scored benchmark responses

The next five questions on the pre-post test dealt with part-whole tasks – two area, two set, and one contextual. Participants were scored based on whether they had a correct answer or not. For the area and set tasks one dealt with fractional parts less than one and the other dealt with fractional parts greater than one. The contextual task required participants to draw a picture of a finished patio given a picture of ¾ of the patio. Examining student responses on the pre-test revealed that on both the area and set part-whole tasks students tended to be able to solve these when given the whole and asked to find an amount less than one; however when the amount given was greater than a one and asked to show one-whole, students struggled (see Table 3). Pre-post test paired t-tests revealed no significant difference for the area and set tasks when the given the whole and asked to find an amount less than one, but a significant difference ($t_{41} = -4.253$, $p = .000$) pre-test to post-test when the amount given was greater than a one and asked to

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show one-whole. Additionally, there was a significant difference (t_{41} = -2.672, p = .011) pre-post on the contextual task.

	Area <1		Area >1		Set<1		Set>1		Contextual	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
M	1.00	1.00	0.43	0.83	0.98	0.98	0.43	0.79	0.50	0.71
SD	0.00	0.00	0.50	0.38	0.15	0.15	0.50	0.42	0.51	0.46

Table 3. Prospective teachers' scores on part-whole tasks

Approximately 12% (n=5) of the students could solve the fraction sequence ___, ___, 4, ___, ___, $5\frac{1}{5}$, ___ prior to the methods course but after exploring the teaching of fraction concepts nearly 60% (n=25) of the participants could solve the sequence. Statistical analysis showed this to be a significant increase ($t(41)$ = -.6.105, p < .001). It was disheartening to note that 40.5% (n=17) of the prospective teachers remained unsuccessful in solving the fraction sequence at the end of the course.

Participants were given the pedagogical scenario "A fourth grader said that 3/4 and 5/6 are the same size because they both have one piece missing. Do you agree? Explain." Students were scored on a scale of 0-3. Analysis of pre- and post-tests showed that about 24% (n=10) more of the participants had a correct answer and explanation. Further statistical analysis did not reveal a significant difference between participants the pre-test (M=190, SD=1.08) and their post-test (M=2.17, SD=1.21) scores.

Score	Pre-Test n(%)	Post-Test n(%)
0	5(11.9)	7(16.7)
1	11(26.2)	6(14.3)
2	9(21.4)	2(4.8)
3	17(40.5)	27(64.3)

Table 4. Prospective teachers' scores on pedagogical scenario

The final portion of the pre-post test required participants to order three sets of fractions (set of unit fractions, set with a common numerator, and a set with each fraction missing one piece). Table 5 shows the percent of participants that successfully ordered each of the sets of fractions. Paired t-tests revealed no significant difference when participants were asked to order a set of unit fractions. Significant differences were found when asked to order a set of fractions with a common denominator (t_{41} = -2.364, p = .023) and when the set involved fractions with one missing piece to make it a whole (t_{41} = -2.693, p = .010). A paired t-test involving the sum of participants scores on all three of the ordering fraction problems also revealed a significant increase (t_{41} = -3.650, p = .001) in students ability to order fractions from the pre-test to the post-test.

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Set	Percent Correct		Mean(SD)		$t(41)$	p
	Pre	Post	Pre	Post		
Unit Fraction	91	98	.90(.31)	.97(.16)	-1.780	.083
Common Denominator	86	98	.85(.37)	.97(116)	-2.364	.023*
Missing Piece	45	71	.46(.51)	.72(.46)	-2.693	.010*
All 3	38	69	2.21(.77)	2.67(.53)	-3.650	.001*

*$p < .05$

Table 5. Differences between prospective teachers ordering fractions pre- to post-test.

Discussion

This study revealed that this group of prospective elementary teachers brought with them a limited understanding of fractions to their mathematics methods courses and that despite a concentrated effort to improve those understandings, only minimal progress was accomplished. Although these prospective teachers had many years of school mathematics, including four college level mathematics courses and one primary level mathematics methods course, their reasoning about fraction concepts was often incorrect and largely based on limited understandings or misconceptions they had previously developed or, at best, understandings of fractions as part of a whole. If the strength of these pre-service teachers understandings lies only in a part-whole understanding of fractions then as mathematics educators we should be concerned. Lamon (2005) suggests that a part-whole understanding of fractions is not sufficient suggesting that when students' experiences have been limited to part-whole relationships they tend to have a limited understanding of fractions. Lamon recommends that students should be provided with a variety of experience to help them develop the multiple meanings of fraction.

Findings from this study demonstrated that many prospective elementary teachers are leaving their coursework with misconceptions such as seen in the ordering fraction problems and the pedagogical scenario a common misconceptions where students believe that when two fractions are missing one piece they are the same size fraction. As has been pointed out by NCTM (2000) "as students work with number, they gradually develop flexibility in thinking about numbers, which is a hallmark of number sense" (p. 80). Extending this to number sense about fractions, or fraction sense, students need to develop flexibility in their thinking about fractions (Reeder & Utley, 2007). The development of fraction sense is gradual and takes time to development, thus, it would too ambitious to imagine that the concentrated efforts during one semester to improve prospective elementary teachers fraction understand would result in significant progress. While prospective elementary teachers did make progress in developing their fraction sense they still need more opportunities to revisit fraction and other rational number concepts using pedagogical tools and teaching practices that help prospective teachers deepen and extend their fraction sense.

Ball (1990) points out that the types of experiences with learning and teaching mathematics that students have had when they reach their mathematics methods course(s) greatly impacts the influence that teacher educators can have in developing them into teachers of mathematics.

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Additionally, related to Ball's question suggesting that mathematics courses be examined, the results of this study further suggest that mathematics faculty responsible for the development and teaching of courses designed for prospective elementary teachers should be encouraged to collaborate with faculty teaching the mathematics methods courses to provide prospective teachers with more opportunities to develop their fraction sense. This study has important implications for mathematics teacher educators with regard to the kinds of challenges we face in improving not only our prospective teachers' content knowledge but their pedagogic knowledge as well.

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CREATING ANALOGIES ABOUT EXPERIENCES IN ORDINARY LIFE WHEN DOING A PROOF BY CONTRADICTION

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This paper investigates a geometry teacher's use of analogies about experiences in ordinary life when doing a proof by contradiction in an urban classroom. The analysis applies Systemic Functional Linguistics for examining how the teacher established connections between the source and the target of the analogies. The results show that the teacher created the analogies by using three different linguistic resources: lexical cohesion, conjunctions, and reported speech. The teacher's use of these linguistic resources intended to teach students how to formulate a mathematical argument by creating a parallelism in the grammatical structure of the analogies.

Introduction

The *Proof and Reasoning Standard* (NCTM, 2000) emphasizes the importance of increasing students' opportunities to engage in reasoning and proving. At the same time, the *Connection Standard* promotes the importance of establishing connections between students' experiences in mathematics classrooms and their experiences in ordinary life. One of the main goals of the high school geometry course has been to teach students to reason logically so that they can apply the same reasoning to situations in ordinary life (Fawcett, 1938). Current documents, such as the *Common Core Standards*, also stress the importance of teaching students to make mathematical arguments. In this paper, I show an example of a geometry teacher's use of analogies that drew upon experiences in ordinary life when doing a proof by contradiction in an urban classroom. The focus on proof by contradiction is relevant because research shows that students have difficulties with this type of proof (Lin, Lee & Wu Yu, 2003; Reid & Doubin, 1998). I apply analytical tools from Systemic Functional Linguistics (SFL) to examine the linguistic resources that the teacher used to create the analogies. An examination of teachers' practices that are intended to enable students to establish connections between arguments in ordinary life and mathematical arguments can lead to identify ways for helping students to develop their reasoning skills.

Theoretical Perspectives

Prior work in mathematics education has examined the use of analogies in mathematics teaching and learning (Bayazit & Ubuz, 2008; English, 1997; Greer & Harel, 1998; Reed, Ernst, & Banerji, 1974; Richland, Holyoak, & Stigler, 2004). Pimm (1981, p. 100) has offered a useful image of an analogy as a mathematical proportion (e.g., $A{:}B{::}C{:}D$) where one knows more information about one set of relationships (e.g., $A{:}B$) than about another one (e.g., $C{:}D$). The literature on cognition describes analogies as mappings from a base, B, to a target, T (Gentner, 1983). The base includes specific prior knowledge that constitutes the foundation of the analogy and the target includes the new knowledge that the analogy is supposed to make explicit.

In a study using data from 8th grade mathematics classrooms, Richland et al. (2004) showed that teachers, for the most part, assume the responsibility of selecting the analogies that they use in their explanations. This finding is important in light of research which suggests that two main components of an analogy affect children's ability to transfer what they know to the new concept that is the target of the analogy: *transparency* and *systematicity* (Gentner & Toupin, 1986). Transparency describes the degree of similarity between the objects that constitute the analogy, and systematicity involves the causal chains that connect the components of the analogy. For example, a teacher could say, "solving a linear equation is like taking off your shoes and your socks." This analogy has a low degree of transparency because a variable and a foot have little resemblance as physical objects. However, the analogy has a high degree of systematicity because the processes involved in the two cases are analogous: inverse operations are performed in the opposite order. The two aspects of the analogy that would need to be explicit for students are the mapping between the objects in the base and in the target and the causal links.

In order to examine how teachers achieve transparency and systematicity in their talk, I focus on the linguistic resources that teachers use to achieve cohesion (Halliday & Hasan, 1976). Cohesion includes the resources for making connections regarding the meanings in a text. I propose an operational definition of transparency and systematicity in relation to the linguistic resources used to achieve cohesion. I use lexical cohesion as a marker for transparency. Lexical cohesion involves making semantic connections in a text between chains of ideas that share similar meanings. A lexical cohesion chain can signal the transparency in an analogy because the types of word choices used when making an analogy can make explicit the relations that are established within the base and the target. In addition, I propose that speakers' use of resources from the system of conjunction is a marker of systematicity. In SFL, the system of conjunction includes words and terms that provide logical connections in a text such as "and" and "or," but also other conjunctions such as "however," "because," and "but" (Martin & Rose, 2003). I assume that the speakers' choices of conjunctions can signal the causal links between components of the analogy, thus showing whether the analogy has systematicity or not.

Research Questions

The research questions of this paper are the following:

1. How does a teacher use lexical cohesion to achieve *transparency* when using analogies?
2. How does a teacher use conjunctions to achieve *systematicity* when using analogies?
3. What other linguistic resources does a teacher use to map the source and the target of an analogy?

Mode of Inquiry

Data Sources

The main data sources for this paper are videos from a geometry lesson taught by Mr. Harrison in three different classes at Bondi High School, a large Midwestern urban school with an ethnically diverse population and with around 50% students from low SES background. For this paper, I selected videos where Mr. Harrison was proving the Angle-Side-Angle (ASA) triangle congruence theorem by doing a proof by contradiction in three regular geometry classes. I segmented the videos according to changes in the activity structure in each lesson, such as teacher exposition or triadic dialogue (Lemke, 1990). I selected the segments that included a discussion of a proof by contradiction to either motivate or *install* (Herbst, Nachlieli, & Chazan, 2011) the ASA theorem (e.g., the statement asserting that if two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent). These segments constituted the corpus of the paper.

Analysis

The selected segments were transcribed and parsed into clauses. I performed a lexical cohesion analysis with the purpose of identifying semantic chains in each segment. Then, I performed a conjunction analysis to identify the purpose of the conjunctions used in relation to the analogy. Finally, I analyzed the teacher's use of reported speech, as a result of my observations that this was another linguistic resource that the teacher deployed. Reported speech has the purpose of attributing to a source what someone would think or say, and it is a resource from the system of engagement in SFL (Martin & White, 2005).

Description of the Class Discussion and the Analogies Used

In the three lessons, Mr. Harrison's discussion followed a similar structure. First, he requested students to provide an example of a situation where they had been falsely accused of doing something, and he elaborated what would be a proof by contradiction of that false accusation. Then, he provided the proof by contradiction to prove the ASA theorem. Finally, he made the analogy explicit by stating how the proof for the ASA theorem was similar to the argument against the false accusation. The following transcript from the 1st period class shows an example of how Mr. Harrison elicited students' experiences:

> *Mr. Harrison*: So, have you guys ever made an argument where somebody says something and you're like [pause 3 seconds] how do I say this? You say, well, if that was true, then, this, this, and this happens. Like, somebody claims, I don't know. Can somebody give an example of that? Like a real life example? Maybe somebody accuses you of, I don't know, cheating on somebody. Or, somebody accuses you of doing something you didn't do. And your argument against that is like, "Well, if that were true, then…". You guys?

In this excerpt, Mr. Harrison provided the case of "cheating on somebody" to model the type of accusations that students could have been subject to. Mr. Harrison also started phrasing what an argument against that accusation would look like by saying, "Well, if that was true, then…". This phrasing provided the basis for the argument in the proof by contradiction.

Table 1 shows a list of the base knowledge in the analogies that the students shared. In some classes, the students hesitated to provide an example from their own lived experiences and the teacher asked students to make up a story (1st and 4th period classes).

Class period	**What is the base knowledge?**	**Evidence for the contradiction**
1	Being accused of fighting someone.	The accused person does not have a black eye.
4	Being accused of initiating communications with a friend's ex-boyfriend.	The date of the text messages shows that the ex-boyfriend initiated the communication and not the accused person.
6	Being accused of shoplifting.	The accused person's bag is empty.

Table 1. Examples provided for the analogies

In the three classes, the teacher provided the same proof by contradiction of the ASA theorem. The proof started by assuming that a pair of triangles (ΔABC and ΔDEF in Figure 1) is not congruent, even though two pairs of angles and the pair of segments included between those angles are congruent. Without loss of generality, $\overline{AB}$ must be longer than $\overline{DE}$. Then, one can assume that there is a point M on $\overline{AB}$ where $\overline{AM} \cong \overline{DE}$. Consequently, the triangles would be congruent by the Side-Angle-Side theorem, since $\overline{AM} \cong \overline{DE}, \angle MAC \cong \angle EDF, \overline{AC} \cong \overline{DF}$. However, there is a problem with this conclusion, because the angles BCA and EFD were supposed to be congruent, and, as a result of the triangle congruency, the angles *MCA* and *EFD* would also be congruent. This would mean that the angles *BCA* and *MCA* are congruent, but one angle is obviously bigger than the other one. Therefore, the original assumption is false and the triangles are indeed congruent.

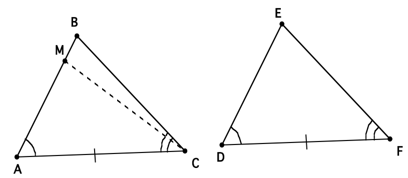

Figure 1. Diagram for the proof of the ASA theorem

Results

Using Lexical Cohesion to Achieve Transparency

The analysis of lexical cohesion intended to look for chains of semantic terms that were used to create the analogy. According to Halliday and Hasan (1976), there are two main types of lexical cohesion: reiteration and collocation. Reiteration involves the repetition of a term, the use of synonyms, the use of a general word (e.g., "Angle-Side-Angle" and "rule") or the use of a superordinate term to denote a broader category of another term (e.g., "proof" and "proof by contradiction"). Collocation involves a relation that is not equivalence, but another type such as complementarity or hyponymy (e.g., class and part). Table 2 shows the results for the analysis of lexical cohesion chains for repetition and synonymy. The coding for general word and superordinate did not yield any cohesion across the two contexts and is not included here.

Class period	Type of reiteration cohesion	Terms used		Frequency in the mathematical argument	Frequency in the non-mathematical argument
1	repetition[1]	argument		1	7
		problem		5	0
		rule		6	0
		true		1	5
1	synonym	argue and prove	argue	0	1
			prove	1	0
		claim and accuse	claim	1	1
			accuse	0	1
4	repetition	accuse		5	7
		show		6	4
4	synonym	argue and show	argue	2	2
			show	6	4

[1] For sake of brevity, I only show terms that were repeated more than 5 times.

Class period	**Type of reiteration cohesion**	**Terms used**		**Frequency in the mathematical argument**	**Frequency in the non-mathematical argument**
		contradiction and wrong	contradiction	3	0
			wrong	3	1
		proof, argument, and strategy	proof	2	0
			argument	2	1
			strategy	2	0
6	reiteration, repetition	accuse		1	5
		assumption		3	3
		show		4	1
		true		0	7
6	reiteration, synonym	argue and show	argue	0	2
			show	4	1
		assume and pretend	assume	0	1
			pretend	1	1

Table 2. Analysis of reiteration cohesion

The lexical cohesion analysis demonstrates that some terms throughout each lesson provided a common reference to talk about the examples that the students provided and also about the mathematical proof. The use of the same terms for non-mathematical and mathematical arguments provided a cohesive effect. For example, in the 4th period class, the teacher used the word "show" a total of 10 times with three main purposes: (1) to announce the proof of the ASA theorem (e.g., "During this proof that I'm about to **show** you I don't want you to write anything down), (2) to model how to provide the evidence of the contradiction (e.g., "And you **show** her your phone"), and (3) to state the contradiction in the proof (e.g., "That's kind of like me **showing**, or you **showing** the text messages…"). By using repetition Mr. Harrison provided a link between the mathematical and the non-mathematical argument when creating the analogy, thus achieving transparency.

Collocational cohesion was useful for naming the different components of a proof by contradiction when talking about the non-mathematical context and, also, when talking about the theorem (see Table 3). For example, in response to the student's example about being accused of initiating a conversation with the ex-boyfriend of a friend, Mr. Harrison said, "So, how do you **argue** that you've been **accused wrong**?" Here, Mr. Harrison framed the formulation of an argument as a strategy for refuting a false accusation. Mr. Harrison used a similar collocation of terms when introducing the proof for the ASA theorem. He said, "You guys wanna know why Angle-Side-Angle works? I can **show** you and you're gonna **show** the same kinda **argument strategy**." Then, later he said, "So, here's the **strategy**… Okay, somebody **accuses** me and says, 'Hey, these triangles aren't congruent." Collocational cohesion enabled the teacher to achieve transparency by labeling different components of the proof by contradiction such as the claim that would be contradicted (e.g., the accusation), the assumption, the contradiction (e.g., the problem), and the proof by contradiction (e.g., the argument strategy). The transparency in the text involved the mapping between the use of one term in the context of the example provided by the student and the use of the same term used in the context of the proof of the ASA theorem. Therefore, Mr. Harrison's explanation of the proof made the strategy explicit by using the same language of an accusation to bridge the two contexts.

Class period	**Terms used**
1	accuse, argue, argument, claim, convince, logic, problem, prove, true, wrong
4	accuse, argue, assume, proof by contradiction, show, strategy, true, wrong
6	accuse, argue, assume, assumption, logic, pretend, problem, proof, show, train, true, wrong

Table 3. Analysis of collocational cohesion

Using Conjunctions to Achieve Systematicity

According to SFL theory, conjunctions can help establish four main types of logical relations: addition, comparison, time, and consequence (Martin & Rose, 2003, p. 127). There are two main types of conjunctions according to their purpose. External conjunctions help to establish logical relationships in terms of the meanings conveyed in a text. Internal conjunctions have the function of organizing a text. Table 4 shows the results of the coding according to the logical relations that the conjunctions established. The teacher used more external conjunctions than internal conjunctions. Therefore, the analysis focuses on the external conjunctions that the teacher used, because these were the ones mostly used to create the arguments in the analogies.

Class period	**Addition**		**Comparison**		**Time**		**Consequence**	
	External	Internal	External	Internal	External	Internal	External	Internal
1	30	13	6	0	0	0	62	0
4	19	13	8	0	0	0	33	0
6	26	12	4	0	0	0	38	0

Table 4. Conjunction analysis

Most of the external conjunctions that the teacher used showed a logical relation of consequence. The conjunctions for consequence (i.e., *because*, *so*, *therefore*, *if*, and *then*) were useful to connect the logical chain of steps in the proof by contradiction. There are four types of conjunctions to denote consequence according to their purpose: cause, means, condition, and purpose. Most of the conjunctions to denote consequence that Mr. Harrison used had the function of showing cause (23 in the 1st period, 16 in the 4th period, and 25 in the 6th period). For example, when Mr. Harrison identified the contradiction in the proof of the ASA theorem in the 1st period class, he said, "**So**, there's a problem with that. There's a problem with the idea that the triangles are not equal. **So**, when this is happening, you say, 'Well, guess what, you were wrong when you said that the triangles are not equal.'" Mr. Harrison used "so" to denote the causal links between the problem (e.g., the contradiction) and the consequences of the problem, and, also, between the consequences of the problem and the conclusion of the proof (e.g., the assertion that the triangles are not equal is false). The use of conjunctions to denote cause strengthened the causal links, thus enabling the teacher to achieve systematicity.

The second most frequent type of conjunction used were conjunctions to denote addition. The predominance of conjunctions with a logical relation of addition is not surprising, since "and" is included within this type of conjunction, which is one of the most common conjunctions in the English language (Martin & Rose, 2003, p. 113). In the segments under investigation, "and" was the conjunction most used with the purpose of adding clauses (23 times in the 1st period, 17 times in the 4th period, and 24 times in the 6th period).

The conjunctions to denote comparison were the third most frequent conjunctions. These conjunctions had the function of establishing the analogy between the examples from experiences in ordinary life and the proof of the ASA theorem. The only conjunction used for comparison was "like." For example, the teacher said, "Okay, so somebody accuses me and says, 'Hey, these triangles aren't congruent.' It's kinda **like** somebody accusing, you know, Lena, talking to this guy first or whatever." The use of "like" here, intended to establish the link between the two examples, thus making the analogy explicit. The use of the conjunction "like" to connect the two proofs is an example of how the teacher used conjunctions to map the causal chains in the base and in the target of the analogy.

Using Reported Speech

Another linguistic resource that enabled the teacher to create the analogy was the use of reported speech. The teacher assumed the voice of an accused person when discussing the example, and, also, when proving the theorem. The transcript below shows how the teacher used reported speech in the 6th period class.

> *Mr. Harrison*: Let's take your example, somebody accuses you of shoplifting. One option could have been to say, "Hey, let's go ahead and pretend that I did shoplift for just a second. If I shoplifted, then you would expect to see something in my bag, or somewhere on me. Well look in my bag; it's empty. So, guess what, I didn't shoplift."

Mr. Harrison also enacted how to argue against the accusation that the triangles are not congruent by ASA theorem, as shown in the transcript below.

> *Mr. Harrison:* So, I'm like, "Alright, let's pretend that they're not equal, for just a second. If they're not equal, then maybe these two pieces would not be equal. So maybe *DE* is shorter."

Table 5 shows the number of clauses stated by the teacher that include reported speech to make the arguments in each class. The use of reported speech helped the teacher to engage students in a hypothetical dialogue that provided evidence against a false accusation using a proof by contradiction. The teacher applied the same structure of a dialogue about an event in ordinary life to a hypothetical dialogue with mathematical content, thus making the analogy explicit.

Class period	Clauses that use reported speech in the mathematical argument	Clauses that use reported speech in the non-mathematical argument
1	12	7
4	21	12
6	8	17

Table 5. Clauses with reported speech

Discussion

The interplay of three linguistic resources, specifically, lexical cohesion, conjunctions, and reported speech, provided the structure to model a proof by contradiction. These three resources allowed the teacher to create a sort of template that he could apply to build a proof by contradiction regardless of the content (e.g., mathematical or non-mathematical). Figure 2 shows a model for the structure of the argument. This model takes into account the different linguistic resources that helped him to establish the analogy by establishing a *parallelism* in the grammatical structure (Martin & Rose, 2003, p. 237). The parallelism was achieved by using repetitions and synonyms, by voicing the "accusations" in a similar way through reported speech, and by using conjunctions to connect causal links within each analogy.

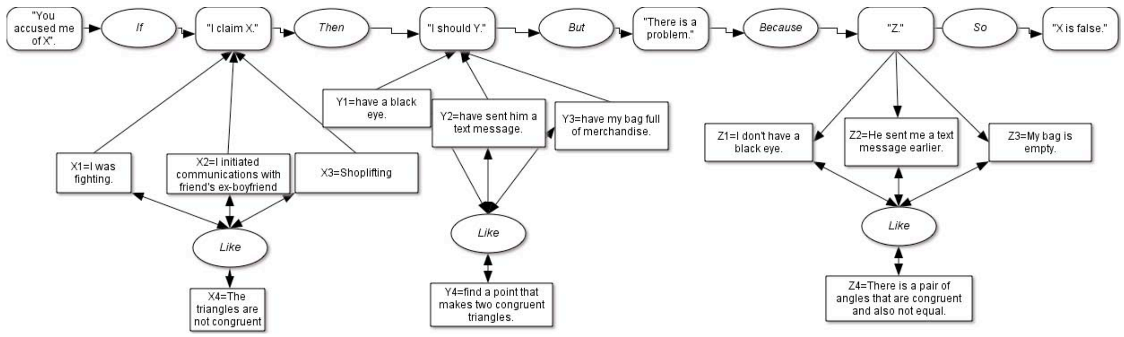

Figure 2. Model for the structure of the analogies

The goal of teaching geometry as a vehicle to teach students how to apply logical reasoning in situations of ordinary life continues to be a goal today. By allowing students to draw upon their prior knowledge of ordinary situations through the use of analogies, teachers can broaden students' opportunities to learn and instill in students an appreciation for mathematical thinking.

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THE EFFECTS OF AN ADVANCED HIGH SCHOOL COURSE IN NUMBER THEORY AND ALGEBRA ON STUDENTS' MATHEMATICAL SELF-EFFICACY

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The purpose of this study was to determine ways high school students' mathematical self-efficacy was affected by participation in an advanced course in proofs and problems in number theory and algebra. In the course, emphasis was placed on students engaging in higher order mathematical thinking and developing mathematical habits of mind. Because self-efficacy affects goal setting and perseverance in the face of challenging tasks, which ultimately affects achievement, we were interested in determining students' course goal self-efficacy at the beginning of the course and whether self-efficacy changed as a result of course participation.

Introduction

In this research report, we describe ways high school students' mathematical self-efficacy was affected through participation in an advanced course titled *Proofs and Problems in Number Theory and Algebra* (PPNTA). In the larger study that supports this brief report, we will examine students' understanding of and skill in constructing valid proofs within the backdrop of number theory and algebra. We are further interested in the ways PPNTA course participation increases students' mathematical habits of mind. The study will ultimately result in three years of data, enabling the authors to track longitudinal trends and a number of different foci, both in pedagogy and content.

In this advanced course, students were expected to engage in higher order mathematical thinking and develop proof construction and problem solving ability, as well as an understanding of the notion of a mathematical habit of mind. Thus, the goals of the course were not content-specific but instead focused on ideas such as elegance in proofs and solutions and metamathematics. In addition, students were expected to increase their skill in engaging in intellectual argument with others about mathematics and to work individually and in teams to solve complex problems in algebra or number theory.

Students in this course are academically gifted and have demonstrated high achievement and high mathematical self-efficacy in courses with content-specific objectives. However, it is unclear whether these efficacy beliefs would be similar in a course such as PPNTA where goals are focused on engaging in mathematical habits of mind (MHM) and the construction of proofs. While there is not an agreement on what the MHM phrase means, it would certainly include 1) to explore mathematical ideas, 2) to formulate questions, 3) to construct examples, 4) to identify problem solving approaches that are useful for large classes of problems, 5) to ask whether there is "something more" (a generalization) in the mathematics on which students are working, and 6) to reflect on answers to see whether an error has been made (Millman & Jacobbe, 2008, 2009). These six traits help define a term that is featured prominently in an important report on the mathematical education of future teachers (CBMS, 2001). In addition, MHM is closely linked to the Polya Principles of Problem Solving.

For all students of mathematics, there is a real difference between studying a topic (such as calculus), which includes both manipulation and conceptual understanding, and the ability to engage in the kind of abstract thinking that is required in writing proofs. Because self-efficacy plays a critical role in the ways individuals approach difficult tasks, set goals for themselves, and persevere when faced with a challenging problem (Bandura, 1994), we were interested in determining students' course goal self-efficacy at the beginning of the course and whether self-efficacy changed as a result of participating in the course.

Theoretical Framework

Self-efficacy refers to the beliefs individuals hold about their capability to achieve a certain level of performance on a given task or goal (Bandura, 1994). These self-referent beliefs influence actions, and, as Pajares and Schunk (2001) explain, self-efficacy is a better predictor of what individuals accomplish than are their actual capabilities. Self-efficacy determines what people choose to do, how much effort they put into a task, and whether they persist when challenged (Pajares & Schunk).

A number of studies have shown a positive relationship between self-efficacy and academic achievement. Multon, Brown, and Lent's (1991) meta-analysis of 68 self-efficacy studies conducted between 1977 and 1989 indicated a positive correlation between self-efficacy and academic achievement. Further, according to Zimmerman (1995), other research reveals a causal link between self-efficacy and academic achievement (see, for example, Barry, 1997 and Schunk, 1981, 1989).

The causal link between self-efficacy and achievement is seen as a reciprocal relationship. As Pajares and Schunk (2001) explain, "According to Bandura's social cognitive theory, behavioral and environmental information create the self-beliefs that, in turn, inform and alter subsequent behavior and environments" (p. 251). Thus, self-efficacy can be positively influenced by engaging in classroom activities that increase students' competence (including modeling, providing feedback, and strategy training), and as self-efficacy is built through these types of activities, academic achievement can also increase. As described in Pajares and Schunk's overview of self-beliefs and school success and Zimmerman's (1995) review of self-efficacy and educational development, this reciprocal relationship has been demonstrated in numerous studies.

Methods

PPNTA Course

The course was developed as a collaborative effort between the Georgia Institute of Technology and a local charter high school for mathematics, science, and technology. The course instructor was a Georgia Tech graduate student in mathematics and computer science who earned a bachelor's degree in mathematics with highest honors. His previous teaching experience consisted of two years as a teaching assistant for undergraduate calculus courses.

Georgia Tech's Center for Education Integrating Science, Math, and Computing (CEISMC) financially supported the teaching of the course through a research assistantship funded through Georgia's Race to the Top award, which was funded by the U.S. Department of Education. The director of CEISMC, who is co-author of this paper, had a major role in planning the PPNTA course. In addition to co-authoring the course textbook, he also made classroom observations over the course of the semester and delivered lectures in the areas of introduction to proofs, the use of groups in geometry, and the notion and use of equivalence classes.

The PPNTA course was conceived as an introduction to mathematical proofs using the subjects of number theory and algebra as context. In the course, students were introduced to mathematics as a living research discipline that can be used to discover new ideas about numbers, space, functions, and other objects as well as their inter-relationships. It was designed as an “explore, generalize, prove, think” environment in which students approached mathematics much differently than in a traditional math classroom. In some ways, this type of environment is similar to the culture of doing mathematical research. The purpose was to help students understand that mathematics is not fundamentally about calculation nor is it based on rote memorization. The philosophy of the course is aligned to the National Council of Teachers of Mathematics (NCTM) Principles and Standards.

Students attended class Monday through Friday for 18 weeks. Class periods were 48 minutes. Class activities included instructor lecture, reviewing problem sets, students working problems independently or in groups, and students making presentations to the class as a whole.

Goals for students were to: (1) identify what makes a mathematical proof correct, (2) identify flaws in fallacious proofs, (3) learn some commonly applied proof techniques, (4) become proficient at reading and writing mathematics in general and proofs in particular, and (5) practice applying problem solving methods to find solutions and demonstrate clearly their correctness. Topics covered during the course were:

- Basic properties of the integers
- Divisibility and prime numbers
- The Fundamental Theorem of Arithmetic
- Diophantine equations
- The idea of equivalence relations and its applications
- Basic properties of polynomials
- Divisibility of polynomials, divisibility methods, and the roots of polynomials
- Applications to combinatorics

One example of how students were to explore mathematics and its ideas was a section (covered in the third week) in which the instructor first worked with students to show that the $\sqrt{2}$ is irrational using the usual proof by contradiction. The students were then asked to fashion a proof of the fact that $\sqrt{3}$ is irrational using the logic of the $\sqrt{2}$ example. From this approach, they were asked to generalize the procedure so that it was valid for the square root of any prime number. In order to understand what is really going on in this proof structure, we asked the students to prove that the $\sqrt{4}$ was irrational. Of course, they all knew it was false, but having the students figure out why the “proof” of $\sqrt{4}$ is irrational must be incorrect was important for truly understanding what a proof is and what it isn’t.

Participants

Participants were 15 students (14 seniors and 1 junior) enrolled in the PPNTA course at a local charter high school with a focused curriculum in math, science, and technology (18 total students were enrolled; 15 agreed to be in the study). Eleven students were male and four were female. Ten students were Asian and five were Caucasian. Students had completed every high school math course available to them except for AP statistics, and all but one had taken two semesters of calculus (Calculus II and III at Georgia Tech) beyond AP Calculus and differential equations prior to enrolling in this course. All students were planning on majoring in a STEM or premedical field once enrolled full time in college. All participating students in this study earned an A as a final course grade in the PPNTA class.

Setting

The school where the study took place is a public, charter high school for mathematics, science, and technology whose first courses were given in 2007. The new campus, opened in 2010, includes high-tech classrooms and project-based work areas for student, university, and business collaboration. All eighth grade students in the county may apply for admission to the school, but due to a high number of applicants, a lottery is used to determine which students will be admitted. Enrolled students choose one of three areas in which to focus their studies: engineering, bioscience, or emerging technologies. Advanced Placement (AP) courses are offered in calculus (AB and BC), statistics, physics (mechanics and E&M), biology, chemistry, and computer science, as well as in the humanities. Math courses are offered in accelerated integrated geometry and accelerated integrated pre-calculus (courses that are aligned to the state mathematics curriculum), in Calculus 2 (which includes linear algebra), and in Calculus 3 (both taught via video conferencing with Georgia Tech). Courses in differential equations and PPNTA are also offered. In 2009, the school's total enrollment was 327 students; 16% were economically disadvantaged and 2% had identified disabilities. Standardized test data indicate high percentages (> 90%) of students meet or exceed standards.

Data Collection

To answer the research question *In what ways does participation in the Proofs and Problems in Number Theory and Algebra course affect students' mathematical self-efficacy?* we measured mathematical self-efficacy using a pre and post self-efficacy instrument. In designing the instrument, we relied on Bandura's *Guide for Constructing Self-Efficacy Scales* and tied items to specific course goals. As Bandura suggested, we phrased items on the pre- and posttest in terms of what students *can* do in order to measure perceived capability. We extended this by asking students on the pretest to provide an additional measure of how capable they were to achieve each goal. Though Bandura cautions against asking individuals to judge potential capabilities, we chose to include the extended items on the pretest because students had not yet had opportunities to develop ability in most of the course goals, which was likely to result in low self-efficacy measures. Measuring both current and potential capability on the pretest allowed us to make additional comparisons that enhanced the meaningfulness of results.

On the pretest, students ranked their self-confidence on a scale of 0 to 100 (with 100 being most confident) on how confident they were that they could already complete the task (e.g., *understand the importance of proofs in mathematics*) and how confident they were that they could learn to do the task. For the posttest, students responded to the same prompts to rate how confident they were in their ability to complete each task. We chose the 0 to 100 scale based on Bandura's suggestion to use a broad scale to increase measurement sensitivity and reliability. The pretest instrument is provided in Appendix 1.

Students also completed a post-course survey about their experiences in the PPNTA course. The survey included self-efficacy items that provided students with the opportunity to describe the ways in which their confidence had changed, if at all, due to completing the course and the aspects of the course that affected their confidence. All students provided open-ended responses to the self-efficacy items.

Results and Discussion

Overall means for (1) the pretest in which students rated confidence in their ability to already complete each course goal (prior to the class beginning), (2) the pretest on students' confidence they could achieve the course goals, and (3) the posttest on students' confidence in their ability to accomplish each course goal (at the end of the course) are provided in Table 1. The difference between the pre measures was statistically significant ($t_{14} = -9.26$, $p < .000$) and indicated that although students began the course with low self-efficacy, their efficacy regarding their ability to learn content and accomplish the course goals was significantly higher. The standardized effect size for the difference was 1.37, indicating a large effect. There was no significant difference between students' confidence they could accomplish the course goals and their confidence at the end of the course. The 29.16-point difference between the pre- and posttest measures of self-efficacy was significant ($t_{14} = 5.59$, $p < .000$), and the effect size (1.70) was large.

Table 1. Student Self-Efficacy Beliefs on Course Goals Pretest

Item	N	Confident in ability to x	SD	Confident can learn to x	SD	Diff	Sig
understand importance of proofs in mathematics	18	63.8	25.6	91.2	9.0	27.3	.000
know how to work in teams to solve problems from number theory and algebra	18	63.4	27.7	92.7	10.1	29.3	.000
know how to use concepts learned about algebra in other courses in this course	18	49.9	29.9	81.8	20.2	31.8	.000
understand the concept of "elegance" in proofs	18	47.7	29.9	78.2	21.6	30.1	.000
know different methods to construct proofs	18	44.2	20.2	78.8	21.3	34.6	.000
know how to use computational math tools in problem-solving and proof construction	18	43.9	30.6	81.8	16.1	37.8	.000
know how to explain ideas that motivate proofs	18	43.1	22.8	75.3	17.0	32.3	.000
know how to work individually to solve problems from number theory and algebra	18	42.8	29.1	81.3	17.9	38.5	.000
know how to develop a mathematical habit of mind	18	39.7	21.9	75.6	18.5	35.9	.000
know how to define what a "mathematical habit of mind" means	18	38.6	26.2	74.7	21.1	36.2	.000
know how to identify the fallacious reasoning in incorrect proofs	17	37.9	24.3	75.9	18.9	38.0	.000
know how to construct valid proofs	18	35.2	17.8	84.2	14.5	49.0	.000
know how to engage in intellectual arguments with others about math	18	29.8	27.6	70.7	25.7	40.1	.000
know how to create examples that provide insight into designing proofs	18	26.8	20.8	73.4	18.3	46.6	.000

The authors would like to thank Mr. Daniel Connelly, a mathematics graduate student, for his contributions to the course and its quality teaching.

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Appendix 1. Mathematics Self-Efficacy Pretest

The table below lists goals for the course in the MIDDLE COLUMN. Read the goal and then, in the LEFT COLUMN, mark how confident you are that you can already do this or have already reached that goal. In the RIGHT COLUMN, mark how confident you are that you can reach the goal. In each column, rate your degree of confidence by recording number from 0 to 100 using the scale given below:

0	10	20	30	40	50	60	70	80	90	100
Cannot do at all					**Moderately can do**					**Highly certain can do**

Here's an example:

How confident are you that you can already do this?	Goal	How confident are you that you can learn to do this?
write in a number between 0 and 100		write in a number between 0 and 100
35	**Arrange a place to study without distractions.**	**80**

This means less than moderately confident one can already arrange a place to study without distractions.

This means between moderately and highly confident one can learn to arrange a place to study without distractions.

How confident are you that you can already do this?	Goal	How confident are you that you can learn to do this?
write in a number between 0 and 100		write in a number between 0 and 100
	Understand the importance of proofs in mathematics.	
	Learn different methods to construct proofs.	
	Understand the concept of "elegance" in proofs.	
	Create examples that provide insight into designing proofs.	
	Construct valid proofs.	
	Identify the fallacious reasoning in incorrect proofs.	
	Engage in intellectual arguments with others about mathematics.	
	Explain ideas that motivate your proofs.	
	Use concepts learned about elementary number theory and algebra in other courses to solve problems in this course.	
	Work individually to solve mathematical problems from number theory and algebra.	
	Work in teams to solve mathematical problems from number theory and algebra.	
	Define what "mathematical habit of mind" means to you.	
	Develop a mathematical habit of mind.	

TRACING THE DEVELOPMENT, DURABILITY AND DISSEMINATION OF MILIN'S INDUCTIVE ARGUMENT

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This case study documents the development, durability and dissemination of an inductive argument built by a nine-year-old student, Milin, over a 13 month period spanning his fourth and fifth grade years. He investigated counting problems whose solutions were represented with a variety of representations that he successfully linked together and used to convince his classmates and researchers of the validity of his arguments. This study provides a detailed analysis of Milin's problem solving. It shows the durability of his understanding over time and the impact his reasoning had on his classmates as his ideas traveled within the class.

Introduction

There has been considerable work done over the last decade that has established the power of children's reasoning (Maher, Powell, Uptegrove, 2010). Given certain conditions and tasks, children naturally engage in thoughtful problem solving and demonstrate their mathematical understandings using a variety of representations. By making use of their representations, learners are empowered to make convincing arguments for their solutions (Sran, 2010). This study adds to the literature by tracing, in detail, the building of an inductive argument by Milin, a nine-year old student. He worked in a variety of settings, first with a partner, then a small group. He shared his solutions with his classmates and during task-based interviews. We present eight episodes to trace how Milin developed, sustained and shared his justification for the solution of a counting problem using an inductive argument and how his ideas spread within the classroom. The research questions that guided this study are:

1. How did Milin build his inductive argument?
2. How did his argument travel to other students?

Theoretical Framework

In the last two decades, increasing attention has been given to how mathematical understanding is acquired (Davis, 1984; Davis, 1992; Maher & Davis, 1995). This has resulted in a shift of attention from rule-based learning to meaningful learning. Davis (1984) indicates that "understanding" involves building schemes and fitting new ideas into existing ones. Learning can take place while working on new problems and making use of personal representational systems such as spoken and written language, physical models, drawings and diagrams, and mathematical notations. According to Davis and Maher (1990), new ideas can emerge from previous knowledge already acquired. Hence, understanding can be achieved when a learner fits the new ideas into already existing schema. Davis (1984) uses a metaphor of a jigsaw puzzle to illustrate how this learning might occur. For problems requiring solutions for which the learner lacks fully developed schema, new experiences are needed. According to Maher (1998), the *building blocks* for new schema come from an individual's experiences, and so when students work with new problems, they have the opportunity to use more elaborate and sophisticated representations to build and expand their knowledge (Maher, Martino & Alston, 1993).

Yackel and Hanna (2003) have emphasized the social aspects of reasoning, pointing to the importance of communal activity where learners have the opportunity to interact with others. In the process of sharing and justifying solutions to problems during small group work, individual interviews, group interviews, or classroom discussions, the students' ideas are made public. Maher and Martino (2000) identified several conditions that promote a culture of *sense-making*.

Methods

Data were collected from three main sources. The video recordings serve as the primary source of data for this study. Cameras were used to capture the conversations and written work of the students. The second data source is students' written work. A third source consists of individual and small group interviews and observer field notes. Multiple data sources allow for triangulation and help ensure the validity of data analysis.

Milin and his classmates engaged in variations of the Tower Problems that (1) asked them find all possible towers of a given height when selecting from cubes of two colors and (2) come up with a convincing argument for the solution.

Analysis for the present study uses components of a model described by Powell, Francisco, and Maher (2003). All video sessions were transcribed, coded and verified by an independent researcher. The video clips that accompany these episodes can be accessed at http://www.video-mosaic.org/.

Results

In the course of Milin's earlier explorations with towers, several strategies were identified. These included guess and check, opposite by color, opposite by inverting, staircases, elevators, and organizations by groups with similar attributes (Sran, 2010). For this report, we describe Milin's problem solving in which he generated taller towers using smaller ones, and building a *family tree.* Using the strategies that took on a variety of representations including physical models, verbal explanations and diagrams, Milin formulated and applied an effective inductive argument to justify an earlier conjecture of a *doubling* rule.

Episode 1: February 6, 1992

Milin and his classmates completed their building of five-tall towers as a classroom activity. The researcher (R1) asked students to think about an assertion made by some children that there were exactly ten towers with two red cubes, displaying the student models in front of the class (see Figure 1). She asked whether two red cubes could be separated by three yellow cubes another way. Milin immediately responded: "On ones there is only three. On two's there is only two. And on threes there is only one." He then commented that no others could be built.

Figure 2. Collection of all five-tall towers with exactly two red cubes

Episode 2: March 6, 1992

A month later, after Milin had worked on five-tall towers, he shared with R2 his continued work on the problem. He shared his record of the number of towers of varying heights (see

Figure 2) and explained to R2 that there were two one-tall towers, four two-tall towers, eight three-tall towers, and so forth.

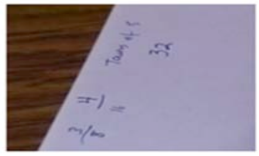

Figure 3. Milin's written record of his towers of various heights

When R2 asked about what the towers of one and two-tall would look like, Milin built all one and two-tall towers. R2 asked Milin to elaborate further, and Milin showed how taller towers could be generated from smaller ones. He placed his one-tall black tower in front of the two-tall towers with black bottoms (see Figure 4). R2 asked about three-tall towers and in response, Milin generated these towers by first placing a cube of one color on top of each one of the four two-tall towers and then made the remaining four three-tall towers by placing the cube of the second color to each two-tall tower. As he placed these new towers next to the matching two-tall tower he responded: "See … That would go into this family." Milin continued to use the term *family* throughout this interview as he arranged taller towers next to the corresponding shorter towers (see Figure 4). He remarked that the same strategy would work for four-tall towers.

MILIN: Yeah. Two for this, two for this, two for this, two for this , two for this, two for this, two for this. *(Milin points to each of the towers of three)*

R2: Yeah

MILIN: And once you get to 16 *(Milin points to the column with 5 and 32 on his paper)* you get all of them and you get 32.

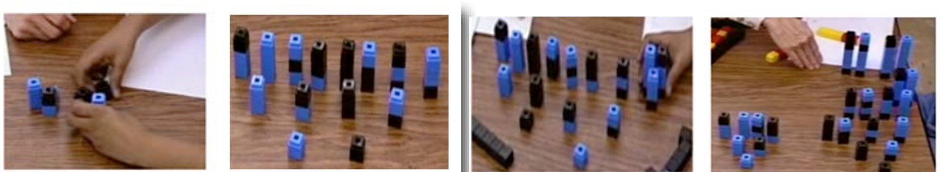

Figure 4. Milin explains and builds his "family" strategy

Episode 3: March 6, 1992

At this point, Milin reported finding 50 six-tall towers using organizations by categories. More convinced of the doubling pattern, he reflects on his earlier work, suggests that he did something wrong, and conjectured that there should be 64 six-tall towers.

MILIN: But, I think I did something wrong on umm from 32 to go to 6[*six-tall towers*]. I think I did something wrong.

R2: Why?

MILIN: Mmm, I don't think that pattern would break down like.

R2: You really don't? You...

R1: *(off-camera)* Ah-ha

R2: If it didn't break how many should there be?

MILIN: Uh, 64

Episode 4: March 10, 1992

Four days later, Milin introduced his doubling argument to justify his solution for towers of varying height to three of his classmates (Maher & Martino, 1996a). In response to the researcher's question about the number of six-tall towers Milin said:

MILIN: Probably 64.
R1: Why do you think 64?
MILIN: Well, because there was a pattern.
R1: What's that?
MILIN: You just times them by two

When the researcher asked why the towers were doubling in each successive height, Milin offered the following explanation: "For each one of them you could add one no two more for on because there is a black I mean a blue and a red." R1 then asked the group to justify the eight three-tall towers, when selecting from two colors. Milin again explained that each two-tall tower could be turned into two three-tall towers as shown in Figure 4.

MILIN: Yeah. You have to keep on putting two for this two for this two for this and two for this and it will work out.
R2: Do you agree with that Jeff?
JEFF: Yes.
R2: Okay. So-
MILIN: Because you can't. There is only two colors you can't put any more on them.
R2: Now imagine we have our eight. Where do we go from eight? Because I heard Michelle say twelve.
MILIN: Sixteen.

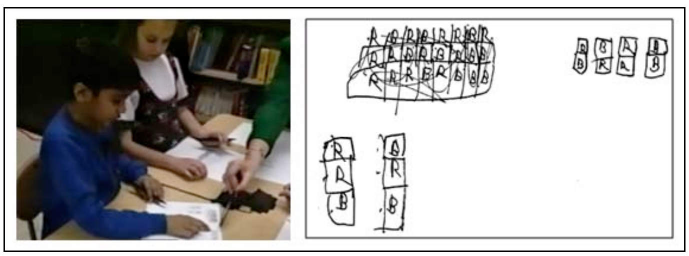

Figure 5. Milin shows how there are only two possible three-tall towers from each two-tall tower

Milin now expressed certainty that this "doubling" pattern would continue, in contrast to his doubt expressed in the earlier session. He continued:

MILIN: And for each one you keep on doing that. And for six you get sixty four.
R1: Does that make any sense?
JEFF: Yeah
MILIN: It follows the pattern to five why can't it follow the pattern to six?

Episode 5: October 25, 1992

Eight months later, Milin's response to the written assessment bears a remarkable resemblance to his *family* strategy (see Figure 6).

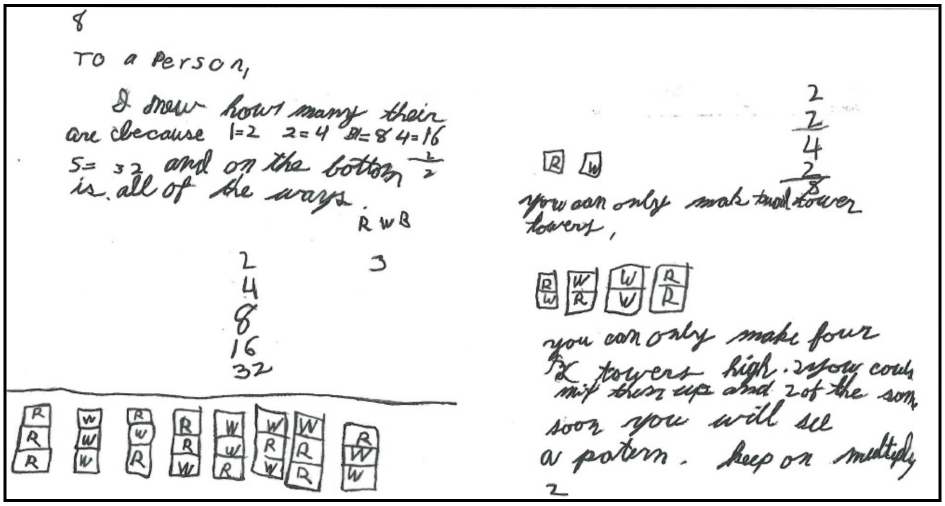

Figure 6. Milin's written response for October 25, 1992 assessment

Analysis of Milin's work showed that he first placed a cube of one color on top of all towers from (*n-1*) stage and then placed the cube of the second color to get all four two-tall towers. He repeated the process to get all eight three-tall towers as shown in Figure 7 below.

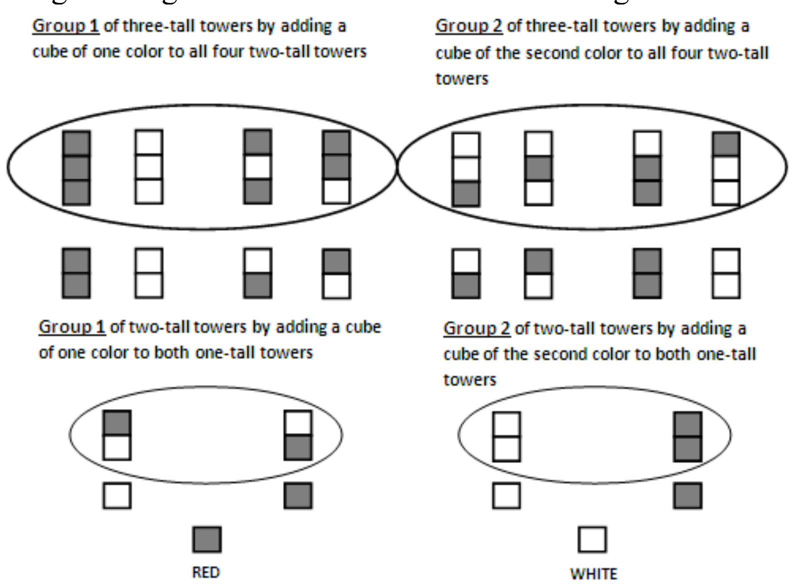

Figure 7. Milin's strategy for generating families of towers

Episode 6: February 26, 1993

Eleven months later, researchers posed a probability problem that required the building of towers of varying heights in order to determine the sample space for the experiment (Maher, Sran, & Yankelewitz, 2010b). This activity provided an opportunity for the research team to follow the durability of students' tower problem-solving ideas and simultaneously to monitor how particular strategies were shared among students. Milin successfully recalled his earlier strategy for building towers n-tall and began by sharing with his partner, Michelle. He explained by building tower models and placing the tops of the two three-tall towers (see Figure 8), and he pointed out that the smaller towers looked just like the two-tall tower. Milin then added a red cube to one tower and a yellow cube to the other (see Figure 8) as he already had before this explanation.

Figure 8. Milin's Explanation of how two-tall towers grow to two three-tall towers

Michelle indicated her understanding of the doubling pattern as she explained:

MICHELLE: You could add yellow on from there and you could also add red on. See, I understand but like its everything times two

R1: What? Tell me.

MICHELLE: Like from here from the two if you times by two you got fours

R1: Why is it that? But why is it four? I can see it is four. I can see you got eight. But can you tell me why that works?

MICHELLE: Because there is two different colors

MILIN: Yeah.

Episode 7: February 26, 1993

During Milin's explanation of his strategy for building towers of varying heights, Milin mentioned that Stephanie should also be aware of this strategy because they had shared this during the small group assessment held on March 10, 1992. This prompted R1 to invite Matt and Stephanie to Milin and Michelle's table. It was Michelle who now demonstrated her understanding of Milin's inductive argument to Matt and Stephanie. An excited Matt joined in the explanation and referred to the collection of towers as "family tree," explaining:

MATT: These are the parents-

R2: Okay.

MATT: -children, their children

Matt then shares this new understanding of the "doubling" pattern with two other students as Stephanie listens intently after she was unable to explain why the doubling pattern worked (see Figure 9).

Figure 9. Matt joins in the explanation with Michelle and then shares it with others

Episode 8: February 26, 1993

Michelle's explanation of Milin's reasoning confirmed Stephanie's conjecture that there would be 16 four-tall towers. However, while Stephanie remembered that there was a doubling pattern, it was not until this session that she demonstrated her understanding of how towers grew. As Stephanie listen to her partner, Matt's explanation, she seemed to make personal meaning as to how the doubling rule worked. An enthusiastic Stephanie eagerly shared her knowledge with her classmates (see Figure 9) referring to a "family" and its members (Maher, Sran & Yankelewitz, 2010b). She explained:

STEPHANIE: Alright, I have one red, okay? And I have a yellow, and from each of these, you can make two because all you you have to do is you add on a r..., you can add on a red to the red or a yellow to the red. And for the yellow, you can add on a red to the yellow and a yellow to the yellow, okay?
MICHELLE: So you don't have to look for duplicates.
STEPHANIE: Then each one of these has two. Like, okay if this is a family tree, thi... the mother, the parents
STUDENTS: Laughter
STEPHANIE: Have kids and then, six kids, okay, well actually, no eight kids. Then they have eight kids, and each one of them has two kids. And this one, you can add one red one yellow, one yellow one red

Figure 10. Stephanie presents her understanding of the "doubling" pattern to the entire group

Discussion

Local organizational strategies can be powerful in learning. In his early work, Milin made use of these to organize groups of towers. He began his problem solving by building random towers with the cubes by using "guess and check" methods, randomly making a new tower and then comparing it those that were already built in order to identify duplicates. He then created pairs and groups of towers that had similar properties. These groupings included opposites by color, opposites by inverting and then, staircase/elevator patterns. He discovered, however, that these strategies were no longer efficient for building taller towers. When these new organizations were inadequate to justify a complete solution, Milin developed a *family* strategy, based on his conjecture of a doubling pattern. This seemed to have triggered his finding a global organizational strategy that he referred to as a *family* (Maher, Sran & Yankelewitz, 2010a, 2010b; Sran, 2010). These organizations and strategies offered Milin important components for building a global organization that worked for towers of any height when choosing from two colors. Further, it eliminated the need of checking for duplicate towers. The progression to the *family* solution was an iterative process in which Milin revisited earlier strategies and built on and extended them.

The activity of problem solving with peers and the opportunity to share progress with researchers provided Milin with the time and resources that were needed to build his inductive argument and share it with classmates. The problem-solving investigations were integrated into regular classroom work. This suggests that there is a potential to engage students in thoughtful problem solving at an early age, providing an opportunity to build and share proof-like arguments.

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UNDERSTANDING STUDENTS' SIMILARITY AND TYPICALITY JUDGMENTS IN AND OUT OF MATHEMATICS

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Research in the field of mathematics indicates that many students struggle with justification and proof. However, in non-mathematical contexts, students are relatively strong at inferential reasoning. Our research presents two parallel lines of investigation—one focused on mathematical domains, the other focused on non-mathematical domains—in order to examine and compare the ways in which students reason in each. We report on three interrelated studies:(i) a small-scale interview study in which students were asked to sort numbers, shapes, and birds according to their characteristics;(ii) a small-scale interview study in which students were asked to determine whether various properties were true or false; and (iii) a large-scale survey to further elaborate the results from the first studies.

Justification and proof serve important roles in the field of mathematics, and consequently, there is increasing concern among mathematics educators regarding the difficulties students demonstrate in learning to reason mathematically (Healy & Hoyles, 2000; Kloosterman & Lester, 2004; Knuth et al., 2002). In non-mathematical domains (e.g., biology), however, the research is strikingly different: cognitive science research highlights considerable strength in students' causal reasoning (e.g., Gelman & Kalish, 2006; Gopnik et al., 2004). These seemingly conflicting findings serve as the driving force behind our research: why is it that students struggle to reason mathematically and yet are simultaneously skilled at reasoning outside of mathematics? In order to begin answering that question, we adopted a methodology often used in cognitive science (e.g. Kalish & Lawson, 2007) to examine students' reasoning in the mathematical domains of number and geometry. The main objective was to examine different types of inductive strategies students use in mathematical domains. Outside the mathematics classroom, children typically develop facts and ideas via empirical generalizations and causal theories (Chater & Oaksford, 2008), and students may therefore rely on and make connections to their non-mathematical ways of reasoning as they encounter ideas and problems in mathematics. This paper reports the results of the first phase of a multi-year study designed to explore the connections between students' reasoning in mathematical and non-mathematical domains.

Leveraging Inductive Reasoning

Researchers in mathematics education acknowledge the difficulties students have with formal proof (Dreyfus, 1999; Healy & Hoyles, 2000; Knuth, Choppin, & Bieda, 2009), and generally consider the inductive reasoning skills that students typically employ to interfere with their development of deductive means of reasoning and justifying. Researchers in cognitive science have found that students have difficulties *in general* with formal inference and deductive arguments, but they—unlike many mathematics educators—have continued to pursue a deeper examination of inductive inference. We agree with the latter that inductive reasoning is a powerful and useful tool, and are consequently beginning to extend cognitive science research on strategies of inductive inference to the mathematical content domains of number and geometry.

Cognitive science researchers have examined the inductive judgments people make and found that peoples' beliefs are frequently based upon category membership status, similarity, and typicality (see Feeney & Heit, 2007). For example, after being told a novel property that a robin possesses, people are more likely to extend that property to other robins than to all other birds in general. People are also more likely to extend that novel property to a blue jay than to a mouse, based upon perceptions of greater similarity between robins and blue jays than between robins and mice. In addition, people believe a robin to be a more typical bird than a penguin, and are consequently more likely to generalize from robins to all birds than from penguins to all birds (Osherson et al., 1990). In mathematics we lack a similar knowledge of students' understanding of the categories, levels of similarity between objects, and typicality of objects that may inform students' inferential reasoning. Therefore, the purpose of this course of study is to understand and map students' similarity and typicality judgments in the mathematical content domains of number and geometry.

This paper follows a three-part study design in which the results of each study were used to design and implement the following study, chaining together in such a way that the third study relied upon as well as further elaborated the results of both the first and second studies. Given this dependent relationship, as well as the quite distinct differences in methodologies and consequent analyses for each study, we outline each study and attendant results individually, focusing particularly on the ways that the studies build upon each other.

Study 1

In order to design a study that would elicit the typicality ratings and similarity relations for numbers and shapes (triangles and parallelograms), we needed to identify what features of each domain were considered relevant. When looking at triangles, for example, what do middle school students, undergraduates, and mathematics and engineering (hereafter STEM) graduate students see as important characteristics? Consequently, we designed a structured but open-ended methodology that prompted interviewees to utilize and share their own feature classifications.

Methods

We adopted the sort-re-sort procedure used by Medin et al. (1997). In one-on-one interviews, participants were presented with a set of cards that contained numbers or geometric shapes, the selections of which were based upon the varying of dimensions identified by previous research (Feldman, 2000; Miller & Gelman, 1983; see Figure 1).

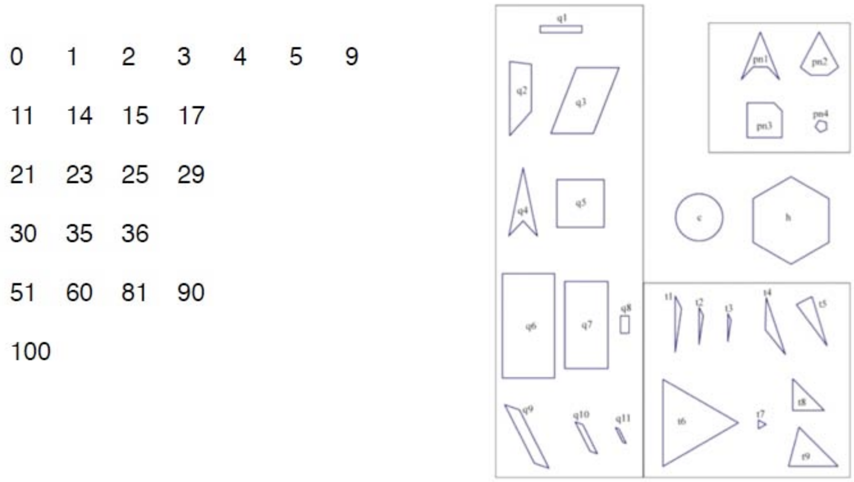

Figure 11: Number and shape cards.

Participants were asked to form groups with the cards, and then encouraged to further separate cards within those groups. Once sorting appeared to be complete, the cards were reshuffled and the participants were asked to form different groups. The sorting-re-sorting was continued until the participants had exhausted their different ways of grouping. The participants included 14 middle-school students, 14 undergraduate students (from a wide variety of majors with calculus the highest level of college mathematics completed), and 14 STEM graduate students).

Results

The results were striking because the different populations (middle-school students, undergraduates, and STEM graduate students) were not particularly different in their sorting and their sorting rationale. The following tables (see Tables 1 and 2) contain a list of several features used in the sorting task, along with the percentage of use by each population. In presenting the results, only those features noticed by at least 50% or more of the middle school students are included as that population is the primary focus of our research.

Table 4: Number Features.

Features	Middle Schoolers	Undergraduates	STEM Grads
Multiples	86%	93%	93%
Parity	79%	93%	87%
Prime	50%	71%	80%
Value of Digit	50%	29%	47%

In the domain of number, participants in all three groups were likely to remark upon features such as number parity, whether or not a number was prime, and the multiples of numbers. Comparing across populations (and not included in the table), the STEM graduate students were most likely to notice square numbers (53%), undergraduate students were most likely to remark upon shared digits between numbers (43%), and middle school students were most likely to mention factors of numbers (21%). Interestingly, in general undergraduate students did not appear, as might be expected, to be very different in their sorting than the middle school students—perhaps an indication that the salience of particular features seems to have a complex relationship with mathematics expertise.

The same trend appeared in the domain of geometry, such that there was not a clearly defined trajectory from middle school novices to STEM experts. Table 2 contains a list of the geometric features used to sort by at least 50% of the middle-schoolers, along with what percentage of each population sorted by that feature.

Table 5: Shape Features.

Features	Middle Schoolers	Undergraduates	STEM Grads
# of Sides	100%	100%	100%
Size	78%	85%	100%
Shape	64%	78%	93%
Angles	64%	85%	80%
Similar	50%	43%	40%

Once again, there were some particularly interesting similarities between the three groups, particularly that all of the participants remarked upon the number of sides. Middle school students, undergraduates, and STEM graduate students were all relatively likely to remark upon the size or shape of a geometric object. In addition, these two features are the only two that showed a progression from middle school students to undergraduates to STEM graduates. Other features was more erratically patterned. For example, 20% of the middle school students and about 50% of the STEM graduate students remarked upon the regularity of shapes, but none of the undergraduates sorted by regularity. In contrast, the similarity of shapes (i.e., the size and shape of geometric objects) was noted the more by middle school students than either undergraduates or STEM graduates, with the latter being least likely to use the feature to sort. (For more detail on Study 1, see Knuth et al., In press.)

Study 2

The results from Study 1 provided insight into the features of mathematical objects that are salient to middle school students, undergraduates, and STEM graduate students. Study 1 reveals something of the structure of students' representations of the domains of number and shape. Study 2 went on to explore whether this structure influences reasoning and proof strategies.

Methods

We designed a two-part semi-structured interview protocol in which we presented participants with both specific and general propositions and then asked them to determine whether the propositions were always true (see Figure 2).

Eric came up with a new mathematical property. He thinks this property is true for every whole number.

Eric's property:
First, pick any whole number.
Second, add this number to the number before it and the number after it.
Your answer will always equal 3 times the number you started with.

Lewis came up with a new mathematical property. He thinks this property is true for every triangle.

Lewis's property:
For any triangle, two of the sides added together are longer than the third side.

Figure 12: Specific propositions.

If students used examples in attempting to determine the truth of a proposition, they were asked about their example choices, including their judgment of the similarity and typicality of each example. For the number proposition above, for example, one participant used the numbers 4, 6, 8, and 5 to check the proposition's validity. When the participant had determined to her satisfaction that the proposition was true, she was then asked whether any of the numbers were unusual or typical. The final series of questions for each proposition asked the participant to share their beliefs about the similarity of the numbers and shapes they tested:

> *Interviewer:* "So are all these numbers similar to each other or different than each other?"
>
> *Participant:* "Well, I guess I think of 4, 6, and 8 as similar. Because they're all even. And I think of 5 as different because it's an odd number."

Next, the participants were presented with general propositions (see Figure 3, as an example) and asked to supply examples that varied in typicality and similarity. In particular, participants were

Maria came up with a new mathematical property. She thinks this property is true for every triangle.

****

If someone asked you to pick a very **typical** triangle to test if this property is true, what triangle would you pick?

If someone asked you to pick a third triangle that is very **different** than your first one, what triangle would you pick?

If someone asked you to pick another triangle to test that is very **similar** to your first one, what triangle would you pick?

Figure 13: General proposition.

asked to generate three different examples to test: a typical example, an example similar to that typical example, and an example different from that typical example.

The participants included 14 undergraduate students (not discussed in this paper) and 20 middle-school students. Further details about the methodology and results from the middle school students' responses to specific propositions are presented in Cooper et al. (2011).

Results and Discussion

Results from Part 2 suggest that the vast majority of middle-school students can use example-based reasoning as a method of proof, that more examples tended to be more convincing than few examples, and that the greater the variety of examples, the more convincing. Variety, here, seems to validate the numbers and shapes we used in Study 1, as students talked about varying numbers and shapes along the very dimensions we tried to systematically represent in the sorting task. Particularly, students frequently reported using typicality and similarity judgments to inform their selection of items as they tested examples. Importantly, students who discovered deductive proofs were less likely to vary examples overall, but on the problems where they did vary examples, they reported using variations in typicality and similarity in an intentional manner. While students who did not discover deductive proofs tested the greatest variety of examples, their variation was not intentional.

Students frequently referenced their everyday experiences—both inside and outside the mathematics classroom—during the interviews. For example, a respondent who talks about squares being typical because all the corners have 90 degree angles is quite different from the respondent who declared a tall rectangle to be typical because it was a common skyscraper shape. This result prompted us to reconsider the idea that typicality is a singular construct, and instead reframe typicality as context-relevant. In addition, as was also found in Study 1 with the non-linear trajectories of participants attending to particular mathematical features, in Study 2 middle school students' typicality judgments for number and geometry categories did not always correspond to expert-based dimensions. In particular, we became interested in how the student-based notions of typicality contrast with expert-based knowledge relating to special mathematical properties.

For instance, in a mathematical contexts, for the purposes of justification, experts may consider mathematically-special numbers like 0 and 1 to be highly unusual or atypical – mathematical properties that hold for these numbers may not be generalizable, and these numbers may be useful to employ in the search for counterexamples. However, in terms of numbers seen in everyday life, these numbers are very typical or common. The analysis of examples given by students and their associated typicality comments led us to tentatively map numbers and shapes by their everyday usual-ness (student typicality) and by their mathematical

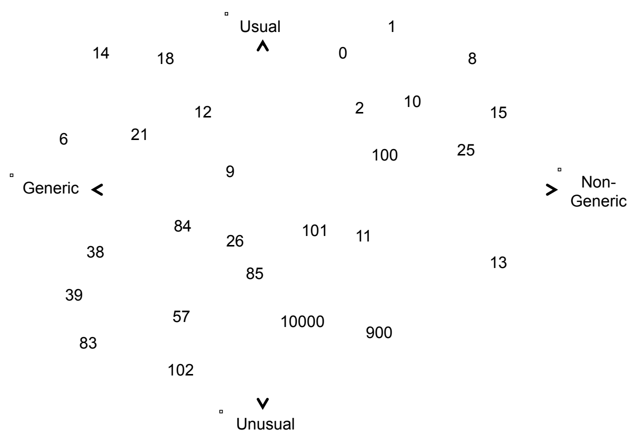

Figure 14: Everyday versus generic number placement.

generic-ness (mathematical typicality, where non-generic indicates a high presence of mathematical properties, such as prime-ness in numbers or congruence in shapes). Figure 4 illustrates our placement of numbers according to these two dimensions.

Study 3

Given the intriguing results from Studies 1 and 2, our next step was to administer a survey to a large enough number of middle school students to validate our results. Given the need to consider the importance of context, we determined that cuing the students to the context of "math class" versus "everyday life" was a necessary nuance. That is, we wanted to find out if students consider the expert-based generic-ness dimension of number and shape typicality if sufficiently prompted to consider these objects in a mathematical context.

Methods

We designed a 7-point Likert-scale survey where students rated the typicality of items in three different mathematical domains (numbers, triangles, parallelograms) and three different contextual cues (neutral, mathematical, everyday) (see Table 3). Through careful variance of features, items covered a wide gamut of types—in triangle, for example, we included isosceles, equilateral, and scalene, and varied both the size and the orientation of the shape (some shapes had a side parallel to the bottom of the survey page, while others did not). Additionally, we always placed the neutral cues first, but otherwise varied the order of the cues and domains.

Table 3: Stimuli prompts in the number domain.

	Neutral Context	Mathematical Context	Everyday Context
Section Prompt	Now we want you to think about how typical different numbers are.	Now we want you to think about mathematical properties – that is, the kinds of things you learn about in math class.	Now we want you to think about numbers you see in everyday life outside of school, for example, around your house, in stores, or outside.
Individual Item Wording	Think of numbers. How typical is this number? [item displayed]	Imagine that we learned a new mathematical property that was true of this number. How likely is it that the property will be true of most other numbers? [item displayed]	How typical is this number of those you see in your everyday life? [item displayed]

A total of 474 middle school students, drawn from a suburban middle school in a Midwestern state, completed the survey. Students were distributed across grades 6 (144 students), 7 (160 students), and 8 (163 students), and mathematics classes used reform texts.

Results

Using the findings of Study 1 and Study 2, we identified a number of mathematical and non-mathematical properties that we hypothesized students may attend to in their typicality rankings (see Figure 5). We computed the mean typicality rating for items falling into each of these categories, depending on whether the item was presented in a mathematical or everyday context. The results suggest that while both mathematical (generic-ness) and non-mathematical (common-ness) properties can influence typicality ratings, the everyday notion of typicality is most salient to middle school students. The results also indicate that contextual cues do not significantly influence students' typicality judgments, and that even cueing the context "mathematics classroom" did not increase the likelihood that students would attend to particular mathematical properties.

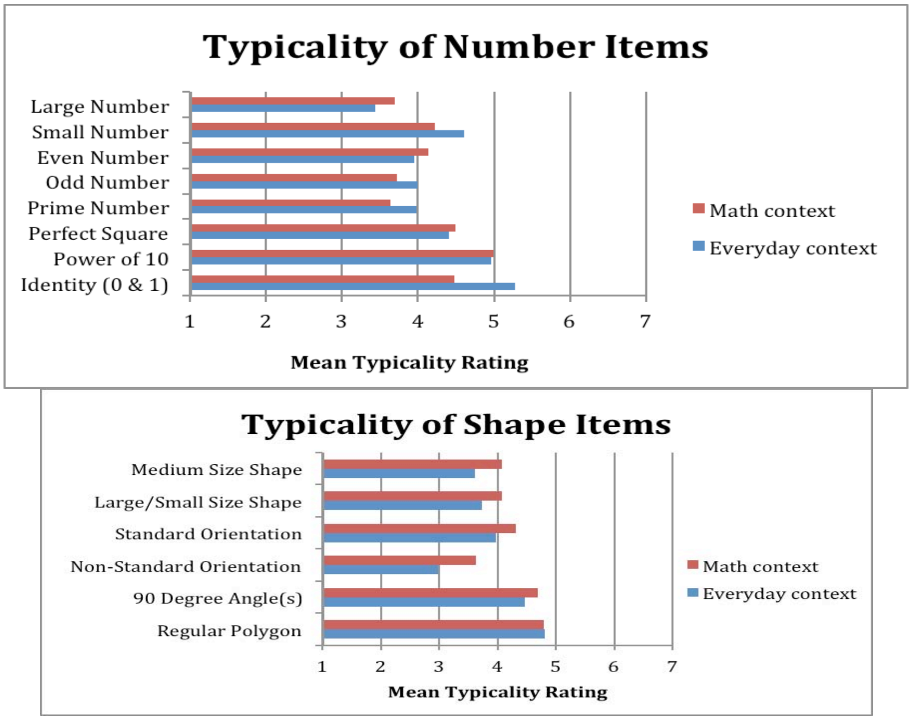

Figure 15: Mean typicality ratings for number and shape items.

Conclusion

Study 1 resulted in an identification of various sorting features of mathematical objects noticed by middle school students, undergraduates, and STEM graduate students, and furthermore, suggested that patterns across the three participant groups are not simple. In the domains of both number and geometry, an increase of mathematical expertise does not necessarily manifest itself in a higher likelihood of noticing particular features, and the three groups varied in the features to which they attended. The results of Study 2 suggest that typicality and similarity judgments may play a role in inductive and deductive reasoning, and that participants believed that varying the typicality and similarity of examples is a good strategy for testing propositions in both number and shape. In Study 3, the survey validated our initial smaller-scale interview studies, providing more confidence in our student typicality (usual-ness) identification of specific numbers and shapes. In the next phase of our work, we are further investigating the relationship between similarity/typicality and the nature of the empirical arguments student generate.

We believe that inferential reasoning can be leveraged to support the development of deductive reasoning, as it seems that applying strategies frequently used for the domain of living things—that is, thinking of examples, typicality, and similarity—can support the development of more sophisticated ways of reasoning in mathematics. Our research highlights the need to more closely and systematically examine inferential reasoning in mathematics, and instead of viewing students' use of examples as inappropriate or limiting, we view such use as a potential means of improving students' abilities to reason deductively. The importance of proof in mathematics, and the struggle of most students to justify and reason deductively, suggests that this examination is of paramount importance, and our research opens a new—and hopeful—line of study.

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QUANTIFYING INSTRUCTIONAL FOCUS ON PROCEDURAL UNDERSTANDING IN ALGEBRA: A DIGITAL OBSERVATION PROTOCOL

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We present a new digital observation protocol that captures the moment-by-moment emphasis of a lesson along a set of customizable dimensions. The result is a time-series graph showing the lesson progression, including the total amount of time allocated to each dimension during instruction. We present the results of a small scale study to illustrate how the tool may be used to characterize instruction along dimensions relevant to the newly released Common Core State Standards for Mathematics. We conclude by outlining our plans for using this tool to study the relationship between procedural understanding instruction and student achievement levels.

The call for mathematical understanding has been voiced by several prominent groups in the past 10 years (e.g., The National Council of Teachers of Mathematics, 2000; the National Mathematics Advisory Panel, 2008; and the National Research Council, 2001). More recently, the introduction of the Common Core State Standards (CCSS) for Mathematics makes the case for understanding succinctly while simultaneously raising important questions about implementation:

> These Standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. *But what does mathematical understanding look like?* One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, *why* a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as $(a + b)(x + y)$ and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness (CCSS, 2010, p. 4; emphasis added).

Our work emerges from the belief that while the CCSS and other documents advocate an instructional focus on understanding, the construct of "teaching for understanding" is rarely defined in a concrete and useful way. What constitutes teaching for understanding, and how are teachers to achieve this goal in the classroom? The research described in this paper is part of an ongoing effort to operationalize the process of teaching for understanding and assess its utilization in the classroom. To address the latter objective, we have developed a time-sensitive digital observation tool, which we describe in detail in this paper. We conclude with a brief review of the results obtained thus far and future efforts to confirm (or refute) the educational benefits of teaching for understanding.

Operationalizing Understanding

Resources for refocusing instruction on teaching for understanding (e.g., Blythe, 1998; Wiggins & McTighe, 2001) provide general strategies for teachers but often lack the content specificity needed to significantly reform high school mathematics instruction. Reflecting on this need and on the qualitative differences between "A-students" and "C-students," Burke (2001) developed a framework of mathematical "literacies" that characterize expert performance in algebra. This framework was eventually incorporated into NCTM's *Navigating through Algebra in 9-12* (Burke, Erickson, Lott, & Obert, 2001) and serves as the theoretical basis of our work. As Table 1 illustrates, Burke's original framework was stated in terms of student competencies, which we have translated into student-centered (and teacher-centered) questions.

Burke's Procedural Literacies	*Student-Centered Framework Questions*
The student understands the overall goal of the procedure and knows how to predict or estimate the outcome.	1a. What is the goal of the procedure? 1b. What sort of answer should I expect?
The student understands how to carry out the procedure and knows alternative methods and representations of the procedure.	2a. How do I execute the procedure? 2b. What are some other procedures I could use instead?
The student understands and can communicate to others why the procedure is effective and leads to valid results.	3. Why is the procedure effective and valid?
The student understands how to evaluate the results of the procedure by invoking connections with a context, alternative procedures, or other mathematical ideas.	4. What connections or contextual features could I use to verify my answer?
The student understands and uses mathematical reasoning to assess the relative efficiency and accuracy of the procedure compared with alternative methods that might have been used.	5. What do I look for to decide if this the best procedure to use?
The student understands why the procedure empowers her or him as a mathematical problem solver.	6. What can I use this procedure to do?

Table 1. Burke's Literacies and the FPU Questions

These student-centered questions (heretofore, the Framework for Procedural Understanding or FPU) are based upon the premise that the dichotomy between procedural skill and conceptual understanding is a false one (Star, 2005; Wu, 1999). Indeed, recent models of mathematical knowledge have challenged the assertion that conceptual knowledge is deep, meaningful, and connected, whereas procedural knowledge is shallow and rote (Baroody, Feil, & Johnson, 2007; Star, 2007, 2005). According to these models, mathematical procedures and mathematical concepts can each be known in either a shallow or deep way; it is the number of connections that defines how deeply one knows, and hence how well one understands, a given procedure or concept (Star, 2005; Hiebert & Handa, 2004; Skemp, 1978). Figure 1 illustrates this conceptualization, depicting two independent dimensions of growth that are applicable to the learning process – depth and practice (Hasenbank, 2006). The illustration is meant to apply equally well for procedural and conceptual knowledge.

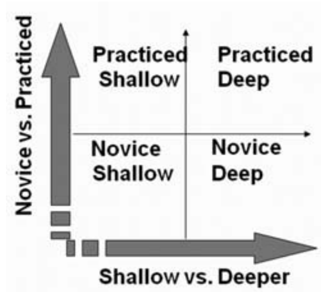

Figure 1. Levels of Conceptual and Procedural Knowledge

The FPU provides a working definition for what the CCSS has termed "mathematical proficiency." Collectively, the questions that constitute the FPU are meant to focus instruction, assessment, and reflective practice on important habits of mind (Driscoll, 1999) that lead to more robust mathematical understanding. In our prior work, we have often illustrated the alignment of the FPU with the recommendations of the "Adding It Up" report (National Research Council, 2001), the Principles and Standards for School Mathematics (NCTM, 2000), and the National Mathematics Advisory Panel report (2008). Given that the CCSS (2010) build upon and in many ways subsume the aforementioned documents, it is not surprising that FPU is also well-aligned with the CCSS. The CCSS Standards for Mathematical Practice, for instance, emphasize that students are to make sense of problems and persevere in solving them:

> Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and **goals.** They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. ... Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches. (CCSS, 2010; from the Introduction, emphasis added)

Likewise, instruction that employs the FPU encourages students to identify the goal of the algebraic procedure (question 1a), identify or estimate the expected answer (question 1b), seek and compare the relative efficiency of alternative procedures or strategies to complete each task (questions 2b and 5), and verify one's solution (question 4). Given national efforts to implement the recommendations of the CCSS, therefore, a focus on FPU (or any effort to establish the impact of CCSS-recommended practices) is particularly timely.

Characterizing Instruction

In a series of studies, we have trained teachers to utilize questioning techniques, classroom activities, and a wide range of assessments that incorporate the FPU, and we have evaluated its impact on student learning. In these studies, we observed statistically significant gains favoring students in the "treatment" condition. Here, we will provide just a brief overview of the results and refer the interested reader to the cited papers if more details are required. In our first study, we found that college algebra students scored significantly higher on a test of procedural understanding versus a matched comparison group (Hasenbank, 2006; Hasenbank and Hodgson, 2007). In a second study, we focused on high school algebra classrooms in southwestern Wisconsin and observed significantly greater gains in procedural skill versus a matched comparison group over a one year period (Hasenbank & Kosiak, 2008; 2010).

One outgrowth of these efforts to assess the impact of FPU-related practices is the need to establish fidelity of implementation. In our second research study, for instance, we observed that teachers implemented the FPU in different ways and to differing degrees. To quantify these differences, we employed a digital classroom observation tool (described in greater detail below) which provided concrete evidence of the variability we observed in teacher practice. The graphs in Figure 2, for instance, illustrate the contrast between two algebra lessons presented by two different teachers. The vertical scale records a timestamp, while the horizontal categories represent the instructional focus at that moment in time. The eight categories represent the eight dimensions of the Framework (note that the labels {1a, 1b, 2a, 2c, 3, 4, 5, 6} have been mapped to {1, 2, 3, ... 8}).

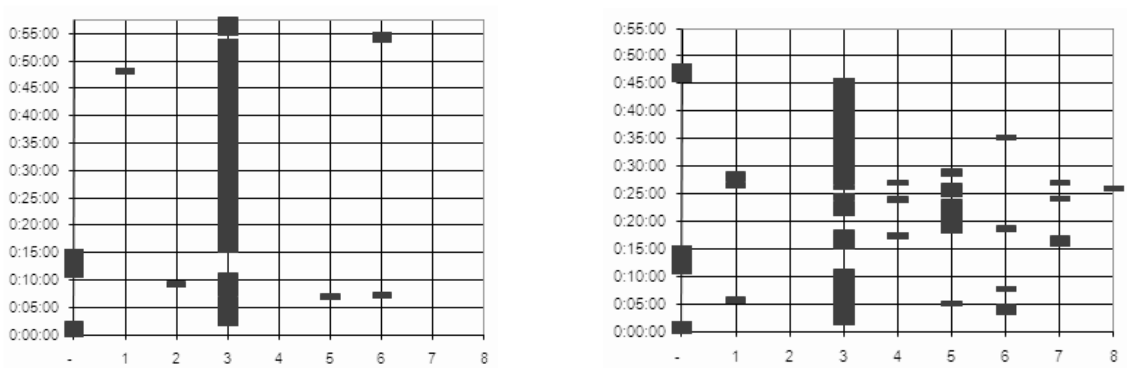

Figure 2. Lesson Progression for Two Different Classroom Episodes

It is clear that both teachers placed a significant emphasis (85% and 61% of class time, respectively) on category 3 (performing the procedure), but the teacher on the right also integrated other aspects of the FPU into the lesson. For instance, she spent nearly 6 minutes, or about 12% of the lesson, on discussions categorized as ‘category 5 - why the procedure works.’ We also categorized another 9% of her lesson on other FPU categories, with discussing the merits of various procedures and questioning the reasonableness of the results making up the next two largest components (4% and 3% of the lesson, respectively). This variability in classroom practice among teachers who were explicitly trained to implement FPU-oriented instruction leads naturally to the questions driving our on-going research agenda. Specifically, “To what extent do algebra teachers already focus on FPU-oriented principles in their daily lessons?” and “Is there a relationship between the instructional focus within an algebra classroom and student performance/achievement?”

Digital Classroom Observation Protocol

While there were a number of written observation protocols in existence that captured information similar to the digital tool we set out to create, none captured and summarized the moment-by-moment focus of instruction on understanding in real time in a format that permitted direct comparisons between observations. Many existing observation protocols (e.g., the Praxis III: Classroom Performance Assessments, ETS, 1993; the Reformed Teaching Observation Protocol, Arizona State University, 2000; or the Inside the Classroom Observation and Analytic Protocol, Horizon Research, 2000) take the form of a written observation protocol that provides space for noting the time, recording comments, and selecting a category or rating that reflects the nature of the entire classroom episode.

Our digital observation instrument improves on those protocols by recording a timestamp whenever the observer selects a different primary or secondary code or records a typed comment. For the purposes of our research, we use the eight dimensions of the FPU (plus “conceptual” and “other”) as our primary codes (see Figure 3). During an observation, the observer switches between the appropriate code categories and enters qualitative descriptions where desired. By recording a timestamp whenever a change is made, the instrument captures the researchers’ perception of the lesson progression over time.

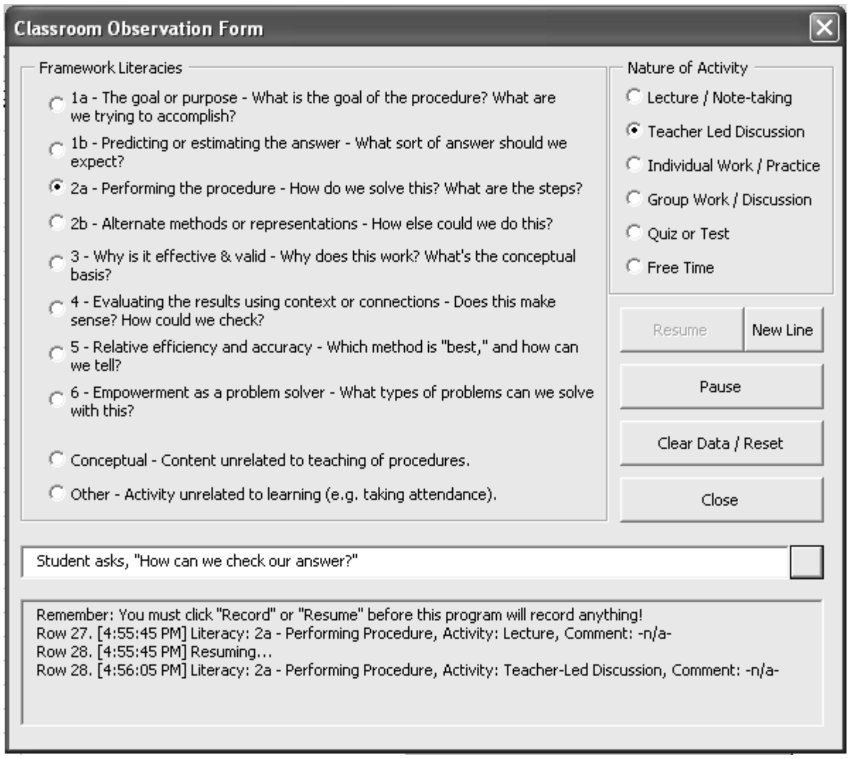

Figure 3. User Interface for the Digital Classroom Observation Form

The observation instrument also generates two visual displays of the lesson progression (see Figure 4). On the left, we see a graph of the chronological progression of the lesson showing how the categories (horizontal axis) shift over time (vertical axis). On the right is a simple bar graph showing the relative frequency distribution for the primary categories. This digital observational protocol allows researchers to characterize the extent to which instruction focuses on procedural understanding and each component of the FPU.

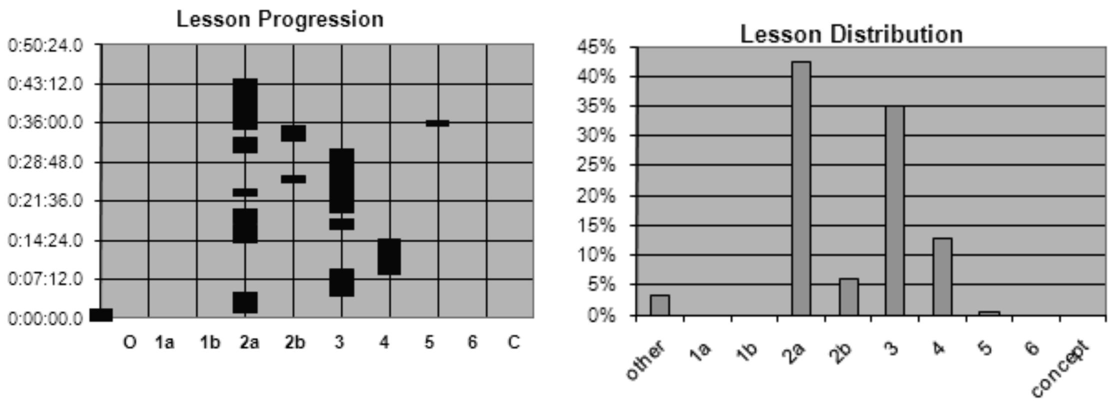

Figure 4. Visual Displays of Lesson Progression

Inter-rater Reliability

Two independent observations may be merged to allow for both visual and numerical assessment of inter-rater reliability. The software discretizes the data by sampling at fixed time intervals (e.g., one sample every five seconds) and generates a table reflecting the pair-wise associations of the primary codes assigned by the two observers (Figure 5). If the observers were in agreement at each discretized moment, all entries would appear on the main diagonal of the table. Summing the entries along the main diagonal gives the percentage of discretized intervals on which the researchers' codes agreed (72% in the example shown). Examining the off-diagonal elements allows us to identify the most common areas of disagreement (e.g., code 5 with code 3 occurred 7% of the time in our example), which can be cross-referenced with the chronological progression plot and the observers' comments for clarification. The code-pairs are used to compute Cohen's *kappa*, which provides a statistical representation of the extent of the observers' agreement beyond that expected by chance alone (Agresti, 1990).

	-1	0	1	2	3	4	5	6	7	8	9	Row:
-1	-	-	-	-	-	-	-	-	-	-	-	0%
0	-	20%	-	-	2%	-	-	-	-	-	-	22%
1	-	-	1%	-	3%	-	-	-	-	-	-	3%
2	-	-	-	-	-	-	-	-	-	-	-	0%
3	-	2%	-	-	31%	-	3%	2%	3%	-	-	41%
4	-	-	-	-	1%	1%	-	-	1%	-	-	3%
5	-	-	-	-	7%	2%	18%	1%	-	1%	-	29%
6	-	-	-	-	-	-	-	1%	-	-	-	1%
7	-	-	-	-	2%	-	-	-	-	-	-	2%
8	-	-	-	-	-	-	-	-	-	-	-	0%
9	-	-	-	-	-	-	-	-	-	-	-	0%
Col:	0%	22%	1%	0%	45%	3%	20%	4%	4%	1%	0%	

Figure 5. Code-Pairs

Results on the Nature of Algebra Instruction

Our observations of the teachers participating in our professional development program can be viewed as a snapshot into the nature of algebra instruction in Wisconsin. These teachers were trained to cultivate classroom norms that promote understanding using the FPU as a guide. In that study, five algebra teacher participants from rural and suburban Wisconsin were observed a total of 21 times using the digital observation instrument. The combined results are displayed in table 2 below. We note teachers devoted an average of nearly 37 minutes per hour of instruction-time on procedural skill (item 2a of the FPU), with a range from as little as 32 minutes up to about 48 minutes per hour of instructional time. By comparison, these same teachers spent an average of only 12 minutes on the non-routine aspects of the FPU, with times ranging from 6 to 20 minutes per hour of instruction-time.

We believe these results demonstrate the utility of our observation protocol; we are able to quantify the focus of instruction in a manner that allows for rigorous statistical analysis. It also affords us with a critical tool for answering the question: how much FPU-oriented focus is enough?

Category	Mean (Median) in Minutes Per Hour	S.D. (Range)	Min & Max
Procedural Skill	36.9 (34.8)	6.28 (16.0)	31.8 to 47.8
Procedural Understanding	12.4 (13.1)	5.13 (13.5)	6.3 to 19.8
Non-Framework categories	10.7 (11.6)	4.78 (10.5)	5.7 to 16.1

Table 2. Distribution of Time across FPU Categories

Discussion and Future Directions

In summary, our digital observation tool allows researchers to characterize the nature of instruction with a degree of precision not previously available. It should be noted that the categories used for observation are easily customized, so the tool is easily adapted to a wide variety of research areas where a predefined set of observable categories of behavior are available. Our own future plans include observational studies with high school, middle grades, and college algebra sites, through which we will attempt to correlate individual teachers' emphasis on FPU with student achievement. With the introduction and widespread adoption of the CCSS, the importance of understanding the variables associated with procedural understanding has never been higher. If indeed malleable factors such as teachers' emphasis on procedural understanding tasks are confirmed to be significant predictors of student achievement, we will have identified key focal points for professional development of teachers. We are hopeful that the data we gather will afford us a better understanding of how—and to what extent—teachers institute the mathematical practices outlined in the CCSS and as operationalized by our Framework for Procedural Understanding.

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MATHEMATICS IN CONTEXT: COGNITIVE ANALYSIS

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A research is reported in which we carry out an analysis of the students cognitive process, whom are asked to solve a contextualized event on systems of linear algebraic equations in the context of matter balance in situations of chemical mixtures. The cognitive analysis is founded on the theories of Conceptual Fields and Mathematics in the Sciences Context. The analysis is focused on the representations carried out by students regarding the invariants in the schemes that they build. During the students acting process emerge different types of representation which are appropriate to the context in which the research develops.

Introduction

The need for a comprehensive training on the student, has led the Red Internacional de Investigación en Matemática en el Contexto de las Ciencias (MaCoCiencias) to perform research of cognitive type that revolves, among others, around the analysis of the cognitive processes of students for the knowledge construction of various sciences who are linked. Addressing science events such as a chemical or physical system or a natural phenomena, where mathematics should be applied for their solution, is a complex task; because it is necessary to integrate the knowledge and carrying out activities of analysis for knowledge which is both explicit and implicit in the event to be addressed (Camarena, 2000: 27-32), playing an important role in the representations which can be formulated about the events to be dealt with, in order to promote the construction of knowledge. Moreira (2002, p. 37) assumes that we do not apprehend the world directly, but we do it from the representations of the world that we build in our minds.

Learning mathematics, among others, implies that the learner is able to use them to solve specific events in his area of professional and labor job (Camarena, 2000: p. 46) Thus in the knowledge construction of various related sciences, it is important to know from the cognitive point of view what happens to students when they work with contextualized mathematics.

Research Problem

There are many actions that favor the knowledge construction on students, among them stands the learning and teaching activities with contextualized mathematics. The research that is carried out regarding the cognitive process of students provides a guideline for the design and redesign of these didactic activities in students at university and technical programs. The research reported makes use of the conceptual fields by Vergnaud for the cognitive analysis.

Thus, the research problem deals with the issues of how the cognitive processes of students is carry out when they are faced with contextualized events of Mathematics in Context, from the perspective conceptual fields by Vergnaud. The *research objective* seeks to analyze the cognitive process of a students group by means of analyzing the representations they make about the invariants in the schemes they build when they address the learning activities associated with contextualized events where systems of algebraic equations are linked to balance of matter; that is to say the events being addressed deal with mixing chemical solutions.

Theoretical Frameworks

Mathematics in the Sciences Context

The theory of Mathematics in the Sciences Context (Camarena, 2009: 14-25, 2000: 1-70) has been developed since 1982 through the research at Instituto Politécnico Nacional of Mexico. The theory of Mathematics in the Sciences Context reflects upon the relationship that should exist between mathematics and the sciences that require it, between mathematics and everyday life situations as well as its relationship to future professional and labor activities.

The educational philosophic assumption of this theory is that the student is trained to transfer mathematics knowledge to the areas which require it, so that they develop competences for their working and professional life, because it aims at contributing to the comprehensive training of the student and to build mathematics for life (Camarena, 2009: 14-25, 2000: 1-70).

The theory of Mathematics in the Sciences Context deals with the problem of learning and teaching of mathematics at university level programs where mathematics is not a goal in itself but a tool to support sciences and a subject that develops the students thinking skills. To this end, the theory conceives the process of learning and teaching as a system with five phases: curricular, cognitive, didactic, epistemological and teachers training, moreover, social, economical, political and human relations aspects are included in the learning environment (Camarena, 2000: p. 12). All the phases are necessary to ensure the completion of the philosophical assumption posed, moreover, all stages are interrelated among them, and none of them is unrelated to the others. As a theory, in each of the phases a theoretical methodology is included, in accordance with the paradigms upon which it is based, which serve as a guide for the steps of curricular design, the didactics to be followed is described, the cognitive functioning of students is explained and epistemological elements are provided on mathematical knowledge relating to the activities of professionals, among others.

The cognitive analysis addressed in this research directly affects the cognitive phase of the theory, where a contextualized mathematical concept acquires sense by means of the activities of the context itself, because the concepts are not isolated, they are constituted as a network and they bear a relationships among them (Camarena, 2000: p. 54). Therefore, for the cognitive analysis it is important to set the contextualized events and define from them the learning activities that lead to the knowledge construction, both of the concepts of each science involved in the event and the linkages between them, stages 4 and 9 of the didactic strategy of Mathematics in Context, shown in the following paragraph, a situation that is consistent with the theory of Conceptual Fields by Vergnaud. Thus, the didactic phase is the means by which cognitive processes are achieved; the didactic phase possesses a teaching strategy that supports the development of competencies in students within the learning environment, which is called Mathematics in Context (Camarena, 2000: 1-70).

With Mathematics in Context the student works with contextualized mathematics in the areas of knowledge of his future professional subject matter, with daily life activities and with labor and professional activities, all that through contextualized events, which can be problems or projects. In general speaking of Mathematics in Context is to develop the mathematical theory to the needs and rhythms that dictate the courses of engineering. Mathematics in Context includes 9 stages, which are developed in the learning environment in teams of three students: academic leaders, emotional leader, work leader.

Stages of Mathematics in Context: 1.- Identify contextualized events to develop the competencies being questioned. 2. Establish the event of the context. 3. Identify the variables and constants of the event. 4. Include the mathematical topics and concepts necessary for development of the mathematical model and its solution. 5. Determine the mathematical model.

6. Give the mathematical solution of the event. 7. Determine the solution required by the event in the context disciplines. 8. Interpret the solution in the event terms and the context disciplines. 9. Present the descontextualized mathematics in the classroom.

Mathematics in Context acts as didactic strategy and allow following a methodical process to contextualize mathematics through stages 2, 3, 5, 6, 7 and 8, thereby linking mathematics with other sciences of professional majors and technical programs being treated. The type of contextualized event to be chosen should have historical background, that is, it must have been previously worked upon by a teachers group in order to identify the type of components of the competencies that come into play as well as the type-questions that the students made at the time to address the events, among others. The success of the contextualized event in order for it to develop skills in students has to do with the adequate selection and with the guidance of the teacher at the time the students are solving the event.

Conceptual Field.

The theory of conceptual Fields deals with the formation of mathematical concepts from a psychological and didactic approach, which leads us to consider the learning of a concept as the set of "problem situations", which constitute the reference of their different properties and the set of schemes brought into action by the subjects involved in those problem situations (Vergnaud, 1996: p.145). The sense of the mathematical concept is acquired through the schemes evocated by the individual subject to solve a problem situation.

Vergnaud (1991: p.147) defines Conceptual Fields as "a set of problem situations, concepts, invariants, schemes and operations of thought that are related to each other for a specific area of knowledge". The theory of Conceptual Fields allows the cognitive analysis in the problem situations proposed to the students through the analysis of the conceptual difficulties, the repertoire of available procedures and possible forms of representation.

From the perspective of the theory of Conceptual Fields understanding of the student's action relative to the concepts involved and the structure of systems of linear algebraic equations in the context of a phenomenon of balance of matter, focuses on studying some aspects of the operations of thought which make the invariants in the schemes constructed by students, which directly or indirectly impact on knowledge about this mathematical structure being immersed in a problem situation where the linking of two contexts take place: mathematics and chemistry.

For representations which one of the thinking operations, Vergnaud mentions that the students transform an action on the mathematical object, establishing control by himself through the relationship and classifications in his reality, in such way that invariants arise in the knowledge development, in this way the representations for Vergnaud are tools, whichever notation or sign or set of symbols.

To finish with this section it is important to mention that the "problem situations" referred to in the theoretical framework by Vergnaud correspond to the "contextualized events" and the "learning activities" proposed by the theory of Mathematics in the Sciences Context.

Method

The methodology to carry out the students cognitive process analysis form the conceptual content of systems of linear algebraic equations in the context of matter balance involves the following three blocks: Contextualization of systems of linear algebraic equations in the matter balance. Determination of the learning activities to be applied to the students group. Analysis of the students cognitive process through the thinking operations they make upon the invariants in the schemes they construct. The last one is the results and their discussion.

The Sample

Being a qualitative analysis, we worked with a group of two students in the first quarter of the Technician in Food Technology program, who are currently enrolled in a mathematics course that includes the topic of systems of linear algebraic equations as well as a chemistry course which addresses the topic of balance of matter by mixing chemical solutions, both courses are not related in a curricular way. The activities are carried out by students in the chemistry laboratory, in different sessions, covering twelve hours.

Observation Instruments

The data collection from the cognitive process is made by obtaining written and film productions that help to refute or confirm the analysis conducted with the written information. The analysis is qualitative focusing on the thinking operations that they perform on the invariant patterns that build in their cognitive processes in the activities of contextualized event.

Development of Research

First block, contextualized event students are to face is contextualized, which is a phenomenon that recurrently appears in specific operations in the area of professional and labor training of the technician in food. It *says:* We have *100 ml of sugar solution to 60% and 100 ml of sugar solution to 35%. From these solutions it is desired to obtain 100 ml of a sugar solution to 50%.*

The contextualization stages described in the theoretical framework, which are immersed in the didactic strategy of Mathematics in Context, is the methodological process that is used for the systems of linear algebraic equations contextualized in the matter balance, generally in contextualized events of mixing of solutions.

In the didactic phase of Mathematics in the Sciences Context, contextualization is set by the teacher prior to implementing the didactic strategy of Mathematics in Context, because this contextualizing provides him with elements for the learning activities design for 4 and 9 strategy stages, similarly, to take time, see the necessities of the cognitive infrastructure and consider the possible paths of solution. Also, it allows determining the mathematical concepts and of the context (chemistry) that are present in the event, establishing the relationship between concepts that belong two different areas of knowledge and observing the close relationship between them. In terms of Vergnaud, the concepts and invariants to which the student must converge for the construction of knowledge are identified. In relation to mathematics: The concept is “systems of linear algebraic equations” and the invariants are mathematical concepts such as algebraic equation, linear algebraic equation, systems of equations and solution methods. In relation to the context: The concept is matter balance and the invariants are inherent concepts such as concentrations management, visualization of the phenomenon as a system with inputs and outputs, mixing substances to obtain a desired concentration and concentration measurements. The schemes and operations of the invariants mediate the action over reality, as well as the organization forms and structuring of the different concepts of interest and criteria for meanings acquisition.

From the contextualization of the first methodological block derives the activities to be determined in the ***second block,*** which are shown in table 1.

The learning activities objective is that the student analyzes the linear behavior by means of the obtaining of concentrations and volumes combinations, each one separately, to later on confront him with the idea of managing the two variables simultaneously, and the pursuit of the mathematical model which represents the context event, see table 1.

CONTEXTUALIZED EVENT	
We have 100 ml of sugar solution to 60% and 100 ml of sugar solution to 35%. From these solutions it is desired to obtain 100 ml of a sugar solution to 50%.	
LEARNING ACTIVITIES	**PURPOSES**
Activity 1. To mix the two chemical solutions in order to obtain as a result 100 ml of solution, do not consider the concentration of the solution only the volume.	To mix chemical solutions, from their volume.
Activity 2. To mix the two chemical solutions in order to obtain as a result a solution with a sugar concentration to 50% (do not consider the volume of the solution only the concentration).	To mix chemical solutions, from their concentration.
Activity 3. With the given solutions, 100 ml to 35% and 100 ml to 60% carry out a mixture in order to obtain 100 ml of a new solution with a concentration to 50%.	To mix solutions considering both volume and concentration.

Table 1. Contextualized event and its learning activities

As it has been mentioned, the conceptual field is constituted by problem situations (contextualized events and learning activities), concepts, invariants, schemes and thinking operations. Now it can be established conceptual field of systems of linear algebraic equations in the context of matter balance.

Results and Discussion

Third block, the behavior of students is analyzed through the operations of the invariants in the schemes they build. It must be pointed out that during activities from the contextualization of systems of linear algebraic equations with matter balance, it was detected in the students actions the recurrent use of different operations of thought, which were considered proper of the linking of two specific areas of knowledge, that is to say, they may or may not arise in other circumstances. These thought operations relate specifically to representations, for it was necessary to define a classification of the types of representations found, which is expressed below.

Propositional type representation: it is considered as the description of the event the made by the student, in the first instance, in his own language (natural language), from the cognitive point of view it is of great use to the student to understand the activity to be performed. He can use some isolated concepts of mathematics, the chemical science or other sciences, without articulating them. The overall purpose is to communicate and try to understand the event to be solved.

Non-operating figurative type representation: it is when the student perceives the information of the event but if asked to represent the situation in written form, he draws pictures, wanting to see the link between the two areas of knowledge; however, it does not allow him for a procedure of solution. When the students group understands the activities en their language they proceed to make some context area drawings (figures). However, do not shift from these figures to a written expression that enables the solution of the given event, that is to say, they still fail in linking the two areas of knowledge.

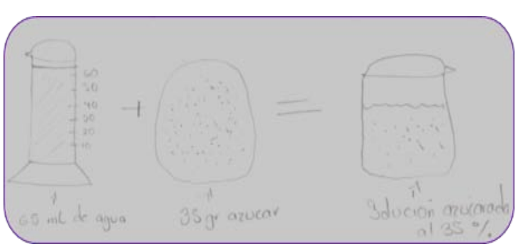
Figurative operating type representation: the student continues to make drawings but he already takes into account the numerical information, the drawing can therefore, serve as a support for the solution of the event without algorithms. Again, the experiment presented in this document, the students recurrently makes schemes (figures) representing the matter balance of the proposed activities; which allow them on the one hand to understand the variables and constants involved, and on the other hand they serve to give way to the symbolic representation that allows the event resolution. Then, at this point begins the comprehension of the link between the two knowledge areas.

Analogous type representation: the student retains the relevant information to solve the event and simplifies the information by using symbols, points, crosses, use of images. He uses the experience of having solved previous events to solve the new one, at least partially, by a very primitive process and of very local scope, that is to say, the similar events processes are considered to verify if they are helpful in solving the new event. The analogous representation can be used after the propositional and even after the non-operating figurative one and it is when the student refers to processes that appear similar to the one he faces and that can be useful for event understanding and resolution. His search for similar events allows him to support the emerging understanding of the previous representation.

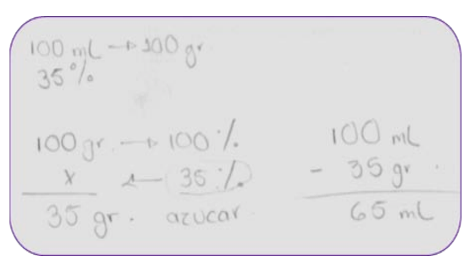
Symbolic type representation: it has been considered that these representation constitutes a means to identify more clearly the mathematical objects for the conceptualization of them and the conceptualization of context. The student translates the sentence of an event to a mathematical representation, arithmetic, algebraic, analytic or graphic, this representation is considered as an expert and allows providing the required response. It is considered that knowledge is obtained that can be used in different contexts. The students group who have gone through the previous representation, makes use of a symbolic representation (it may be graphic, arithmetic or algebraic), it permits them to integrate both mathematical end context knowledge to provide an appropriate solution to the event posed. This representation, in the case of research, has been considered the one that allows the proficiency of the interest concepts.

In general terms, it is observed that the representations different types of the invariants play an important role in the contextualized event resolution and the learning activities that have been posed to the students. The approach to learning activities was characterized by because they commuted by different types of representation to until reaching a symbolic representation and obtaining a satisfactory result. In the acting of the students it was detected that when the group addresses the learning activities and builds knowledge, the representation types, found during the research and considered as being proper to the linking of two sciences, are developed gradually at the same pace as the conceptualization of systems of linear algebraic equations in the context of mixing chemical substances. Additionally it enables the student to develop skills to solve new contextualized events that they may be posed.

As it has been mentioned, the students group moves through the representations different types until reaching the symbolic one, which enables the students to construct their own knowledge about the study phenomenon that links mathematical concepts and sciences. That is why, the generation of symbolic representation was discussed in great depth, identifying three related processes, which other types of representation are present. Interpretation and selection process, Structuring process and Operationalization process.



Figurative operating representation (drawing).



Symbolic representation (arithmetic)

Figure 1. Process of selection and interpretation.

Interpretation and selection process, the context is considered as a fundamental part of the event, from the contextualized event the students group performs a selection of information that seems relevant, in it the available prior knowledge is considered and students translate the information into a representation type, such as algebraic (linear equations), arithmetic (simple rule of three), tabular (tables of data) or graphics, even in a figurative operating representation, see Figure 1.

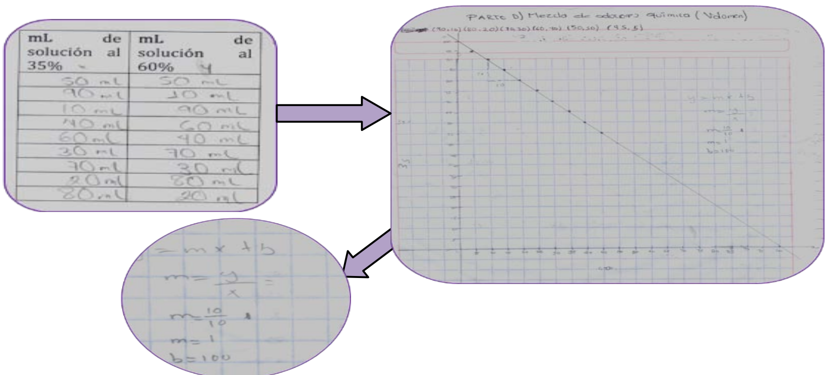


Figure 2. Structuring process.

Structuring process, in this one intervene again the previous knowledge, we resort to analogies with events which have been previously solved and which appear to be similar to the pose done; it seems that the previously solved events are stored in the memory and constitute a part of the knowledge brought into play by the students group, similarly, the procedures and strategies which were followed undergo a restructuration and advance progressively as attempts to solve the event are made. See figure 2.

Operationalization process, it takes place when representations are manipulated and the students group attains the solution of the posed contextualized event. It is identified that during this process the students group applies the operating knowledge that come from their experience and with that the group formulates procedures or strategies. In this process the previous knowledge of the students group plays an important role as they allow modeling the event and working with the mathematical model by symbolic modes of representation that are more operational than the propositional, see figure 3.

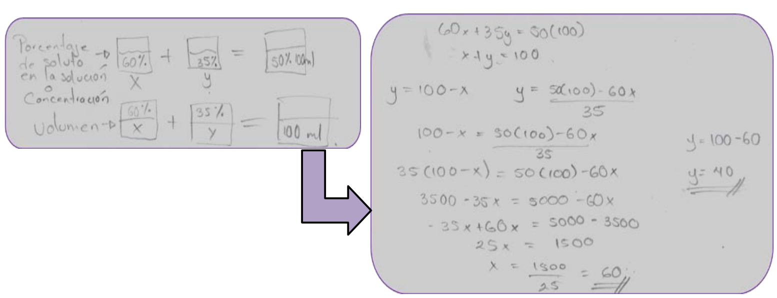


Figure 3. Operationalization process.

Conclusions

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

It can be observed from the research that it has been sought to contribute to the understanding of the knowledge construction of the student in a conceptual content derived from the link between two different contexts, mathematics and technical area of the working field of a Senior Technical in Food Technology. The shown analysis has been addressed paying attention to the action of the student in the solution of contextualized events and teaching activities with systems of linear algebraic equations in the context of matter balance, which constitute a means of analysis upon which the knowledge construction is described. It is a cognitive type study which uses the Conceptual Fields by Vergnaud as a theoretical framework, initially developed to carry out research in elementary education and that has been retaken to explain a contextualized phenomenon at Senior Technical level.

During the analysis of the activities by the students group it was necessary to define the types of representations of the invariants in the schemes, which emerged in a spontaneous way during the students performance. It is truth that the students, at the end of the sessions, have acquired appropriate schemes to face contextualized events that require systems of linear algebraic situations, but, it is necessary to further explore into it, above all in different contexts.

Another important element to be mentioned is that if more diverse contexts are used and more contextualized events are addressed by the student, he could be constructing his knowledge in a more everlasting way and will be able to carry out the transference of mathematical knowledge to other sciences.

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SECONDARY MATHEMATICS TEACHERS' CONCEPTIONS OF WORTHWHILE TASKS

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With a growing emphasis on task based learning, it is important to examine what constitutes worthwhile mathematics tasks. In this study we examined secondary mathematics teachers' conceptions regarding what constitutes a worthwhile mathematics task. The 27 teachers in this study participated in both focus group and individual interviews. Teachers' conceptions were compared and contrasted with conceptions about worthwhile tasks found in mathematics education literature. Many of the teachers' conceptions aligned with definitions of tasks found in the literature, but we identified additional nuances and pragmatic conceptions. Teachers' conceptions were analyzed and classified. These classifications characterize distinctions among the uses of mathematical tasks that may inform future research.

In his 1994 essay titled *What we Know About Mathematics Curricula*, Schoenfeld explained the benefits of problem solving in mathematics classrooms over the traditional learning of facts and theorems stating, "Large amounts of mathematics can be learned as sensible answers to sensible questions—that is, as part of mathematical sense making, rather than by mastery of bits and pieces of knowledge" (p. 59). This idea of problem or task based learning continues to gain momentum in the United States. National Science Foundation (NSF) funded curricula such as *Connected Mathematics* and the *Interactive Mathematics Program* (IMP) are examples of curricula that build mathematical understanding through the use of mathematical tasks. The National Council of Teachers of Mathematics' (NCTM) recent document *Focus in High School Mathematics: Reasoning and Sense Making* (2009) promotes curricula focused on student reasoning and problem solving, suggesting curricula centered on rich mathematics tasks can help to support this goal. Similarly, the Common Core State Standards' Mathematical Practices' (2010) foci on mathematical problem solving and reasoning seem to suggest that rich mathematical tasks should play a central role in today's mathematics classrooms.

Researchers classify tasks in a variety of ways. Tasks may be sorted by cognitive demand, purpose, or even whether or not they can be considered worthwhile. Though a review of the literature related to tasks revealed a great number of studies examining tasks, teachers' perspectives on tasks were lacking. In this study we sought to examine teachers' conceptions of tasks, more specifically, we sought to understand what teachers consider a worthwhile task.

Prior to a discussion of worthwhile tasks, we must first define the word task as used in this study. We have chosen to use Stein and Smith's (1998) definition of a task, "A segment of classroom activity that is devoted to the development of a particular mathematical idea" (p. 276). With this definition a task may be a single problem or could be a week long project. The following section discusses ways researchers have chosen to classify such tasks and the idea of what precisely constitutes a worthwhile task.

Tasks in the Literature

The literature is rife with discussions of mathematical tasks (e.g. Hiebert et al., 1997; Henningsen & Stein, 1997; NCTM, 2007; Stein & Smith, 1998; Swan, 2008). These studies focus on a variety of aspects of tasks such as enacting tasks (e.g., Stein, Smith, Henningsen, & Silver, 2009) and using tasks for assessment. There have also been several efforts to classify tasks by various criteria. We discuss several of these classifications in the following paragraphs.

Swan (2008) classified tasks according to the processes that they help the learner develop mathematical understandings. The five classifications of task types correspond to the processes of classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analyzing reasoning and solutions. This focus on the task outcomes departs from the approach of classifying tasks by the cognitive demands of the task, an approach taken by several researchers.

Stein and Smith (1998) classified tasks by cognitive demand. The classification system consists of four categories of tasks, two related to high cognitive demand and two related to low cognitive demand. Stein and Smith did not perceive of these categories as hierarchical in nature. The use of a variety of tasks from different levels of cognitive demand is supported by the following claim,

> (The authors do not) suggest that all tasks used by a teacher should engage students in cognitively demanding activity, since there may be some occasions on which a teacher might have other goals for a particular lesson, goals that would be better served by a different kind of task. (Stein, et al., 2009, p.5)

This view allows for the flexibility of choosing tasks of varying cognitive demand depending on the desired learning goals; however, recent research has found that students in classrooms that enacted high cognitive demand tasks performed better on assessments that measured problem-solving and mathematical reasoning (AERA, 2006).

In addition to classifying tasks by the aforementioned attributes, there have been efforts to classify tasks by their worth. Hiebert et al. (1997) discussed appropriate tasks. The authors characterized appropriate tasks as those able to be understood by students but also problematize mathematics for the students, and engage students in thinking about important mathematical concepts. These three aspects are also evident in NCTM’s notion of worthwhile tasks.

NCTM’s *Principles for Teaching Mathematics* (1991) discussed yet another classification of tasks, those considered worthwhile. We have used this concept of worthwhile tasks as our conceptual framework for examining teachers’ conceptions. NCTM characterized worthwhile tasks as those that facilitate student interest and intellect, develop students’ mathematical skills, create connections among mathematical ideas, promote problem solving and communication, and take into account students’ backgrounds and dispositions. More recently, NCTM (2007) summarized these attributes stating, “Worthwhile mathematical tasks are those that do not separate mathematical thinking from mathematical concepts or skills, that capture students’ curiosity, and that invite students to speculate and to pursue their hunches” (p.33). While NCTM advocated the use of such tasks, they also acknowledge that teaching with such tasks requires a great deal of planning and knowledge.

Each of these descriptions of tasks in the literature help to build an understanding of tasks and their various uses in the classroom; however, teachers’ voices regarding tasks are noticeably absent from the literature. This study’s purpose is to illustrate the similarities and differences in teachers’ conceptions of mathematical tasks. We also sought to extend the comparison of the teachers’ conceptions with perspectives from the literature on mathematical tasks.

Methods

This study is part of a larger study in which we examined teachers' conceptions of an integrated mathematics curriculum. The participants were 27 secondary mathematics teachers in the first year of state standards implementation. The teachers were identified through referrals from colleagues and recruitment at local professional development workshops. The new standards departed from the traditional secondary mathematics course sequence (algebra, geometry, algebra 2, etc.). The new courses contained algebra, geometry, and data analysis units within a single course. This organization of the content led some people to refer to the courses as integrated. The new standards also led to changes in teaching practice. The state developed mathematical frameworks to reflect its goal of reform oriented teaching practice. These frameworks contain collections of mathematical tasks grouped by curricular units. These tasks were meant to bring together mathematical processes, especially mathematical connections. This new focus was a dramatic change from the prior, more traditional state standards.

The research team conducted 6 focus groups consisting of 4-6 participants for a total of 27 teachers from 9 school districts in northeast Georgia. The purpose of the focus groups was to examine participants' experiences with the implementation of the new state standards. After partially transcribing the focus group data and coding for emerging themes, we purposefully selected 9 participants for in-depth interviews in order to obtain maximum variation (Patton, 2002). These interviews allowed us to gather more information about topics that arose in the focus groups. The interviews were fully transcribed and analyzed using the constant comparative method decoupled from grounded theory (Glaser & Strauss, 1967). A minimum of two researchers participated in each of the rounds of coding to provide internal validity.

Results

To motivate results regarding teachers' conceptions of worthwhile mathematical tasks, it becomes necessary to first consider how our participants described and defined mathematical tasks. During focus groups and in-depth interviews, members of our research team asked participants to draw from their experiences teaching the new standards and to describe a mathematics lesson they considered to be particularly successful with their students. These discussions typically included examples of mathematical tasks that teachers found to be worthwhile for their students. When the idea of task was introduced in the interview setting, often times, the researcher probed for the participant's meaning. Interestingly, one teacher seemed to define a mathematical task by detailing what a task was not:

> If there was only one way to solve a problem, nothing open-ended about it, like word problems in Algebra I. They're not Tasks …There's nothing to engage in, there's nothing to investigate. It doesn't generate any good conversation, and there aren't multiple paths to the same solution.

The negation of this teacher's description seems to imply her definition of a mathematical task. More commonly, teachers described tasks as mathematical problems with multiple paths to a particular solution, allowing students to engage in mathematical processes.

Drawing from participants' definitions, the authors analyzed specific portions of the transcripts when teachers' discussions related to mathematical tasks. We inductively coded the transcripts and developed emergent themes within participants' discussions of worthwhile tasks. In our final stage of analysis, we organized the themes into three broad categories: themes related to the mathematical content, mathematical processes, and practical considerations.

Themes Related to Content

This first category of emergent themes is related to the mathematical content developed within the tasks. Often, teachers talked about the importance of embedding the mathematics within a context, and teachers appreciated tasks that promoted students' construction of mathematics. Both of these themes refer to the ways in which the content should be incorporated into a worthwhile task as well as how the content is presented to students.

Embeds the Mathematics in a Context

During the in-depth interviews, teachers often provided concrete examples of tasks where the mathematical content was situated within an interesting context. For example, several teachers favorably referenced a mathematical task that develops the various points of concurrency. In this task, the mathematics develops within the context of finding an appropriate location for a cell phone tower. Students are asked to determine where to place the cell phone tower in order to provide maximum coverage for three cities (Georgia Department of Education, 2008).

Teachers explained that incorporating a meaningful context was a particularly important characteristic of a worthwhile mathematical task for several reasons. These types of tasks provided their students with opportunities to connect their mathematical knowledge to what teachers called the "real world" and "real world thinking". The context surrounding the mathematics motivated students to engage in and learn essential mathematical content in the task. As one of our teachers explained, "A good task has … good mathematics in it. And…it's a context that kids would be interested in." Many teachers discussed the context as a tool to allow the mathematics to become more relevant for students. The context not only provided motivation for students but also encouraged a deep exploration and construction of mathematical ideas.

Promotes Students' Construction of Mathematics

Within teachers' descriptions of worthwhile tasks were commendations for mathematical tasks that allowed students to construct their own mathematical understandings. Some of the tasks within Georgia's new mathematics framework allowed students to think independently about the mathematics, to struggle with the mathematics, and to construct their own mathematics. For example, one teacher explained, "I think the advantage is that [the task] certainly gives the kids an opportunity to discover things for themselves and lets them be more independent and be more independent learners." A majority of the teachers in this study shared this notion of allowing students to discover mathematics through the tasks and make mathematics their own.

The teachers contrasted worthwhile tasks with tasks that were inundated with leading questions. In general, teachers frowned upon leading questions, stating that the questions did not permit students to freely explore the mathematical ideas posed by the task. Teachers instead favored asking appropriate questions to help students "figure out what is going on." One teacher stated, "They're (students) discovering that you're not telling them… They're actually figuring out what the theorem says without you just coming out and telling them the information." These quotations emphasize the teachers' expectations that worthwhile mathematical tasks should encourage students to explore and investigate the mathematical ideas presented by the task.

Themes Related to Mathematical Processes

The second category of themes relates to the mathematical processes involved as students engage with the mathematical tasks. These processes align to several of NCTM's (2000) Process Standards, namely problem solving, communication, and connections.

Fosters Problem Solving

Worthwhile mathematical tasks provide students with opportunities to engage in problem solving (NCTM, 2000). Therefore, it is not surprising that teachers commonly described worthwhile mathematical tasks as "multi-layered problems" through which students can engage in "real world thinking." These kinds of tasks allow students to use a variety of approaches in order to solve the problem. One teacher said, "I like problems [tasks] in which you can, I guess, when you really have to slow down and think about all different angles." Teachers explained how worthwhile tasks require students to think deeply about the mathematics, because "the biggest difference within those tasks is that all of the sudden they [students] are expected to think on their own and process." One teacher described worthwhile tasks as prompting metacognitive strategies, "I think the task gets them to think about thinking more and to kind of rack their brain… how would I figure that out?" Many times, teachers' conceptions of worthwhile mathematical tasks were located within the intersection of tasks that foster problem solving and tasks that allow students to communicate mathematically, because they claimed mathematical communication aided students in problem solving.

Encourages Mathematical Communication

The participating teachers believed that worthwhile mathematical tasks should be a vehicle to engage students in mathematical communication. In describing the characteristics of tasks that enable such communication, one teacher said, "The task part of the class can focus on the open-ended questions and the open-ended discussion type of thinking that needs to go along with that." Because teachers often considered worthwhile tasks to include multiple paths to the same solution, they expected students to discuss the various approaches used to develop the mathematical concepts within the task. Students were encouraged or expected to work on tasks in groups to develop this mathematical discourse and a depth of understanding. One teacher reported asking students, "Can you tell each other how you did it?" stating that this strategy fostered communication across students. She also commented that worthwhile tasks "just seemed to provide a lot of that conversation." When describing the difference between the new standards and their old method of teaching, the richness of classroom communication was a frequently and positively discussed byproduct of the new standards. Teachers appreciated how students' communication detailing the multiple solution paths within a worthwhile task also generated rich mathematical connections across these conversations.

Provides Opportunities to Make Connections

Teachers' discussions of worthwhile mathematical tasks included the importance of affording students multiple opportunities to make rich mathematical connections. Teachers often equated mathematical learning to making connections. One teacher stated, "I mean to me that is mathematical learning, when I can look at four groups presenting four different paths to the same solution and I can see how your geometric representation is the same thing as what I did algebraically." Making comparisons across the representations help students to develop and recognize mathematical relationships. These connections, facilitated by the tasks, enhanced their students' conceptual understanding.

> If they [students] did it algebraic(ally), geometric(ally), and then another way to look at it. Being able to have that task and summarizing it at the end. You know now, okay, here's little picture, little picture, now let's look at a big picture.

The connections made by students when working through a worthwhile task help students develop a unified picture of mathematics.

Themes Related to Practical Considerations

Teachers frequently focused their discussion on more practical considerations of worthwhile tasks. These practical considerations perhaps arose from the dramatic change in practice that was demanded from implementing task based curriculum. Although the impetus for the discussion of these issues may be a byproduct of the teachers being in the midst of this implementation, these practical considerations were addressed by each of the teachers.

Achievable

Teachers explained worthwhile tasks should be accessible to all students, and tasks need to be achievable in a variety of ways. First, a worthwhile task must take into account students' mathematical backgrounds, allowing them to progress through the task successfully. More importantly, teachers shared the need for students to recognize that they had the ability to work though the task with success. One teacher said, "A good task would be achievable, meaning that the students really think that they can do it." Second, many of the tasks provided by the new mathematics framework were rather lengthy. So long, in fact, these tasks seemed impossible to complete, leading teachers to proclaim that the length of the tasks needed to be achievable as well. Echoing this concern, a teacher shared, "Achievable. They can't be 25 pages long and when the kid looks at it, say, 'Oh, this is impossible.'" Finally, all of our participating teachers expressed a desire for all of their students to experience success in this new task based environment. In order to achieve success, several teachers emphasized that worthwhile tasks need to allow for differentiation among students. One teacher remarked, "A good task has to be accessible to all students at some level. I believe that strongly. What do they call it… like multiple entry and exit points for a task, everybody should be able to do something."

Standards Based

The teachers wanted their students to be successful on the end of course exams administered by the state; thus, teachers characterized worthwhile mathematical tasks as tasks that focused on the content standards students were required to master. One teacher stated, "Good ones (tasks) are focused on the standards that we are trying to teach." Another teacher addressed her concern for the state standards stating,

> I have certain standards that I'm supposed to teach, and then the tasks need to reflect those standards. You can't just pick it [tasks] because it's a fun problem to engage it. I mean by law I have to teach certain standards, you know, so I do have to choose task that will…that I can use to pull out specific mathematical ideas.

The majority of teachers echoed this concern, and this concern influenced their selection of tasks. The notion of standards based, achievable tasks encompassed the practical considerations important to our teachers' conceptions of worthwhile tasks.

Discussion

After conducting our analysis, we returned to the literature on worthwhile tasks. Typically, there exists a gap between literature in mathematics education and practice. As researchers, we wondered to what extent were teachers' conceptions of worthwhile mathematical tasks reflected in the literature. The juxtaposition of the participating teachers' conceptions with classifications provided by the literature in mathematics education regarding worthwhile mathematical tasks proved to be rather illuminating, providing implications for research and practice.

We used NCTM's (1991) classification of defining characteristics of worthwhile tasks to make our comparison. We created a Venn diagram and labeled one set as Classifications from Literature and the other as Conceptions from Practice (see Figure 1). As seen in the intersection of the two sets, many of the classifications and conceptions overlap. We believe several similarities exist between research and practice as well as some interesting differences for further discussion and analysis. For example, when reviewing the similarities across sets, we noticed that NCTM suggests worthwhile tasks facilitate student interest whereas our participating teachers appreciated embedding a task within a context to make the mathematics relevant to students. As is evident in Figure 1, several additional similarities exist; yet, two important differences emerge through this comparison. First, one of NCTM's classifications identified worthwhile tasks as tasks that develop students' mathematical skills. However, our participating teachers saw the development of skills as a completely separate activity from tasks in the classroom. Second, several participating teachers emphasized the necessity for tasks to focus on the content standards they are expected to teach from the state's curriculum frameworks in order for the task to be worthwhile, whereas this characteristic is not located within NCTM's classifications. This difference seems to capture the important and practical realities of teaching. As mathematics educators, we must consider possible implications these findings have for both preservice teacher education as well as professional development.

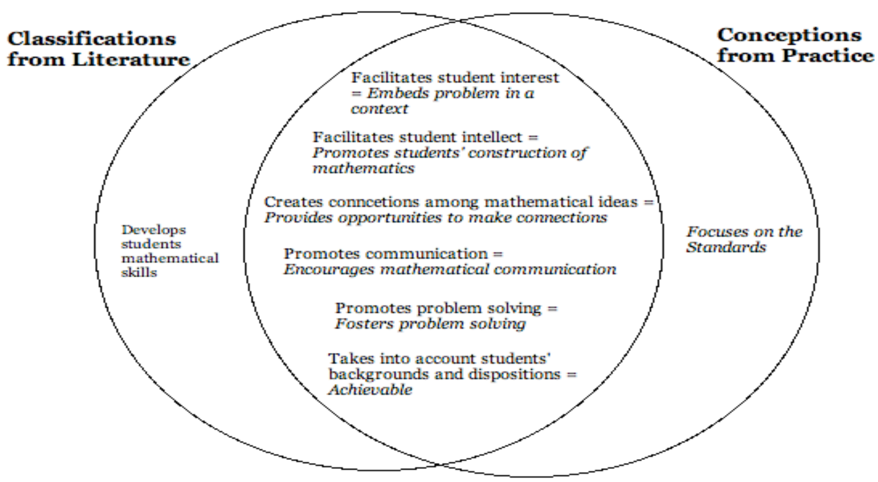

Figure 1: Worthwhile tasks as classified in research and conceived of by teachers

In the Teaching Principle, NCTM (2000) concludes their discussion of worthwhile tasks by stating, "Worthwhile tasks alone are not sufficient for effective teaching" (p. 19). Similarly, our research findings do not provide a conclusion regarding worthwhile tasks, but, rather, a beginning. Many of our participants discussed the difficulty of either finding or developing these worthwhile mathematical tasks. One participant shared,

> An appropriate task or word problem, which to the maximum extent included multiple disciplines and gave as many chances as possible to make the learning relevant … Those are really hard to write. If you find one, you hang on to it, forever.

It is important for both professional development and teacher education programs to support teachers in developing such worthwhile tasks. This study provides an important initial step toward the inclusion of teachers in discussions of worthwhile tasks. Further research is needed to understand how teachers develop and modify tasks to make them worthwhile. As research in mathematics education continues to study worthwhile mathematics tasks, research is needed to inform practice. This statement assumes that research is relevant to practice. Therefore, as researchers, in order for us to make research more relevant to practice, we believe carefully listening to teachers describe their practice is an excellent place to start.

Endnotes

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TEXTBOOK IMPLEMENTATION IN RURAL SECONDARY INTEGRATED MATHEMATICS CLASSROOMS

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Although professional development can help teachers become familiar with new curricular materials, implementing the materials can be difficult for teachers. This paper presents results from a study of teachers' implementation of an integrated mathematics curriculum in rural secondary schools following their participation in a professional development. Results show teachers are primarily using the textbook for their instruction, but that they implement less of the textbook content in higher level mathematics and struggle to find a balance implementing the content across content strands. These results have important implications for researchers conducting curricular evaluations and for the creation of effective professional development.

Introduction

It is important that claims made about curricular effectiveness include measures to judge the adequacy of implementation (National Research Council (NRC), 2004; Senk & Thompson, 2003). The NRC further noted the importance of documenting the faithfulness of implementation and recommended researchers document the extent of coverage or "implementation fidelity" of the curricular materials. Implementation fidelity measures the extent to which textbook materials are used for instruction and are not indicative of the quality of teaching (McNaught, Tarr, & Grouws, 2008; NRC, 2004).

Reform efforts in mathematics have lead to the creation of curricular materials that focus on strengthening the mathematical knowledge of all students and are guided by instructional practices that promoted problem solving, communication, reasoning, and creating mathematical connections (Senk & Thompson, 2003). However, implementing reform mathematics curricula represents a challenging transition for many teachers (Ziebarth, 2003). Teachers' beliefs and backgrounds greatly influence how they implement curricula, potentially causing the implemented curricula to be significantly different from the developers' intended curricula (Ball & Cohen, 1996; Remillard, 2000) and from a teacher's own intentions (Stein, Remillard, & Smith, 2007; Stein & Smith, 2010). Therefore, it cannot be assumed that access to teachers' guides and curriculum materials will ensure proper implementation (Scott, 1994), nor that teachers within the same school will implement curricula materials in the same way (Bowzer, 2008; Chávez, 2003; Stein, et al., 2007).

The purpose of this paper is to report findings on teachers' implementation of a reform mathematics textbook in rural high schools following their participation in a professional development. This research is an important step towards determining the extent to which the textbook impacted student achievement and the extent to which the textbook is used for instruction.

Theoretical Perspective

The Comparing Options in Secondary Mathematics: Investigating Curriculum (COSMIC) research team was designed to evaluate high school students' mathematics learning from different curricular programs (COSMIC, 2005). They have provided methodological approaches

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and instruments to document and measure implementation fidelity. The COSMIC team has made great strides in the field of curricular evaluation and, in particular, in creating innovative ways to measure implementation fidelity. They have created indices for *opportunity to learn* (OTL), *extent of textbook implementation* (ETI), and *textbook content taught* (TCT).

OTL, ETI, TCT Implementation Indices

The COSMIC team measured the OTL, ETI, and TCT through Table of Contents Records (TOC-logs), which were self reported by the teachers and customized for the textbook they were using (McNaught, et al., 2008). For each lesson of the textbook, teachers indicated if they taught the content (a) primarily from the textbook, (b) primarily from the textbook with some supplementation, (c) primarily from an alternative source, or (d) not at all (McNaught, et al., 2008). The OTL index measured the percentage of textbook content that was taught, either solely from the textbook or through supplemental materials. The ETI index weighted the options in the TOC-logs, giving the content taught primarily from the textbook the most weight (a weight of one), content with some supplementation a weight of two-thirds, content mostly from alternative sources a weight one-third, and content not taught a weight of zero. The weights were then summed and divided by the total number of lessons contained in the textbook (for additional information: McNaught, Tarr, & Sears, 2010; McNaught, et al., 2008). This measured the degree to which the textbook was used directly to teach the content. Similarly, the TCT index used the same weighted sum but divided by the total number of lessons taught through any means. This is a measure of how the textbook was used to teach content in the textbook and ignores the topics students were not taught. Each of these three indices was measured at the course level.

McNaught et al. (2010) analyzed data from 174 TOC-logs collected from teachers using both an integrated and subject specific curriculum. Across the teachers using integrated materials, they found the mean OTL index was 60.81 with a standard deviation of 19.91, the mean ETI was 50.73, with a standard deviation of 20.20, and the mean TCT was 81.96 with standard deviation 14.50. While teachers using an integrated curriculum are covering just over 60% of the textbook content, they rarely use alternative sources for their instruction.

Context

The North Carolina Integrated Mathematics Project

The North Carolina Integrated Mathematics Project (NCIM) was developed to create and support a community of teachers using the *Core-Plus Mathematics* (CPMP) (Coxford, et al., 2001) integrated curriculum materials, particularly in high needs schools. Spread throughout rural parts of the state, the seven partner schools in the NCIM project were identified as low-performing based on North Carolina accountability measures. At the time of this study, the average student population for the project schools ranged between 110 to 163 students and the ethnic make-up consisted of: 1% American Indian or Asian, 6% Hispanic, 16% White, and 77% Black. Approximately 72.8% of students at each school qualified for free and reduced lunch.

To prepare teachers to implement CPMP, in order to strengthen STEM education at these schools, the project directors and evaluation team designed four components for the NCIM professional development. The four main elements of the professional development model were (1) a summer workshop providing in-depth education on use of curricular materials (one or two weeks), (2) a web-based environment supporting information exchange, (3) two face-to-face follow-up conferences, and (4) instructional coaches who visited each site monthly.

Methods

Participants

The NCIM project has grown since its inception in the spring of 2008 to include new teachers and has lost two due to teacher turnover. At the time of this study, 12 teachers in the NCIM professional development received instructional coach visits each month and took part in the other components of the project. Each of these 12 teachers began teaching CPMP for the first time based on their schools' participation in the NCIM project.

Data Collection

To measure the implementation fidelity of teachers using the CPMP materials, NCIM project teachers completed TOC-logs (McNaught, et al., 2008) for each course they taught. These logs were replicas of the table of contents in the CPMP textbooks so that teachers readily recalled the material they taught. These were provided to teachers at the beginning of the semester, and teachers returned portions of the logs throughout the school year.

Analysis

This analysis was based on the 15 TOC-logs that were completed during the 2008-2009 and 2009-2010 school years. OTL, TCT, and ETI indices were computed using the COSMIC approach (McNaught, et al., 2008). It is important to note that teachers completed one TOC-log for each unique course they taught. For example if they taught two sections of Course 2 and one section of Course 3, then they would have one measure of OTL, TCT, and ETI for Course 2, with both sections having the same measures, and one measure of each for Course 3.

Due to the sample size, to determine quantitative differences in teachers' implementation across courses and content strands in the textbook, the Kruskal-Wallis distribution free test (Hollander & Wolf, 1999) was used. The Jonckeere post-hoc test (Hollander & Wolfe, 1999) was used to test for ordered alternatives to see if there was a decrease in the indices across the course sequence or content strands. Finally, Dunn's Test (Elliott & Hynan, 2010), a nonparametric multiple comparisons test based on pair-wise rankings, was used to determine differences among specific groups.

Results

Opportunity to Learn

The mean OTL index for NCIM teachers was 52.18 with a standard deviation of 16.15 (Figure 1). These data indicate on average just over half of the content in the CPMP textbooks was covered, though there was considerable variation among teachers' OTL indices.

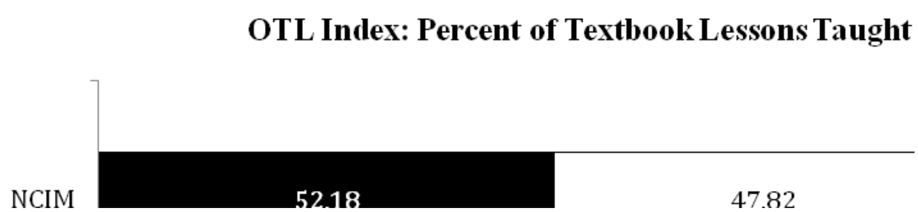

Figure 1. Summary of mean OTL indices for NCIM project teachers.
Note. The heavy shading indicates content taught primarily from textbook, with some supplements, or primarily from an alternative source. White space indicates content not taught.

Extent of Textbook Implementation

Recall the ETI index is a weighted measure describing the degree that the textbook, rather than other materials, was used to teach the content. The NCIM teachers had a mean ETI index of 49.30, with standard deviation 16.41. It is also helpful to look at the ETI index disaggregated by the weighted options: content taught primarily from the textbook, with some supplementation, primarily from alternative sources, or not at all (Figure 2). From reviewing the figure, it appears the NCIM project teachers rarely supplement the textbook material.

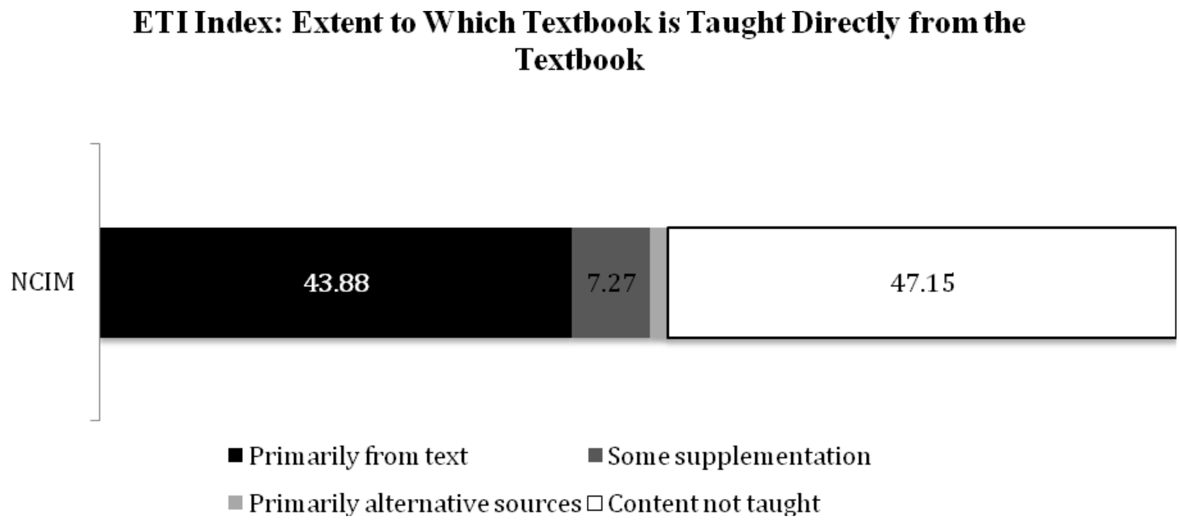

Figure 2. Summary of ETI indices for NCIM project teachers.

Textbook Content Taught

Recall that the TCT index restricts the focus to only the CPMP content that was taught. The mean TCT index for NCIM teachers was 91.64 with standard deviation 7.52. The differences in the TCT indices became more prevalent when considering the TCT indices for content taught primarily from the textbook, with supplementation, and from alternative sources (Figure 3). This suggests that NCIM teachers primarily used the textbook for their instruction and very rarely used alternative sources.

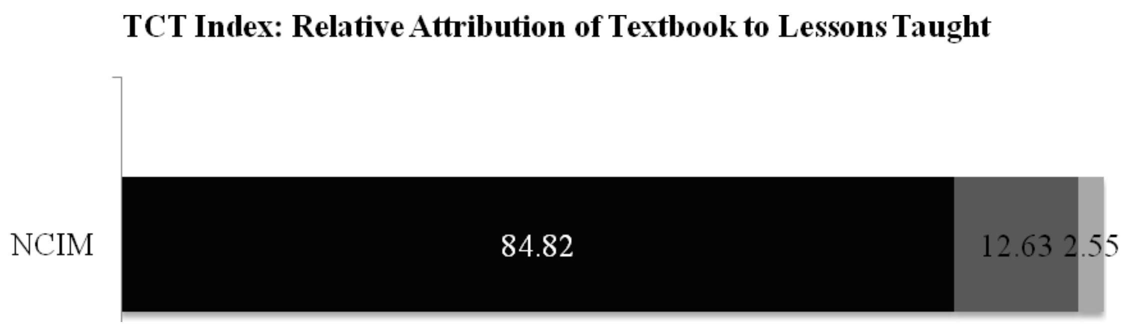

Figure 3. Summary of TCT index for NCIM project teachers

Across Courses

The schools involved in the NCIM project began as ninth grade academies and added an additional grade level each year until they had their first senior class. There was only one mathematics teacher at each school until the second or third year the school was open. As a result it is interesting to track teachers implementation of the curriculum as they progress through the course sequence as instructors. The mean for each implementation index across the first three integrated courses is given in Table 1. There was a significant decrease for the OTL $(p = 0.008)$ and ETI $(p = 0.03)$ indices with each subsequent course that was taught. Thus, as students continued through the integrated sequence, they had less opportunity to learn the material in the text. The TCT is still high, above 85% for each course. From this we can conclude that when NCIM teachers taught content in the textbook, they were primarily using the textbook and not supplemental materials. These findings also suggest that teachers of Courses 2 and 3 need continuous support throughout their implementation of CPMP.

Table 1. Average Implementation Fidelity Ratings Across Courses Taught by NCIM Teachers

	OTL	ETI	TCT
Course 1	63.21	60.12	94.92
Course 2	55.03	52.68	87.89
Course 3	43.38	39.22	91.89

Similar findings can be seen in the implementation indices of individual teachers, Nicole, Maria, and Felicity, as they transitioned to teaching higher sequenced courses (Table 2). It is interesting to point out that Nicole's OTL and ETI indices dropped more than 20 points each from Spring 2009 to Fall 2009 though she was teaching the same course. She completed the Course 2 workshop during the summer of 2008 and the Course 3 workshop during the summer of 2009. Perhaps her indices were higher the semester following her participation in the Course 2 workshop. Another plausible explanation is that the first Course 2 class was a yearlong course (two semesters), while the course she taught in the fall of 2009 was a 90-minute block course (one semester). Having the material spread throughout an entire school year provided her with more time to get through the material.

Table 2. Individual NCIM Teacher's Implementation Ratings Over Time

Teacher	**Year**	**Course**	**OTL**	**ETI**	**TCT**
Nicole	Fall 08	Course 1	74.29	68.57	92.31
Nicole	2008-2009*	Course 2	66.15	64.10	96.90
Nicole	Fall 09	Course 2	44.62	41.03	91.95
Maria	Fall 08	Course 2	65.00	75.38	75.38
Maria	Fall 09	Course 3	36.76	33.82	92.00

Maria	Spring 2010	Course 4	26.09	26.09	100.00
Felicity	Fall 09	Course 1	48.57	43.33	89.22
Felicity	Spring 2010	Course 2	38.46	31.79	82.67

* This was a yearlong course.

It is evident from these results that NCIM teachers struggled to teach the content in the textbook as they progress through the sequence of courses.

Content Strands

In her dissertation, McNaught (2009) identified which units in CPMP corresponded to four different content strands, to determine the differences in implementation across major topics. The four strands were: Algebra and Function, Probability and Statistics, Geometry and Trigonometry, and Discrete Mathematics. Recall the fourth component of our professional development model included visits from an instructional coach. Instructional coaches completed reports following each classroom visits, and these reports highlighted that our project teachers were skipping units of the textbook. To determine if there were differences in the topics taught by NCIM teachers, an approach similar to McNaught's was utilized. Using the table of contents in the second edition of the textbook, each unit of each course was aligned to one (sometimes two) of the content strands. The OTL, ETI, and TCT indices were recalculated for each teacher for each of the four content strands (Table 3).

Table 3. Average Implementation Indices Across Content Strands

Content Strand	OTL	ETI	TCT
Algebra	84.17	76.83	91.20
Statistics	38.98	37.84	82.49
Geometry	37.50	36.94	77.43
Discrete Math	16.37	15.77	26.19

The nonparametric Kruskal-Wallis distribution free test (Hollander & Wolf, 1999) was used to determine if there were differences in the implementation indices across the four content strands. There were significant differences in textbook use across teachers and content strands for each of the indices, OTL $(p < .0001)$, ETI $(p < .0001)$, and TCT $(p < .01)$.

The Jonckeere post-hoc test (Hollander & Wolfe, 1999) was used to test for ordered alternatives to see if there was a decrease in the indices across the ordered strands: Algebra and Function, Probability and Statistics, Geometry and Trigonometry, and Discrete Mathematics. Results show evidence that all three indices decrease along the four content strands, OTL $(p < 0.05)$, ETI $(p < 0.0001)$, and TCT $(p < 0.05)$. This would imply that the average OTL for the Algebra and Function content strand is greater than the other three, yet the OTL for the Probability and Statistics strand is greater than the Geometry and Discrete Mathematics OTL.

Dunn's Test, a nonparametric multiple comparisons test based on pair-wise rankings, was used to determine which content strands differ in location, when $\alpha = 0.05$. The OTL for Algebra $(\bar{x} = 84.17)$ was significantly different than each of the other content strands; Statistics $(\bar{x} = 39.98)$, Geometry $(\bar{x} = 37.50)$, and Discrete $(\bar{x} = 16.38)$.

Using the ETI index, there were significant differences between the Algebra $(\bar{x} = 44.18)$ strand and all other strands; Statistics $(\bar{x} = 27.21)$, Geometry $(\bar{x} = 26.86)$, and Discrete $(\bar{x} = 15.75)$. These results were favorable to Algebra, suggesting students were exposed to more Algebra content than the other strands. The results from the TCT index showed differences between Discrete Math $(\bar{x} = 16.5)$ and both Statistics $(\bar{x} = 34.79)$ and Geometry $(\bar{x} = 33.54)$. The TCT for Algebra $(\bar{x} = 29.19)$ was not significantly different than any other content strand. These findings suggest students had more of an opportunity to learn Algebra than the other content areas. The Algebra topics are currently covered on state assessments and seem to be the main focus of teachers' instruction. It is evident from workshop evaluations that many of the statistics and discrete math topics are unfamiliar to these teachers.

Taken together these findings point to the wide variance in implementation across the four main content strands in the textbook. This also provides empirical evidence to suggest teachers were skipping major units in the textbook. These data need to be further triangulated with interviews from the teachers and the instructional coach reports to determine *why* or *how* they chose to skip sections of the textbook.

Discussion/Conclusions

Teachers in the NCIM project used the content from the textbook and primarily taught the content from the textbook, however this does not imply their instruction was consistent with the written text. Next steps in this research will be to conduct classroom observations to understand *how* the content is being taught, to conduct interviews to determine how teachers make decisions about what content to teach, and using these indices in measures of student achievement. Only after fully understanding how teachers implement curriculum can we begin to measure the impact on student achievement. The research conducted in the paper provides evidence that just over half of the content in the textbook is being taught, with the major focus on the Algebra units. There is also indication that teachers need on-going support in their first three years of implementation and as they move to teaching higher courses in the CPMP sequence.

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ANALYSIS OF THE MOTIVATION AND ACHIEVEMENT OF SECONDARY STUDENTS ENROLLED IN A SUMMER MATHEMATICS ACADEMY

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Achievement and motivational data was collected from 465 students enrolled in a Summer Mathematics Academy designed to provide enrichment and accelerated courses for secondary students. Academy courses were designed taking into account motivational variables associated with the development of intrinsic motivation, a mastery goal orientation, and personal choice and challenging tasks. Results show that Academy participants scored at an extremely high level on the Terranova and PSAT assessments. Motivation was assessed using Personal Constructs Theory. Results are consistent with earlier work that emphasizes Interest and Challenge/Control tradeoffs in determining intrinsic motivation in students.

Summer Mathematics Programs and Student Motivation

This paper examines the academic achievement and motivation of secondary-age students enrolled in a summer credit advancement program in mathematics. In contrast to compulsory academic-year coursework, there is practically no research conducted about specialized summer programs in mathematics. In particular, these programs' effects on academic achievement and motivation have not been extensively studied even though many schools are now implementing summer programs to in hopes of boosting academic success. This paper will focus on credit-advancement programs and not necessarily remedial or enrichment-only programs. Being that there has been little research done on these credit advancement programs, other academic programs that have been researched have been used to help make some of the theoretical and empirical arguments in this paper.

Many studies have found that summer vacation influences a direct decline in academic achievement (Cooper & Nye et. al., 1996, Entwistle & Alexander, 1992). In fact, summer vacation accounts for about a month's loss of academic learning or about a tenth of a standard deviation on students' academic achievement (Cooper & Nye et. al., 1996). Not surprisingly, the effects of summer are more damaging on mathematics scores than other subject areas tested. Moreover, the summer months may have a particularly negative effect on gifted students who become bored and then underachieve the subsequent school year (Li et. al, 2009).

A small but growing number of studies show that bridging students' academic year experiences through appropriate acceleration and/or enrichment programs in the summer can increase students' mathematical understanding and their skills (Tichenor & Plavchan, 2010). These studies show that an important part of a camp's design should be to provide a learning-enriched atmosphere, allow for risk-taking, and a demonstrated passion for mathematics. With this type of design for the summer camps, the students have shown to appreciate, enjoy, and learn math thereby increasing their own confidence in math.

Some would argue that because of the pacing required of these summer programs, lasting retention would not be evident, and thus no real overall increase in academic achievement would persist. In fact, the impact of summer programs tend to be long lasting (Barnett, 1993), involving increased knowledge and achievement in mathematics subject matter (Knapp, 2007), and in an increase in the probability of taking advanced mathematics courses in the future. (Li et. al, 2009).

Motivation Related Variables

Part of the puzzle when designing summer programs is to build in appropriate motivational appeal. Particularly for acceleration and other non-compulsory programs, students and their parents have choice in enrolling and persisting. The challenge of the program, its utility and the kinds of interests it fosters can be critical for its continued success. So it becomes important to examine the reasons students have for enrolling in off-calendar offerings, and the degree to which the offerings fit their goals and values. Students who are intrinsically motivated, for example, will show resiliency towards the given task even though the activity may be difficult (Gottfried, 1983; Schunk, 1990) without the need of any type of reward to begin or finish the task (Beck, 1978; Deci 1975; Woolfolk, 1990). Additionally, intrinsically motivated students will be more likely to retain the concepts learned and gain confidence about engaging in unfamiliar learning activities (Poonam, 1997). Of course, a student who is assigned a task in which they have little interest value is less likely to exhibit excitement and persistence than a task that strikes curiosity and interest.

Overall, research shows that a higher level of intrinsic motivation should lead to retention of concepts, higher confidence, and thereby, higher academic achievement. Significant and positive correlations have been found between intrinsic motivation and achievement as measured by standardized achievement tests in specific subjects, such as mathematics and reading for early-elementary, late-elementary, and junior high school students (Adelman, 1978; Gottfried, 1985, 1990). It would seem likely then that academic intrinsic motivation would also be positively related to achievement in these specialized summer mathematics programs at the high school level.

The remainder of this manuscript will describe the design of a summer acceleration program for middle- and high-school students in the greater Phoenix Arizona area, and analyze the achievement and motivational profiles of students enrolled in the program.

Method

Summer Math Academy

The Summer Math Academy was offered at a large, urban high school by its mathematics faculty. At the time of the research, the summer program was in its tenth year running with three years of previous achievement data collected. We focus only on these three years from 2007 to 2010. Prior to 2006, the Academy only offered Algebra courses. In 2007 that the Academy began to offer Pre Algebra and a limited number of Geometry courses to advance students through these basic courses in high school. The few Geometry students in 2007 were organized in a more guided, but self-taught environment. These students were in the same room as one of the Algebra classes. In 2008, the Math Academy decided to offer 6th grade and 7th grade mathematics as well as more Geometry courses. Starting in 2008, the Geometry courses were not organized as independent studies but had their own classroom with a geometry-specific teacher.

Many of the students who sign up for the Summer Math Academy report preparing themselves for tough courses and academic goals. A good portion of the students are preparing to take Advanced Placement Calculus and Advanced Placement Statistics which are required for many college majors. By 2010 it became possible through the academy, for students to take AP Calculus I, AP Calculus II, AP Vector Calculus, AP Differential Equations, and in the near future, possibly Linear Algebra. In addition, many students were preparing to exceed the

Arizona Instrument to Measure Standards mathematics tests to qualify for college scholarships, be competitive in the future job market, and complete the state's four year high school mathematics requirement.

For middle schoolers, the curriculum in the algebra and honors geometry courses sufficed to receive high school math credit. Overall, there was a wide variety of students as the youngest enrolled were about to enter into the fourth grade while the oldest were going into the tenth grade.

There are a total of 28 days of instruction in the Summer Math Academy each year. The first day begins four days following the end of the regular school year. In the three years this study focused on, the Academy ran from Monday through Thursday with every Friday off for a total of seven weeks. The mathematics program would start at 7:30 a.m. and finish at 11:30 a.m. each day. The students had a five-minute break every hour.

Participants

In total, there were 465 students that had attended the Academy from the first three years while a total of 444 students were present the summer of 2010. The breakdown of the number of students enrolled in each course by year can be found in Table 1. Note that some students enrolled in the summer Academy for more than one year and thus the table does not reflect unique students. However, the 465 total is the number of unique students that completed the Academy. Just about every year in the camp's history, students were denied enrollment because there were a fixed number of teachers. This was to ensure that no classroom had too many students in it.

	6th Grade Math	7th Grade Math	Pre Algebra	Algebra	Geometry	Total
2007	-	-	62	59	7	128
2008	22	17	35	71	12	157
2009	27	43	45	110	64	289
Total	49	60	142	240	83	465

Table 1. The number of participants by year in each course at the math camp

Design of Camp Experiences

Since intrinsic motivation generally increases student engagement and achievement the Summer Math Academy attempted to provide students choices of topics to choose from (Patall et. al, 2008). We attempted to provide contextualization, personalization, and problem choice to the extent possible in each course (e.g., Cordova & Lepper, 1996). Moreover, students in most classes could choose their own pace when working individually (Patall et. al, 2008).

Students are also more likely to perceive successful outcomes when they are in environments that promote mastery motivation (Gottfried, 1983; Weiner, 1979). We felt that helping students create mastery goals and fulfilling them through applied effort would intrinsically motivate students (Gottfried, 1983; Harter, 1974, 1977). Because intrinsically motivating tasks are challenging, the academy encouraged students to choose advanced mathematics courses, in order to push them to the extent of their abilities. But, there needed to be a safe environment that encouraged students to fail positively. We build this in, for example, by allowing students to attempt the tasks multiple times to approach mastery. The flexibility of the Academy's camp environment also allowed students to monitor themselves, go beyond the presented content and discover deeper connections between concepts,. All of those aspects have been shown to influence motivation positively.

Achievement Measures

The students took a pre-test and a post-test to help determine the effect of the Academy camp on their academic achievement. The Arizona Instrument to Measure Standards Grade 8 test scores were collected as students took this test prior to entering the Math Academy. As a post-test, scores on the Terra Nova and PSAT were collected as most students took these after entering the Academy. Any student who took the PSAT and/or Terra Nova before entering the math academy had their scores excluded from the data.

Measures of Motivation: Repertory Grid Techniques

This study used Repertory Grid techniques to determine the content and structure of students' motivations (e.g., Middleton, 1995). Motivational constructs for the algebra and geometry courses were elicited separately as these were distinct classes that students took—there was no overlap in students taking the courses—and because the content of those courses was hypothesized to motivate students differently.

Construct elicitation

A Motivational Repertory Grid was developed for the Algebra course and Geometry course. For each course, ten different major algebraic or geometric topics emphasized in the course were chosen. Students were presented with 45 (${}_{10}C_2/2$) pairs of these topics in random order. They were asked to report what made one topic more preferable than the other topic. Students reasons for choosing one topic over the other are considered their *personal constructs* related to motivation. Students' personal constructs were tabulated and rank ordered. The top ten identified constructs were placed onto rows on a Repertory Grid, and the 10 topics used to elicit constructs were arranged on the columns of the Grid. Students were then asked to rate, on a scale of 1 to 10, the degree to which each *construct* described each *topic.* The final grid for each student then, was a 10 by 10 grid of ratings with each construct being assessed across each topic—for a total of 100 ratings.

Students' ratings were summed and the mean score in each cell computed. These mean scores were then used to create a Group Repertory Grid, representing an average motivational set for the group of students in each course. Table 2 lists the key topics used to elicit constructs, and the motivational constructs obtained, from algebra and geometry courses respectively.

Geometry Topics	Algebra Topics	Motivation Factors
Perpendicular Bisector and Theorems	Ratios and Proportions	It is more Fun to Learn
Converse/Inverse/Contrapositive	Finding the Equation of a Line	It is more Interesting
Parrallel Lines with Transversal make Angles Congruent or Supplemental	Order of Operations (PEMDAS)	It is Easier to Learn
Addition/Subtraction/Multiplication/ Division Properties	Multiplying Polynomials (FOIL)	It is more Challenging
Writing Two-Column Proofs	Graphing Linear Equations	I can Visualize the Solutions
Perpendicular Line to a Plane	Factoring Polynomials	I get to Decide How to do Problem
Similar Triangles	Finding Slope	I can Apply it to Real Life
Congruent Triangles	Inequalities (> <)	I can Memorize How to do It
Properties of Polygons	Word Problems	It Makes Sense to Me
Angles of Triangles and other Polygons (Exterior angle, Interior Angles)	Solving Systems of Linear Equations (Substitution/Elimination)	It Helps Me Understand more Concepts in Math

Table 2. The course topics and motivation factors found in the Motivation Repertory Grid

Results

Analysis of Achievement Scores

To analyze the effect of mathematics academic achievement, scores from the AIMS, Terranova and PSAT standardized tests were collected and analyzed. SThe Terra Nova is taken in the spring of the ninth grade and the PSAT is typically taken the fall of their junior year.

The rank of the students who attended math camp on the standardized tests are compared. It would seem reasonable that within this small group, the ranking of the students should stay consistent from year to year i.e. the top/low ranked students should stay in the top/low ranking. Then the percentiles on the tests of these students are compared to the remainder of the population to see if there is an overall jump for the math Academy students as a group in the mathematics achievement scores.

Table 3. *Students who went to camp multiple times and their scores on PSAT and TerraNova*

	Number of TN students	Average TN Percentile	Number of PSAT students	Average PSAT Percentile
All Student's Average	179	81.62	97	78.20
Camp Twice	13	81.08	9	79
Camp	166	81.70	88	78.20

Once				

Table 3. Scores of students attending multiple times on PSAT and TerraNova

It is unclear from our analyses, whether or not the Summer Academy greatly enhanced students' achievement. The extremely high average performance across the board on the Terranova, and the PSAT constitute a kind of ceiling effect. Moreover, because only a handful of students had taken the Summer Academy more than once, any group differences are not detectable statistically.

Cluster Analysis of Repertory Grids

Group Repertory Grids were subjected to hierarchical cluster analysis (Wards Method), using a squared-Euclidean distance metric. This analysis arranges the constructs in the grid in a 10-dimensional space, using a linear combination of the topics to determine their relative positions in the space and their relative distances from each other. PSAW (SPSS) version 18 was the software package used for the analysis.

Results are displayed using a dendrogram, which illustrates the degree to which constructs are similar to each other. Constructs most similar (most close in squared Euclidean distance) are arranged most proximal on the dendrogram, and constructs least similar are arranged farther apart (See Figures 1 and 2).

From the dendrogram, three clearly defined clusters of constructs emerged from the analysis. The first and most tightly organized cluster deals with the interest level of the topics. Similar to Middleton (1995) findings with gifted elementary students, our analysis shows that Interest is a major factor in determining students' beliefs about the motivational value of their programs. Also consistent with their work, we found that the tradeoffs between challenge and control are a critical factor. Sense making, not surprisingly, rounded out the three clusters. Because the summer Academy emphasized personal choice and control, sense making, multiple attempts and other motivation-related design choices, we have some confidence that the Algebra program was, by and large, appealing and effective for participants.

Figure 1 shows the structure of students' motivational constructs for the Algebra courses.

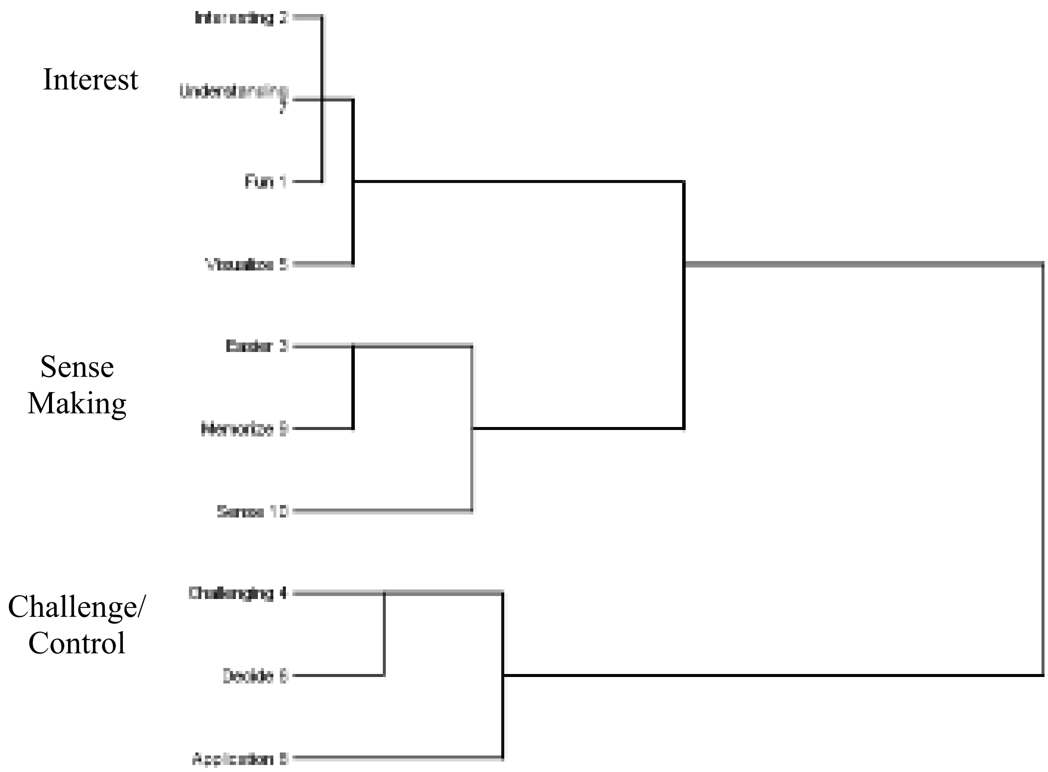

Figure 1. Organization of students' motivational constructs re: Algebra Course

Results for the Geometry course are similar (See Figure 2). Again, there were three defined clusters of constructs, organized around Interests, Sense Making, and Challenge/Control tradeoffs. For Geometry, however, the division between Interest and Sense Making was less pronounced than that for the Algebra Courses.

When the actual content of the courses was analyzed, one interesting result was manifest: For Algebra, Word Problems was rated as an entirely different topic, motivationally, than any of the others in the course. Solving word problems, in essence is seen as something separate from the regular features of Algebra courses, and was rated accordingly. Similarly for Geometry, Proof was somewhat separate, only related to properties of polygons. Neither of these findings is surprising, as we can merely peruse the tasks assigned students in the courses to find that the vast majority of Algebra problems were symbolically-presented, while in Geometry, proofs were primarily associated with properties of triangles and their applications and not with the other, more construction-oriented, or visually-presented exercises.

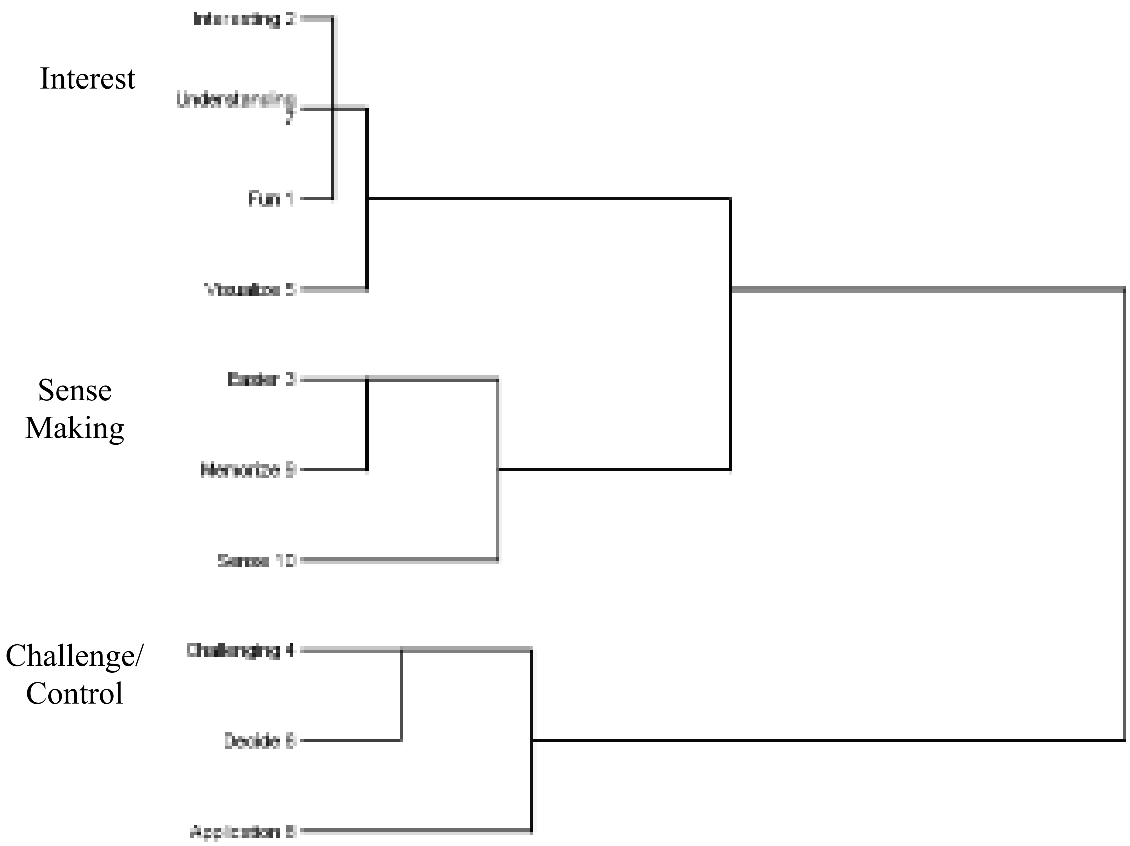

Figure 2. Organization of students' motivational constructs re: Geometry Course

Discussion

Results show that the Summer Math Academy provided an environment where students could achieve at a very high level, accelerating through Algebra and Geometry courses. Their motivation for the courses was focused on Interest, Sense-Making, and appropriate trade offs between Challenge and Control conditions. Overall, the Summer Math Academy is growing in numbers and in impact. More research needs to be conducted on how achievement is impacted, as the small numbers of students who have taken the Academy over multiple years has limited this study's ability to make longitudinal claims of impact. Nevertheless, the present study demonstrates that high challenge, motivationally relevant programs in the summer can fill an important niche in the lives of a relatively large number of students, providing enrichment and acceleration opportunities in a time that is normally educationally dormant.

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COMPARING SECONDARY TEACHERS' ABILITY TO MODEL SOLUTIONS IN PROBLEM CONTEXTS INVOLVING FRACTIONS AND ALGEBRAIC STRUCTURES TO THEIR PERCEPTIONS OF THE ADEQUACY OF THEIR MODELED SOLUTIONS

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We build on our previous research indicating that preservice secondary teachers' encounter difficulties when attempting to model fraction problems in situational contexts where algebraic manipulations are often the expected solution method. We extend our prior research on preservice teachers by providing data gathered through investigations into the nature of in-service secondary teachers' difficulties related to similar fraction situations. Specifically, our study challenges the field to focus on the mathematical knowledge needed for teaching elementary mathematics concepts from a conceptual perspective to struggling secondary students, particularly when solution methods involving algebraic manipulations do not initially prove mathematically fruitful.

The degree to which elementary teachers struggle with conceptual understandings of fraction concepts and computation is well documented (e.g., Ma, 1999). Recently researchers have been investigating the misconceptions associated with secondary mathematics teachers and the models they create to facilitate conceptual fraction development with their students without relying on computation strategies associated with manipulating and solving algebraic expressions (e.g., Olson & Olson, 2010, 2011; Sjostrom, Olson, & Olson, 2010). This recent research offers compelling insight that preservice secondary mathematics teachers do not provide robust models for ratio and proportion situations involving fractions, nor do they accurately arrive at solutions without resorting to algebraic manipulations.

Based on our previous work and the implications thereof, the research presented in this paper represents an intentional focus on the "next steps" we previously identified to meaningfully extend our work to the examination of the ways in which *both* preservice *and* in-service secondary teachers conceive of fraction models in contextual situations. As such, the primary research question addressed in this paper is: What are secondary teachers' abilities, and perceptions of their abilities, to provide meaningful models to represent the mathematics in situated and contextualized fraction problems without explicitly relying on algebraic symbolism and manipulation?

In this paper we investigate the degree to which secondary teachers can provide a meaningful model for students lacking a robust understanding of algebraic manipulations (particularly with respect to such problems involving fractions). For our purposes, we refer to preservice and in-service teachers of Grades 7-12 as "secondary teachers." We argue that to work with students who struggle with algebra and fractions, secondary teachers must be able to provide meaningful models and representations for situations involving fractions without strictly relying on symbol manipulation and algorithmic computational strategies.

Moreover, it is our argument that the ability to present students with, and facilitate students' development of, solution methods that involve models for situations involving fractions is critical for all teachers. Such ability is specifically important for secondary mathematics teachers to effectively work with students, many of whom lack deep conceptual or computation understanding of fractions themselves. Furthermore, our study is of particular relevance to the call described in the Common Core State Standards for Mathematics ([CCSSM], 2010) Standards for Mathematical Practice for students to have the "ability to contextualize." Specifically, the second CCSSM Standard for Mathematical Practice is to, reason abstractly and quantitatively. In this standard, CCSSM authors state:

> Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships…the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of the quantities, not just how to compute then; and knowing and flexibly using different properties of operations and objects. (p. 6)

This paper extends recent research on secondary teachers fraction understandings in problem situations that allow for solutions to be both modeled (e.g., area models, pictures, number lines, and so forth) and arrived at through algebraic symbolism and manipulation. This research is significant because while researchers have examined how students and teachers understand fractional concepts (Charalambous & Pitta-Pantazi, 2007; Izak, 2008; and Sharp & Adams, 2002), Lee and Orrill (2009) indicate that most studies have not looked at teachers' knowledge any closer than simply saying that teachers' reasoning is insufficient.

Theoretical Perspectives

Lamon (2007) is widely considered the most up to date resource related to research on rational number concepts and proportional reasoning. Regarding children's understandings of fractions, Lamon states, "One of the most compelling tasks for researchers has been to discover how instruction can facilitate the joint development of rational number understanding and proportional reasoning" (p. 8). We address Lamon's idea by examining how teachers' understanding of fraction concepts in word problems involving multiplication and division situations compares to teachers' perceptions about the adequacy of their explanations of those problems. In other words, our research is positioned to understand teachers' abilities to, and perceptions of abilities to, facilitate instructional situations that foster the development of rational number understandings and proportional reasoning.

Borko, Eisenhar, Brown, Underhill, Jones, & Agard (1992) highlight certain issues underlying our research in their identification of a prospective elementary teacher with a major in mathematics who could apply and make sense of the division of fractions rule but who could not conceptually explain the situation to sixth-grade students. What we found in our previous work is consistent with NCTM (2000); the invert and multiply strategy is perhaps the most mechanical and least understood procedure in the elementary school curriculum. When looking at middle school and high school curricular topics, similar concerns should be raised about ratios and cross-multiplication. Smith (2002) reports that most students rarely use cross-multiplication. While efficient and universally applicable, students find it difficult to learn, and resist using cross-multiplication as it does not seem to match their mental operational view of ratio. Furthermore, when confronted with alternative strategies, Tirosh (2000) indicated students might consider such strategies as a type of error.

Campbell, Rowan, & Suarez (1998) argue that because algorithms are important, teachers should know and be able to use various strategies for finding a solution, and assist students in making sense of processes and procedures to determine if their work is reasonable. Related to the need for connections among algorithms and conceptual understandings related to algorithms, Ma (1999) identified Chinese elementary teachers as having a more robust understanding of the methods and techniques for comprehending and explaining concepts with fractions, especially division of fractions. Chinese teachers had a repertoire of four ways to explain division of fractions while teachers in the United States usually had only one way.

Although Ma examined elementary teachers, it is also imperative to examine secondary teachers, including ways in which their students conceive of mathematical concepts. In their work, Cramer, Wyberg, and Leavitt (2007) offer support for experiences such as those examined in our study, "Many middle school students are able to apply the algorithm for multiplying fractions, but most are unable to describe the reasons why the algorithms work. Fewer are able to make sense of the division algorithm" (p. 533). Cramer et al. further note that while middle school students are able to make connections between models, images, and symbols for addition and subtraction problems, "Developing mental images related to multiplication and division of fractions proved to be much more complex" (p. 535).

Furthermore, Kieran (2007) provides a way to conceptualize the difficulties that students have with the development of fraction concepts and computational strategies, and the connections to generating models and equations from word problems involving fractions. In particular, Kieran reports that, "Generating equations to represent the relationships found in typical word problems is well known to be an area of difficulty for algebra students" (p 271). However, there is little work done in examining how secondary students make use of modeling to solve these problems if they cannot represent the ideas with algebraic notation, including the use of ratio and proportion relationships to solve these problems.

Although the focus of Lamon's (2001) discussion is related to elementary students, we argue that many of the concerns directly relate to students of all levels, and particularly secondary students. Specifically, Lamon suggests that, "Current instruction in fractions grossly underestimates what children can do without out help. They have a tremendous capacity to create ingenious solutions when they are challenged." (p. 153) She further indicates that, "Like many constructs in mathematics, the rational numbers can be understood only in a whole system of contexts, meanings, operations, and representations." (p. 150)

Our work responds to the connections we identify that need to be made in this previous research to better understand algebra students' (or in general, secondary students') understandings of word problems involving fraction concepts and algebraic manipulations. However, more than that connection, our work responds to Son and Crespo's (2009) suggestions that it is important for the mathematics education research community to, "…continue to investigate prospective and also in-service teachers' reasoning and responses to students' thinking." (p. 259). Our work takes this line of reasoning one step farther. We ask teachers to provide an explanation to a problem, and then have them reflect on and rate the adequacy of their response to provide an explanation.

Furthermore, our research builds upon prior work examining both children's and teachers' understandings of the same fraction problem as that presented in this paper (Olson, Slovin, & Zenigami, 2009; Olson, Zenigami, & Slovin, 2008; Olson, & Olson, 2010; Olson, & Olson, 2011; Sjostrom, Olson, & Olson, 2010; Slovin, Olson, & Zenigami, 2007). In general, the investigations examined fraction word problems that have been solved by fifth-grade students with limited prior experiences related to multiplication and division of fractions, let alone the use of ratio and proportion as a solution method. Yet, these children build upon their conceptions and words in the problems to provide insightful solutions, often using linear models or number lines.

In framing strategies for solving problems, Ni (2000) discusses the use of number lines, which includes the difficulties student have in conceptually understanding the meaning and representation of a rational number. Specifically, Ni states that, "the graph representation of a given algebraic problem is not necessarily less abstract than the equation for the problem" (p. 140). However, what we have previously found is that not only do children in grades 6 - 8, find it difficult to model the solutions for these same questions, but so do secondary preservice teachers. Gilbert and Coomes (2010), in discussing a problem similar to the ones discussed in this article, offer the following argument:

> Any reasonably numerate adult should come up with an answer to this task. Teachers, however, not only must be able to answer this question correctly; the tasks of teaching require that they must also understand and interpret multiple approaches to the same answer. (p. 421)

Methodology

The same four problems used in previous studies (Olson & Olson, 2010; 2011; Sjostrom, Olson, & Olson, 2010) were used in this study, as were the same overall prompt and effectiveness rating prompts (Olson & Olson, 2010; 2011). However, only data from Problem 1 is used in this paper. The data from Problem 1 are reported for two reasons. First of all, the fifth grade children were successful in solving this problem with a process that can best be described as "an application of 1-1 correspondence." Second, for some reason, this problem often appears mysterious to teachers, especially when asked to provide an explanation. As stated, Problem 1 read: It takes 3/4 liter of paint to cover 3/5 m^2. How much paint is needed to paint 1 m^2? Explain your reasoning and support your answer.

Written instructions were provided as well as verbally given to emphasize the importance of the written work used to explain the solution process. These instructions were: For each question, show how you would explain how to solve this problem to a student who does not understand, or is having difficulty understanding, how to solve it algebraically. That is, you are to provide your solution to the problem 'according to how it is written' by using *models and sense making rather than direct computations and algebraic symbolism.*

As with prior work with these problems, Explanations were coded using the five-point explanation rubric from the State of Illinois extended response format (Illinois State Board of Education, n.d.). The complete rubric is explained in other papers (e.g., Olson & Olson, 2011), but the description of work to receive the highest rating (4) is, "Gives a complete written explanation of the solution process; clearly explains *what* was done and *why* it was done. May include a diagram with a complete explanation of all its elements." Decreasing levels of explanation (e.g., 3, 2, 1, or 0) essentially lack one or more of these components. While the explanation is consider a higher priority than only a correct answer, the answer was also rated as correct (2), incorrect (1), or no response (0). Comparing correctness of response with quality of explanation allowed us the ability to examine how well teachers could explain the problem, as well as *if* they could solve it.

After the secondary teachers had provided a solution to the problem, they were asked to rate the effectiveness of their explanation according to a five-point Likert-like scale. These data were gathered to obtain how well they felt they responded so that the explanation would be beneficial for someone who struggles with algebraic representation and symbolism. The scale for the effectiveness rating was: Highly Ineffective (1), Ineffective (2), Neither Ineffective or Effective (3), Effective (4), and Highly Effective (5).

Findings

Data were gathered from thirty-four teachers from four groups at two institutions of higher education during the 2010 – 2011 academic year. Two groups were preservice teachers and two groups were in-service teachers. For our purposes, data from the four groups were combined and are reported in Table 1.

Answer	Explanation	Rating
0	1	-
2	1	4
2	2	4
0	1	-
1	2	5
2	4	4
0	1	1
2	2	2
2	2	1
2	2	4
2	2	3
2	2	4
2	4	4
2	2	3
2	1	1
2	4	4
0	2	4
0	2	4
2	4	2
2	2	3
1	3	4
2	2	3
2	4	4
1	2	4
2	4	4
2	3	2
0	1	2
0	1	-
2	4	4
2	4	3
2	4	4
1	1	3.5
2	2	2
2	1	4

Table 1. Individual ratings of answer, explanation and effectiveness rating

As indicated in Table 1, there were more correct answers than highest ratings for the explanations. In particular, 68% of the teachers answered correctly, but 21% did not provide an answer. Regarding the ratings for the explanations, only 32% of the teachers received a rating of 3 or 4. However, about 55% rated the effectiveness of their explanation as either a 4 or 5. Consequently, there appears to be a clear disconnect between a high quality explanation (as determined by our coding scheme), and what the *teachers* view as an effective explanation. In particular, two teachers' explanations were scored as a 1, but they rated their effectiveness as a 4 (i.e., "effective"); six teachers' explanations were scored as a 2, but they also rated their effectiveness at 4 ("effective"); and one teachers' explanation scored a 2, but this teacher rated his effectiveness at 5 (i.e., "highly effective").

We argue that these disparities are reflective of people who were able to algebraically solve the problem using an equation (or equivalence of two ratios) but they provided very little explanation beyond the equations. Certainly, the argument can be made that these teachers did not "contextualize" the fraction concepts and computations, as specifically suggested in the CCSSM. On the other hand, there were a few examples of teachers who either did not know their explanations were well presented, or otherwise displayed a lack of confidence in their explanations. Specifically, one teachers' explanation was rated 4. However, she rated her effectiveness as 2. Similarly, another teacher with an explanation rated 3, rated her effectiveness as 2. We maintain, that although these teacher's explanations were well presented, the disparity among the ratings is still indicative of a mismatch between knowing how to solve a problem and knowing if the process used will be effective in developing students' understandings.

Discussion

The results shown in this paper have many implications for teacher education. The CCSSM (2010) expects students to have computational competency, conceptual competency, and to employ standards of mathematical practices. In particular, a Grade 6 expectation for ratio and proportion is that students should be able to, "Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities" (p. 42). Moreover, in the notes on courses and transitions the document reads, "This body of material includes powerfully useful proficiencies such as applying ratio reasoning in real-world and mathematical problems." (p. 84.) The data from our study indicate teacher education must focus on developing teachers who can address students' understandings, particularly related to the ideas outlined in the CCSSM, namely that, "One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from." (p. 4)

An underlying concern we identified in discussing the results of this study is the question of when do teachers gain the knowledge they need to effectively teach? More specifically, if teachers are to be prepared to model solutions to this problem for a student who does not already know, where should that teacher be expected to learn such knowledge? With an ability to make sense of the question and sketch a model, a solution can be obtained that does not depend on either complexities of division or of computations involving ratios.

Lamon (2001) states that what is needed is, "better understanding of the role that presenting and representing fractions in instruction plays in enabling or disabling the development of rational-number understanding" (p. 164). While a teacher can enable or disable the development of rational-number understanding, our study shows that our teachers' own understandings arguably were not enabled during the course of *their own* learning of rational number concepts. Consequently, these teachers will arguably not enable, nor provide the opportunity for enabling their students' development of rational-number conceptual and computational understandings.

In light of the evidence presented of secondary teachers' difficulties in modeling and explaining solutions to word problems involving fractions, as teacher educators we must ask important and timely questions of ourselves. How do we help secondary teachers learn to provide better mathematically and conceptually grounded explanations of their reasoning? How do we help secondary teachers understand if their explanations of "elementary" concepts will potentially be effective? We argue that the lack of facility to choose an effective model to represent a fraction problem involving ratios and unknown quantities will set a secondary teacher up for failure when they are faced with students who cannot reason abstractly about such problems. It is essential that secondary teachers develop multiple methods for assisting students to solve problems including the use of mathematically and conceptually rich models and multiple representations that match the context of the problems.

While these ideas can be an emphasis in teacher education coursework, we argue, like Wu (2011), that they need to be addressed in mathematics *content courses* for preservice *secondary* mathematics teachers, as well as professional development opportunities. Previously, we have stated:

> We strongly argue that the role of instruction in undergraduate mathematics classes must be adjusted to value, encourage, and emphasize deep mathematical thinking, connections, and reasoning that, along with methods courses, allow for PSSTs [Preservice Secondary Teachers] to develop more robust mathematical understandings related to facilitating the development of students' mathematical knowledge at a higher level than solely algorithmic. (Olson & Olson, 2011, p. 175)

We suggest at each phase of their professional growth, secondary teachers should be nurtured to learn the *content of mathematics* more deeply, and also be provided with experiences that enable them to develop a rich repertoire of ways to explain mathematics to students.

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DEALING WITH ABSTRACTION: REDUCING ABSTRACTION IN TEACHING (RAT)

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Reducing Abstraction in Teaching (RAT) is one of the theoretical frameworks that provide a window for looking at how teachers deal with abstraction in teaching. By analysing mathematics classroom practices captured in the public release video lessons of TIMSS 1999 through the lens of RAT, I illustrate various tendencies of teachers dealing with mathematical abstraction. I, then exemplify some instances where 'reducing abstraction' seems to be an effective teaching strategy while in other cases it may go unsupportive for the development of student's mathematical understanding.

Introduction

Abstraction is often seen as the fundamental characteristic of mathematics; it has been a central topic of discussion since the days of Aristotle and Plato. More recently, "abstraction has been recognized as one of the most important features of mathematics from a cognitive viewpoint as well as one of the main reasons for failure in mathematics learning" (Ferrari, 2003, p. 1225). As such, in the recent years, abstraction has received a growing interest in the research community among psychologists and mathematics educators. In fact, when teachers plan, one of their most important challenges is to figure out ways of translating abstract concepts into understandable ideas. If teachers understand more clearly what mental process their students go through while coping with mathematical abstraction they are attempting to teach, teachers may be able to teach more effectively and students may learn better.

Reducing Abstraction (Hazzan, 1999) is one of the theoretical frameworks that examine mental process of learners while coping with abstraction of new mathematical concepts. It has been used to examine the mental process of learners in different areas of mathematics and computer science (Raychaudhuri, 2001; Hazzan, 2002; Hazzan & Zazkis, 2005). I am however, not aware of any study that specifically looked at how teachers deal with mathematical abstraction in teaching. Hence, a question worth exploring is 'how do teachers handle the mathematical abstraction in their teaching practices? Do teachers reduce abstraction level in the first place? If yes, what is the nature of reducing abstraction?

In this paper, I first briefly visit the notion of reducing abstraction as propounded by Hazzan (1999). Second, I look at the notion of reducing abstraction from teachers' perspective rather than learners' perspective. In so doing, a new theoretical framework, which I call *Reducing Abstraction in Teaching* (RAT), has been developed. Third, using the RAT framework I analyse the mathematics classroom practices captured in the public release video lessons of TIMSS 1999. I, then exemplify some instances where reducing abstraction in teaching seems to be an effective teaching strategy, while in other cases; it may go unsupportive for the development of students' mathematical understanding. Finally, some comments and concluding remarks will follow.

Theoretical Framework

Reducing abstraction framework was first introduced by Hazzan (1999) in order to examine the mental processes of undergraduate students while learning abstract algebra. Her finding is that learners usually do not have the mental constructs or resources '*to hang on to'* to cope with the same abstraction level of the new (unfamiliar) concept and hence, they tend to reduce the level of abstraction in order to make the concept mentally accessible. This usually happens *unconsciously* (cf. Hazzan, 1999 for detailed discussion).

In this paper I, however, attempt to look at the notion of reducing abstraction in teaching. This is a shift in perspectives because a learner's goal is to learn mathematics for themselves whereas teachers are the mediators and their goals are to help their students to learn mathematics. This shift in perspective demands some modification of the original framework. For example, the choice of the words and phrases such as *'unconscious'*, '*lacks of the mental construct 'to hang on to'* are problematic in most cases. The assumption here is that teachers are the experts of the subject and they have sufficient mental resources to deal with the abstraction of the mathematics they are attempting to teach. This follows then, that the act of reducing abstraction in teaching, in most cases, are intentional and of pedagogical value. Hence, the notion of reducing abstraction in teaching is fundamentally different in its characters, goals and process from that of reducing abstraction by the learners. Hence, a modified version of the framework is required if one wants to look at the notion of reducing abstraction in teaching.

The framework of *Reducing Abstraction in Teaching* (RAT) is the result of this necessity which, I believe has "the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice" (Dreyfus & Gray, 2002, p. 113). Because of the space limitation, detailed discussion of the framework is not possible here. I, however, provide a brief overview of the RAT framework.

Reducing Abstraction in Teaching (RAT)

According to Hazzan & Zazkis (2005)*, Reducing Abstraction* refers to the tendency of the students to make sense of the unfamiliar mathematical concept by relating it to their previous knowledge and experience. This usually results in students' working in a lower level of abstraction of the concept. They maintain that all modes of students' learning can be interpreted as some way of *reducing abstraction* level.

This follows, then that while introducing new mathematical concepts, it is necessary for the teachers to maximally use the learners previously acquired knowledge, experience and level of thinking as well as their familiar contexts. In fact, as Cornu (1991) states, "for most mathematical concepts, teaching does not begin on virgin territory" (p.154), all students come with certain ideas, intuition, and knowledge already formed in their mind.

This idea is in line with many other psychologists and educators (see Piaget, 1970; Hershkowitz et al., 2001). For example, Piaget's idea of developmental psychology and genetic epistemology says that children develop abstract thinking slowly, starting as concrete thinkers with little ability to create or understand abstractions. Safuanov (2004) suggests:

> "Strict and abstract reasoning should be preceded by intuitive or heuristic considerations; construction of theories and concepts of a high level of abstraction can be properly carried out only after accumulation of sufficient supply of examples and facts at a lower level of abstraction" (p.154).

From this perspective, effective teaching should involve the process of introducing new abstractions, concretizing or semi-concretizing them, then repeating at a slightly more advanced level. That is, the concepts are concretized and presented to the students in a lower level of abstraction in order to go to the higher level of abstraction using the lower level as a stepping stone. Hence, the notion of reducing abstraction in teaching comes into play.

Building on the work of Hazzan (1999), Wilensky (1991) and Sfard (1991), three interpretations for abstraction level have been identified, all of which interpret teacher's action as some way of reducing abstraction of the concept. These three categories have been further divided into subcategories in order to incorporate different nature of the teachers' act of reducing abstraction:

Category 1: Abstraction Level as the Quality of the Relationships Between the Mathematical Concept and the Learner

On the basis of this perspective, the level of abstraction is measured by the relationship between the learners and the concept (mathematical object). It is based on Wilensky's (1991) interpretation that whether something is abstract or concrete is not an inherent property of the thing, "but rather a property of a person's relationship to an object. Concepts that were hopelessly abstract at one time can become concrete for us if we get into the 'right relationship' with them…" (p. 198). From this perspective, reducing abstraction refers to the situation where an attempt has been made to establish a *right relationship* (in the sense of Wilensky) between the students and the abstract concepts. Reducing abstraction in this category has been further divided in to three subcategories in order to incorporate different ways a teacher attempts to make unfamiliar concept more familiar to the students.

1.1) Reducing Abstraction by Connecting Mathematical Concept to Real-life Situations

Reducing abstraction in this kind refers to the situation where a teacher develops mathematical concept by connecting unfamiliar (therefore abstract) mathematical concept to the real-life situation (familiar therefore concrete). It is based on the idea that real life problems "connect better with students' intuitions about mathematics, show relevance of mathematics and provide a sense of familiarity to the students" (Hiebert et al., 2003, p. 84).

1.2) Reducing Abstraction by Using Familiar but Informal Language Rather than Formal Mathematical Language

Language plays a key role in mathematics classroom. But with regard to the use of informal language in mathematics classroom, some argue that it may give a different meaning by placing it into a different context and hence may be detrimental to formal education. Others argue that appropriate use of everyday language in mathematics teaching may be pedagogically helpful. To what extent the use of everyday language is a help or a hindrance for the learning of mathematics is debatable and is not the focus of this paper. I however, agree with Munro (1989) when he says, "Children construct new ideas and communicate these in language ... with which they are familiar,… and learn later the conventional language formats for expressing these ideas (p. 121). As such, teachers use informal everyday language in mathematics classroom before or alongside formal mathematical language with a view that use of everyday language in teaching would make the abstract concept mentally more accessible to the students thereby reducing the level of abstraction of the concept.

1.3) Reducing Abstraction by the Use of Pedagogical Tools

Reducing abstraction in this kind refers to the act of teachers in which new mathematical concept (often unfamiliar and therefore abstract) is presented to the students in fairly basic level by the use of pedagogical tools and artifacts (often familiar and therefore concrete) such as models, manipulative, metaphors, gestures, etc. These tools act as a vehicle to lead students mentally from the abstract mathematical ideas to the familiar (source) domain and then back to the realm of abstract concepts (target domain) they are attempting to teach.

Category 2: Abstraction Level as Reflection of the Process-Object Duality

Skemp (1976) introduced the idea of relational understanding and instrumental understanding in mathematics education. According to him, relational understanding involves “knowing both what to do and why” (p. 2), and “building up a conceptual structure” (p. 14). Instrumental understanding, on the other hand, shifts the focus to algorithm and end-product (answer) without any understanding. In Skemp’s words, instrumental understanding involves “rules without reasons” (p. 2) and usually can be carried out in lower level of thinking.

Along the same line, Sfard (1991) talks about the theory of process- object duality where she maintains that “abstract notation such as a number, function etc. can be conceived in two fundamentally different ways: structurally- as objects and operationally- as processes” (p 1). According to this theory, the process conception is less abstract than an object conception. In other words, the focus on procedures or the end product (answer) rather than meaningful understanding are considered to be in lower cognitive level and less abstract. The following tendencies of teacher reducing abstraction fall under this category.

2.1) Teacher Reducing Abstraction by Shifting the Focus on Procedure

Reducing abstraction in this level refers to the situation where the teacher shifts the focus on canonical procedures (how to do it) in a way that leads them away from conceptual understanding even though the original mathematical problem includes a focus on concepts, meaning, or understanding.

2.2) Reducing Abstraction by Shifting the Focus on End-product (Answer).

Reducing abstraction in this level involves the situation where the teacher shifts the focus on end-product (answer) and its accuracy even though the original mathematical problem or discussion implies a focus on concepts, meaning, or understanding.

Category 3: Degree of Complexity of Mathematical Concepts

In this category, abstraction level is determined by the degree of complexity. The working assumption here is that “the more compound an entity is, the more abstract it is (Hazzan, 1999, p. 82)”. Reducing abstraction of this kind is further divided into the following subcategories in order to incorporate the different ways of teachers act in classroom practice.

3.1) Reducing Abstraction by Shifting Focus on Particular Rather than General

Reducing abstraction in this kind refers to the situation when the presented materials and discussions limit in their meaning and scope by focusing on particular case rather than general one thus making the problem less complex for their students. Such act of teachers however provides a partial picture of the concept rather than the complete one.

3.2) Reducing Abstraction by Routinizing the Problems

Reducing abstraction in this level involves the situation where teacher takes over the challenging aspects of the problems by telling student how to solve or by solving the problem for students. In doing so, the complexity of the concepts may be reduced, but takes away the opportunities for students to discover on their own.

3.3) Reducing Abstraction by Stating the Concepts Rather than Developing It

When the concepts are involved in the lesson, they can either be developed or be stated. Reducing abstraction in this level refers to the situation when complexity of the concept or problem is reduced by stating the concept rather than developing it.

3.4) Reducing Abstraction by Giving Away the Answer in the Question (Topaze Effect)

Reducing abstraction of this kind refers to the situation when the teacher gives away the answer in the question (or provides more hint than necessary) resulting in a situation of 'Topaze effect'. The name of the 'Topaze effect' comes from a play by Marcel Pagnol written in 1928 in Paris. In the play, Topaze is a school teacher. He wants his student to succeed; be able to find an answer of the problem by the student himself. When the student cannot find the answer easily, the teacher gives away the answer within the question itself in a slightly indirect way thereby lowering the intellectual demandingness of the tasks (cf. Brousseau, 1997). Later the student produces the correct answer but without having learned anything. This is what Brousseau (1997) calls 'Topaze effect'.

I want the reader to note that these three categories of abstraction should not be thought of as hierarchical or disjointed, they are rather intersecting, or one may even emerge from the other. For example, a teacher trying to deal with a concept in a less complex manner (category 3) or a shift of focus to procedures (category 2) can be interpreted as an attempt of the teacher to make the concept more familiar (category 1) for their students. So, based on the perspectives one takes, one category of reducing abstraction can be thought of as reducing abstraction in the other category. I, therefore, assign the teacher's act of reducing abstraction to the categories that I deem they fit best.

Methodology

The research questions that guided this work are: 1) how do teachers deal with abstraction in mathematics teaching? 2) Can the Reducing Abstraction in Teaching (RAT) framework suggest a plausible explanation for the action of teachers and sources of teaching activities? 3) What are the impacts on student's understandings or misunderstanding of the concept due to the act of reducing abstraction in teaching? In this paper, I however, focus on the first two questions, leaving the third one for the future project. For the purpose of this paper, I used TIMSS 1999 public release video lessons to gather the data.

TIMMS 1999 video study was conducted by LessonLab, Inc. under contract to the U.S. Department of Education (Hiebert et al., 2003). The study involved 638 eighth-grade mathematics lessons that were selected randomly to be the representative of teaching in the seven participating countries- USA, Australia, the Czech Republic, Hong Kong, Japan, the Netherlands, and Switzerland. The videos were analysed by TIMMS videos study team itself and others using different theoretical frameworks (Hiebert et al., 2003) with a focus on various aspects of teaching and learning. I am however, not aware of any study with its focus on teachers' dealing with abstraction in teaching. Hence, my present study took place.

My analysis was based on transcripts of the public release video lessons that have been translated into English by TIMMS 1999 video study team. Public release video lessons consist of four lessons from each of the seven participating countries. I read the transcripts repeatedly with an eye towards identifying key teachers' actions and searching and developing the meaning for each of their action. For lesson from English speaking countries, I also watched videos repeatedly. Because of the dynamic nature of video, they were very helpful in my attempt to understand what is going on in the classroom that might otherwise be impossible to see (from the transcript only).

Results and Discussion

Because of the space limitation, I provide examples and discussion on some of the categories and subcategories only.

Category 1: Abstraction Level as the Quality of the Relationships Between the Mathematical Concept and the Learner

Example 1

Consider the following example:

00:04:33	T	We know that the edges of a triangle- or any figure- are called "sides".
00:04:38	T	In a right-angled triangle, this side is attached to a right angle. So what should we call this side? A right-angled side.
00:04:47	T	Yes? Because this side is attached to a right angle so you call that a right-angled side.
00:05:00	T	Do we have any other right-angled side in there?
00:05:02	SN	Yes.
00:05:03	T	Yes, all the way on the other side. That one is attached to the right angle as well, therefore you call that a right-angled side as well.
00:05:19	T	Then I still have one side left. It isn't so obvious because it is lying flat. But if you see this triangle, what can we call that side?
00:05:28	SN	The long side.
00:05:29	T	The long side. That is correct. Or in a different way?
00:05:33	SN	The right side?
00:05:34	T	It is actually at an angle. If you see it in such an- like a diagonal- so you call this the sloped side or the hypotenuse, is what you call this one.
00:05:45	T	These are just names, you know, you may also keep calling this "the long side", no problem. (NTL-PRL 02)

Here, the teacher uses words from everyday language such as ‘the side lying flat” ‘the long side’, side ‘like a diagonal’ and ‘sloped side’ to represent hypotenuse. The use of such informal, kid- friendly language to introduce the concept is an attempt on the teacher’s part to make the unfamiliar (abstract) concept more familiar to the students thereby reducing the level of abstraction of the concept (Category 1.2).

Example 2

Teacher assign the task of solving equation 4x = 4x -1.

00:38:57	T	what do we have that's tricky? We have four X is equal to four X... minus one. So how are we going to see that?
00:39:04	SS	It's not possible!
00:39:09	T	It's impossible. Valentine?
00:39:11	SN	Because it need a place, for example four X is equal to three X minus one. Otherwise we can't take away the X's. We can't take away one.
00:39:21	T	Let's try to take it away, let's try to be as methodical as we can, then let's see what happens. Let's see, what do we find, so-

Dialogue continues…

00:39:35	T	Therefore zero is equal to minus one. Is that possible?
00:39:42	SS	No!

00:39:43 T No. Therefore when you get to a case like this...

At this stage, some of the students seem to be struggling to make sense of what it means when they get 0 = -1 while solving the equation. At this time, the teacher refers to the graphical method in which case students saw clearly that the lines are parallel and so there is no point of intersection. That is, there is no solution. The dialogue continues:

00:41:33 S It is parallel.
00:41:33 T It is, well, parallel. Therefore when do they meet each other, lines?
00:41:37 SS Never.
00:41:37 T Never. Therefore the- what we're looking for as a solution, it's the points that meet each other. Therefore there's none.

Teachers act of mapping the unfamiliar (solution of equations as a null set) target domain to the students familiar source domain (visual representation of the parallel lines) in order to make the concept mentally accessible can be interpreted as an act of reducing abstraction in by the use of pedagogical tools (Category 1.3).

Category 2: Abstraction Level as Reflection of the Process-Object Duality

Example 3

Following dialogue from an Australian lesson illustrates the behavior of reducing abstraction in this category.

00:00:34 T All right, this one is A B C, and this one is D E and F. All right, they are congruent. So I'm going to ask you- ask you 10 quick questions on that.
00:00:49 T All right, number one, which angle is equal to angle A?
00:01:04 T Which angle is equal to angle D?
00:01:15 T Number three, if angle A is 30... and angle D is 60... uh, what size is angle E?
00:01:35 T And number four, what size is angle D?
00:01:49 T Which line is equal to B C?

Teacher continues asking all 10 product questions that do not require any explanation.

00:04:25 SN And 10 is A B.
00:04:26 SN Why?
00:04:28 T Ten is A B. Just to show the congruence show relationship between them…
00:04:33 T Who got them all right? That's pretty good. Who got nine right?
00:04:40 SN Yeah I did.
00:04:42 T Yeah, right. Number 10 was A B…
00:04:46 T All right, okay, I've got a worksheet for you to do... *(AU-PRL2)*

The original problem could lead to the discussion about congruent triangles that might push them to learning with understanding, but the teacher opted to focus on the answer rather than clarifications and justifications from the students. This act can be interpreted as reducing abstraction by shifting the focus on end-product or answer (Category 2.2)

Category 3: Abstraction Level as Degree of Complexity of Mathematical Problem/Concept

Example 4

Consider the following dialogue that focuses on the concept polygon and its interior angles.

00:06:40 T Okay, what are the differences between convex and concave polygon?
00:06:45 T We look at the concave polygon. At least- at least one or more of the interior angles is greater than 180 degrees.

00:06:56 T	We call this concave polygon. Understand?
00:06:59 T	But in this chapter, we don't discuss the concave polygon; we just only discuss the convex polygon. Okay? … *[Dailogue continues.]*
00:07:15 T	Sorry. It is uh, uh, you have learned it in Form One. It is the... equilateral polygon and also... equiangular polygon.
00:07:32 T	What is meant by equiangular polygon? And equilateral polygon?
00:07:58 T	Equilateral polygon? Do you still remember it?
00:08:02 S?	Yes.
00:08:03 T	what does it mean? I'll give you some hints. All the sides..?
00:08:10 SN	Are equal.
00:08:10 T	All the sides are equal. Okay? All the sides of the polygons are equal.
00:08:16 T	How about equiangular polygon?
00:08:17 SS	(inaudible)
00:08:18 T	All the sizes of the angle…?
00:08:21 SN	Are equal.
00:08:21 T	Equal. Get it? *(HK – PRL3)*

At 6:40, and at 7:32, teacher seems to open up the dialogue on the concept of concave and convex polygon, and equilateral and equiangular polygon respectively. He however, took over the challenging aspects and complexity of the problem by telling them the meaning of concave polygon at 6:45. This act of teacher can be interpreted as reducing abstraction by routinizing. *(Category 3.2).*

Further, at 8:03 the teacher asks the question about equilateral polygon and at 8:18 about equiangular polygon. However, teacher choice of questions such as equilateral polygon means, "all the sides are ?"… and equiangular polygon means, "all the angles are?", actually contain the answers within the questions in a slightly disguise form. This type of question in teaching may reduce the complexity of the problem for the students and that students may be able to produce correct answer, it however does not lead students to learning with understanding. (Topaze effect-c*ategory 3.4)*.

Conclusion

In this paper, my aim was to explore how teachers deal with mathematical abstraction in teaching. Do teachers reduce abstraction level in the first place? If yes, what is the nature of reducing abstraction in teaching? In order to answer these questions, a new theoretical framework, which I call Reducing Abstraction in Teaching (RAT), has been developed. Using the RAT framework, I analysed some of the classroom practices captured in the TIMMS 1999 public release video lessons. I found it helpful to use RAT framework to explore the actions of teachers and sources of teaching activities in regard to dealing with mathematical abstraction in teaching. I identified various tendencies of teacher reducing abstraction in teaching, which I summarized under three main categories and subcategories.

As has been exemplified, reducing abstraction in teaching, in some cases, proved to be an effective teaching strategy (mostly in category 1). However, in other cases, it does not seem to be supportive of the development of mathematical knowledge for the students (mostly in category 2 and 3). One important question then, is: what are the reasons behind reducing abstraction in teaching that do not promote learning with understanding (as in category 2 and 3)? Because this study was based on the TIMMS video lessons, and interview with the teachers was not possible, it is hard to infer why the teachers reduce abstraction in the categories that do not appear to promote learning with understanding. Further research will address this issue. Finally, the results emphasize the importance of paying attention to the nature of students' understandings that may arise as a consequence of reducing abstraction of the concepts they are attempting to teaching.

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VOICES FROM THE TRENCHES: TEACHERS' VIEWS ON TEACHING MATHEMATICS TO GIFTED AND TALENTED STUDENTS

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This study reports the views of teachers of gifted and talented students on perceived characteristics of mathematical giftedness as well as well as what current instructional strategies are being used with mathematically gifted and talented students in the general education classrooms. Interviews were conducted with five teachers who teach mathematics to gifted students. Some of the findings include distinction between gifts and talents, student subgroups, and instructional needs of mathematically gifted students.

Introduction

Mathematically gifted students learn differently from their peers. They require special attention and differentiated curriculum to meet their specific needs (Greenes, 1981; Jolley, 2005). When these students do not receive the learning experiences that are appropriate for their unique abilities, they can become bored and frustrated and eventually become underachievers (Croft, 2003; Hoeflinger, 1998). Requiring mathematically gifted students to participate while concepts are repeated for the rest of the class is not a good approach for the gifted student. Studies have shown that curricula in general education classrooms are frequently inappropriate for gifted students. For example, a national study by Heacox (2002) found that an average of 35 to 50 percent of the regular curriculum at the elementary level could be eliminated for gifted students. Krutetskii (1995) also cites research showing that mathematically gifted students can complete a year of grade-level mathematics in three to six months. Mathematically gifted students are more likely to retain mathematics content accurately when taught two to three times faster than the average student (Krutetskii, 1995). Thus, schools need to make curricular changes to account for the learning needs of mathematically gifted students. This study was therefore designed to examine teachers' perceptions of mathematically gifted students and how to meet the instructional needs of such students.

Theoretical Framework

Tomlinson's (1999) work on differentiation informs this study because differentiation is frequently recommended as a strategy to meet the needs of gifted and talented students in the general education classroom. Modifications can be made to differentiate instruction for the gifted learner in the general education classroom according to the interests of the child, the pace of the curriculum, or the depth of the curriculum. Tomlinson (1999) offered four characteristics of the effective differentiated classroom. The first characteristic is that instruction is concept-focused and principle-driven. The next characteristic is that ongoing assessment of student readiness and growth is built into the curriculum. A third characteristic of the effective differentiated classroom is that flexible grouping is consistently used. Tomlinson (2003) defined flexible grouping as "students consistently working in a variety of groups based on readiness, interest, and learning profile, and both homogeneous and heterogeneous in regard to those three elements" (p. 84). A final characteristic of the effective differentiated classroom is that students are active explorers with teachers guiding the exploration.

According to Tomlinson (2003), differentiation is "responsive teaching, [that] stems from a teacher's solid (and growing) understanding of how teaching and learning occur, and responds to varied learners' needs for more practice or greater challenge, a more active or less active approach to learning, and so on" (p.2). Tomlinson explains differentiation of instruction as a teacher's response to students' needs, guided by general principles of differentiation. The principles of differentiation include the following: focusing on essentials; attending to student difference; aligning assessment and instruction; modifying content, process, and products; working and learning together; respectful tasks; flexible grouping; and ongoing assessment and adjustment. Tomlinson recommends that teachers differentiate content, process, or products based on student readiness, interests, or learning profile. Readiness refers to a student's knowledge, understanding, and skill that designate their entry point within a particular sequence of learning. It can be influenced by factors such as prior experience, attitudes, and habits of mind. Interest refers to topics that are of interest and are a passion for the learner. The learning profile refers to how students learn best and includes learning style, intelligence preference, culture, and even gender. Tomlinson adds that differentiation can occur through various instructional and management strategies. Some of the more critical strategies she suggests for differentiation include literature circles, tiered lessons, learning contracts, independent study, varied questioning strategies, compacting, and varied homework.

Method

Design and Theoretical Framework

The aim of this study was to examine teachers' views on teaching mathematics to gifted/talented students. More specifically, the study addressed the following questions: a) What student characteristics make a student gifted/talented in mathematics? b) What instructional approaches are important for teaching mathematics to gifted/talented students? The methodological orientation of the study was phenomenological. This means that teachers' accounts of their experiences may be understood as descriptions of the essential features of pedagogical interaction, allowing us to comprehend the phenomenon in a new way (Van Manen, 1990).

Participant

Five research participants, Julia, Janet, Jane, Joel and James (pseudonyms), were selected on a voluntary basis. The group of participants was fairly heterogeneous in terms of their discipline, age, and experience as a teacher. Three of the participants were female; two were over fifty years of age, two were between thirty and fifty, and one less than thirty years. All five participants were Caucasian. In order to protect the participant's identity, more specific demographic information cannot be provided.

Face-to-face, semi-structured interviews were conducted with each participant. The interviews lasted from thirty to forty-five minutes. The questions were open ended to allow participants to tell their stories freely. A typical question, for example, went as follows: "What teaching techniques do you believe are especially important for teaching mathematics to gifted and talented students?" The interviews took place at a mutually agreeable, private place and time. Interviews were audio taped and transcribed.

Limitations

The limitation of the current study is the representativeness of the participants. Thus, the findings and conclusions from this study might not necessarily reflect the views of all teachers of mathematically gifted and talented students.

Data analysis

After the data were transcribed, the transcripts were read while listening to the tapes, and initial codes were developed. The participant's story was the real guide to coding the data, and codes were developed as necessary to reflect the themes that appeared in the interview transcript. Once the coding scheme reached a point where it seemed to capture the relevant parts of the participant's story, the interview were re-coded.

Results

The main themes that were constructed in the data were characteristics of gifted students, distinction between gifts and talents, struggles/challenges for gifted students, instructional needs, and student subgroups.

Characteristics of Mathematically Gifted Students

The characteristics of mathematically gifted students mentioned by participants were understanding of numbers and patterns, communication difficulties, and lack of organization. Sample comments are as follows:

- Some of the characteristics include their understanding of numbers and patterns; their ability to think about mathematics in a variety of ways, so that mathematics comes naturally to them. They don't struggle at all. They just seem to understand the patterns. Now, sometimes, that doesn't translate, they are not able to show how they understand it. They just understand it, but then they struggle with showing how they got their answer and explaining their thoughts (Janet).
- I think that when we hear the word 'gifted', it means they are exceptional. They have a depth of understanding of the mathematical concepts, rather than the mathematical tasks. So you can give them application problems and they jump on it right away. Or they see a clever way or a much more convenient way of proving something, or just analyzing something. So they are operating at those really high levels of understanding and they always do very well on the application problems, extra-credit problems, things like that (James).
- I find that mathematically gifted students tend to do very well on math tests and quizzes. They tend to be lousy and disorganized. Some of them have find it difficult paying attention in class, especially if the lesson is not challenging enough (Julia).

Distinction Between Gifts and Talents

On whether mathematical giftedness and mathematical talent were the same or different, participants' responses were again inconsistent:

- I always assumed mathematical giftedness and mathematical talent were interchangeable.
  I would say gift is what you have naturally and talent is what you do with that gift (Julia)
- Gift is when you visualize math without having to be trained; talent is when you can learn and be efficient at it. (Janet)
- Talent is innate; talented students communicate well, write out their solutions. Gifted

> students on the other hand are bad at process and communication though functioning at high level in relation to concept understanding (Joel).

Struggles/Challenges for Gifted Students

All three participants noted that mathematically gifted children do in fact have areas of struggle that tend to hinder them from reaching their full potential. These difficulties include difficulty in showing/explaining solution processes, answers, and concepts; tendency to become bored more quickly and thus be unmotivated; reduced confidence with general education students due to being with intellectual peers, and fear of failure. Here are sample participants' responses:

- Showing their understanding and going through the little steps to get the solutions seems to be a huge problems for most mathematically gifted students. They are used to getting the right answers with less effort, and so they struggle with: How did you get that answer? And when they get the answer wrong, they get easily frustrated because they never did the 'baby' steps to get it (Jane).
- One of the biggest challenges facing mathematically gifted students is speaking the language of mathematics. They have been so good at math that they are kind of able to bully their way through a math class. They understand the concepts, especially at the lower levels. Once you start getting into the higher-level concepts and higher level thinking, they ran into trouble because they can't communicate in mathematics. And that has to do with symbols, for instance being able to tell the difference between an equal sign and an implied sign. You have to teach them how to express their ideas on a piece of paper. Otherwise, they are just ideas inside their heads. The other challenge is being organized. In my classes, I force them to keep a three-ring binder, where they keep their assignments, homework, exams, syllabi, schedules, calculators, pencils, and so forth. (Joel)
- They are often are unmotivated and need to be pushed a bit. It is difficult to keep them motivated in a linear fashion. You have to always keep them motivated and interested else, they lose focus entirely. Also, they tend to be perfectionists and fear to attempt a problem that presents a risk of failure (Janet)

Instructional needs

Participants noted that mathematically gifted students should be taught in separate classes from general education students. They further contended that the gifted students must be taught at an appropriate level of challenge - neither too low nor too high. This would require pre-assessing to see what they need to learn and compacting accordingly. Participants also mentioned that teachers should teach mathematics in varied ways to keep students creative; use real-world applications; and learn to think in a different way that values process and problem solving. Sample participants' comments follow:

- The teacher of the mathematically gifted needs to be organized. You need to have a clear path. You need to have assessment ahead of time. Pretest students to know where they are, what they understand and what they don't understand. I also think it is important to use questioning strategies to build student understanding (Julia)
- Generally, I think it is best to teach them in their own classroom. The teacher needs to keep them at the appropriate level of challenge (neither too much nor too little; don't want to bore or frustrate them) (Janet)

- The teacher needs to compact materials to suit their needs after pre-assessing to only address what they don't know. This would help eliminate unnecessary repetitions (James).

Student Subgroups

Participants noted that actual ability differences were not noted by gender or race/ethnicity. They said performance differences could occur based on background factors such as linguistic background (for English Language Learners) or what students have been exposed to and encouraged in. Here are some typical responses from participants:

- There is this general notion that somehow boys are better than girls at mathematics. But I think once you get all of them together, you realize that it is not necessarily true. One of my most gifted and talented students that I have ever taught was a Black female, so that breaks the stereotype right away. So I think with the gifted kids, you start seeing those stereotypes fall apart (Joel).
- I think race/ethnicity does not necessarily have an direct impact on the mathematically gifted student. It depends on the student. And I think that if their first language is not English, that could hold them back a little bit. With African-Americans, I don't think there is anything holding them back. I have several gifted and talented students that I taught who were African-Americans, and several who were Latinos. I noticed with the Latino students that if their first language is not English and they are still acquiring the language, they might be hesitant because they are afraid of coming our perfect in the way they say their answer (Julia).

Discussion and Implications

The teachers in this study mentioned many of the characteristics that are associated with mathematical giftedness. Many of these characteristics are described in the literature by Diezmann and Watters (2000), Hoeflinger, 1998, and Sriranam (2003). Straker (1983) also describes many of the same characteristics and suggests that one should look for indicators such as "a liking fnumbers including use of them in stories and rhymes; an ability to argue, question and reason using logical connectives: if, then, so, because, either, or…, and patter making" (p. 17)

Many scholars have noted that the way giftedness is defined and perceived affects how parents, teachers, or school administrators handle the gifted child (Assouline, 2003; Callahan, Cooper & Glasscock, 2003; Roedell, 1984; Rotigel, 2004; Sternberg, 2003; Van Tassel-Baska & Johnsen, 2007;). The teachers in this study did not have a comprehensive, articulated concept of mathematical giftedness. Their knowledge about some of the characteristics was based primarily on teaching experience and observation of their students' behavior. Potentially, it means that many students could be overlooked in the identification process if teachers do not have knowledge about these types of mathematical giftedness.

Participants from this study mentioned that mathematically gifted students should be presented with materials that are rich in content and that can arouse their natural curiosity. Teaching them material that is below their level causes them to feel bored and disengaged in class. This is consistent with the assertion of Leder (1991) that teachers who teach mathematically gifted children should ensure that their content is problem based, and rich in content.

The findings from this study provide intriguing initial insights into teachers' views on teaching mathematics to gifted/talented students. These findings also point to the fact that stakeholders still have an unfinished business of shedding more light on what constitutes giftedness and how to identify a gifted child

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PRE-SERVICE TEACHER CONCEPTIONS OF REAL NUMBER

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This study examines pre-service teacher conceptions of real numbers within three contexts: distance, a point on the real line, and set membership. The study involved 25 pre-service mathematics teachers, in various stages of education, at a large university in the southeastern United States. Participants were administered a survey to assess their understanding and notions of real numbers. Analysis of participant responses from six survey items (two per context) revealed ten distinct conceptions of real number. Implications of these findings as they relate to nonstandard infinitesimals are also discussed.

Research has shown that many students struggle with the notions of real numbers (Tall & Schwarzenberger, 1978; Fischbein, Jehiam, & Cohen, 1995; Peled & Hershkovitz, 1999; Sirotic & Zazkis 2007). According to Olive (2001) a child's number sequence is a dynamic structure progressing as the child develops. Although Olive is addressing only part of the real numbers (i.e., the counting numbers), it is not farfetched to imagine that the same is true for the reals. But how do learners think about and work with real numbers? What types of concepts do they use and within what contexts do they use them? The purpose of this study is to investigate and document pre-service teacher conceptions of real number within three contexts: distance, a point on the real line, and set membership. For the purpose of this study conceptions of real numbers are not limited to the classical standard construction of the reals, but also include non-standard constructions involving infinitesimals.

The Real Numbers

Standard Analysis

To describe the real numbers in great detail would involve one or more textbooks. For the purpose of this study, the standard real numbers are defined in the classical sense, axiomatically, as a Cauchy Archimedean ordered field (Rudin, 1976). Casually, the reals refer to the set of rational numbers (i.e. numbers that are the ratio of two integers excluding zero from the denominator) and the set of irrational numbers (i.e., Archimedean numbers that cannot be represented as a ratio of integers). Informally, the Archimedean property of the real numbers implies that there can be no largest number and no smallest number.

It is noteworthy to mention that standard analysis as we know it today began to take shape in the early 1800's with the introduction of Weierstrass' epsilon-delta proofs of limit and continuity. These notions found favor among the formalists and their quest for formal mathematical foundations (Nunez & Lakoff, 1998; Tall, 1980; Keisler, 1976). Original notions of the calculus, up to that time, involved infinitesimals (i.e., a small unmeasureable cloud around a number, smaller than any real number) which were done away with because mathematicians had not yet developed a proper treatment for infinitesimals (Keisler, 1976).

Non-standard Analysis

In the early 1960's Abraham Robinson showed that infinitesimals were valid mathematical objects using methods from model theory developed in the 1950's. Robinson referred to his infinitesimal analysis as a non-standard analysis due to its use of non-standard methods (Keisler, 1976). In doing so, Robinson established a rigorous foundation for the infinitesimal notions that had proved useful to Newton, Leibniz, Euler, Hilbert and others (Robinson, 1966; Goldblatt, 1998; Nunez & Lakoff, 1998; Tall, 1980; Keisler, 1976).

The Hyperreals

Succinctly stated, Robinson, operating from the notions of Leibniz, formally defined infinitesimal numbers and extended the set of real numbers to include them, thus forming the hyperreal system. According to Keisler (1976), “one can form the hyperreal number system by adding infinitesimals to the real number system and obtain a powerful new tool in analysis” (p. iii). Goldblatt (1998) defines an infinitesimal number as follows: “a nonzero number ε is defined to be infinitely small, or infinitesimal, if,

$$|\varepsilon| < \frac{1}{n} \text{ for all } n = 1,2,3 \ldots$$

Conversely, 1/ |ε| is considered to be infinitely large or infinite. In this way the hyperreals form an ordered field *R containing R as a subfield” (p. 3).

Real Number Conceptions

Real number conceptions in this study are not limited to the classical standard construction of the reals, but take into consideration the notions of non-standard analysis as well. From this expanded perspective, the context of distance (a variation of the measure context; Fischbein, Jehiam, & Cohen, 1995), the context of set membership, (Peled & Hershkovitz, 1999), and the context of a point on the real line (Peled & Hershkovitz, 1999; Sirotic & Zazkis, 2007) were examined. Below we review the literature related to different conceptions of real number.

Fischbein, Jehiam, and Cohen (1995) conducted a study involving 30 ninth-grade students, 32 tenth grade students and 29 pre-service teachers. The researchers administered a questionnaire that examined formal knowledge of mathematics, real number hierarchy, and intuitive knowledge regarding continuity, infinity, and measure. According to the study, several pre-service teachers struggled with rational and irrationals in terms of set membership and some equated the term irrational with non-whole numbers. Results also indicated that participants in general did not have a problem with intuitive notions involving incommensurable segments (segments that cannot be divided evenly by a common unit), an infinity of rational numbers over a segment, or an infinity of irrational points over a segment. This study did not overtly investigate or involve intuitive notions related to infinitesimals.

Peled and Hershkovitz (1999) conducted a study involving a group of 55 students in their 2nd and 3rd years of college working toward mathematics teaching certification. Their study investigated student’s knowledge of irrational numbers, including location of irrational numbers on the real line and set membership. According to Peled and Hershkovitz, of the 55 students surveyed, 22 students were not able to locate $\sqrt{5}$ on the real number line. Several students believed that a measurement involving $\sqrt{5}$ could not be made because of the never ending decimal. It was hypothesized that students held a belief that numbers such as $\sqrt{5}$ cannot be reached because of the non-periodical never ending decimal expansion. Peled and Hershkovitz’s study (1999) was geared toward the exploration of pre-service teacher knowledge of irrational numbers within the system of standard analysis.

Building on the work of Peled and Hershkovitz (1999), Sirotic and Zazkis (2007) administered a questionnaire to 46 pre-service teachers as part of a course designed for professional development in the area of secondary mathematics. The study included an analysis of follow-up interviews with 16 of the 46 participants. With regard to the location of $\sqrt{5}$ on the real line, Sirotic and Zazkis (2007) found that the participant's answers fell into five categories. In the first category were those who made use of the Pythagorean Theorem to identify the exact location of the point. The second category consisted of those who used at least some precision in a decimal approximation. The third category included those using somewhat rougher decimal approximations to identify the point (e.g. somewhere between 2 and 3). The fourth category included those who used the graph of a function or a similar graphical aid to locate the number on the real line. Finally those who claimed that the task was not possible comprised the fifth category. Much of their analysis was qualitative and based on participant interviews. These five pre-service teacher conceptions emerged within the context of locating a point on the real line. Like Peled and Hershkovitz (1999), these researchers found that participants conceptualized a number as not being able to be reached or as having an approximate location on the real line. This study did not take into consideration infinitesimals or a non-standard approach.

Tall and Schwarzenberger (1978) describes learner conflicts involving the idea that $1 = 0.99\overline{9}$. According to Tall and Schwarzenberger, learner conflicts may result from a lack of understanding of the concept of limit, and the "intrusion of infinitesimals" (p. 6). They suggest that learners often feel that there "should be a one-one correspondence between infinite decimals and real numbers. They are confused when they see that two different decimals can correspond to the same real number" (p. 6). Confusion between decimal representation and other types of number representation has been noted by several researchers since this study (Oehrtman, 2009; Sirotic & Zazkis, 2007; Peled & Hershkovitz, 1999; Richman, 1999).

Oehrtman (2009) conducted a study involving 120 students from an introductory calculus class. Sixty-four of the students had taken calculus in high school. Oehrtman found that mathematical metaphors significantly influenced the way in which students justified certain claims. In terms of inconsistent responses regarding the standard definition of real numbers, Oehrtman points out that several students used what he refers to as an "approximation metaphor." For example, when talking about the equality $1 = 0.99\overline{9}$ several students described an "arbitrarily small," "infinitely small," "infinitesimal" difference or that 0.9 was the "next number" or that it "would touch one" (p. 417). These findings suggest a relationship between human intuitions of infinitesimals. For instance the number next to the number 1 makes sense in non-standard analysis whereas in standard analysis it cannot exist due to the Archimedean property.

Considering participant conceptions from both standard and non-standard analysis, Ely (2010) conducted a case study involving one college student selected from a larger group of 233 college-level calculus students. The larger study involved a questionnaire regarding conceptions of calculus, limits, functions, continuity, and the real number line. From the original sample of 233, six students were chosen for follow-up interviews. From these six, one female student was selected for the case study due to the interesting nature of her responses given during a follow-up interview. Based on this study Ely suggests that some misconceptions about real numbers and infinity may not be a misconception, but instead may be thought of as intuitive conceptions that are consistent with the non-standard system. Ely found that 31% of the participants in the larger study held non-standard views.

Richman (1999) suggests that two different conceptions of infinity form the basis for two distinct interpretations of the statement $1 = 0.99\overline{9}$. The two notions of infinity are related to what Aristotle described as potential infinity and actual infinity. According to Richman potential infinity conveys a sense of a process, as if the number $0.99\overline{9}$ were somehow on its way to the number 1 or in the process of getting there. Actual infinity suggests a completed process, that is, 1 is the same as $0.99\overline{9}$. Richman makes the point that if two numbers are considered the same in terms of limits, a distinction should be made between equality of numbers and convergence at a limit point. In this way, learners might learn why and under what conditions their intuitions are supported.

From the literature it is clear that people struggle with various concepts of the real number system. Much of the research has been conducted from the point of view of standard analysis. Recent research has suggested that some intuitions leading to misconceptions in standard analysis may be valid conceptions when considering nonstandard analysis. The Sirotic and Zaskis (2007) study revealed five conceptions of the standard real numbers within the context of locating a number as a point on the real line. The Peled and Hershkovitz, (1999) study revealed similar findings. Several of the studies revealed issues related to irrational numbers and their non-periodic never ending decimal representations. Nowhere in the literature is there a study investigating knowledge of real numbers from both the standard and non-standard perspectives over several contexts. This paper reports on both standard and non-standard conceptions within three contexts: distance, a point on the real line, and set membership.

Methods

Participants

This study involved 25 pre-service mathematics teachers in various stages of their education. All were juniors or seniors or graduate students at a large university in the southeastern part of the United States.

Procedures

The participants were given a survey including 25 items related to the real numbers. From this survey six items were selected for further analysis: two items related to distance, two related to location of a number as a point on the real line, and two related to set membership (see Table 1). D1 and D2 are related to distance and were presented in a True/False format requesting an explanation for the given answer. The two location items L1 and L2 and one of the set membership items, S1, were also presented in this format with L2 requiring a yes/no answer with explanation. For set membership item S2, participants were asked to circle all the irrational numbers in the given set and asked to explain the reason for their selection.

The first phase of the analysis involved the coding of participants' answers and explanations within the context of real number conceptions. We were not interested in the idea of participant misconceptions and focused solely on participant conceptions, in particular, we recognized that conceptions of real number may appear in both standard and non-standard ways. Although we did not intentionally marginalize concepts that did not fit either one of these constructions, our focus was centered on conceptions that were consistent with one or the other. As a result there may have been other conceptions that were not categorized within this analysis. Since two of the items in this study were closely related to the question posed by Sirotic and Zaskis (2007), we expected to find similar conceptions within the context of locating a number as a point on the real line. We were also interested in learning whether or not any of the five conceptions cited in their study would emerge from within the other two contexts. For these reasons, the five conceptions revealed in Sirotic and Zaskis (2007) were considered as a "start list" as discussed in (Miles & Huberman, 1994, p. 58).

Context	**Label**	**Item**
Distance	D1	The distance between two points is necessarily rational.
Distance	D2	To express a distance in terms of an irrational number makes no sense.
Location of a point	L1	The $\sqrt{5}$ has a precise location on the real number line.
Location of a point	L2	Can the exact location of $\sqrt{7}$ be found on the real number line?
Set membership	S1	The number $0.33\overline{3}\ /5$ is a rational number.
Set membership	S2	Circle all irrational numbers in the given set of numbers: $\{\ 0,\ \ \sqrt{5},\ \ \frac{3\frac{1}{2}}{7},\ \ 0.123152687943,\ \ \sqrt{\frac{1}{16}},\ \ 3i+5,\ \ 13,\ \ \frac{\frac{1}{2}}{\frac{2}{3}}\ \}$ Explain briefly why you believe these numbers are irrational.

Table 1. Items by context

Results and Discussion

Within the context of locating a number as a point on the real line, this study confirmed two of the Sirotic and Zaskis (2007) conception categories: a) use of the Pyhtagorean Theorem and b) Impossible situation. However, during the early stages of analysis it became necessary to expand the conception categories to accommodate the three different contexts as well as non-standard conceptions. Modifying the categories was also necessary due to the difference in focus between this study and the Sirotic and Zaskis (2007) study. For example, we did not ask participants to show "how" they would locate the $\sqrt{5}$ on the real line and therefore graphing or use of a function was not a conception documented in this study. Our items were focused more on pre-service teacher beliefs about its location and "why" they believed what they believed. As a result, several of the conceptions that we found, although related to the Sirotic et al. categories, were differentiated from 4 of their 5 categories. From our perspective, the Sirotic et al. categories represented conflations of conceptions. One example of this idea can be observed in the pre-service teacher response to L2: "yes, but not its decimal approximation." In this example, the pre-service teacher conceived of the $\sqrt{5}$ as having a precise location but believed its decimal approximation did not. Ultimately, in our study, five conceptions emerged from the location of a point on the real line context, a) precise location, b) approximate location, c) dynamic location, d) non-standard conception, and e) formal conception. The analysis of all six items, spanning the three contexts, revealed additional conceptions for a total of ten conceptions. These conceptions are listed in Table 2. In the following sections we present some of the participants' explanations of their answers and a discussion of the nature of the conceptions that seem to be represented by their responses.

Precise location
Dynamic location
Approximate location
Use of irrationals other than Pythagorean irrationals
Decimal representations
Ratio of integers
Use of the Pythagorean theorem
Impossible situation
Non standard conception
Formal conception

Table 2. Participant conceptions

Location of a Number as a Point on the Real Line

The following are examples of students' explanations associated with items L1 and L2 with associated category in parentheses:

> *Every real number has a precise location on the real line.* (precise location)
>
> *Yes. Between 2 and 3 closer to 3.* (approximate location)
>
> *Not exact but close.* (not able to locate)
>
> *Yes, through properties of triangles.* (Pythagorean Theorem)
>
> *It is the limit point of a sequence of rationals.* (formal conception)

In addition to these conceptions, some participants were found to use dynamic conceptions and non-standard conceptions of real number. The following two responses to item L1 are examples of these conceptions:

> *No it only theoretically exists. Because it is irrational it is impossible to pin down.*
>
> *We can never actually get there. We get infinitely close.*

We considered "impossible to pin down" to mean that the participant saw the point as moving. For this reason this response and others like it were categorized as a dynamic conception. The phrase "never actually get there" is a notion referred to in the Richman (1999) study dealing with potential infinity. Following this phrase was the phrase "we get infinitely close." Any explanation that used the term "infinitesimal" or "infinitely close" was considered a non-standard conception.

Distance

In response to items D1 and D2, several participants used the Pythagorean Theorem, drawing graphs and constructing right triangles in an effort to demonstrate an irrational distance.

In addition to the use of Pythagorean irrationals (e.g., $\sqrt{5}$), several students used irrationals other than Pythagorean irrationals. This conception was employed by participants who chose to use an irrational number such as π or a description not explicitly based upon the Pythagorean Theorem. For instance two participants described irrational distance this way:

> *Two numbers can be π distance apart.*
>
> *The distance between 4 and π is irrational.*

Within the context of distance the most common conception of real numbers involved the use of the Pythagorean Theorem; however, the use of irrationals other than Pythagorean irrationals was also common.

Set Membership

One participant's response to S1 suggested a perturbation involving the equality of 1/3 and $0.33\bar{3}$. She clearly had doubts about whether or not 1/3 = $0.33\bar{3}$ and as a result she answered that she was not sure. Her response was as follows:

> *We had this crazy long debate in [math] class. I'm not sure who won...1/3 =* $0.33\bar{3}$

Based on a discussion with the instructor (A. Norton, personal communication, January 17, 2010), the debate was between those who took the position of standard analysis, (i.e. that the two numbers are equal) and those who took the position of non-standard analysis, (i.e. the use of infinitesimals) suggesting that the numbers were not equal. This participant did not specify set membership related to the number $0.33\bar{3}\ /5$ possibly because she was unsure about what the number represented. Her response was nevertheless considered as a non-standard conception since she was on the fence as to whether or not 1/3 = $0.33\bar{3}$. Her response was the only non-standard response to item S1. The majority of participants simply confirmed that the number was rational.

When asked to "Circle all irrational numbers in the given set of numbers" (see S2, Table 1), of those participants who correctly identified $\sqrt{5}$ as an irrational number, many used a "ratio of integers" conception. What this means is that in the process of explaining why they thought $\sqrt{5}$ was irrational, several participants stated that $\sqrt{5}$ was irrational because it could not be written as a ratio of integers. For instance three responses were as follows:

> $\sqrt{5}$ *cannot be expressed as a ratio of integers.*
>
> *Cannot be written in rational form.*

Because you cannot express it as p/q a fraction.

Participants also employed a "decimal representation" conception to explain their reasoning for their choice of the $\sqrt{5}$. The following three responses are typical of this type of conception:

You cannot represent this number as a terminating decimal.

Never ending decimal.

The number is a non-repeating decimal.

Within the context of set membership we documented participants' use of three conceptions: non-standard, ratio of integers, and decimal representation.

Conclusion

Recent research has suggested that many learners, even at the college level, struggle with different aspects of number, particularly with concepts related to the real numbers. The purpose of this study was to document pre-service teacher conceptions of real numbers within three contexts: distance, point on the real line, and set membership. The findings of this study established ten conceptions of real number within these three contexts. From location of a point on the real line this study confirmed two of the five conceptions found in Sirotic and Zaskis (2007) and found similarities relating to two other categories from their study. This study also found that participants described real numbers using irrationals other than Pythagorean irrationals, dynamic conceptions, and non-standard conceptions. From the set membership context, participants employed decimal representation, ratio of integers, and nonstandard conceptions. With respect to the context of distance participants employed the use of the Pythagorean Theorem and the use of irrationals other than Pythagorean irrationals.

Although this study illuminated ten conceptions of real numbers, many questions remain for further study. How and when do real number concepts begin to develop? How are the real numbers presented to learners in textbooks? What is the relationship between intuitive conceptions and the hyperreal system and does the nonstandard system enhance classical mathematics?

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UNVEILING MATHEMATICS TEACHERS' PROFESSIONAL AND PERSONAL IDENTITIES USING PHOTO-ELICITATION INTERVIEWS

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How do mathematics teachers see themselves? For years, educational researchers have been correlating mathematics teaching with what a teacher knows or believes. But the construct of identity–how teachers see and position themselves–is equally important. This study investigates mathematics teachers' identity through the Photo-Elicitation Interview (PEI). Mathematics teachers choose photographs of the various parts of their lives, and then discuss them during an interview to elicit stories of who they are and how they see themselves. This paper details the use of this method to explore the mathematics teacher identity of two Algebra I teachers.

Once upon a time, Karla was the young teacher filled with potential. Karla routinely worked 70-hour weeks, cheered her students on at almost every single sporting event, facilitated an after-school mathematics tutoring club, and built relationships with local media outlets so that her students' class projects would be featured on the nightly news. She was even promoted to assistant principal in her second year of teaching.

At the time, Karla thought good mathematics teaching meant showing her students how much she cared. She positioned herself as an advocate for students who could not help themselves, chose to teach the low-track students in a low socioeconomic, Latino/a-dominant, urban school. On her lunch periods, students filled her classroom, not just because they wanted help on their math homework, but also because they liked talking to her about their lives.

Fourteen years later, Karla still teaches the low-track Algebra I course at a low socioeconomic, Latino/a-dominant, urban school. However, she has since changed schools twice and relinquished all administrative responsibilities. She leaves the campus shortly after students are dismissed, has not been to a single student sporting event, distances herself from extra-curricular activities, and has no plans to bring any news reporters into her classroom.

She sees her primary responsibility as a mathematics teacher to get as many of her students to pass the high-stakes standardized mathematics exam as possible. In order to be on pace with all the other teachers in the mathematics department, her teaching has become more predictable and structured. Her decisions for what content to teach hinges on whether she thinks it will appear on the high-stakes exams. She knows little about her students' lives, and readily admits that she purposely hides herself, her personality, and her background from her students. She even says she would be surprised if any of her students knew about her ethnicity; they all just assume that she is "Mexican too" even though she self-identifies as Indian-American.

Objective of the Research Study

While Karla does not represent every mathematics teacher, the struggle she has in merging her changing professional identity with her personal identity is an issue that affects all teachers (de Freitas, 2008; Enyedy, Goldberg, & Welsh, 2006; Gee, 2000; Van Zoest & Bohl, 2005; Walkington, 2005). While Karla's mathematics teaching practice looks very different today than it did fourteen years ago, she could be construed by some measures as being a better teacher since she is following a common scope and sequence and her students' scores on the high-stakes exams are consistently high. However, the part that makes her unique–her personal identity–is buried deeper and deeper beneath her professional identity.

In this study, I address the problem of the wavering relationship between personal and professional identities of mathematics teachers through the following research questions:

1. What makes an algebra teacher unique? Outside of defining themselves as mathematics teachers, what is it about where they come from, how they situate themselves, and how they see themselves that makes them special? And, how does this personal identity interact with their professional identity as a mathematics teacher?
2. How do algebra teachers' personal identities reveal how they see themselves in terms of power? In other words, within their own identity narratives, what role(s) do they see themselves playing? How do these role(s) unveil the amount of control they have in forging their own identities?

Theoretical Framework

Mathematics educational research generally splits influences to mathematics teachers into the constructs of knowledge (Ball, 2003; Hill, Rowan, & Ball, 2005; Hill, Sleep, Lewis, & Ball, 2007), beliefs (Leatham, 2006; Pajares, 1992; Philipp, 2007), or both (Simon & Tzur, 1999; Simon, Tzur, Heinz, Kinzel, & Smith, 2000). But beyond just what a teacher knows or believes, the construct of identity–how teachers see themselves–is an equally important component into understanding how people teach mathematics (Drake, 2006; Van Zoest & Bohl, 2005). Most of the existing work exploring the identity of mathematics teachers has been largely theoretical as it is has been a difficult construct to research until recent advancements in technology that allows cheap and easy ways to capture photo, video, and audio data (Clark-Ibanez, 2004; Enyedy, et al., 2006; Harper, 2002).

Teacher Identity

Teacher identity is defined in a number of ways within the educational research literature. Van Zoest and Bohl (2005) present a framework for *Mathematics Teaching Identity* as a way of understanding the various social spaces mathematics teachers encounter during their career, specifically in regards to supporting the development of 'reform-oriented' mathematics teachers. These social spaces emanate from Wenger's (1998) idea of *communities of practice*, in which identity is forged through multiple membership within these communities. Gee (2000) sees teacher identities as defined through power, and his four categories of *natural, institutional, discourse, and affinity-based teacher identities* intersect to unveil how much power teachers have in relation to the institutions they work within.

Other researchers see teacher identity as entrenched within the construct of narrative. Holland, Lachicotte, Skinner, and Cain's (1998) definition of identity involves self-authoring; identity is what teachers tell others about who they are, and the self-understanding that comes from acting to be this person. Sfard and Prusak (2005) use the idea of the narrative to frame identity as a collection of reifying, significant, and endorsable stories about a person. And de Freitas (2008) shows mathematics teacher identity as revealed in the spontaneous moments in which a mathematics teacher uses personal anecdotes during procedural moments in classroom instruction.

Listening to Mathematics Teachers' Narratives

Common to these narrative-based definitions of teacher identity is the idea that teachers only reveal their identities if someone is actually listening to what they have to say (de Freitas, 2008; Holland, et al., 1998; Sfard & Prusak, 2005). Within the work of Cognitively Guided Instruction, the simple act of listening to students is shown as a revolutionary teaching practice (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Duckworth (2001) also argued that deep learning happens in the classroom when teachers stop viewing themselves as authorities of knowledge and instead position themselves as questioners and redirectors to listen to students.

This listening works with mathematics teachers as well. Drake (2006) found that listening to pre-service mathematics teachers write about their experiences as mathematics students led to deep reflections of how they taught mathematics. And de Freitas (2008) found that when teachers revealed their *vulnerable 'I'*–their personal identity–it forced her as a researcher to truly listen and form a relationship with the teacher. While most existing research listens to teachers through an analytic construal that emphasizes what a teacher knows or believes, a deeper listening comes through a narrative construal that emphasizes self-authorship and storytelling (Bruner, 1996; Gee, 2000; Hiebert, Gallimore, & Stigler, 2002).

The Photo-Elicitation Interview (PEI)

When researching such a broadly defined idea such as identity, specific anchoring structures are needed in order to unpack a teacher's notion of their own identity. Vygotsky (1978) wrote about the need for physical anchors, images or artifacts that helped situate the exploration of a child's development from an inner speech to a social speech. Using only observations and interviews with no anchoring structures, Van Zoest and Bohl (2002) were only able to use their construct of Mathematics Teacher Identity to analyze a single teacher because of the complexity of framing the teacher's multiple communities. The use of a structured math story/autobiography helped Drake (2006) unpack the identity of the pre-service teachers she worked with. And in their exploration of science teacher identity, Enyedy, Goldberg, and Welsh (2006) found that identity was best studied when analyzing *teaching dilemmas*–conflicts that come up in the classroom with no easy resolution. Therefore, any good study of mathematics teacher identity requires a formal anchoring structure.

The Photo-Elicitation Interview (PEI) is a structured research method that can anchor this exploration into identity through listening to teachers' narratives. The PEI is a qualitative research method in which researchers introduce photographs, either selected by the participant or the researcher, into the interview context (Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006; Harper, 2002). The PEI successfully elicits identity for a number of reasons. First, visual and creative methods like the PEI are especially useful for studying identities (Brown, Wiggins, & Secord, 2009; Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006). Second, visual and creative research methods open up authentic and socially imaginative spaces in a non-invasive way that honors teachers' busy schedules and responsibilities (Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006; Harper, 2002). Third, the PEI holds a strong history of studying identity within other social sciences, such as nursing and anthropology research (Hansen-Ketchum & Myrick, 2008), and is only recently being used within education research (Clark-Ibanez, 2004). Finally, the PEI excels in generating a narrative authored by the research participants themselves (Brown, et al., 2009; Clark-Ibanez, 2004; Gauntlett & Holzwarth, 2006).

This moment in time is perfect for utilizing the PEI as current technology has finally caught up to the theory–almost all teachers own mobile phones with attached digital cameras and have little issue in using them throughout their everyday teaching routine. In my own experience, I have found that PEI opens up narrative in ways that are respectful of teachers' time and responsibilities (Chao, 2010, 2011).

Methods and Procedures

To investigate mathematics' teacher identity, I gave two Algebra I teachers digital cameras and a loose prompt to "capture your world as a mathematics teacher" in at least 20 photographs. I then set a date three weeks later to sit for a formal Photo-Elicitation Interview. One teacher had taught for fourteen years and identified as female and Indian-American. The other teacher had taught for three years and identified as female and white. During the three weeks, I visited and observed each teacher's classroom to get a feel for her classroom teaching practice and to answer any questions they might have regarding the research protocol. The day before each interview, each teacher chose her ten most important photographs. I then sat with each teacher individually for a Photo-Elicitation Interview in which they shared their photographs one at a time while I asked non-judgmental and non-evaluative questions (Johnston, 2004).

Each interview was videotaped and transcribed for deeper analysis using a grounded theory coding structure to elicit emerging themes of mathematics teacher identity (Corbin & Strauss, 2007; Merriam, 2009). The main data source was this transcribed interview of each teacher, which I analyzed in Transana using a qualitative coding scheme built upon the emerging themes of narrative construction, self-authoring, power, positioning, and hegemony (Bruner, 1996; Corbin & Strauss, 2007; de Freitas, 2008; Gee, 2000; Hiebert, et al., 2002; Merriam, 2009). While the actual photographs that teachers captured as well as the notes I took during classroom observations helped to fill in my knowledge of each teacher, I did not use either of these data points as a source for direct analysis.

Both teachers were selected from a larger Academic Youth Development research project (Bush-Richards, et al., 2011). I had previously worked with both teachers through professional development workshops and classroom observations I helped facilitate, and selected them because they were interested in exploring and talking about their identities and mathematics teaching. Basically, I chose teachers who wanted to tell their story.

Results

Because of space limits, I will only show a snippet from the interview with Karla, the teacher from the introduction, and generalize the overall results from both teachers. The third photograph Karla presented was a school hallway display that showcased academic success, called the *Hall of Honor* (Figure 1). After pulling this photograph up, the dialogue over the next four minutes shifted quickly from a teacher talking about her pride in showcasing good academic work to her feelings of sadness that her career as a mathematics teacher had robbed her of an opportunity to start a family. During the following exchanges, I only asked non-judgmental and non-evaluative questions (Johnston, 2004), such as, "Tell me more about that," or, "How does this relate to your math teaching?"

Figure 1. **Photograph 3 in Karla's Interview. This photograph of a 'Hall of Honor' window display elicits a deep narrative that reveals feelings that the educational institution forced Karla to sacrifice personal happiness for her career, even threatening her ability to start a family. All identifying features of students or the school have been blurred out.**

In just four minutes, the discussion gets very personal and reflective of the way Karla is positioned within the various institutions she interacts within. She moves quickly from discussing the hard work and sacrifice it takes to be a "good" teacher, her feelings of society's expectation of mathematics teachers, to how her professional identity has negatively affected her ability to get married and become a mother.

[00:17:57] Karla: And bringing the news camera in my room and stuff. **But I don't want to work 18 hours a day anymore, and I just didn't feel like I wanted to change the system myself.** So, like that's a systemic issues. That's a cultural issue. That's . . .

. . .

[00:19:40] Karla: **I need a life. Otherwise, I'm going to regret like . . . I would spend a decent amount of hours here and do a pretty darn good job.**

[00:19:56] Me: Uh-huh <agree>.

[00:19:56] Karla: I would say I'm an above average teacher with the amount of hours I put in. But not an excellent teacher. **And I could be an excellent teacher, but then my personal life suffers so much.** That's why I started working at the gym.

[00:20:13] Me: Uh-huh <agree>.

[00:20:13] Karla: **That's why I started doing other stuff. Because I wasn't meeting men. I wasn't like . . . I was working, working, working. It got ridiculous.**

. . .

[00:20:32] Karla: **That's not fair for society to ask us to do that. You hear all the stories about "hero teachers" <finger quotes>, about this and that. You know what? I don't want to be the hero, because it's just sacrificing yourself. <laughs> I just want to be good. And that's it.** So . . .

. . .

[00:20:57] Karla: **I wanted to get married. I wanted to, you know. I feel like that's why I got married so late in life.** Like, because <laughs>

. . .

[00:21:29] Karla: Yeah. But, **there's a big difference in trying to have kid when you're 35 and when you're 30.**

This snippet showcases something that happened in both interviews, the elicitation of a deep identity narrative in a relatively short amount of time. First, the PEI opened up emotional wells quickly; both teachers cried during the interview and repeatedly stated that this was the most reflective and personal professional development they had ever encountered. Second, both teachers realized specific practices they would change immediately as a result of the interview. Third, the interview helped the teachers situate the frustrations they felt into a formal structure of power. They saw how they were positioned as mathematics teachers in society and started to explore the relationship between their personal and professional identities. Finally, the PEI drew me into each teacher's narrative through the opening up each teacher's *vulnerable 'I'* personal identity (de Freitas, 2008). I moved from the role of researcher into privileged observer of each teacher's life, listening and responding with authenticity and compassion as the teachers articulated their true feelings of powerlessness. The more room I gave teachers to talk about themselves and how they saw themselves, the more the teachers wanted to, and needed to, talk about who they were. I had to cut each interview off after two hours even though both teachers wished they could keep going.

In answering the first research question, this study showed how the PEI was able to quickly reveal each algebra teacher's uniqueness in her own personal identity. However, both teachers revealed that they felt that what made them special was rarely brought into their professional identity. Both teachers expressed how they felt locked into playing up their role of "mathematics teacher", leaving little room for their real selves to emerge. This outwardly defined role encouraged them to build an "anonymous" professional identity built not upon how they saw themselves, but false stereotypes of what it meant to be a mathematics teacher (de Freitas, 2008; Van Zoest & Bohl, 2005).

In answering the second research question, the PEI helped each teacher self-author their narrative and recognize their own access to power. In their narratives, they positioned themselves as having relatively little power through their profession. Even worse, both teachers revealed that the less they tried to incorporate unique attributes into their teaching, the easier it was for them to teach. Conformity is not only encouraged, but also rewarded with less stress and a more stable career. This reveals tremendous information about the amount of power these algebra teachers felt they held: none.

Discussion and Conclusion

This investigation help me realize that the idea of mathematics teacher identities, while highly theorized in the literature, is something that only fieldwork and a narrative-based, creative methodology can answer. I found that understanding teachers' identities came from allowing teachers to author their own narrative using their own words, ideas, and visual images. Additionally, it was important for me as a researcher to meet teachers within their space and to approach them as owners and authorities of this space, rather than asking them to come into my world and play by my rules. I cannot fully understand exactly what a teacher means when they tell me something. But when they create a visual glimpse into their world, something they can both define and describe, the interactions between teacher and researcher become real–we end up looking through the same window together. They are the tour-guide and I am the tourist. However, the fact that I worked with both teachers prior to this study complicates things. I selected each teacher personally, and my relationship with each teacher affected their openness and trust with me. During analysis, I could not help but use this prior knowledge to filter and add depth to their narrative.

I hope this study adds to the existing research on understanding mathematics teachers through the lens of identity. I have experimented with a tool that might help us better explore mathematics teacher identity in a minimally invasive and easily implementable way. I hope this paper spurs further exploration of the PEI method to help us better understand the conflicting personal versus professional identities of our mathematics teachers.

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Teachers making sense of algebra problems:

"IT depends what the students did"

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How do teachers determine what a mathematics problem is about? Groups of teachers, mathematicians, and mathematics educators were separately tasked with categorizing a set of twelve algebra problems. Teachers appealed to student activity, real or hypothetical, as a means of making sense of and justifying claims about the problems. Both the scale and nature of the role of students in teachers' discussions contrasted with the discourse of the other professional groups. This has potential implications for cross-disciplinary interactions including teacher training and communication through curricular materials.

Introduction

A teacher's interactions with mathematics curricular materials are shaped by varied and complex factors such as her/his knowledge and beliefs as well as by often subtle features of the curricular materials. This paper reports on research which addresses questions about how teachers make sense of curricular materials. In particular, it examines the means by which algebra content was analyzed at two workshops in which algebra teachers, mathematicians, and mathematics educators were separately tasked with discussing and categorizing middle/high school algebra problems. The teachers, as compared to the other professional groups, made more frequent reference to students, real or hypothetical, as they negotiated the categorization. Furthermore, student actions served as justification for the teachers' claims about the nature and focus of specific algebra problems; this was not observed within the other two professional groups.

The sites for the present study were two 3-day workshops comprised of mathematicians, mathematics educators, and teachers from geographically diverse locations. The teachers were middle and high school teachers of algebra; the mathematicians and mathematics educators were university professors (See Table 1). The workshops were organized around a set of 12 algebra problems spanning content from pre-Algebra to Algebra II which were compiled by the workshop organizer Chris Norris, a mathematician, and administered in the participating teachers' classrooms. Norris was the only person who participated in both workshops. The data presented here is primarily drawn from a particular task in which the workshop participants, separated by professional group, analyzed and categorized the 12 algebra problems. This categorization task required each professional group to negotiate and justify categories for the set of 12 problems; the instructions for the task were summed up by Norris as, "So what do you think is the sort of mathematical content of the problems and do they fall naturally into different groups?"

Workshop	# of Teachers	# of Mathematicians	# of Math Educators	Total
1	4	6	4	14
2	8	5	0	13

Table 6. Number Workshop Participants and their Professional Groups.

Perspective

The categorization task can be thought of as mathematicians, mathematics educators, and mathematics teachers *interacting with* the workshop algebra problems. As such, there are three overlapping categories of research which comprise the theoretical framework of the study: (1) research on interactions with curricula, (2) research on the learning and teaching of school algebra, and (3) research on the beliefs, knowledge, and dispositions of the three professional groups. Herein, I use Remillard's (2005) definition of curricular materials as "printed, often published resources designed for use by teachers and students during instruction" (p. 213) and the present study is, in part, framed by Remillard's description of curriculum use as a participatory relationship between teacher and curriculum. This relationship is influenced by characteristics of the individual (beliefs, pedagogical content knowledge, etc.) and by features of the curriculum (voice, representation of concepts, etc.). Her framework acknowledges that teachers construct their understanding of curricular materials yet are also influenced by these materials.

The inclusion of research on school algebra is motivated by the acknowledgement that some view of school algebra influenced the workshop participants' interactions with these problems. Kieran (2007) described the content of school algebra as ranging between reform-oriented and traditional. Reform-minded algebra often is characterized by an emphasis on functions, real-world problems, and a value on multiplicity of techniques (such as the use of technology). Included in this the view of algebra are the multiple ways of representing functions including graphical, tabular, and symbolic representations. A traditional view conceptualizes algebra as generalized arithmetic and often focuses on symbolic manipulations, recognition of forms, simplification of expressions, solving equations, and factoring polynomials. The traditional view puts an emphasis on letter-symbolic and structural aspects of algebra whereas reformed algebra emphasizes a combination of mathematical representations.

Many factors influenced the workshop participants' interactions with the problem set. These factors include the features of the problems as well as less tangible characteristics of the workshop participants. Thus, the theoretical perspective which frames this study also includes research on these disciplinary groups' beliefs about and dispositions toward mathematics in general and algebra in particular. The few studies about mathematicians' and mathematics educators' beliefs about mathematics indicate a diversity of views (Mura,1993, 1995; Burton, 1999). Some mathematicians (e.g., Bass, 2006; Cuoco, 2001), however, have endorsed a more structure-oriented approach to algebra. Similarly, there is not much conclusive to say about teachers' beliefs about algebra. Doerr (2004) noted that "there has been little research about teachers' knowledge and practice and its development with respect to the teaching and learning of algebra" (p. 285). Doerr's survey of such research on teachers' algebraic knowledge revealed "no research evidence in the USA that would suggest that teachers see the concept of function as an integrating theme for algebra instruction across the curriculum despite this being envisioned in the curriculum standards of the National Council of Teachers of Mathematics" (p. 278).

The discussion which follows also draws upon the Mathematical Knowledge for Teaching (MKT) framework of Hill, Ball, and Schilling (2008). By distinguishing subject matter knowledge from pedagogical content knowledge (PCK) and then further unpacking each of these into smaller components, it provides a finer-tuned means for describing the knowledge which workshop participants drew from as they discussed the algebra problems. Of particular relevance is the notion of knowledge of content and students (KCS), a component of PCK, as "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (p. 375).

Methods

Data included audio and video recordings from both workshops, audio recordings of interviews with all but one participant from the second workshop, and observational data/field notes from the second workshop. The open-ended interviews investigated participants' reactions to the workshop and their beliefs about algebra. Transcripts of the categorization task and the interviews were coding in three distinct phases as codes were refined. Codes were based both on the theoretical framework and on themes which emerged in the data. In general, I collected multiple data sources and subjected them to methodical, iterative analyses while simultaneously documenting my reflections and decisions. Member checks were conducted with three of the participants (one from each professional group).

Results

Analysis of the categorization task revealed that all three groups drew from diverse types of knowledge in order to justify claims or observations about the 12 algebra problems. However, in both workshops, teachers, more than members of the other two professional groups, used ideas about real students or their views of typical students to frame their discussion of the algebra problems and to justify their claims about the problems. During both workshops, the teachers referenced students during their discussions of every problem except for problem number one during the first workshop (they spoke about this problem for less than 30 seconds total). The teachers at times referred to their own students in their discussions of the algebra problems; for example, Nadya King, in discussing number four (See Figure 16), said, "So my kids didn't know what it meant 'for increasing values of *a*". Teachers also drew upon knowledge of content and students (KCS) to talk about students in general or about their own conceptions of typical students. Tom Luft, for example, said "I also think that kids don't understand that multiplying by one fifth is dividing by 5 very well" when discussing problem number five, a problem which the teachers subsequently categorized as being about "rational number operations" in the first workshop.

In the expressions below, *a* and *x* are positive numbers. For each expression explain the effect of increasing *a*: does the value of the expression increase, decrease, or remain unchanged?

	Increases	Decreases	Remains unchanged
$ax+1$			
$x+a$			
$x-a$			
$a+x-(2+a)$			

For each expression, explain why you made the choice you did.

Figure 16. Problem Number 4.

The mathematicians and mathematics educators also engaged in discourse related to students and student work. They, of course, were less familiar with the work which students did in the participating teachers' classrooms and they referenced these students or classroom experiences less frequently. Indeed, the data and coding revealed that teachers had a greater focus on students; this was particularly notable in pronoun usage across groups. The teachers would talk about what "they", the students, would or should or did do for a particular problem. Table 7 shows the frequency with which members of each professional group used certain words which reference the students. A caveat of this word-count pertains to the use of the words "they" and "them"; the word count was not sufficiently sophisticated to detect the noun which each instance of these words referenced, however sampling indicated that the overwhelming majority of the usages of these words referred to "the students". Likewise, there were instances where students were referred to but none of the four words included in the word-count were used; for instance, phrases like "mine did" or "a couple did" were not detected. The word count presented below is an imperfect metric but the scale of the difference in rates of usage of words referencing students is quite telling. Teachers used these words more than 500 times per hour. In contrast, mathematics educators used these words 205 times per hour and mathematicians used them 159 times per hour.

	they	them	student(s)	kid(s)	total
Teachers WS1	474	130	26	129	759
Teachers WS2	308	75	50	72	505
usage per hour	312.8	82	30.4	80.4	505.6
Math Educators WS1	119	79	73	37	308
usage per hour	79.3	52.7	48.7	24.7	205.3
Mathematicians WS1	103	51	58	10	222
Mathematicians WS2	79	46	51	0	176
usage per hour	72.8	38.8	43.6	4	159.2

Table 7. Word counts from the categorization tasks, 90 minutes in the 1st workshop (WS1), 60 minutes in the 2nd (WS2). "Usage per hour" is calculated for each professional group using totals from both workshops.

The difference in how professional groups referenced students was not limited to a difference in quantity. In contrast to the two other professional groups, there were instances amongst the teachers in which students were used as a means to justify claims about particular problems. An illustrative example is the teachers' discussion of problem number two (Figure 17 below) during the first workshop; Ursa Harper justifies her interpretation of the question by referring to the mistakes her students made. She is a bit dismissive of Sheila Eastman's claim that the problem is about the distributive law because that is not what Ursa perceived her students to have struggled with. It is worth noting that Sheila was the only person in the teachers' group who did not administer the problems to her own class.

Sheila Eastman	You know, with what [the question writers] did here truly it's just the distributive property because they wanted them to just come up with one [possible first step].
Ursa Harper	But what the kids made mistakes with were order of operations. They also made mistakes on like terms was the big one for my kids.
Sheila Eastman	Okay, order of operations, like terms, distributive property.
Ursa Harper	Bigger than distributive property was order of op, was the like terms.

Sheila Eastman	Oh, okay.
Ursa Harper	I was shocked when I looked at it [the student work]. I was like "excuse me".

This discussion lead directly to the categorization of number two as a problem about "properties of real numbers" which, for this group of teachers, included concepts such as order of operations and like terms.

You are simplifying *7-2(3-8x)*. Which of the expressions are possible results after the first step?

	Possible	Not Possible
5(3-8x)		
7-2(-5x)		
7-6-16x		
7-6+16x		

For each expression, explain why you made the choice you did.

Figure 17. Problem Number 2.

A similar discussion took place about problem number four (Figure 16 **Error! Reference source not found.**). Ursa cites student errors with the distributive law and number sense as the justification that the problem is about these properties. Tom Luft and Tammy Vonce contributed to the discussion by suggesting other ways in which students could approach the problem, namely ideas about graphing, functions, and signed numbers.

Tom Luft	I would, um, extend that to, um, the idea of a function because I think the way my kids approach it is more of an input-output style…
Tammy Vonce	Put in different inputs, look at the outputs.
Tom Luft	Input-output, they looked at in slope-intercept form. You know, they saw *ax+1* and they said, well, this slope's positive, it's going up, this slope's negative it's going down, that kind of thing. So I mean I think that's an underlying principle. Whether or not all kids would see it that way, depending on their experience and stuff, I would think that most kids would just plug in and just see what happens.
Tammy Vonce	Well, you know, my kids, now that you're saying that it reminds me, they looked at it as signed numbers.

This problem was ultimately categorized as being about "signed numbers". The categorization of this problem was negotiated largely through an examination of what students did with the problem.

Generally, the teachers discussed multiple ways in which students could and did solve each of the problems. This was, in part, what lead the teachers, in both workshops, to place individual problems in multiple categories (something which the other professional groups did not do). Their interactions with the workshop problem set were framed to a large extent by their experiences with the problems in their classrooms. This was not the sole influential factor but it was prominent in their discussions as they engaged in the categorization task.

Mathematicians and mathematics educators also referenced (real or imagined) students as they interacted with the problem set. However, this happened less frequently as indicated by

Table 7 and was rarely a sustained focus of their discourse. Perhaps it would have been a larger focus if they had administered the problem set in classrooms of students but this is unclear from the present data. The teachers based their analysis of the problems, in a large part, on what their students did.

Discussion

The present study is largely framed by Remillard's (2005) idea that teachers *interact with* curricular materials. This idea has been extended in the present analysis in order to view the workshop categorization task as mathematicians, mathematics educators, and teachers interacting with algebra problems. By framing these activities as interactions, my intention has been to highlight that the nature of the problems was determined by the participatory relationship (as Remillard calls it) between the workshop participants and the problem set. Her framework for teacher-curriculum relationship is intended as a framework for studies on curriculum use. It should be noted that the workshops were not exactly about curriculum use. Although the participating teachers had implemented the problem set in their classrooms, the decisions they made regarding implementation were neither the focus of the workshops nor of the research and analysis described herein. Furthermore, the problems were presented absent of the organizational structure (e.g., section or chapter headings) typical of classroom materials (though perhaps typical of standardized tests). However, the results described above are relevant to curriculum use; though the categorization task was removed from the typical environment and context in which teachers interact with curricular materials, the workshops provided insight into the ways in which teachers interact with curricular materials. For example, teachers made sense of the problem set through frequent reference to and guidance by their perceptions of students.

This contrast between the professional groups in how they made sense of the algebra problems may have also manifested itself in the groups' differing emphases on graphs. As he revealed in his interview, Chris Norris, the compiler/author of the problem set, did not consider graphing to be a prominent theme in the 12 problems or an important piece of algebra in general, though he did acknowledge that graphs can be useful tools for doing algebra. However, both groups of teachers spoke about graphing in relation to five of the problems. For Chris, a major theme within the workshop problem set was the structure of expressions and equations. This theme was discussed amongst the mathematicians and mathematics educators but was generally not discussed amongst the teachers. It is perhaps necessary to clarify here that I do not present the opinions of Chris Norris in contrast to those of the teachers in order to make a statement about the correctness of the teachers' interpretations of and interactions with the problem set. On the contrary, I wish to highlight the inevitability that interactions with curricular materials are interpretative acts; an individual encounters curricular materials through her or his own subjective filter. In light of this, questions about fidelity or correctness of interpretation are not relevant to my analysis and are potentially inconsistent with my perspective that workshop participants *interacted with* the 12 problems.

The teachers' discussions of the 12 problems differed from Chris' intentions (and from the discussions by mathematicians and mathematics educators) as they pertained to graphing and to structure. Is it possible that this is connected to the prominence of the roles of students in their analyses of the problems? For instance, structure was a focus for both the mathematicians and the mathematics educators in the first workshop as they discussed and categorized problems 10 and 11. Teachers from both workshops discussed graphing in relation to these problems which they perceived as difficult for the students. The teachers often spoke about graphing in terms of its pedagogical efficacy and as a strategy for students to approach a problem. Tom Luft discussed how students "have to have a visual connection". Ursa Harper described number 11 as "straight-forward" for students who have experience with graphing calculators. Thus, it is possible that the prominence of teachers' perceptions of students in their interactions with the workshop problems was, in part, responsible for these differences in focus with the mathematicians and mathematics educators. That is, their greater emphasis on graphing and their lesser emphasis on structure may have been related to their focus on how students would or did approach the problems.

In the interviews of the second group of teachers (none were conducted with the first group), six out of seven described algebra in a traditional way (e.g., generalized arithmetic) as did Chris Norris and all of the interviewed mathematicians. However, graphs are generally associated with reformed approaches to algebra. By emphasizing graphs, were the teachers infusing the workshop problems with a view of algebra different from that intended by the problems' author? The answer to this is unclear especially since the interviews did not reveal particularly reformed views of algebra. There is little evidence that the teachers' emphasis on graphing was a sign of a reformed, function-oriented view of algebra; rather, it may have been motivated by a concern for how students did or should approach the problems. This is certainly consistent with Doerr's (2004) report that there is "no research evidence in the USA that would suggest that teachers see the concept of function as an integrating theme for algebra instruction across the curriculum" (p. 278). In categorizing these algebra problems, the teachers' perceptions of students, and their knowledge of content and students (KCS), may have played a larger role than their stated beliefs about algebra.

Chris' intentions to focus on structure and to deemphasize graphs were not acknowledged within the teachers' categorization tasks at both workshops. This, once again, highlights the interpretive nature of interactions with curricular materials. Furthermore, there are associated implications for developers of curricular materials. Chris' development of the problem set was guided by his opinions about the nature of algebra. The problems were distributed to the participating teachers with no explanation of his intentions or viewpoints and with no instructions for implementation. For the most part, his intended foci for the problem set were not implicitly communicated to the teachers through his presentation of the problems. In order to have successfully communicated his intentions *through* these curricular materials to the teachers it would have been advisable to have considered the role that student activity plays in teachers' appraisals of algebra problems. At least within the context of the workshop, a domain-centric lens was insufficient for consistently communicating ideas to teachers through curricular materials.

Of the three professional groups at the workshops, the teachers were the only group to have administered the problem set to students and were the most familiar with the student work. This surely was a factor in their greater emphasis on student activity during the categorization task. Perhaps the results of this study would have been different had the teachers not implemented the problem set. However, this potential caveat is not particularly relevant to the discussion herein; by the nature of their craft, teachers have access to information about student activity and are concerned with implementation of curricular materials. It is important for developers of curricular materials and for instructors of pre-service and in-service teachers to acknowledge that teachers' interactions with algebra content may be largely framed by their relationships with and perceptions of students; that is, they may draw heavily from their knowledge of content and students (KCS). It would be of use to study this idea further within the context of classroom curriculum use.

During the categorization tasks, perceptions of students played a larger role for teachers than for the other two professional groups. I hypothesized that this was at play in teachers' greater emphasis on graphing; this focus may have resulted from a perceived appreciation for the pedagogical efficacy of graphs more than from views of algebra which embrace functions and graphs as central themes. Furthermore, I have presented a view across professional groups which may inform collaborations and communication across professional groups. It may move us closer to the interdisciplinary cooperation which Bass called for in 1997:

> As mathematical scientists, as mathematics education researchers, and as teachers in universities, colleges, community colleges and schools, we must begin to see our concerns for graduate, undergraduate, and K–12 education as parts of an integrated educational enterprise in which we have to learn to communicate and collaborate across cultural, disciplinary, and institutional borders, just as we are called upon to do in mathematical sciences research. (p. 21)

Endnotes

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STATE CONFERENCE PRESENTERS' CONCEPTIONS OF REFORM IN MATHEMATICS

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This study reports on how the presenters at a conference sponsored by a state affiliate of the National Council of Teachers of Mathematics aligned with literature on reform mathematics. Nonparametric methods were used to compare conceptions on three domains: nature of mathematics, learning mathematics, and teaching mathematics. Significant differences in conceptions were found within groups factored by gender, institutional position, and number of years presenting at a state conference. The findings suggest that teacher educators align more with reform literature on conceptions of learning and the nature of mathematics, while female and male presenters differed in their conceptions of teaching.

The modern standards movement began with the National Council of Teachers of Mathematics' (NCTM, 1989) publication of the first curricular standards: *Curriculum and Evaluation Standards for School Mathematics*. Smith (2001) claimed the intent of the document was to develop students who could engage in complex mathematical problems, alone and with peers, and who could communicate convincing arguments and justifications for their work. To support the vision outlined in the standards, NCTM published the *Professional Standards for Teaching Mathematics* in 1991 and the *Assessment Standards for School Mathematics* in 1995 (NCTM, 2000). These standards documents have had a substantial impact on teacher preparation programs but have not impacted classroom teaching to the same degree (Price & Ball, 1997). Changing classroom practice to be more aligned with the reform movement in mathematics education remains problematic for teacher education faculty, professional developers, and school district personnel. To continue the conversation about reform-based teaching and standards, to respond to criticisms to the 1989 standards, and to address relevant research on teaching and learning since the original publication of the standards, NCTM published revisions of the documents: *Principles and Standards for School Mathematics* and *Mathematics Teaching Today* (Martin, 2007).

In order to move toward the vision of mathematics teaching outlined in the NCTM standards documents, teacher preparation and professional development opportunities must address not only the mathematics content to be taught, but also elements of effective teaching. But how do practicing teachers come to experience such professional development? One important source of professional development for mathematics teachers is conferences, especially the state and regional mathematics conferences sponsored by NCTM affiliates.

Other disciplines also hold state conferences affiliated with their national organizations. Koballa, Dias, and Atkinson (2009) questioned how the messages teachers receive at a conference sponsored by the state affiliate of the National Science Teachers' Association [NSTA] were aligned with national reform efforts. The researchers found that, even though the session leaders were members of the NSTA state affiliate and were likely considered leaders in their schools, most of their conceptualizations of inquiry did not align with the meaning of inquiry promoted by the NSTA.

After reading these results, we began to question how session leaders at a state mathematics conference, sponsored by the state affiliate of NCTM, conceptualize reform in mathematics education. The present study sought to investigate conference presenters' background knowledge about reform and how they relay their knowledge to session participants. A question of interest in considering such knowledge is how the presenters' beliefs align with those promoted by NCTM.

To this end, four research questions were developed for the study, two of which are addressed in this report:

1. What conceptions of reform-oriented mathematics teaching do presenters hold?
2. How are the presenters' conceptions of reform, modes of presentation, and intended impacts related to each other with respect to the mathematics education reform literature?

We will address some of the qualitative results in our presentation.

Conceptual Framework

Philosophers (e.g., Ernest, 1991; Lerman, 1998) discuss two major theories about the nature of mathematics: absolutist and fallibilist. Whereas absolutists attempt to provide an authoritative account of how mathematics should be understood, fallibilists attempt to describe the field and provide a means of growing the body of mathematical knowledge. Reflecting on the nature of mathematics advocated NCTM documents, Sfard (2003) explained that the documents promoted keeping both the student and the mathematics at the forefront, ensuring that "the needs of the learning child never disappear from the reformers' eyes" (p.353). Reformers promote a view of mathematics as relative, fallible, and expanding rather than a mathematics that is static and absolute. As such, mathematics is a human endeavor that children should actively construct during their school mathematics experiences.

Learning theories of mathematics are rooted in specific perspectives of the nature of mathematics. In mathematics education reform literature, learning mathematics is an active process in which students build knowledge through social interaction, discourse, conjecturing, problem-solving, and validating. NCTM (2000) identifies problem solving, reasoning and proof, communication, connections, and representation as processes for promoting the development of mathematical understanding. In their support of the NCTM Standards, Ball and Bass (2003) argue that mathematical reasoning is a basic skill necessary for learning mathematics with understanding. All students can learn mathematics when they are invited to investigate its nature in their own way, when it is situated in culture, and when it is scaffolded by individuals who understand the tacit and concrete dimension of mathematical knowledge (Lampert & Cobb, 2003; Martin, 2007; NCTM).

Because learning mathematics should be an active process wherein students build their own understanding, the teacher's role in helping students develop that understanding cannot be underemphasized. The teacher's role in reform-oriented teaching is active, reflective, and cognitive and must shift from the dispenser of pre-determined knowledge to a facilitator of creative and generative processes. The teacher must be active in managing the discourse in the classroom, which requires "planning that involves capitalizing on what students do and directing their activity toward important mathematical issues" (Lampert & Cobb, 2003, p. 244). Teaching should result in students learning mathematics with conceptual understanding and should be "centrally guided by students' ideas" (Ball & Bass, 2003, p. 30). Furthermore, student differences should be respected and valued; reform teachers modify their instruction based on the needs of the learners in their classrooms. They use worthwhile tasks that reflect a high level of cognitive demand on the part of students (Stein, Smith, Henningsen, & Silver, 2009) and should require students to engage in authentic problem solving. Reform teachers view teaching as problematic and reflect on their practice, often in systematic ways in communities of practice (Martin, 2007).

The NCTM's impact on standards-based reform in learning and teaching mathematics is profound and ongoing. Professional communities that provide discourse and professional development for mathematics educators also contribute to the changing landscape of reform-oriented mathematical content and instructional practices. Specifically, educational leaders who present reform-oriented practices to other practitioners help to form, model, and foster accepted truths of mathematical knowledge within such professional communities (Ernest, 1999). This study hopes to explore how such presenters at the state conference conceptualize and demonstrate reform-oriented mathematics during workshop presentations.

Methodology

To help us determine how presenters' beliefs align with reform literature, qualitative methods and a survey were used to collect data.

The study was conducted during fall of 2010 at a state mathematics conference which annually hosts approximately 1900 attendees. There were 188 sessions spanning grades P-16, held over three days. The conference program was used to identify lead presenters. Session descriptions were used to categorize presentations that addressed various aspects of the nature of mathematics, learning mathematics, or teaching mathematics. One hundred forty sessions met the categorization criteria, and the lead presenters of those sessions comprised the survey sampling frame. Sixty presenters completed the survey for the study. We verified that the sample proportionally represented the presenter demographic position (7 resource personnel = exhibitor or publisher; 12 support personnel = instructional leader, coach, or administrator; 13 teacher educators = professional developer or pre-service program faculty; and 28 active classroom teachers).

A 15-item survey was designed to measure conceptions about reform mathematics, based on findings from Raymond's (1997) study on teacher beliefs. Items contained phrases contrasting more traditional (choice *a*) statements, the negative pole of the scale, to more reform (choice *b*) statements, the positive pole, regarding dispositions on the nature of mathematics, and mathematics learning and teaching. Presenters responded to a five-point Likert-type scale (1 = *Agreement with (a)*, 2 = *Towards (a)*, 3 = *Equal agreement with (a) and (b)*, 4 = *Towards (b)*, 5 = *Agreement with (b)*). The survey was pilot tested with mathematics educators and teachers to determine the appropriateness and clarity of the items. It was then administered online, opening approximately one month prior to the conference. Consent for participation in the study was gained and assured by electronic signature using email addresses. To address non-response, follow up requests to participate were sent electronically, along with personal reminders to complete the survey during the conference.

We disaggregated the data by presenter's gender, institutional position, highest degree earned, and conference presentation experience because prior research revealed that the *type* of provider was not an adequate predictor of session quality; rather, "the skills, background, and preparation of professional development providers" (Banilower, Boyd, Pasley, & Weiss, 2006, p. 35) were greater determinants of effective professional development. Graphical displays, within each domain, were first examined holistically and then disaggregated by the above factors, to examine the overall alignment of survey takers with reform literature.

Results

Reliability for the conceptions of reform mathematics scale was measured by computing Cronbach's Alpha (0.602). Utilizing "Cronbach's Alpha if Item Excluded" analysis, Cronbach's Alpha (0.720) increased with the elimination of four items from the instrument. The adjusted coefficient demonstrated sufficient reliability for the scale. Questions (Q1-Q3) were related to the nature of mathematics, (Q6-Q8, Q10) comprised the domain on learning mathematics, and the teaching of mathematics was assessed using questions (Q10-Q13, Q15). The item means are presented in Table 1. Higher means for an item indicate a conception towards reform mathematics. For comparative analysis, nonparametric tests were used because these methods are better suited for ordinal data, they are appropriate for unequal or small groups, and they do not require the assumption of normality for the population distribution from which the sample was drawn. An alpha level of 0.05 was used for all Kruskal-Wallis Rank Sum Analysis of Variance test. Significant results are presented below.

Domain	**Nature of Math**			**Learning Math**				**Teaching Math**			
Question	Q1	Q2	Q3	Q6	Q7	Q8	Q10	Q11	Q12	Q13	Q15
Response Mean	3.68	2.97	2.95	4.05	3.23	4.19	4.24	4.66	3.64	3.14	3.78
No. of Responses	60	60	60	60	60	59	58	58	59	58	60

Table 1. Item Response Averages

Conceptions of the Nature of Mathematics

Rank orders pertaining to the nature of mathematics were found to be significantly different among groups when factored by number of years presenting at the state conference (see Table 2). For all groups, ranking tended towards mathematics as changing, fallible and relative, as opposed to fixed, absolute and predictable. A consistent trend across all items within the domain was a higher score mean for the group 2-3 years (M_{Q1} = 33.12, M_{Q2}= 34.08, M_{Q3} = 38.31) than that of the group First year (M_{Q1} = 30.4, M_{Q2} = 26.55, M_{Q3} = 23.40). One possible rationale for this finding relates to exposure to the conference settings and to others who discuss effective teaching and learning. Some presenters may have presented as a class requirement or as a way to get more involved in their state organization. After that first experience, presenters may express perturbations in their conceptions based on feedback from participants in their sessions or they may consider their practice in greater depth before they present at the conference in the future.

Level	Count	Score Sum	Score Mean	ChiSquare	DF	Prob>ChiSq
First	20	468.000	23.4000	8.0680	3	0.0446*
2 – 3	13	505.500	38.8846			
4 – 6	9	230.500	25.6111			
>6	17	566.000	33.2941			

Table 2. Kruskal-Wallis Test: Nature of Math by Years Presenting at Conference

The respondents struggled with the dichotomies provided that sought to tease out the conceptions of the nature of mathematics: “I struggled with the relative/absolute statement and the doubtful/certain statement. My faith has certainly swayed my opinions, but because I understand math, I tend to be in the middle about some of my answers.” Additionally, the responses were influenced by their classroom practices: “I think that mathematics had a definite body of knowledge...however, the way we teach and learn mathematics varies. Exploring mathematics can lead to new discoveries and understandings about how and why mathematics works the way it does.”

Conceptions about Learning Mathematics

The four items assessing conceptions of learning mathematics showed closer alignment with the reform literature. For the learning domain, a significant difference in score means by position (see Table 3) prompted an inspection of item means. On item Q6, which contrasts learning through skill and drill with learning through investigation, a significant difference was detected ($\chi 2 = 10.48$, $p = 0.015$), where the greatest differences was between teacher educators (M = 41.31) and teachers (M = 24.13). Although no other significant differences were found, on two of the remaining items, teacher educator score means ($M_{Q7} = 33.15$, $M_{Q8} = 35.38$) were greater than teacher ($M_{Q7} = 28.68$, $M_{Q8} = 24.72$). For teacher educators, conceptions of learning mathematics tended towards choice *b*, where learning by investigation, group work, with a good teacher, and by doing challenging tasks were ranked higher than learning by skill and drill, independent work, being a strong student, and memorizing. Teachers tended to rank items in this domain as being equally important for learning mathematics.

Level	Count	Score Sum	Score Mean	ChiSquare	DF	Prob>ChiSq
Resource	6	182.500	30.4167	9.1998	3	0.0267*
Support	10	273.000	27.3000			
Teacher	22	425.500	19.3409			
TEducator	13	445.000	34.2308			

Table 3. Kruskal-Wallis Test: Learning by Position

The following quotes highlight some of the differences in these two groups’ conceptions:

Teacher Educator: “I firmly believe that learning and understanding mathematics requires students to be actively engaged in mathematics. They need the opportunity to develop their conceptual understanding of mathematics. Drill, lecture, and practice don't work.”

Teacher: “Mathematics is a combination of memorization/ skill & drill to learn the basics and exploration once the basics are concrete.”

This finding seems contrary to those by Banilower et al. (2006) that type of presenter was not an indicator of quality; however, we did not analyze the backgrounds of each teacher educator, which is what Banilower et al.’s research claims is the greater determinant.

Conceptions about Teaching Mathematics

The results for conceptions about teaching mathematics were in the middle of the traditional-reform scale (see Table 1). The exception was the question that generated the most consistent results, Q11: “Good mathematics teaching is (a) guided primarily by a textbook and its supplemental materials or (b) by a variety of instructional materials from various sources.” Strong agreement with choice *(b)*, ($M_{Q11}= 4.66$) may be explained by the present curriculum and materials being used in the state in which the conference was being held. The state is rolling out an integrated curriculum based on the process standards for which quality texts were not readily available. Teachers saw this as a challenge to implementing the new curriculum: "one challenge was pulling together the materials; no textbooks exactly aligned to the standards and materials were *everywhere* on the internet; careful planning for the tasks is required" (Edenfield, 2010, p. 174).

The overall scale analysis showed a significant difference in score means by gender (see Table 4). Females ranked items as "towards *b*", while males tended to rank items as "equally both *a* and *b*". The greatest point of departure for the two groups was on the teaching domain. Although the score means in the domain were not significantly different, the most noticeable variations were on items Q11 ($M_{female} = 30.52$, $M_{male} = 26.81$) and Q15, which contrasts the importance of using multiple examples with using problem solving tasks in teaching, ($M_{female} = 32.76$, $M_{male} = 24.28$). The following quotes by gender illuminate some of the thoughts participants had about teaching:

> *Male1:* "Although the student has to choose to participate in the learning process, the teacher must also meet them halfway by doing their best to provoke them."
> *Male 2:* "Since students have different learning styles and readiness levels, there must be a balance of what they see and what they do. This usually changes with each content, so teachers must constantly assess their students to find the balance."
> *Female 1:* "When teaching mathematics the instruction should not be driven by any particular resource. It should be driven by the teacher. The teacher can plan for flexibility within the lesson and should make explicit any information that does not come up in the student work or classroom discussions."
> *Female 2:* "Problem solving tasks are most beneficial in the learning, practicing, retention, and mastering of math. I equate math problem solving to be quite parallel to life. It is in doing, making mistakes, and analyzing those mistakes that we truly learn."

Level	Count	Score Sum	Score Mean	ChiSquare	DF	Prob>ChiSq
Female	40	1234.00	30.8500	4.6818	1	0.0305*
Male	15	306.000	20.4000			

Table 4. Kruskal-Wallis Test: Overall by Gender

Discussion and Implications

Across all subgroups in the study, the means for the questions related to the nature of mathematics were centered around three on the five-point scale. The participants, therefore, viewed mathematics as balanced between absolute and fallible. Watson (2008) contrasted the discipline of mathematics with school mathematics:

> Mathematics as a discipline, by contrast to school mathematics, is concerned with thought, structure, alternatives, abstract ideas, deductive reasoning and an internal sense of validity and authority…. The concerns of school mathematics pull learners in directions that differ from these. The core activity in school mathematics is to learn to use mathematical tools and ways of working so that these can be used to learn more tools and ways of working later on. (p. 6)

These differences in mathematics as a discipline and mathematics in schools may partly explain our results. We posit that it is possible for a person to view school mathematics as absolute, while viewing the discipline of mathematics as fallible. Our questions did not address this difference, but it is an area for future research. While the mathematics education literature promotes a view of mathematics as fallible, none of the subgroups in the study showed a tendency toward viewing mathematics in this way. Raymond (1997) found that teachers' conceptions of the nature of mathematics are more influenced by their early learning experiences; as most mathematics teaching in the US is not considered standards-based (Stigler & Hiebert, 1999), it is not surprising that the presenters' conceptions of reform fail to align with the reform literature.

The findings on gender parallel similar findings in the field of workplace psychology (deMarrais & LeCompte, 1999). Men and women conceptualize the purpose and function of their jobs differently, and teaching is no exception. Boaler (1997, 2008) also identified gender differences in the students at Amber Hill and Phoenix Park that paralleled findings on brain research on gender differences: Male students valued separate knowledge involving logic and abstraction whereas females preferred making connections involving creativity, intuition and experience. Additionally, women preferred more cooperative environments that encourage sense making, while men preferred more competitive work or learning environments. Given that teachers' beliefs about teaching and their practice are highly influenced by their own learning experiences, their preferred methods of learning might influence their ideas of effective mathematics teaching. Because our sample size was small, we refrain from generalizing our results. Our purpose was not to privilege a particular conception of teaching but rather to describe presenters' conceptions of effective mathematics teaching. We will continue mining literature on gender differences, paying particular attention to studies conducted in STEM fields, to contribute to our understanding of this finding.

For the sample, those conference presenters who were teacher educators were more aligned with reform literature concerning the learning of mathematics than were the presenters who were classroom teachers. Battista (1999) explained that most educators (including teachers, professors of education, and administrators) hold faulty views of constructivism, not the research-based constructivist theory. Some

> conceive of constructivism as a pedagogical stance that entails a type of nonrigorous, intellectual anarchy that lets students pursue whatever interests them and invent and use any mathematical methods they wish, whether these methods are correct or not. Others take constructivism to be synonymous with "discovery learning" … others even see it as a way of teaching that focuses on using manipulatives or cooperative learning. (p. 429)

In mathematics education reform literature, learning mathematics is an active process in which students build knowledge through social interaction, discourse, conjecturing, problem-solving, and validating. One possible explanation for the differences between teachers and teacher educators is that the teachers were not as aware of research on how children learn mathematics or held misconceptions about those theories, as Battista described; whereas the teacher educators may have read more research literature as a function of their graduate work or their jobs as teacher educators. The teacher educators at the conference were probably mathematics teacher educators, and not generalists who teach foundations of education courses, and were hence likely to know more about learning theories specific to mathematics. Another possibility is that many teachers go through teacher preparation programs wherein they discuss student thinking as theoretical. They may not have the opportunities to engage in discussions about research on learning and actual student thinking and learning in the context of their practice. We see this finding as particularly relevant for professional development, including initial teacher development. If, through further research, we see a general trend that teachers' conceptions of reform are more aligned with research literature with respect to teaching than to learning, then professional development is needed.

We know that the conceptions revealed by presenters via the survey may not align with their actual practice or the messages they portray in their presentations. We are in the process of analyzing a subset of presentations and interviews with those presenters. We will search for confirming or disconfirming evidence about the conceptions communicated in the survey but we also expect to get a better understanding of the messages conference attendees receive. We hope to report on these extended findings at the conference. From our observations of sessions at the conference (not having yet completed the analysis), we noticed a lack of focus on worthwhile mathematics in the actual presentations, with the focus being more on using engaging activities with students. Most session presenters were not clear about what they wanted students to learn, and how they wanted that learning to occur when the activities presented in their sessions were used with students. As the findings unfold, we will be able to offer other implications for professional development. For example, in all professional development materials – including workshops and presentations – not only does the goal of the session need to be emphasized but also the mathematical goals for students.

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LIVED AND LIVING MATHEMATICAL EXPERIENCES OF PRE-SERVICE ELEMENTARY TEACHERS: AN EXPLORATORY INVESTIGATION

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The "Developing Investigations of Mathematical Experience" (DIME) research program is focused on building a research-based understanding of the interiorized experiential world of idiosyncratic mathematics. We seek to identify and relate expressed indicators of intended and actual mathematical experiences, as they appear to occur in the complex relationships of participants involved in mathematics teaching and learning. The goals of this case study are to characterize individual experiences in ways that acknowledge a person's active and reflective thinking efforts within problematic contexts linked to emotive dimensions of their lived and living mathematical experiences, in order to inform improved mathematics teaching practices. To probe these largely unexplored complex domains, we are using both phenomenological and constructivist theories and methods to build our investigations.

Today, there is surely an implicit and implied emphasis upon high quality mathematical experiences within the goals and expectations for a sound mathematical education. Yet, within our literature and advocacies for curriculum, for teaching, for learning and development, and for research knowledge there seems to be very little explicit attention to the phenomenal data of an individual's "lived/living mathematical experiences." This frames an important goal of our research efforts.

What is the nature of lived and living mathematical experiences? What are the particular qualities (essences) of human experience that make it a "mathematical experience?" Are these experiential qualities that are unique to mathematics, or are they also found more broadly in other human experiences? In what ways do "lived/living mathematical experiences" vary or differ among humans, and why? What factors related to the individual or the experiential context might frame or affect or impact upon the actual phenomenon of a high quality "lived mathematical experience?" In what ways do the experiences of individuals vary within mathematical contexts, such as classroom situations that are intended to engender the "same" kinds of experiences for all participants? How do "lived mathematical experiences" accumulate within differing individuals, and with what more general intellective and emotive consequences? In what ways do thinking and feeling aspects of "mathematical experience" interact within a lived experiential context, and with what kinds of impacts or outcomes? In what ways might a mathematics educator use what can be learned about "lived/living mathematical experiences" of students, parents, teacher educators, mathematicians, or others in society to improve educational practices?

These are some of the questions that serve as "starting points" for our research program. In addition, we are confronting many research methodological questions that are framing much of our initial struggles to conceptualize what we see as a largely undeveloped research domain for the field of mathematics education.

Purpose of the Study

This study is being conducted with four selected volunteer students enrolled (Spring semester 2011) in a third undergraduate mathematics course (Geometry and Measurement) in a sequence specifically designed to prepare future elementary teachers. These students are also participating with all of the enrolled students in a companion investigation of the impacts of differentiated course instructional approaches on students' mathematical understandings.

The primary research question for this study is: What is the nature and quality of self-reported and analyzed mathematical experiences of students in differentiated classroom situations and in individually presented supplementary geometry and measurement situations?

This individual case-study research will allow us to develop a deeper understanding of the ways that selected students perceive important elements of their course experiences, including their activities in learning mathematical concepts and processes, how the differentiation has impacted their experiences in learning and achieving the course objectives, and what ideas from the course will be useful in their future teaching of mathematics to children. In addition, a variety of geometric and measurement problematic situations will be posed for them to solve, with the purpose of investigating the nature and quality of their living mathematical experiences and their mathematical thinking and feeling within extensions to the content they are learning in the course.

Prior Research on Mathematical Experience

Brown (1996) described the mathematics classroom from the perspective of social phenomenology where it is seen as an environment of signs, comprising things and people, which impinge on the reality of the child. His framework sees mathematical ideas as contained and shaped by the child's personal phenomenology, which evolves through time. He introduces the notion of "personal space" as a model for describing how children proceed through a classroom environment of phenomenal experiences toward their idiosyncratic "sense making." The report provides interesting data from children solving symmetry tasks, and demonstrates how Brown uses his framework to interpret the child's functioning in experiential terms.

Roth (2001) critically questioned some of these views and approaches in his review of Brown's (1997) book, an extensive attempt to relate many important aspects of mathematics learning and teacher education to phenomenology, hermeneutics, post-structuralism, and semiotics. In particular, Roth's central concern is whether all mathematical activity and experience can be reduced to text, based upon Brown's unequivocal assertion that "There is no experience outside the text" (p. 226).

Among the few investigations we've found, Francisco (2005) involved the elicited reflections of five high school students in their 12th year of a longitudinal study in which their school mathematics from grade one had been developed in ways claimed to be consistent with a constructivist approach (called "Rutgers Math"). Maher (1987) had described these learning conditions to include: collaborating in work, exploring patterns, conjecturing, testing their own hypotheses, reflecting on extensions and applications of the concepts, and explaining and justifying their reasoning. Steffero's (2010) study of one female student's beliefs and mathematical activities that occurred over a seventeen-year period constitutes an enormous record of her mathematical experiences from grade four through graduate school. Handa (2006) examined ways of knowing and doing mathematics using phenomenological reflections to construct views of experiential relationships with the subject matter of mathematics.

Perhaps there are good reasons why so little explicit focal attention is given to "mathematical experience." Thompson (1991) observes that while the paper authors (in an ICME conference on epistemological foundations of mathematical experience) are concerned with mathematical experience, they offer no information about "what is a mathematical experience." Indeed, in DIME we fully acknowledge a serious lack of clarity of constructs in this domain of "mathematical experience," but through systematic efforts to study "lived mathematical experience," we will seek to build new, grounded clarities of meanings. Here again, we may easily fail.

It is the goal of this research program to begin to study intentionally the phenomena of "lived mathematical experiences." Why bother? Are not the traditional views emphasized in intended curriculum, instruction, learning, assessment, evaluation, and research on learning and teaching adequate? Indeed, it is exactly because the goals and strategies of a sound mathematical education today embody a major emphasis upon stimulating, nurturing, and demonstrating high quality thinking arising from particular kinds of intended experiences that we must now seek to understand more clearly the nature of what students and teachers are actually experiencing in their "math lives" (Csikszentmihalyi, 1990).

If we want educational outcomes that mirror the empowerment that a deep knowledge and proficiency of mathematics can afford, we must now seek to move beyond our current perspectives and approaches that still seem to produce (for too many of our students) much less than we seek. If we are to understand deeply why so many of our students still achieve very poor understandings of the mathematical ideas and processes that our reformed curricula and our improved teaching methods, materials, and tools seek, we must begin to penetrate beyond more superficial indicators, such as test scores or written work, or even observed classroom behaviors into the interior world of the student's actual "lived mathematical experiences" where we may be able to identify deeper explanatory aspects of the individual's progressive growth (or deficits) toward greater mathematical knowledge and proficiency.

There is one other rationale that seems increasingly important. Too many of our citizens bear negative feelings toward their own past mathematical education (Hersch & John-Steiner, 2011), but for us educators to help our current students avoid such debilitating beliefs, feelings and attitudes we must begin to understand the nature of their experienced emotions that occur and develop within the mathematical contexts we offer. That is, not only must we understand the intellective dimensions of a person's constructions and reconstructions of their mathematical knowledge and proficiency, but we must also understand the emotional dimensions of their experiences, and how the interplay of thinking and reasoning actually functions within their live experiential feelings, emotions and affective schemas. It is time to seek to understand the whole "mathematical life" of our clients.

The goal of such new understandings is the assumed promise or potential that from such knowledge we mathematics educators will all be helped to engage in improved forms of mathematical education in which enhanced qualities of "lived mathematical experiences" occur for all persons. We seek a pedagogy that understands and honors the experiences of the other (surely a primary intention of a constructivist epistemology).

Theoretical Perspectives: Lenses upon "Mathematical Experiences"

We each know the centrality of the "lived experiences" we have throughout our lives. Even modest personal reflection can lead an individual to a sense of realization that it is specific experiences that shape "who they are, what we know, how we think" in powerful and fundamental ways. This includes what is experienced "outside," in the so-called "real world" that involves interactions within our environment (our physical experiences) and within our interactions and relationships with other minds (our social experiences). But, this also includes interior experiences that occur in our mind within our "inside" constructed world, where our thinking and our feeling "being" (person) is shaped and functions.

In Project DIME, we acknowledge the significance of the typical "mathematical" environment that includes students, teachers, parents, discourse, textbooks, technology tools, classrooms in schools, tasks and tests, the structured contexts of lessons, local, state and national curriculum frameworks, professional preparation and development, mathematicians, societies and cultures, governance, politics---all of the usual cultural and social elements in and around a person's mathematical education today. In doing so, we also accept that the meaning structure for any or all of these elements is a totally idiosyncratic construction for any of the multitudes of persons attending to matters of mathematical education. Now, we seek to "look into" the interior phenomenal world of "lived experiences" where each individual dynamically encounters and processes their idiosyncratic "mathematical experiences" that lead to "who they are and become," mathematically.

Phenomenology begins in the lived world, and seeks to bring to reflective awareness the nature of the events of "lived experience" (Hegel, 1977; Husserl, 1970, 1982). The principle of intentionality acknowledges an inseparable connection to the world (in our focus, the world of mathematical education) wherein "…we question the world's very secrets and intimacies which are constitutive of the world, and which bring the world as world into being for us and in us" (van Manen, 1990, p. 5). We are choosing to study thematic meanings, adopting themes and conducting thematic analyses within our particular orientation to the phenomenon of "mathematical experience" as "people of mathematics:" all being teachers, teacher educators, students of learning and teaching, mathematically educated, and interested is pedagogic theories. In DIME, we are trying to be explicit about our individual and shared intentions and orientations as a preparatory anchoring step in our research process.

As such, phenomenology accepts the curriculum of being and becoming (paideia), pursuing understanding of the personal, the individual, set against the background of an understanding of the other, the whole, the communal, or the social. It seeks to explicate phenomena as they present themselves to consciousness---the only access humans have to the world. But, consciousness cannot be described directly (the fallacy of idealism); the world cannot be described directly either (the fallacy of realism); real things in the world are only meaningfully constituted by conscious human beings, and these constructed meanings can only be revealed by the constructing human as inferences.

In our formative research approaches we presume the nature of "lived experience" to be fundamentally an internal construction/re-construction that emphasizes a consciousness of "sense-making," attempted within the unique idiosyncratic mental operations, schemas and constructive mechanisms as they exist and function in the mind of the individual within that "lived experience." To emphasize Piaget's (Piaget & Inhelder, 1969) theoretical inclusions that both intellective and affective aspects are involved within experience leading to development, so we seek to study both as a seamless whole, even as we reject many other dichotomies as false (such as "thinking versus feeling") typical of a strictly modernist structuralism.

Among the constructivist focal constructs we want to consider in our study of "lived mathematical experience" are representation and re-presentation, reflection and reflective analysis and abstraction, intuition and intuitive reasoning, and perturbation and equilibration, and particularly to search for where and how they may be found to function in, impact upon, and in turn be affected by particular experiential contexts. For example, in today's curricular frameworks one sees a major attention given to "representations" and representational activities; students are expected to learn to understand, use, and make various canonical mathematical representations. Yet, what from a study of "lived mathematical experiences" can we find about a student's actual conceptions and views of representations, and especially how they experientially use such images in problem contexts to re-present the conceptual ideas they are presumed to represent? Moreover, von Glasersfeld's (1991) view of cognitive functioning sees representation, re-presentation, and reflective abstraction as inseparable aspects; can we find this exhibited in "lived mathematical experience?"

Methods

On the first day of class, both research projects were explained to all nineteen students; most agreed to participate fully in the course-based investigation. Additionally, nine students expressed interest in participating as one of four selected paid volunteers. Over the next several days, the Principal Investigator (Hatfield) met each prospect individually for about 45 minutes, to explain further details related to the conduct of the ten 90-minute weekly (extra out-of-class) interview sessions, and to engage in a sample geometric problematic situation that illustrated what might be expected in each interview. By the time these preliminary sessions were completed, two students had withdrawn from the course (due to personal circumstances); all others confirmed their interest to participate if selected. Four of the remaining volunteer prospects were then selected based upon PI perceptions of these factors: expressed interest in the research, personal goals for learning from the individual sessions, openness, qualities of "thinking aloud" during sample problem solving, qualities of reflective analysis about their past mathematical experiences, and survey profiles of views of mathematics, mathematics learning, mathematics teaching, and school mathematics. Two women and two men were chosen; one female and one male are both older ("non-traditional") students and both expressed greater insecurity (less confidence) in relation to their mathematical knowledge and prior experiences.

The constructivist/phenomenological research methodologies used in this study require dynamic spontaneous interactions with each subject, functioning within a specific task-oriented framework. Data used in analyses include interactions and observational field notes of the researcher-interviewer (Hatfield) and a researcher-observer (Belbase), written or computer-based productions of the student, and video-recorded behaviors and verbal responses constructed by the subject, prompted by a "thinking aloud" expectation and questioning and probing protocol. Additionally, both the researcher-interviewer and the researcher-observer constructed post-session written descriptions of their intended and lived session experiences, these to be used as further data for analyzing and interpreting the co-contextualized (collaborative intellective and social dynamic) experiences of the student participants.

In general, all 90-minute interviews are structured into the following sequence of events (with approximate time allocations to be completely flexible and determined by the real-time decisions of the researcher-interviewer to proceed forward in the sequence, or to dwell longer to pursue productive interactions on behalf of the research question).

Part 1. (~20 minutes) Experiences in the course activities---Reflective interactions first focus on the student's conceptions and perceptions of what they have been experiencing in the class sessions, and how those experiences (from their point-of- view) are shaping their mathematical knowledge, their views of "doing" mathematics, and their ideas related to their future teaching of geometry and measurement to elementary students. These are prompted by questions such as the following: "Talk to me about what you've been experiencing during class sessions. Which of these experiences have seemed especially significant to you? Why? How have you benefited from those experiences, and why do you think you did? Tell me what you thought about ______ (a particular activity or event). Have there been any course events that were not particularly helpful or useful to you? Do you see ways that your course activities might impact your future teaching? Why? How might that happen?"

Part 2. (~50 minutes) "Living/lived mathematical experiences"---A non-routine geometric problematic situation is posed to the student, with the request to try to solve it as they share what they are experiencing via a "thinking aloud" approach. These geometric situations are challenging applications beyond the course content, solvable by good precollege problem solvers, including upper elementary students. An example: "Three towns, A, B, and C, are all on the same road. Town A is 12 kilometers from town B, and town B is 6 kilometers from town C. There is an additional town X on the same road that is also twice as far from town A and from town C. How far is town X from town B?"

There is not an a priori interview protocol, but efforts of the student constructively develop within an approach that promotes solving efforts (in which the researcher-interviewer will encourage student questions, and may include carefully placed problem-related questions or suggestions he poses) while frequently asking that the student describe her current thinking. At appropriate points he may ask her to pause ("take time out") in her solving efforts to reflect upon and describe qualities of her ongoing "living/lived mathematical experiences" in the unfolding situation. The fundamental goal is to reveal and study mathematical experience in terms of a dynamic interplay among cognitive and emotive thinking and feeling represented in the interiorized events of another person.

Part 3. (~20 minutes) "Looking back" and "Looking ahead"---Near the end of the interview, the subject is asked to reflect upon the session and to ponder anticipations beyond the session. This "looking back" phase of discussion reflectively focuses on her reported experiences of the session, emphasizing the problem-solving episode. In this, attention is given to potential connections to the nature and quality of experiences occurring in the class sessions. The "looking ahead" aspects then requests the student to anticipate potential connections of what she has experienced in the interview to how she views her future teaching of elementary mathematics, and in particular geometry and measurement, with children. In this discussion one focus is upon how elements of her personal philosophy and epistemology for mathematics are possibly being transformed by the course and interview experiences.

In the final interview session, the student will be asked to collaborate to construct a reflective analysis of their lived mathematical experiences across all course sessions and interview problem-solving episodes. Their written productions will be used for stimulated recall and analysis; selected segments of video may also be used to stimulate and support their recall and analysis. They will be guided to discuss what they perceive as the essences of the mathematical experiences they have had across the sessions, and how these relate to the qualities of their other lived mathematical experiences.

Results

Data are currently being collected and formatively analyzed, and this will continue to the end of the scheduled individual interviews (late April 2011). Formative data analysis has been initiated, in part to reflect a "constructivist teaching experiment" in which ongoing analyses and interpretations yield idiosyncratic inputs to influence potentialities within the subsequent interview dynamics. Identified elements of expressed and observed "lived" and "living" experiences are used to inform the intentional questions and interactions of the researcher-interviewer with a particular student during a subsequent interview session, all framed within the contexts of the three-part dialogue.

Using discourse analysis supported by video reviews (Hurlburt & Akhter, 2006; Petitmengin, 2006; Wolcott, 1994), thick descriptions for each session for each student are being constructed independently by the researcher-interviewer and the researcher-observer; these will then be jointly analyzed and contrasted, leading to consensual descriptions. After the course ends (by research protocol agreement) the observational and analytical voices of the other two project researchers [Chamberln (course supervisor and collaborative investigator) and Schnorenberg (course instructor and collaborative investigator)] will be added to the detailed analysis of all data from the forty sessions. In particular, all consensual descriptions will be collaboratively reviewed and analyzed, leading to a written interpretation for each student to portray the composite essences and overall qualities and to characterize a phenomenological model of mathematical experience for each of the four students. Particular attention will be given to identifying ways their mathematical experiences may have changed or developed across the class sessions and interview episodes in terms of their understanding and reasoning for "sense making" in geometry and measurement, their emotions and feelings for learning and solving problems, and their conceptions and anticipations for their future teaching children mathematics.

Some very preliminary results from our mid-study analyses include:

- There are differences among the students in their lived mathematical experiences that deeply affect their beliefs and self-efficacy about mathematics.
- There are strong connections between, and mediations by, qualities of prior lived mathematical experiences and the ways they seem to be able to engage in, and benefit from, new living mathematical experiences.
- The qualities of their current mathematical experiences are strongly shaping their views about how they will engage elementary students in their future teaching.
- A developing idiosyncratic "theory" of a student's experiential relationship to doing mathematics provides an important meditational template for teaching interactions.

Final interviews (in the Fall) with each student will be conducted in which the written interpretation (characterizing model) of the student will be shared and discussed. In this context we will encourage the student to agree or disagree on particular elements of interpretation, to explain and elaborate differences, to offer alternative views and ideas, and express the degree of agreement they feel they have for the characterization of their lived mathematical experiences during the Spring studies. These interview discussions may lead to revisions in the written interpretation, and will be data also summarized and reported.

Discussion

Little theoretical or empirical research with explicit attention to "lived or living mathematical experience" has been reported to date. Yet, there is a pervasive and deep focus on implied or intended mathematical experience across the history of mathematics education. The DIME (Developing Investigations of Mathematical Experience) research team seeks to establish a research program that will begin to build toward a deeper understanding of what a variety of participants in mathematical education can reveal about their interiorized worlds of "lived and living mathematical experience."

The students (prospective elementary teachers) who participated in this study have been invited (and all have given preliminary commitments) to participate in follow-up investigations of mathematical experiences during their enrollments in two "methods of teaching elementary mathematics" courses, in "student teaching," and then into their first year of classroom teaching of mathematics. Through this sequence of studies with the same students, we will build accounts of the accumulated impacts and consequences of their idiosyncratic lived mathematical experiences upon their development and modifications of personal theories and practices for teaching children mathematics.

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(RE)SHAPING ELEMENTARY PRESERVICE TEACHERS' ATTITUDES TOWARD MATHEMATICS

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The purpose of this study was to investigate how preservice elementary teachers' past schooling experiences and their participation in a mathematics methods course influenced their attitudes about the teaching and learning of mathematics. Pre- and post-surveys were administered to preservice teachers (n = 75) enrolled in mathematics methods courses at a university in the northeastern United States. Findings indicated that while prior schooling experiences influenced preservice teachers' initial attitudes about mathematics, the methods courses were conducive to shaping attitudes in a positive way, suggesting that experiences that build on preservice teachers' prior experience can, in fact, shift attitudes.

Introduction

Learning to teach mathematics is a multifaceted process beginning as K-12 students, continuing into teacher preparation programs, and extending into teachers' own classrooms (Wideen, Mayer-Smith, & Moon, 1998). Similar to K-12 students who enter the classroom, preservice teachers (PTs) enter teacher education programs with certain conceptions regarding the roles of teachers and students in mathematics classrooms. These conceptions are based upon particular forms of knowledge, resources, experiences, and conceptions of what it means to be a competent mathematics student and teacher (Ball, 1989; Ball, Lubienski, &Mewborn, 2001). Lortie (1975) suggested that the process of becoming a teacher involves an "apprenticeship of observation." From this perspective, teachers' socialization into teaching starts not when they enter teacher education programs, but as participants in K-12 school settings. Lortie claimed that the thousands of hours spent as a pupil in school create a "latent culture" that surfaces when one becomes a teacher. Additional research has shown that this apprenticeship of observation is influential in shaping preservice teachers' ideas about teaching and learning (Ball & Cohen, 1999; Feiman-Nemser, 1983; Wideen, et al., 1998). Once participating in teacher education programs, courses and field experiences typically expose PTs to multiple views that can help reshape their conceptions of teaching (Ball & Cohen, 1999; Ebby, 2000). However, the lenses through which preservice teachers make sense of these course and field experiences are developed by prior knowledge and experiences (Ball, 1989).In this paper we argue that just as teachers are encouraged to access students' prior knowledge and use their experiences as resources, teacher educators should take the same asset-based approach with preservice teachers.

Purpose of Study

Although several studies make the case that past schooling experiences are influential, few have specified how to build upon the existing attitudes and experiences preservice teachers bring with them to teacher education programs. A first step in building upon PTs' prior experiences is to understand the attitudes with which PTs enter mathematics education coursework, how those attitudes are a reflection of prior school experience, and how attitudes change through participation in a mathematics education course. Correspondingly, we asked, *How do preservice elementary teachers' past schooling and their mathematics methods course influence their attitudes about the teaching and learning of mathematics?* Regardless of the nature of teachers' prior experiences in mathematics, those experiences can provide an effective backdrop for developing attitudes towards mathematics teaching and learning aligned with reform recommendations in mathematics education (National Council of Teachers of Mathematics, 2000).

Perspectives

It has long been argued that teachers' affect is an important part of the way teachers understand mathematics (Ball, 1990). In Phillip's (2007) review of the literature on mathematics teachers' beliefs and affect, he argues that "for many students studying mathematics in school, the beliefs or feelings that they carry away *about* the subject are at least as important as the knowledge they learn *of* the subject" (p. 257). He defines "affect" as "[a] disposition or tendency or an emotion or feeling attached to an idea or object" (p. 259). He then goes on to explain that while "[a]ffect is comprised of emotions, attitudes, and beliefs….[a]ttitudes are more cognitive than emotion but less cognitive than beliefs" (p. 259). Much of the research on elementary teachers' affect has focused on the anxiety they have about mathematics; however not much, if any, has focused on teachers' affect and the relationship to their expected practice (Phillip, 2007).

Our study follows a well established line of research that has attempted to examine PTs' attitudes towards the teaching and learning of mathematics whether they adopt a more reformed view of teaching mathematics (Ebby, 2000; Eisenhart, Borko, Underhill, Brown, Jones, &Agard, 1993; McGinnis,and MacNab& Payne, 2003). Methods courses that expose teachers to reform practices tend to positively influence PTs' attitudes towards mathematics teaching and learning. However, Eisenhart et al (1993) found that while their participant had a desire to teach for in a reformed way, observations indicated that she rarely taught mathematics using approached aligned with reform documents in mathematics education. McNab and Payne (2003) had a similar finding with PTs in Scotland, where once they were out in the classrooms were "relatively unadventurous in their teaching" (p. 66). While field based experiences can contribute to PTs' attitudes towards mathematics, we chose not to include this factor in our analyses here. This was due to the varied field based experiences of participants included in this study; thus, we focus on the mathematics methods course.

Modes of Inquiry

Research was conducted at a private university in the northeastern United States. The teacher education program offered both a traditional four-year undergraduate degree and a graduate degree that could be completed in a twelve-month period. As part of the teacher education program, preservice teachers were required to take one mathematics methods course. This course was typically taken during the fall semester before student teaching.

All participants were undergraduate or graduate level preservice elementary school teachers enrolled in one of four sections of the mathematics methods course. Three professors taught the four sections of the elementary mathematics methods course. The mathematics methods courses at this university emphasized a reformed view of teaching mathematics. At least half of the class sessions used manipulative materials where the instructor emphasized links between concrete models and abstract mathematics concepts. Reflecting upon those experiences was an important part of connecting past experiences and prior knowledge to new ideas. The course was organized around the NCTM Standards (2000) where the instructors purposefully used process standards as a means for teaching content standards. Participating PTs completed both a pre-survey, which was administered the first week of the mathematics methods course, and a post-survey, which was administered during the last week of the course during the same semester.

Instrumentation

The pre-survey sought to capture participants' entering attitudes about mathematics and inquired about their experiences as K-12 students. The purpose of the post-survey was to examine preservice teachers' attitudes the teaching and learning of mathematics, field experience, and mathematics methods course experience. The population size was 102 and the total sample size for those who took both the pre- and post-survey was 75, a 73.5% response rate. The surveys, which were based on existing instruments, were first created and piloted in the first author's dissertation. Overall factor analyses indicated that approximately 67% of the variance was accounted for within seven scales. The sub-scales of each of the surveys had moderate to high reliability, with Cronbach's alpha levels from .612 to .921.

The pre-survey included the following four sections about mathematics: attitude and past experiences, teaching and learning, methods course expectations, and diverse learners. The post-survey included the following four sections about mathematics: attitudes and practicum experiences, teaching and learning, diverse learners, and future teaching. Most of the 48 items (per survey) were on a four-point Likert scale ranging from "strongly agree" to "strongly disagree." However, the section about the practicum experience included the fifth option, "not applicable."In addition to the pre- and post-surveys, which were administered during the first and last weeks of the course, observations of the mathematics methods courses were conducted and course artifacts (i.e. syllabi, assignments, and assessments) were collected to provide contextual information related to survey and interview data related to the mathematics methods courses. However, this paper solely focuses on the survey data.

Data Analysis

Multistage survey data analyses were carried out using SPSS. First, descriptive statistics were applied to analyze overall item response percentages and note any possible trends in responses. Second, we used correlations to examine the relationships among past experiences, the methods course, attitudes about mathematics education, and confidence to teach. Lastly, a multiple regression model was created to examine how past schooling and the mathematics methods course accounted for preservice teachers' attitudes towards mathematics and perceived level of preparation to teach mathematics.

Results

Considering the influence past experiences have on PTs conceptions of teaching and the desire of teacher education programs to help shape these conceptions (Ball, 1989; Lortie, 1975; and Scott, 2005), we became interested in responses on three unique items on the post-survey that directly asked preservice teachers about the perceived impact of their past K-8 schooling, practicum, and mathematics methods course on their future teaching practices (see Figure 1). The percentages were based on the total n of 75 to avoid an inflated percent due to missing data. The stem for the three items stated, "The following will have a major impact on the way I teach math in the future." The PTs were then asked to respond to this statement specifically about their past K-8, practicum, and methods course experiences. 85% percent either strongly agreed or agreed that past experiences would have a major impact on their future teaching practices. 81% percent strongly agreed or agreed that they believed their practicum experiences would be influential, and 96% thought that the mathematics methods course would have a major effect on their future teaching. Further analyses examined the relationships and influences of past experiences and the methods course.

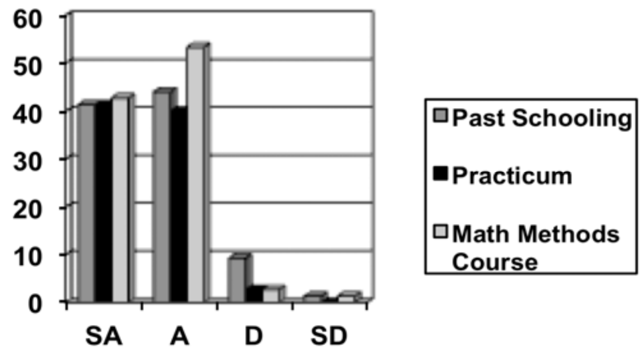

Figure 1. Elements influencing preservice teachers' anticipated practices by percentages

Results displayed in Table 1 indicate a positive relationship between PTs' attitude towards mathematics and positive prior schooling experiences in mathematics at the K-8 grade level (r= .599, p< .01). Participants' *positive math attitude* had an especially strong positive relationship to their perceived *proficiency* and positive high school experiences with mathematics (r= .719, p< .01). Undoubtedly, these constructs are all interrelated. If preservice teachers perceived themselves as highly proficient in mathematics, they were likely to be more confident (r= .585, p< .01), look forward to teaching mathematics (r= .563, p< .01), and have a higher positive attitude towards mathematics (r= .713, p< .01). Similarly, preservice teachers' attitudes and self-perceptions were related to their own experiences as students learning mathematics.

Table 1.Relationships among Attitudes and Prior Schooling Experiences in Mathematics

Pre-Survey Items	2	3	4	5	6
1. Positive math attitude	.599**	.719**	.713**	.661**	.553**
2. Positive K-8 math		.539**	.526**	.455**	.368**
3. Positive 9-12 math			.599**	.508**	.440**
4. Perceived Proficiency in math				.563**	.585**
5. Looking forward to teaching math					.504**
6. Confidence in ability					

** Correlation is significant at the 0.01 level (2-tailed)

To examine the relationships among preservice teachers' experiences in mathematics methods course, attitudes about mathematics, anticipated approaches to teaching mathematics, and perceived preparation, bivariate two-tailed Pearson's correlations were run at the .05 alpha level (see Table 2) among items pertaining to these topics on the post-survey. Results indicated a moderate positive relationship between participants who had a more positive attitude towards math and whether they learned a variety of strategies in the mathematics methods course (r= .273, p < .05), planned to teach mathematics in a conceptual manner (r= .326, p< .01), were going to require their students to memorize facts (r= .274, p < .05), and agreed that the mathematics methods course would have a major impact on their future teaching (r= .268, p < .05). Preservice teachers who indicated that they learned a variety of strategies in the methods course showed an increased desire to teach mathematics (r= .371, p< .01), confidence (r= .277, p

< .05), and believed that the course would have an impact on their teaching practice (r= .440, p< .01). An increased agreement that the mathematics methods course would have an impact on teaching practices was also significantly related to an increase in looking forward to teaching mathematics (r= .360, p< .01) and confidence (r= .291, p < .05). Participants' level of confidence was also associated with whether they would encourage students to use multiple strategies (r= .279, p< .05), a characteristic of teaching with a conceptual focus.

Table 2.Relationships Among Attitudes and the Mathematics Methods Course Experiences

Post-Survey Items	2	3	4	5	6	7	8	9	10
1. Positive math attitude	.273*	.306**	.794**	.566**	.326**	.168	.066	.274*	.268*
2. Learned a variety of strategies		.192	.371**	.277*	.149	.043	.142	-.013	.440**
3. Prepared to teach math			.397**	.438**	.139	-.149	.227*	.047	.210
4. Looking forward to teaching math				.711**	.311**	.011	.140	.061	.360**
5. Confident in ability					.412**	0.031	.279*	.137	.291*
6. Teach conceptual math						.275*	.382**	.155	.014
7. Teach procedural math							.051	.601**	-.083
8. Encourage multiple strategies								.016	.119
9. Require students to memorize facts									-.109
10. Methods course, major impact									

* Correlation is significant at the 0.05 level (2-tailed)
** Correlation is significant at the 0.01 level (2-tailed)

Following a confirmatory approach, we hypothesized that the variation found in preservice teachers' feelings of preparation and anticipation to teach mathematics after being in the mathematics methods course for one semester could be explained in terms of the variables listed above. In statistical terms, the hypotheses can be expressed as: $H_0 = \beta_{positiveK8math} = 0$, $H_a = \beta_{positiveK8math} \neq 0$ and $H_0 = \beta_{mathcoursestrats} = 0$, $H_a = \beta_{mathcoursestrats} \neq 0$. The significance level was set at the 0.05 two-tailed level.

The overall regression of *preplookfwd* on positive *K-8 math* and *math course strategies* was statistically significant [R^2 = .208, F (2, 75) = 9.441, p< .001]. Overall, the variance explained by the two predictors differed significantly from zero; thus, we rejected the null. Table 3 shows the overall model summary and significance levels. This model had a higher F-value and was statistically significant. The *positiveK-8 math*variable accounted for approximately 12.5% of the variance in *preplookfwd*, while *math course strategies* accounted for an additional 8.3% of the variance. Taken together, the predictor variables could explain approximately 20.8% of the variance of *preplookfwd*. Although the model was significant, nearly 80% of the variance was unaccounted for in *preplookfwd*, which supports the argument that a multitude of variables influence preservice teachers' attitudes and preparation to teach mathematics. The unstandardized coefficients of the model were .192 for *positive K-8 math* and .330 for *math course strategies*. The regression solution for this model was:

$\hat{Y}_{preplookfwd} = 0.942 + 0.192X_{positiveK8math} + 0.330X_{mathcoursestrats}$ .To evaluate the effect size of the regression model, we computed a post-hoc power analysis. The model with two predictors (see Table 5) had a very high level of power (1 - β = 0.97) with a medium effect size (f^2 = 0.26) at the alpha level of .05.

Table 5.Model Summary and Significance of Two Predictors

Predictors	R^2	ΔR^2	F	p	DW
1. Positive K-8 Math	.125	.125	10.447	.002	
2. Positive K-8 Math Math Course Strategies	.208	.083	9.441	.000	2.164

Note. Dependent Variable (constant): PrepLookfwd to teach Math

Discussion

Descriptive statistics from the post-survey clearly showed that more than 80% of participants perceived their prior schooling and the mathematics methods course experiences as having a major impact on their anticipated teaching practices. The multiple regression model confirmed that the two variables accounted for a significant proportion of preservice teachers' perceived level of preparation and their attitude towards teaching mathematics. Nevertheless, the two factors combined accounted for only 20.8% of the desired outcome variable including preservice teachers' looking forward to teach mathematics and viewing themselves as prepared. Thus, other factors beyond those in our model account for almost 80% of PTs' preparation to teach mathematics and attitudes about teaching mathematics. In this section we present an interpretive summary of the two main themes from our data.

Evolution of Attitudes

Findings indicated a strong relationship between PTs' attitudes about mathematics and their prior schooling experiences. A positive increase in participants' attitudes towards mathematics was related to positive experiences in K-8 prior schooling (r= .599, p< .01) and to high school (r= .719, p< .01). Similarly, an increased response to perceived proficiency in mathematics had a strong positive relationship with attitude towards mathematics (r= .713, p< .01). In addition, participants' perceived proficiency was related to both their prior K-8 (r= .526, p< .01) and high school (r= .599, p< .01) experiences in mathematics. Thus, correlation results indicated that those with more positive prior schooling experiences had more positive attitudes towards mathematics and considered themselves as more proficient. These findings support a qualitative study conducted by Ellsworth and Buss (2000), who examined preservice teachers' attitudes towards mathematics by analyzing their autobiographies. They found that past teaching models was the most salient theme because preservice teachers' commonly reported that their interest in or attitude towards mathematics was positively or negatively affected by past teachers. By bringing past experiences to the surface, preservice teachers may be more cognizant of how their own attitudes about mathematics are shaped.

Influential Factors

It is particularly important to acknowledge that preservice teachers enter teacher education programs with a wealth of knowledge from their prior schooling. Although in some cases, the goal of a course is to reshape or challenge entering assumptions about the role of teaching, PTs can also have positive perspectives about teaching which complementary ideas can be built upon. The mathematics methods course could build upon PTs'attitudes, which are often more positive and fertile than expected. While the mathematics methods courses observed in this study did not appear to have an overt agenda or strategy to build upon PTs' past experiences, questions were raised about their view of the teaching and learning of mathematics. As the instructors taught methods for different mathematics topics such as multi-digit subtraction, PTs would use the standard algorithms in many cases and connect their prior knowledge about the procedure with the concrete materials. It would have been interesting for PTs' to explicitly compare different strategies and discuss the benefits of alternative algorithms.

While the two experiences we focused on, prior schooling and the mathematics methods course, can be of great importance in preparing elementary teachers to teach mathematics, our study found that the two only accounted for 20% of the variance of PTs' attitudes towards teaching mathematics. This suggests that the past experiences that we often try to work with may not account for as much as we thought it did. While important, past experiences might play a smaller part than we expect. We suggest that explicit efforts still be made in the mathematics methods courses to connect to PTs' prior knowledge. Many scholars who support a social justice approach to teaching have made a strong case for the importance of adopting an asset model by using students' prior experiences as resources (Cochran-Smith, 2004; and Ladson-Billings, 1995). In a similar way, we believe that teacher educators need to use preservice teachers' entering characteristics as resources and adopt an anti-deficit model. However, there are many additional factors that need to be explored over time, such as field experiences, student teaching, mentors, family members who are educators, and mathematics content courses, which could potentially influence changes in participants' attitudes and confidence. In addition, continued professional development opportunities can continue to build upon preservice teacher education experiences to increase teachers' confidence and knowledge. In the final section we make recommendations for teacher education and future research.

Recommendations

Although several scholars have argued that beginning teachers' socialization into teaching takes place when they are students, empirical work has not explored the extent to which past experiences influence preservice teachers (Scott, 2005). This study explored that issue as it pertains to mathematics teacher education and showed that past experiences only accounted for 12.5% of the explained variance in PTs' attitudes and confidence to teach mathematics. We believe that this could actually be a very encouraging finding. While past schooling experiences are a significant factor and need to be taking into consideration, there are many additional factors that account for and influence teachers' attitudes that need to be explored.

Teacher educators should engage PTs in worthwhile and authentic activities that help them to bridge their own personal factors with contextual factors to adopt and practice that can support student learning (Goos, 2005). It is particularly important to acknowledge that preservice teachers enter teacher education programs with a wealth of knowledge from their prior schooling. Harkness et al. (2006) suggested that mathematics methods courses should provide opportunities for PTs to engage in meaningful problem solving tasks to make sense of the mathematics and make connections to improve upon their future practices. Ebby (2000) argued that methods courses should shift from mastering practical skills to developing "habits of mind" such as "making sense of children's understanding and learning to take a reflective stance towards one's own teaching" (p. 93).

Future research should follow PTs longitudinally across teacher education programs and into their entry into the profession. In this study, the pre- and post-surveys were confined to one semester long mathematics methods course. Based on the factor analysis, the instrument also had room for improvement, as surveys do not fully capture the variables of interest due to self-reporting and restricted Likert scales. The sample was also restricted to the mathematics methods courses at one university during one semester. Multiple data sites over time and across institutions would allow for stronger comparisons. In addition to survey results, qualitative interviews that elaborate on these experiences would help us further investigate how PTs' attitudes are reshaped.

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BUILDING COMMUNITIES: DEVELOPING A SHARED SENSE OF ELEMENTARY MATHEMATICS PEDAGOGY WITH GRADE LEVEL TEAMS

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Introduction

This study examines the effects of a professional development (PD) experience on second and third grade teachers at a Title I school in southeastern United States as they worked together to develop a shared understanding of teaching mathematics. This case study analysis (Yin, 2003) investigates participants as they collaborated with a university-based mathematics teacher education group within a semester long PD experience. This paper discusses the process of change for this group of teachers as they worked to break down barriers and question beliefs about instruction and student ability in mathematics.

Theoretical Framework

The literature relating to mathematics PD is extensive. Most empirical studies document the implementation and evaluation of PD models that are transformative in nature, meaning that they are dedicated to improving teacher quality in mathematics (Smith, 2001). These models differ from PD that is additive in nature, meaning that the purpose of the experience is to add new ideas or material to existing instructional practice (Smith, 2001). Recommended PD models described in the research literature contain what can be considered the four essential elements of mathematics PD:

1. PD should be ongoing and grounded in practice: Teachers require extensive experiences geared towards developing both mathematical content knowledge and pedagogical content knowledge (Hill, Rowan, & Ball, 2005). These experiences must connect to everyday teaching practices (Garcia, Sanchez, & Escudero, 2006).

2. PD should focus on mathematical content: Research indicates that teachers are inadequately prepared to deal with the discussions that can occur in a reform-minded mathematical classroom (Hill, Rowan, & Ball, 2005) and many elementary school teachers lack sufficient content knowledge to teach their students.

3. PD should have student learning as the ultimate goal: Guskey (2000) emphasized the need to determine the effectiveness of PD by the impact it has on student learning. Many studies conducted on models of PD measure effectiveness by teacher reaction or through district approval.

4. PD should create communities of teachers: Cobb, Stephan, McClain, and Gravemeijer (2001) describe the need for PD to enable teachers to collaborate with each other. Teaching is typically an isolated profession where teachers have little opportunity to collaborate to meet the needs of all students. Sustainable PD requires that teachers take ownership of their development through these communities of practice (Gellert, 2008).

The model developed for teachers in this study was grounded in the four essential components of mathematics PD outlined above. Prior to working with these teachers, the PD team spent two years observing classroom settings in four elementary schools as part of a state funded grant to gain a better understanding of their needs in terms of mathematics instruction. This information was used to create a PD model that was unique to the needs of these teachers. The PD team developed a model that had two main objectives: (1) To develop the pedagogical content knowledge necessary to implement an inquiry curriculum in the manner in which it was designed and (2) To develop a shared sense of purpose by analyzing a common instructional issue amongst participants. Although a focus on mathematical content was an absolute priority, it was also necessary to problematize participants' current instructional methods through the introduction of an action research agenda that would enable them to analyze student work as a team and, as a result, evaluate pedagogy to improve student achievement.

Methods

Research Question

The central research question guiding this study was: How does an ongoing mathematics PD model focused on developing an action research agenda impact the instructional practice of elementary school teachers? This question was examined through a case study design where multiple forms of data were collected during the experience and then analyzed and compared to determine overall results (Yin, 2003).

Sampling

Teachers were selected to participate in the PD model and concurrent research study through snowball and criterion sampling methods. When contacted by the PD team, all participants volunteered to participate in both the PD and the research study. The team of 11 participants (five second grade teachers and six third grade teachers) met two times monthly after school with one member of the PD team (who acted as both facilitator and researcher for this study) for a total of ten hours of formal, content-focused PD. In addition to these experiences, teachers read and critiqued articles relating to mathematics instruction, developed a research question and design, and collected and analyzed data from their individual settings.

Building a Community of Researchers

In order to scaffold participants as they conceptualized a research agenda, the facilitator/researcher first divided the group into grade-level teams so they were better able to consider issues relating to their students. The facilitator/researcher then asked participants to work together in each team to identify the specific needs of their students in mathematics. Once each team created a list of needs, the facilitator/researcher then asked each team member to write about their beliefs regarding how students learn mathematics and what it means to be an effective teacher of mathematics. The facilitator/researcher then extended this examination of beliefs by asking participants to compare their beliefs and their current teaching practices. Although this was an examination of teacher perceptions, the facilitator/researcher was able to scaffold participants towards the goal of investigating both student understanding in mathematics and instructional practice in mathematics. The culminating research question developed for both grade-level teams as a result of this examination was: What are the best ways to teach basic facts effectively?

Building Mathematical Understanding

The facilitator/researcher asked the 11 participants to complete a mathematics assessment prior to attending the first PD session to determine their needs in terms of building pedagogical content knowledge. This assessment was comprised of a selection of nine released items relating to number concepts from the National Assessment of Educational Progress (NAEP). The assessment was limited to items related to number concepts because second and third grade student achievement on a standardized assessment given quarterly showed weaknesses in this area. The PD team altered the nine items by including a reflection component where each participant was asked to explain the strategy used for solving each problem in writing. It was this reflection component that provided the most information for the facilitator/researcher to develop each PD session.

Data Collection and Analysis

Consistent with case study research, four forms of qualitative data were gathered throughout this study to determine changes in teacher beliefs and instructional practice related to mathematics pedagogy. Participants were asked to complete a reflection paper describing the results of their grade-level team research and their overall reactions to the PD experience. The facilitator/researcher provided a list of writing prompts to assist participants and to ensure information was provided relating to teacher beliefs about teaching mathematics. In addition, data was gathered through participant selected student work and classroom observations examined during PD sessions, anecdotal notes written by the researcher during PD sessions, and the reflection component of the NAEP assessments administered prior to and following the overall PD experience. Data analysis began with repetitive readings of each form of data to ascertain an overall impression and to identify overarching patterns or themes. From there, constant comparative methods (Strauss and Corbin, 1998) were used in a secondary analysis to examine themes across data.

Results

In the beginning of the PD process, participants expressed their concern that they would be forced to implement the inquiry-based curriculum the rest of the school had already been using. The facilitator/researcher worked to build a positive relationship and establish a sense of trust by seeking permission from administration to allow participants to teach mathematics in whatever method they felt was most effective. As a result, participants were more open to questioning their own practices. While they did not implement the curriculum fully, they did begin to ask open ended questions in their classrooms and provided students with tasks that were developed as a result of analyzing student work during PD sessions. Participants also began planning together and developing hypotheses regarding the most effective ways for students to understand basic facts. They quickly realized that summative assessments of student learning would be insufficient in determining student understanding.

Participants reported the value of listening to their students and increasing student discussions as part of mathematics lessons. They also recognized the need to “talk less” when teaching to allow students time for engaging in meaningful discussions. “I have learned that I need to let go of some of the control in my room and let my students come up with their own thoughts and conclusions about the problems. They might have a better way of looking at something or even be able to teach each other better than I could” (Participant 3). Their view of student abilities in mathematics was also challenged throughout the research process, “Many times I would give them a problem and we would work it out exactly the way I had in MY mind. I realized during these sessions that my students are capable of using different strategies to solve the exact same problem…I need to allow my students time to explore and try different mathematical strategies to figure out the answer” (Participant 6).

Even with these positive changes, participants still viewed themselves as transmitters of knowledge rather than facilitators during lessons. Often participants described how they needed to present information to students in order for them to build conceptual understanding through tasks. “I gained some insight into how my own team explains their mathematical processes to their students (Participant 1)”. They began to change their instructional practices to allow for open-ended questioning, conceptual tasks, increased communication, and student autonomy, however many continued to view the beginning of the lesson as the time when they “taught” students and the second part of the lesson as the time when these reformed practices would be implemented. Although perceptions of practice were beginning to change by the end of the PD sessions, as is evident in the examples reported above, participants in this sample did not completely transform their beliefs about the teacher’s role in mathematics instruction.

Discussion

Research has shown that while building teacher content knowledge in mathematics is a priority, without problematizing instructional practices and building a sense of community among colleagues, little change will occur in the classroom (Cobb & McClain, 2001). Professional learning and teaching communities have long been recommended for improving teacher quality in mathematics. The use of action research in the mathematics classroom holds potential as a means of solidifying and sustaining these communities. By developing a shared sense of purpose that holds practical implications for instruction, teachers will be more motivated to collaborate in PD opportunities. This study presents a model of PD that works to simultaneously build teacher pedagogical content knowledge and build a shared sense of community among colleagues engaging in action research. Teachers in this setting showed the ability to question their own mathematical understandings and their students' mathematical understandings. As a result, they were able to design a research agenda built around analyzing student work as a means to develop instructional practice.

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EDUCATIONAL METAPHORS IN TEACHERS' CONSTRUCTION AND FRAMING OF PROBLEMS OF PRACTICE

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In this paper we study how teachers construct problems of the mathematics teaching practice. Identifying problem design as an important and understudied aspect of problem resolution, we consider conceptual metaphors as particular resources employed by teachers as they construct and frame problems of practice. We focus on a specific conceptual metaphor, the metaphor of education as a journey, and illustrate how teachers use variations of this metaphor in their constructions of a specific problem of practice. We examine how the framing of a constructed problem affects first the potential for resolving the problem and ultimately, the opportunities for succeeding at teaching and learning mathematics.

Researchers have worked to positively impact the teaching and learning of mathematics by studying problems of teaching. These studies have carefully catalogued the problems of teaching practice (as perceived by researchers and by teachers) and considered remedies for those problems (e.g. Ball & Bass, 2000; Junk, 2005; Lampert, 2001; and Little, 2003). While this is important and useful work, it fails to examine an important aspect of problem resolution: the construction of the problem. The design of a problem directly impacts the steps to (and even possibility of) a solution (Getzels, 1979, 1975). For example, the problem of a student's failure to learn could be framed by a teacher as either a student's innate cognitive deficit or as a lack of appropriate experiences. If the problem of learning is seen as a biological deficiency in the student, there is little a teacher can do to encourage learning. On the other hand, if the problem of learning is seen as a need for additional experiences, there is much for the teacher to try. Thus, the ways in which the teacher constructs the problem of teaching has immense consequences for his or her work and for the potential resolution (or at least mitigation) of the problem.

While we know some about what problems of practice teachers notice, we know very little about the tools and resources that teachers draw upon to frame problems and how those tools and resources might result in crafting problems that are more or less resolvable. In this paper, we will consider how teachers use conceptual metaphors to construct and frame problems of practice.

Our current work focuses on a specific conceptual metaphor, the metaphor of *education as a journey*, and illustrates how teachers use variations of this metaphor to construct and frame a particular problem of practice. We examine how the framing of a constructed problem affects first the potential for resolving the problem and ultimately, the opportunities for succeeding at teaching and learning mathematics.

Conceptual Metaphors as Resources for Constructing Problems of Practice

Problems, which we operationally define as perceived misalignments between what exists and what is desired, arise as people consider what is wrong, what counts as part of the problem, and what might be done to fix the situation. Thus problems are not immutable, universal Truths, but rather human creations which depend largely on the lens we use to frame a given reality.

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Conceptual metaphors are one type of resource we often draw upon as we interpret realities and hence frame particular situations.

Conceptual metaphors are more than figures of speech. Instead, they are the use of one idea to frame or make sense of another (Johnson, 1987; Lakoff & Johnson, 1980/2003). Conceptual metaphors allow us to describe abstract domains, processes, and experiences by relating them to physical or tangible domains, processes and experiences. For example, we may describe life by relating it to a physical journey or frame teaching as conducting an orchestra or planting a garden. According to Lakoff and Johnson (1980/2003, 1999), Yero (2002) and others, the metaphors we use create, guide, and reflect our perceived realities and experiences. In a similar sense, the metaphors teachers use in connection to the teaching and learning of mathematics at once affect and reflect their classroom experiences and perceived realities. In particular, teachers sometimes use conceptual metaphors as resources for constructing or framing the problems of their practice. In this paper, we focus on one particular conceptual metaphor of education, the *Educational Journey* (EJ) metaphor, and illustrate how teachers use this conceptual metaphor as they frame and construct a specific problem of practice.

Researchers have documented the use of *Educational Journey* in teachers' talk (Munby, 1986) and have explored implications of this metaphor in a variety of teaching contexts (e.g. Milne & Taylor, 1995; Sumsion, 2002; and Thornbury, 1991). The *Educational Journey* metaphor conceptualizes education as a journey in which teachers are guides and learners are travelers (Clark & Cunningham, 2006). Such a journey may be further characterized in a various different ways: The learner-travelers, for example, may autonomously venture forth in a journey of discovery, making paths as they explore the mathematical landscape. Alternatively, the learner-travelers may all follow a pre-determined path in a journey planned and led by the teacher-guide, attempting to travel at a specific common pace, towards the same specific places. Such possible characterizations of the EJ metaphor give rise to different versions or strands of this metaphor.

Clark and Cunningham (2006) described one such version of the EJ metaphor, which they identified as the predominant metaphor in the No Child Left Behind (NCLB) Act of 2001:

> All students are following the same path towards the same destinations defined by shared standards. […] Standards can serve as ideal prototypes relative to which all individuals can be compared. (p. 278)

We will refer to this particular strand of the EJ metaphor as the EJ-NCLB metaphor. We note that an implicit entailment of Clark and Cunningham's characterization of EJ-NCLB is that the journey along the common path must happen at a certain pace, and hence progress is measured by timely student achievement on standardized tests (Clark & Cunningham, 2006, p. 279).

Another possible strand of the EJ metaphor is one we have termed the *Educational Journey Landscape* metaphor (EJ-L). This strand of EJ conceptualizes learning as exploration and discovery of a mathematical landscape: Students are autonomous traveler-explorers who journey along a variety of paths towards a variety of destinations. Students self-monitor their progress and re-direct their journey by learning from mistakes and looking for landmarks. Teachers are expert travelers who journey along with their students helping them notice primarily the features and landmarks of the paths they choose to travel, not just the destination.

For example, when teachers use the EJ-L metaphor, they say statements like the following:

- "Students' mistakes are learning opportunities that *redirect* their thinking"
- "We are looking at the powers of 10 but if we never *explore* the powers of ten children don't know how big those numbers are"

- "That approach is going to be more efficient, but first kids have to mess with the messiness of numbers to understand why the efficient formula works. When I can see that in fifth graders, there's no *stopping them* if they are given the appropriate experiences."

The first statement conceptualizes students' mistakes as opportunities for learning, and implicitly positions students as individuals who may themselves re-direct their journey by noticing their mistakes. The second sentence suggests that progress in one's education journey is tied to exploration, while the third sentence highlights students as self-propelled, persistent traveler-explorers.

Other authors have also used landscape metaphors to suggest learning happens as we traverse the landscape of ideas. For example, Yero (2002) conceptualized knowledge as a landscape, while Davis and Hersh (1982) conceptualized mathematics as a landscape.

These two strands of the EJ metaphor (EJ-NCLB and EJ-L) frame our analysis of how teachers use this metaphor when constructing problems of their practice. We explore which of the two versions of this metaphor is foregrounded in the way teachers construct problems. We also propose how the specific strand of the EJ metaphor foregrounded in the framing of a problem may contribute toward resolving, mitigating, or re-constructing the problem in productive ways.

Data Sources and Analysis Methodology

The data set used for this work consists of over 80 hours of video data documenting regular meetings of a teacher study group. The study group was supported by the Center for the Mathematics Education of Latino/as (CEMELA), a National Science Foundation Center for Learning and Teaching. The purpose of the group was to support elementary and middle school teachers in schools with large Latino/a student populations in thinking about how to teach mathematics in ways that could best support the learning of their Latino/a students. Ten teachers participated in the teacher study group: two middle school teachers (from the same school) and 8 elementary school teachers (from two different schools). All schools were located on the outskirts of a mid-size city in the southwest. The three schools serviced student populations that were predominantly Latino/a and had many students who received free or reduced lunch and/or were classified as English language learners. The study group met 25 times over two years for a total of 66 hours. Each meeting was videotaped (both whole group and small group activities), resulting in the 80 hours of video data.

In this paper, we limit our analysis to data from one meeting of the group. While we have extensive transcriptions and analysis of data from other days, we are choosing to focus on this particular data not only because it is rich in teacher talk about students and teaching, but also because it contains all instances documenting the voice of Patricia, a nationally known mathematics educator who visited the CEMELA teacher study group on the day we are analyzing.

We began our analysis by transcribing the video data from this meeting. This transcript documented Patricia's talk about the work of teaching and student learning which served as the basis for our characterization of the EJ-L. In our initial pass through the data, we noted that Patricia's talk embodied a distinct conceptualization of teaching and learning. In order to capture this conceptualization, we systematically explored Patricia's talk for evidence of conceptual metaphors. We first coded all instances of the EJ metaphor used in her talk. This coding proceeded by noting all instances of path schema identifiers: words and phrases indicative of physical movement such as that which may happen in a physical journey. Path schema identifiers include phrases such as go forth, come back, go over, beyond, along, to, and from (e.g. Clark & Cunningham, 2006, p. 272). This first layer of coding enabled us to capture all instances of Patricia's talk where the EJ metaphor could arise, since path schema identifiers flag this metaphor. We further coded the passages containing path schema identifiers to eliminate all instances in which the use of such identifiers was either literal (e.g., a teacher talks about a student moving a physical shape: "brought it over here") or otherwise not tied to a conceptual metaphor about teaching and learning (e.g., a teacher references a personal experience unrelated to teaching). The educational metaphors contained in all remaining passages marked all occurrences of the EJ metaphor. These where then collected, sorted by themes, and summarized (Miles and Huberman, 1994). The final summary constitutes our characterization of a new, specific EJ strand which we have termed EJ-L given its overall conceptualization of education as exploration of a landscape.

This same method (first coding for path schema identifiers, then refining the coding to arrive to all instances of the EJ metaphor) was used to identify all instances of EJ metaphor in teacher talk within this data source transcript. When possible, we then sorted the occurrences of EJ metaphor into one of the two strands: EJ-NCLB or EJ-L.

Finally, we identified, within all coded instances of EJ, those in which the metaphor appeared in conjunction with the construction of a problem of practice. In other words, we identified instances in which teachers perceived what existed as different from what was desired. This resulted in several illustrations of how teachers use the EJ metaphor as they construct specific problems of practice.

Teachers Construction of Problems of Practice and the EJ Metaphor

In this paper, we focus on a particular problem of practice: Students don't remember/retain mathematical knowledge/information. This problem was constructed and framed independently by three different teachers in the teacher study group, at three different points in time during the transcribed meeting.

The first teacher, Matilde, framed this problem by proposing that students don't retain information because the teacher needs to move quickly through the topics in the curriculum and this prevents students from having enough practice with the concepts. The teachers were discussing students who seemed to understand the concept of area, but were unable to accurately determine the area of a polygon.

> *Matilde:* Probably what happened... Because [the students] got that concept [area] and, for example, (incomprehensible), they got the concept and then *we need to pass, the teacher needs to pass to another thing*, so they just got the concept, but they don't have enough mechanical practice, so next year, they already touched the same thing and they kinda remember the same concept but they still don't have enough practice and this goes year by year.

In this instance, the problem is that students are unable to accurately determine area even though they appear (to the teachers) to understand the concept of area. In her problem construction statement, Matilde's use of the metaphor (we need to pass, the teacher needs to pass to another thing) conceptualizes the need for teachers to move relatively quickly from one topic in the curriculum to the next. Matilde also indicates that this kind of teacher journey is recurrent: it goes on year after year.

Matilde is predominantly framing the problem of students' knowledge retention through the EJ-NCLB strand of the educational journey metaphor. Her words foreground the need for all students to travel at the same, relatively quick pace, moving from one topic to the next in a teacher-led mathematical journey. Above all, Matilde's framing of this problem seems to identify the need for the teacher to move, and to move fast, through the curriculum as the reason for the perceived misalignment between what is and what is desired of the students.

A second teacher, Olivia, constructed this problem by stating that students don't commit certain facts to memory because they don't get lots of practice: teachers do enough and then *move on,* a pattern that repeats at each grade level:

> *Olivia:* But now, I mean, we've had classes where we, where it's OK for them to not to know by memory because there's different methods and different tools for them to learn it or to figure it out, but you're right: certain things are just kinda dedicated to memory... and that happens with lots and lots of practice, and without the content we don't do that. We do enough and then we *move on and the next* grade level, they have to start all over again and then we *move on* and then it's just kinda... 'Cause that's a complaint in every grade level with every teacher.

In Olivia's problem construction statement the EJ metaphor appears twice, as we have indicated through italics. Olivia's use of the metaphor also highlights the movement of the teacher away from one topic or objective towards the next, in an implicitly timed, teacher-led journey. This emphasis in teacher movement suggests again that Olivia's use of the EJ metaphor is aligned with the EJ-NCLB strand of the metaphor. Hence Olivia is also framing the problem of student retention of knowledge through the EJ-NCLB metaphor, and the teacher's movement is again related to the perceived misalignment between what is desired of the students and what actually happens. It is interesting to note, however, that in her construction of this problem Olivia remarks that a teacher moves when she has done *"*enough*"* rather than when the "need" to pass arises, as remarked by Matilde.

Liliana, a third teacher, constructed this problem by proposing that students don't retain information because students don't spend enough time explaining how *they go there*, how they go about solving problems:

> *Liliana:* […] Because just by coincidence we did the same, similar type of activity, and she says, hers is in sixth grade and mine is in fourth grade and it's two years and they're having the same difficulties. What's going on with these kids…that they're not retaining that information? And I think in my class, one of the things that contributes to that, is the fact that they don't spend enough time really explaining how *they go there*. So I've been trying to really give them that extra time.

In her problem construction statement, Liliana use the EJ metaphor only once, as indicated through the italicized path schema identifier. Liliana's use of the EJ metaphor foregrounds student movement, omitting any reference to teacher movement. Liliana's words "how they go there" seem to recognize that there is more than one way to "go there" and that students could retain information better by being able to each describe their own journey towards a particular mathematical destination. This suggests Liliana's use of the EJ metaphor is best aligned with the EJ-L strand of the Educational Journey metaphor. In particular, Liliana seems to be framing the problem of student retention of knowledge through the EJ-L strand of EJ. She foregrounds a lack of opportunity for students to explain how they go about solving problems, as the primary reason for the misalignment between what is and what is desired.

Working Towards Solutions: Alternate Framings of Problems of Practice

As argued and illustrated in our introduction, the design of a problem directly impacts the steps towards, and even possibility, of a solution (Getzels 1979, 1975). In this section we discuss how the previous illustrations of problem construction and their respective proposed framings through EJ-NCLB and EJ-L impact the possibility of resolving this problem and suggest mechanisms for more productive problem construction.

Framing through Educational Journey-NCLB

In our previous illustrations of how teachers constructed the problem of students' lack of retention of knowledge, the most notable characteristic of the framings through EJ-NCLB (Matilde's and Olivia's) was the foregrounding of the teacher movement, without explicit references to student movement. This prominent focus on the relatively fast-paced teacher movement has been identified as a feature of the Educational Journey Metaphor of NCLB as used teacher talk. More specifically, teachers using this version of the Educational Journey metaphor often emphasize their movement as teachers or their responsibility to continue moving students through content (Wood and Lozano, 2010).

Also, the framings through EJ-NCLB, in particular the identification of fast teacher movement as a reason for the perceived misalignment between what is and what is desired, led to problematizations of the actual problem, rather than to the consideration of possible solution. Matilde's problematization, for example, explicates the original problem of students' lack of knowledge retention by linking it to the new problem of the teacher's fast-paced movement. In this problematization Matilde seems to exchange the original problem for a less solvable one, one whose perceived solution hinges upon altering national educational policies over which she has little control. In this way, Matilde's problematization seems to position her further away from, rather than closer to, the possibility of considering attainable problem solutions.

Olivia's reference to a teacher who moves once she has done "enough" could suggest that Olivia perceives lack of student ability as an additional barrier to considering attainable problem solutions. From this perspective, Olivia's problematization seems to constrain the possibility of solving the problem perhaps more than Matilde's: Olivia may perceive she has little control over educational policies, and even less control over students' natural abilities or dispositions.

Both of these illustrations depict how the construction of a problem and its framing through EJ-NCLB, may be unproductive if it leads to a problematization of the original problem that further removes a teacher from the possibility of considering attainable problem solutions. Such framings can constrain (rather than enable) possibilities for solving the problem and, ultimately, a teacher's opportunities to be successful at teaching and learning.

Framing through Educational Journey-Landscape

In Liliana's construction of the problem of students' lack of retention of knowledge, the most notable characteristic of her framing through EJ-L was the immediate foregrounding of the students' lack of movement and their need to move, without explicit references to teacher movement. This central focus on students and their autonomous, individualized movement along various possible paths in the mathematical landscape, lies at the core of our characterization of EJ-L and was the basis of Patricia's distinct conceptualization of mathematics teaching and learning.

Liliana's framing through EJ-L, in particular her focus on the students and her identification of the need for *individual* students to explain their *own* way of "going there" seemed to present a tangible option for addressing and possibly resolving the problem. Liliana's construction of the problem through the framing EJ-L seem productive, as it renders the problem accessible and enables (rather than constrains) opportunities to be successful a the work of teaching and learning.

We caution, however, that purposefully framing the construction of problems of practice through EJ-L, may not always lead to clearly productive or readily solvable problems. Other factors, besides the use of various conceptual metaphors, may also contribute to rendering a constructed problem productive or unproductive, useful or not useful. For example, a teacher who constructs the problem of students' lack of retention of information through EJ-L, like Liliana, but concludes that his students' deficient natural abilities will prevent them from explaining "how the go there," has in fact constructed an unproductive problem. More research is needed on these types of problem constructions.

Conclusion

In this work, we have illustrated different ways of constructing a particular problem of practice, shown how conceptual metaphors may serve as key resources for undertaking the construction of such problems. Different variants of the conceptual metaphor of Education as a Journey served as tools to parse out different framings of constructed problems and, at the same time, suggested specific (more or less productive) perspectives to frame a given problem. Through Liliana's talk, for example, we saw how the EJ-L metaphor strand proved useful in allowing her to focus on changing student experiences, rather than immutable students abilities, or educational policies.

Our analysis also showed that new problematizations of a given problem can constrain opportunities for teaching and learning, especially if they exchange the original problem for a less solvable one, further removed from a teacher's sphere of influence or control. We saw, for example, how Matilde's problematization of student knowledge retention shifted her focus away from considering a change in students' classroom practices, and led her to focus on the less tractable realm of national educational policies, characterized by the same problematic metaphor foregrounded in her framing.

We believe, however, that certain problematizations of perceived problems can be productive for problem resolution and may render certain problems of practice more, rather than less, accessible.

These findings begin to suggest (to teachers, teacher educators, professional developers, and policymakers) how different resources might lead to more (or less) productive problems and eventually to better learning outcomes. Furthermore, this work suggests that policymakers play a central role in teacher problem construction: The metaphors they use as foundations for policy become resources for teachers as they consider how to frame the work of teaching. As a consequence, policy makers might consider how to craft policy with care so that it provides teachers with framings and resources that lead toward useful and productive problem construction.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' REALIZATION OF THE LOGICO MATHEMATICAL ACTIVITY TRHOUGH ANALYSIS OF A CASE STUDY

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In this study I studied the nature of prospective secondary mathematics teachers' thought processes coming to understand how a high demand task differs from other tasks with respect to the mathematical activity it focuses on. In a teaching mathematics method course twenty- eight prospective secondary mathematics teachers were asked to reason on a case study for two three-hours- long class period. Data have shown that not only these students' recognition of the differences between the empirical activity and logico mathematical activity have developed but also their understanding of what a mathematical concept is and how it is learned have progressed comparatively through their focused analysis.

Background

Current reform in mathematics education has emphasized the importance of prospective mathematics teachers' awareness of teaching mathematics effectively (NCTM, 2000). Some researchers have provided useful frameworks for teacher educators for supporting prospective mathematics teachers with ways to learn to teach mathematics for understanding (Hiebert, Morris, & Glass, 2003; Simon, 1994,1995,2006; Simon & Tzur, 2004). These researchers agreed that teacher educators might take prospective teachers' attention to the tools by which they are to learn to teach rather than providing them with the end products of teaching mathematics. This paper is a preliminary report of how the integration of these frameworks (Hiebert et.al., 2003; Simon, 1994, 1995, 2006) can be useful for pre-service secondary mathematics teacher education. Particularly, this paper relates to the part of effective teaching on selecting and/or creating tasks effectively to teach mathematics for understanding (NCTM, 2000) at the secondary grade level. In this paper, I provide data showing how prospective secondary mathematics teachers' thought processes regarding what a mathematical concept is has developed through their recognition of the differences between logico mathematical activity and empirical activity. Their focused analysis of a case study allowed these teacher candidates to reflect on the mathematical activity on part of the students. In the following paragraphs, I discuss the following: Theoretical framework (Simon, 1994,1995,2006) this study is mainly based on, what logico mathematical and empirical activity is and how the results of this study might have potentially important findings regarding mathematics teacher education.

Simon (1995) proposed Hypothetical Learning Trajectory (HLT) as a framework for teachers and teacher educators to use in their classrooms. He stated that HLT has three main components: "the learning goal that defines the direction, the learning activities, and the hypothetical learning process-a prediction of how the students' thinking and understanding will evolve in the context of the learning activities (p. 136). Simon asserted that the teacher's hypotheses of students' thinking and conceptual development is dependent on the generation of learning activities and the nature of the learning activities depend on the teachers' hypotheses of the students' learning process (Simon, 1995). HLT presupposes that the teacher has some knowledge about her students from earlier experience such as teaching other students in similar situations, through pre-assessment, through knowledge of the curriculum students have gone through before taking the course etc. HLT assumes that the teacher/researcher will revise and re-plan her teaching accordingly based on the ongoing assessment of students' understanding. That is, the teacher/researcher creates hypotheses about where her students' current knowledge is depending on the assessment, verbal or written. This helps her to create more questions/tasks on which students can think further. This then allows her to have more data on what students know from which the next lesson period is constructed. In general, then, the teacher/researcher has a learning goal for her students to have. However, for each teaching session, she creates (sub)learning goals for her students depending on what they know. Thus, HLT is a recursive process from which each lesson is constructed on its own but toward the general goal of conceptualizing mathematics. In that sense, each lesson is an experiment.

As important as it is for teachers and teacher educators to have a learning goal for their students, in another study, Simon also stated that having a particular learning goal is not sufficient; teachers also need to be able to create or select tasks/ activities to bring about the desired thought processes on the students' part (Simon &Tzur, 2004). Similarly, research has emphasized that teachers' knowledge of curriculum (Shulman, 1986; Ball, Thames, & Phelps, 2008) is pivotal because their awareness of cognitively demanding tasks and their use of those tasks effectively in their classes has the greatest impact on both students' learning and their perceptions of mathematics (e.g. Ball, 2000; Lappan, 1993; Lappan & Briars, 1995).

Simon &Tzur (2004) as an elaboration on HLT emphasized that teachers' learning goal(s) for their students should not be confused with the students' goals. That is, as important as it if for teachers to have a learning goal for their students, the students' goal-directed activity is the main issue from which the conceptual learning develops. Teachers might have a learning goal for their students to understand a concept, however it is the students' goal-directed activity that brings about the desired change in their conceptions. Simon (2006) further emphasized that teachers' awareness of the students' goal-directed activities might provide them with useful ways to create or select among tasks. Simon (2006) introduced empirical activity and logico mathematical activity. He defined empirical activity as the process by which students collect a set of results and identify a pattern in those results without any insight into why that pattern is produced (the logical necessity). In contrast, the logico mathematical activity refers to the learned anticipation of the logical necessity of a particular pattern or relationship. Simon stated that the distinction he made is fundamental to what is meant by a mathematical concept and the process by which concepts in mathematics are learned (through logico mathematical activity). He further argued that the distinction he is making has important pedagogical implications in the sense that effective mathematics teaching might be established by selecting or creating tasks triggering the logico mathematical activity on the students part.

Simon (2006)'s emphasis on logico mathematical activity and Thompson (2008)'s emphasis on conceptual analysis has correspondence. Thompson (2008) proposed a way to identify and articulate the key mathematical ideas inherent to the high school mathematics. He hypothesized that once one answers the question of " what mental operations must be carried out to see the presented situation in the particular way one is seeing it?"(Glasersfeld, 1995,p.78 cited in Thompson, 2008), s/he might describe the mathematical idea that is foundational to the subsequent ones. He called this conceptual analysis and stated " …when conceptual analysis is employed by a teacher who is skilled at it, we obtain important examples of how mathematically substantive, conceptually-grounded conversations can be held with students" (Bowers & Nickerson, (in press) cited in Thompson, 2008,p. 46). In that sense, once prospective teachers have identified the logico-mathematical activity on the students' part, they might be able to create/choose lesson designs that can bring about students' learning.

The goal of this study is to investigate how prospective secondary mathematics teachers' analysis of a case study might help develop their awareness of logico mathematical activity and its distinction from empirical activity. This might contribute to their interpretation of tasks focusing on important mathematics. As Thompson (2008) mentioned, even curricula claiming that their focus is on conceptual understanding are full of tasks/problems with little reference to significant mathematical ideas. This means that, it is imperative for teacher education programs to help their student teachers construct useful ways to identify tasks focusing on significant—core—mathematical ideas that constitute basis for other important ideas. Findings might have the potential to strengthen existing mathematic teacher education programs (Thompson, 2008).

Since I was the teacher/researcher, when I am referring to the teacher, I will be using the phrase "she", in the following sections.

Methodology

In this study, I embrace the idea that knowledge is constructed via one's own experiences. We have no access to the objective reality independent of our own knowledge (Von Glasersfeld, 1988). Though, we have the capacity to adopt to the changes around us as long as those changes are made persuasive or preferable. Although knowledge construction is an individual act, teachers can design situations in which students can engage in informed explorations and reflective inquiry (Cobb, 1999). This way, the social interaction will stimulate questioning, conjecturing and justifying.

While planning each session throughout the whole semester, involving eight- three hours long class periods, I used the hypothetical learning trajectory (HLT) (Simon, 1995). First, I assessed the students' understanding of the specific mathematics idea that I would use in the case discussions. Secondly, I collected data on their thoughts on learning and teaching mathematics through written responses at the beginning of the method course. In all the classes I taught, I revised and re-planned my teaching based on each lesson in which an ongoing assessment was held and pre-assessing and post assessing for each lesson was conducted through written responses. Students were asked to write reflections based on the questions I provided them after each class. In particular, depending on the small group and whole group discussions, I had hypotheses about where my students' current knowledge is. This helped me create questions on which they can think further. This provided me with more data on what they know from which I planned my next lesson period. For the general course, I had the overarching learning goal for my students to conceptualize the mechanism of teaching, components of which were discussed previously (Confrey, 1992; Hiebert, et.al, 2003; NCTM, 2000; Simon, 1994,1995, 2004,2006). However, for each teaching session, I revised the (sub)learning goals for my students depending on what they know. This allowed me to create each lesson on its own but toward the general goal of conceptualizing the teaching mathematics for understanding. In that sense, I treated each my lessons as experiments such that I designed the kinds of tasks, hypothesized and described why certain activities might trigger the change in my students' thinking and hypothesized how they will be thinking given the certain instructional tasks. (Hiebert, et.al., 2003).

The case study that I used for this paper was on multiplication of fractions (Stein et.al, 1998). I used this case study on purpose because 1) it was one of the basic concepts I knew from pre-assessment that they did not know the reasoning behind the procedure " multiply the numerators and denominators in each fraction", 2) it would be a relatively basic concept about which they would be able to talk about the key developmental understandings as opposed to other concepts such as functions (Simon, 2006), 3) it would provide the most opportunity for my students to be able to think about the parts of effective teaching mentioned above and 4) the most importantly, it would provide a rich context based on which the empirical activity and the logico mathematical activity could be discussed for the following reasons. By analyzing such a case study might have create an opportunity to enable my students to compare and contrast the reasoning of students on the specific concept from which they could think about the distinctions between the evidence of empirical activity and logico mathematical activity.

I analyzed my each of my students' thinking one-by-one and in juxtaposition to each other before and after this lesson using the artifacts from their written responses to the questions I posed them. I also analyzed their reasoning during the lesson using the transcript from the videotapes of the teaching episodes. In addition, I analyzed the relationship between the specific steps of the lesson and students' responses to them such that the cause-effect relationship between the proposed lesson and the hypotheses I held would be evidenced. To make sure that hypotheses and conclusions drawn from the data were plausible, other researchers were consulted to challenge the conjectures and /or to affirm their reasonableness.

The following sections present excerpts from transcripts of the videotaped lessons and the students' reflections on the kind of questions I posed them before and after the lessons on discussing the case.

Results

At the beginning of the course, I assessed my students' thinking on what learning and teaching means. In general their thoughts were clustered around the idea that learning means " taking the information in from an authority" and teaching is "transferring the knowledge from the teacher to the student". This is not surprising news given the fact that these students were taught mainly with a focus on procedures rather than with a focus on conceptions before coming to the course.

After the first session, I wanted my students to write on: "what does conceptual understanding mean to you?" and "what does connections in mathematics mean to you?" Their written work clustered around the two interpretations of the meaning of connections at the beginning of the course. First one was that connections meant only to think about the general relationship among objects (such as function, derivative etc.). Second one was that real life phenomena can be explained using mathematics. However, what were not clear from the data was how the students were thinking about the nature of the connections (if they were thinking about the connections deeper within a topic among different core mathematical ideas or not) and how the connections are established. Thompson (2008) also mentioned the issue of "the connections among topics" in his plenary speech at PME, 28. He regretfully stated that the emphasis on the connections among topics has the potential danger of taking the attention from the importance of foundationally core mathematical ideas of high school mathematics. Then, during the next session, I asked my students to talk about what they think about the meaning of connections in mathematics without "re-stating the word connection". The following discussion has occurred:

> S_2: When we make sense of the new knowledge using our previous knowledge then this means we make connections.
>
> R: How are you making sense of the new knowledge using your previous knowledge?
>
> S_2: You learn the first topic and then you learn the second topic. Then, you connect them by realizing that they are related to each other.

S_3: Yes, for instance, when we learn about how to solve pool- filling problems, the teacher explains how we are using the formula for finding the solutions then it make sense to us.

S_4: But I think making sense of real life issues such as problems in physics etc. using mathematics is important too. When we learn for instance about quadratic functions, if we can apply that knowledge to the physics topics then it means we have made the connection.

And other students agreed with all what is said. Then, I decided that for students to realize the nature of connections as making relationships among key mathematical understandings (key mathematical ideas) centering around quantities from which new quantities arise (Thompson, 1994), I needed to provide them with a concept basic enough from which they can deduce the meaning of connections in mathematics at the finest level. My hypothesis was that realizing how the connections are made and between what mathematical constructs those connections are made would allow them to realize the logico mathematical activity so that they would be able to set their learning goals for their students at the finest level.

At the beginning of the lesson focusing on the case study, I asked my students to work on an re-designed form of the task on multiplications of fractions individually and then I asked them to categorize the level of the task. All of them said " a high level task". Then, I re-grouped them, three in each group, and provided each group with one of the text in which two middle school teachers', namely Kevin and Fran, preparation period, implementation and their reflections were discussed. Then I asked them to think about the teachers and students' actions in terms of their goals. They mentioned that teachers are trying to teach multiplication of fractions conceptually and the students are learning fraction multiplication with understanding. Then, I asked my students to focus on the "reason behind the teachers' goals in asking the particular what and why questions and students' focus in answering those questions. They first found the questions started with "what and why" terms and then we started talking about them one by one.

R: Why do you think Kevin is asking the question why and Fran is asking the question why? Is there a difference between them asking the question? what are their goals?

S_3: He [referring to Kevin] is kind of taking their [referring to the students] attention to the connections in their solutions. Like, what connections did you make while finding your solution.

S_4: He [referring to Kevin] is making the students realize their solutions. Like, I think, he is making them realize their way of solution rather than the result itself. Like, how they were reasoning while there are solving the problems.

S_5: For instance, in Kevin's class, when one of the students did not take one hexagon as the whole, then Kevin wanted his student to realize his mistake. But he did not tell him that. He asked the "what" question on page 169, to take his student's attention to the two hexagons as a whole rather than one hexagon. However, in Fran's case it is different. She asks "what " question to just get an answer.

R: Do you agree with what $S_5$is asking guys? [no hands yet]

Who agrees with her? [some hands up]

Who does not aree with her? [no hands up]

Who is not sure about it? [no hand up]

R: all right what about those not raising their hands? [laughs in the class]

S_1: I think in Kevin's case students are feeling as if they have to answer the teacher. Students are feeling as if they have to get over with the result as soon as possible. I mean Kevin is not allowing for his students' understanding.
R: okay. You seem to be disagreeing with S_5? You are saying Kevin is not after students' understanding of the concept? [looking at S_1].
S_1: Nods his head.
S_4: But anyway, Kevin is asking the question "what" to get at how his student is thinking. Like, he did not get whether his student was solving the problem procedurally or conceptually so he asked the question to understand whether his student is thinking procedurally or conceptually. I mean, he asked the question to get how his student was reasoning while solving the problem, how he thought about it.
S_6: I think in both cases [referring to Kevin and Fran's classes] those students know that ½ of 1/3 is 1/6. But the issue is that Fran is giving away the whole in her class but Kevin is not doing that. Both of them had the goal that their students would understand the "concept of whole " but Fran is directly giving away the answer to her students.

The excerpt above is important because of the following: when S_3 mentions the "connections while finding the solution" S_4takes it from there and elaborates on in such a way that the focus of students were on their reasoning rather than the result itself. However, this is not even well articulated that S_5 elaborates on the "reasoning and the connections" in a deeper way and brings the discussion on the key understanding " the concept of whole". Though she does not explicitly say that, she only mentions that the "what" question in Kevin's class was on the meaning of the whole. Then, S_6 kind of finalizes the issue as "whole as the interest of concept such that when students find the ½ of 1/3 they can go back to the whole and think about the result in terms of the whole" in that discussion between the teachers and the students. This shows that, these prospective teachers' attention was on the logico mathematical activity that the students have been engaging in. The empirical activity, although these prospective teachers do nor mention explicitly is that in Fran's class, although the task is the same as the one used in Kevin's class and the questions are kind of centered around "why" and "what" questions in both classes, the ways these teachers are engaging their students differs in terms of their attention to their students' focus on the logico mathematical activity versus empirical activity. That is, the deduction made in Fran's class is independent of the quantities of interest, though in Kevin's class students' deduction of ½ of 1/3 is through the quantities of interest. The claim that I am making in this discussion was even more evident in the following discussion happened during the next class.

S_4: Fran is not doing, is not asking questions to take students' attention to their reasoning, remember like we said before, we mentioned something like called specific math idea, she is not asking further questions that would allow her students to focus on specific math idea, she is not doing that. But in Kevin's case, when he is not satisfied with the student's answer he is asking more questions to take their attention to the conceptual thinking…
R: hihim
S_7 : I would like to mention the same kinds of things, in Fran's question, students do not need to think about what they did during the task, like she is directly focusing their attention to the question itself , like instead of thinking " what did I do during the task" , they are for instance in the question "what is 1/6 of 2 hexagons" the student the thing in that task was to make them think about the two hexagons as a whole, but the student does not need to think about it, like he is focusing on the question itself and he is finding 1/6 of two hexagons only. But in Kevin's

question, he is focusing on whether his student understood the two hexagon as a whole or not, he is examining that and he is making his students go back to their reasoning while engaging in the task. But in Fran's case, Fran is focusing only on the question and the student do not need to think about how they engaged in the task.
R: okay, what are the students' goal when the teacher asks the questions?
S_7: In Kevin's case students are focusing on their way of thinking. Like what I did here, what was my thinking here they are focusing on that with Kevin's questions. But in Fran's case it will be the same thing I said before but they are focusing only to the question, they are not focusing on what they thought.
The excerpts above show that these student teachers are not only able to think about the key mathematical ideas but also they are able to think about students' focus of attention, namely the quantities.

A few weeks later, I asked my students the question, "When you think about the Kevin and Fran's teaching and the task they have used in their classes, how would you compare and contrast their teaching in terms of empirical and logico mathematical activity?". The following excerpt is especially important because it shows how S_1 thinks about the question. As depicted elsewhere, at the beginning S_1 had thought that Kevin was not after understanding. However, the following data shows that his thinking on it has changed after the discussions during the classes.

> S_1: We have discussed Kevin's and Fran's tasks in terms of questioning and students understanding. We agree on that Kevin was trying to emphasize conceptual understanding better than Fran since he was asking high level questions that are not about numeric values but the connections and relations between concepts…In the Fran's case the teacher's focus is on one student and other students are expected to model that, students to go on procedures... Fran's "what" questions focus on numbers or results in each step and he helps his students to come up with the right numbers but Kevin's "what", "of what" questions requires students to focus on the relationships between the steps and he wants from them to internalize the concept. So Kevin's task is more prone to be considered as a logico mathematical task. The students in Kevin's class think of logical necessity among the steps but Fran's students try to model a student and focus on the results in each step.

Conclusion and Discussion

Data show that at the beginning of the course the line of reasoning these student teachers hold corresponds with what Thompson(2008) mentioned in his speech: that connections in mathematics involve the connections between topics. Also, they agree on the idea that the connections are made with the teacher's statement after the two topics are learned. This shows that they are not aware of the meaning of logico mathematical activity through which conceptions are learned. However, with the analysis of the case study, data show that these student teachers' understanding of logico mathematical activity has evolved. Particularly, with the teacher-researcher's attempts at taking these student teachers' attention to the focus of teachers questioning and students' answering in the case study, these student teachers' understanding of the meaning of key mathematical ideas and how those ideas are developed has progressed. In particular, taking these student teachers' attention to focus of the teachers' questions "what and why" asked during the classes in the case study, and taking their attention to the students' focus while answering those questions helped these prospective secondary mathematics teachers to become conscious about the nature of the logico mathematical activity.

Results have important implications for teacher education. As Simon (2006) proposed and the results of this study showed, prospective secondary mathematics teachers' awareness of what we mean by mathematical concepts and as Thompson (2008) emphasized, their awareness of the idea that connections should be made among core mathematical ideas has evolved.

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FROM METHODS COURSES TO STUDENT TEACHING: EXAMINING THE EFFECT OF AN INNOVATIVE FIELD EXPERIENCE

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The main goal of the Iterative Model Building (IMB) approach is to improve professional development programs for future elementary mathematics teachers, and thus improve student learning in mathematics. The innovations of the IMB approach occur during mathematics field experience for pre-service teachers (PSTs) and include a central focus on children's reasoning through teaching experiments (Steffe & Thompson, 2000) and purposeful reflection on practice through Lesson Study (Lewis, 2002). Indicators of teacher quality were examined through analysis of lesson plans and lesson enactment comparing student teachers who had participated in the IMB approach with student teachers who had participated in traditional forms of field experience. Initial findings suggest that teaching aspects fostered by the IMB approach during the field experience had a positive residual effect during student teaching. Findings from the study will advance knowledge of how to prepare more effective elementary teachers.

Background

The Iterative Model Building (IMB) approach focuses on children's thinking and reflective teacher practice through an intensive field experience approach to improve and measure the quality of teachers. In the experimental field experience placement, pre-service teachers (PSTs) begin each class session by interviewing children through Formative Assessment Interviews with a focus on children's thinking (Steffe & Thompson, 2000). The PSTs then take the information learned from the interviews and create models of student thinking that explain student reasoning. Following Model Building, the PSTs consider their knowledge about student thinking and plan a mathematics lesson for the following week. The PSTs present their lessons in a host teacher's classroom and then engage in Lesson Study groups formed by 6 PSTs, the host teacher, and one field experience instructor, to reflect on the previous lesson and plan for the next lesson (Lewis, 2000). Each week, all PSTs in the field experience are involved with the cyclical process of interviewing students, building and discussing models of students' thinking, teaching or observing lessons, and participating in lesson study. Together, these components complete Phase One of the IMB cycle.

One year later, students that completed the mathematics field experience begin student teaching in elementary classrooms. All experimental components of the IMB approach are complete at the onset of student teaching, so from this point, teacher quality can be measured to compare student teachers that were in the IMB approach for field experience with students that participated in traditional forms of field experience. During student teaching, teams of university observers visit the host classrooms of the student teachers to observe mathematics lessons and interview the student teachers about teaching mathematics. These measures of teacher quality form Phase Two of the IMB project.

Theoretical Perspectives

Effective teaching requires knowledge of how individual students in the classroom think about specific content (Akerson & Hanuscin, 2007; Carpenter & Fennema, 1992; Carpenter et al., 1988; Carpenter, Fennema, & Franke, 1996; Fennema et al, 1996). In addition to considering the mathematical reasoning of students, effective teachers are reflective about their practice and often work collectively with others to improve their practice (Cochran-Smith & Lytle, 1999; Lewis, 2000). Teachers that embody these characteristics are able to engage their students in learning mathematics through the implementation of investigative tasks with specified cognitive complexity that is appropriate for the students that are being taught (Stein, Smith, Henningsen, & Silver, 2009). These two innovations, noticing student thinking and reflecting on teaching, are key tenets for effective mathematics teaching. The following describes these innovations in greater detail.

Teaching Experiments: Model Building

Effective teachers develop pedagogical content knowledge by noticing and attending to students' thinking and reasoning (Hill, Ball, Schilling, 2008; Jacobs, Lamb, & Phillips, 2010). Teachers who reflect on student knowledge and consider students' understandings when teaching provide children with a better mathematics education (Carpenter & Fennema, 1992; Fennema et al., 1996; Jacobs et al., 2010). Components of pedagogical content knowledge, such as Knowledge of Content and Students (KCS) can be developed through engaging in teaching experiments with children that orient teachers to student reasoning (Hill et al., 2008; Steffe & Thompson, 2000; Steffe, 2002).

Teaching experiments require a specific methodological approach in which the teacher must continually attempt to make sense of student explanations and actions (Steffe & Thompson, 2000). As the teacher works to continually interpret student behavior, he or she begins to develop hypotheses about student cognition and becomes familiar with the student's ways of operating. As the teacher engages in this process, he or she begins to create viable models of student thinking by making predictions based on observed student actions. When teachers understand how children are thinking about a specific mathematical topic, they are better prepared for teaching because they are able to formulate predictive models about student thinking as they enact mathematics lessons. Teachers that are able to anticipate student responses are able to formulate lesson plans and deliver lessons that consider models of student thinking.

Reflective Practice: Lesson Study

In addition to focusing on student thinking and reasoning, effective teachers develop pedagogical content knowledge by learning in professional communities (Lewis, 2000, 2002; Wenger, 1998). Lesson study is one format that can be used for professional growth in a community of practice. In lesson study, teachers develop, teach, and analyze whole classroom lessons to build an understanding of students' knowledge. When teachers reflect on their teaching and engage in discourse with others about their practices, the teaching community fosters an ongoing venue for teacher learning (Akerson & Hanuscin, 2007; Grossman, Wineburg, & Woolworth, 2001).

Lesson Planning

Teachers that have increased their pedagogical content knowledge by considering student thinking and reasoning, and reflecting on teaching practices, are able to formulate lesson plans that include components of effective teaching (Jacobs et al., 2010; John, 2006; Lewis, 2000). During the lesson planning process, effective teachers decide what will happen when teaching mathematics content and they formulate instructional decisions for the plan based on student understanding (Bumen, 2007; Gilbert & Musu, 2008; John, 2006; Panasuk & Todd, 2005). These teachers often consider how they will launch the lesson, provide an investigative task, and summarize student learning in the lesson to develop student reasoning (Van de Walle, Karp, & Bay-Williams, 2010). Effective mathematics teachers structure lessons that consider students' prior knowledge and reasoning to promote students "doing mathematics" (Stein et al., 2009, p. 6). *Doing mathematics* ensures that the cognitive level of the planned tasks are sustained when teaching. This process of lesson planning is one component of the Lesson Study cycle (Lewis, 2000).

Lesson Enactment

Once the plan has been created, teachers must sustain the cognitive level of the task to ensure students are thinking complexly, exploring mathematical concepts, and analyzing tasks as planned (Stein et al., 2009). The quality of this sustainability for maintaining the cognitive task of the lesson is dependent on the pedagogical content knowledge the teacher has previously developed from noticing student thinking and reflecting on practice in a teaching community. As the lesson is enacted, the attention to student reasoning becomes a considered component in the decision making process. Likewise, the reflective learning that took place from the community of practice should also be considered to deliver lessons that are effective.

Application to Practice

For this project, PSTs in the experimental group engaged in the IMB field experience approach that directly focused on student thinking and reasoning with teaching experiments and included a lesson study component for reflective practice. The majority of the research on these innovations in the past has focused on in-service teachers, with little attention being focused on the role of these innovations with PSTs. It would be beneficial to know how the innovations used in the IMB field experience affected teacher quality one year after the innovative approach when the PSTs were student teaching. As a result, this study seeks to compare student teachers that participated in the IMB field experience with student teachers that did not participate in the IMB field experience. The research will answer the following research questions:

1. What effect does the IMB approach during field experience have on PSTs lesson planning and lesson enactment one year later during student teaching?
2. How do lesson plans and lesson enactment during student teaching differ between PSTs who participated in the IMB approach for field experience and PSTs who participated in a traditional field experience?
3. How do the teaching characteristics of student teachers who participated in the IMB approach for field experience compare with the teaching characteristics of student teachers who participated in traditional field experience?
4.

Method

Data were analyzed using both qualitative and quantitative methods. Data sources include written mathematics lesson plans, the Mathematics Lesson Observation and Analytic Protocol, and the Classroom Teacher and University Supervisor Observation Form. An equal number of participants from both the experimental and control groups were invited to participate in the study. The number of willing participants from the experimental IMB group was significantly higher than the number of willing participants from the traditional control group. Participants consisted of seven student teachers that had participated in the IMB approach during field experience and three student teachers that participated in control sections of the field experience.

Each of these student teachers was formally observed teaching mathematics two times by university personnel and at least once by either his or her University Supervisor or Host Classroom Teacher.

Written lesson plans were collected from each teacher for each of the two observations. The written lesson plans were blinded and scored independently by two researchers according to a Mathematics Lesson Plan Rubric (see Table 1), based on the Launch, Investigate, Summarize lesson structure for effective mathematics teaching (Van de Walle et al., 2010). The two raters reconciled score differences to arrive at a consensus. After a consensus had been reached for each category on the rubric for each lesson plan, the mean of the scores from the IMB approach were compared with the mean of the scores from the traditional approach.

The Mathematics Lesson Observation and Analytic Protocol was used during classroom observations by trained university personnel and completed after each lesson observation. The protocol format was adapted from Inside the Classroom: Observation and Analytic Protocol from Horizon Research, Inc. This instrument focuses on reformed mathematics teaching and documents the Design, Implementation, Mathematics Content and Classroom Culture of the lesson; the protocol concludes with an overall capsule rating of the lesson. For each of these categories, Key Indicators and a Synthesis Rating were recorded numerically and Supporting Evidence for Synthesis Rating was recorded in written format. Each observer initially scored the lessons independently using the protocol and then the two observers met to reconcile differences in scores. Means were calculated to compare the IMB group to the traditional control group. All written comments were typed and analyzed qualitatively.

The second instrument, the Classroom Teacher and University Supervisor Observation Form, was completed by University Supervisors and Host Classroom Teachers. This instrument is comprised of six sections: Content Organization, Content Knowledge and Relevance, Implementation, Teaching Learning Interaction and Assessment, Active Learning, and Overall Assessment. A six-point scale was used to describe the extent to which effective teaching practices were evident during teaching. This protocol elicits numerical and written responses from the University Supervisors and Host Classroom Teachers. The numerical data from the protocols were analyzed to find the mean scores for the IMB group and control group for each indicating section. The written data were analyzed to discover evidence of effective mathematics teaching.

Data Analysis and Findings

The first source of data used to examine the effect of the IMB approach on lesson planning and enacting of the lesson was the written lesson plans. The rubric used to score the written lesson plans included four categories to rate different aspects of the lesson, namely: Objectives and Adaptations (2 points), Launch (3 points), Investigate (2 points), and Summarize (3 points). Scores obtained in each of these categories were totaled up for a possible rating of 10 points. Table 1 shows a portion of the rubric that includes the description of the highest level of performance for each category.

Category	Description of Highest Level of Performance
Objectives and Adaptations 2 points	Lesson includes specific prerequisites. Lesson includes specific measurable objectives. Lesson includes State Standards addressed. Plans for Gearing Up are appropriate. Plans for Gearing Down are appropriate.
Launch 3 points	Launch provides some context for the topic and to motivate students to work on the investigation. Includes ways to make sure that the problematic task is understood and that goals and expectations are clear.
Investigate 2 points	Investigation includes a problematic task at the cognitive level of "doing mathematics." Considers students' prior knowledge and leaves residue of mathematical value. Equity and accessibility are addressed. Includes possible hints the instructor will provide when needed, and things to pay attention to in order to observe students thinking and to assess students understanding.
Summarize 3 points	Summarize includes a description of how to organize the sharing to make sure the class is engaged in a productive discussion and reflection in which students learn ideas and strategies from each other. Includes possible questions to ask to encourage generalizations and to make sure students justify their answers.

Table 1. Section of the rubric used to rate written lesson plans.

After two researchers independently rated the blind lesson plans and reconciled scores, ratings for the two groups were analyzed. Given that the samples are small, statistical analysis cannot be used to determine if the differences found are significant. We decided to calculate the difference in ratings for each category as a percentage of the total possible points for each category. Given that the rubric can differentiate between 4 levels of performance in each category, we thought that differences greater than 20% of the possible points would be considered significant. Table 2 shows the scores for the written lesson plans. Written lesson plans for the IMB group consistently scored higher than those of the control group for most categories. On average, written lesson plans for the control group had a total rating of 1, whereas lesson plans for the IMB group had a total rating of 3 and thus the difference was 20% of the maximum possible score. We then examined the differences in rating by category. Three categories showed a difference in rating greater than 20 % for at least one of the lesson plans. These categories were objectives and adaptation, launch, and investigate. Thus written lesson plans of participants in the IMB group tended to do a better job of addressing the expectations for most aspects of the lesson plan. We are aware that average total scores for lesson plans for both groups were relatively small. We attribute this to the fact that the participants were not submitting these lesson plans for a grade in a class, but rather they included the level of detail they felt necessary for them to be able to teach their lessons. Which is not uncommon of inservice teachers too, they no longer write the elaborate lesson plans they were required to do when in methods classes.

	First Lesson Plan					Second Lesson Plan					
	O&A	L	I	S	Tot	O&A	L	I	S	Tot	2 L
IMB	1	.67	.92	.42	3	.88	.88	.75	.5	3	3
Control	.5	.33	.5	.5	1.83	.5	0	.5	0	.17	1
Difference	.5	.3	.4	-.1	1.2	.4	.9	.3	.5	2.8	2
% Diff	25	11.1	20.8	-2.8	11.7	18.8	29.2	12.5	16.7	28.3	20

Key: O&A–Objectives and Adaptations, L–Launch, I–Investigate, S–Summarize, Tot–Total, 2L–Average rating of two lesson plans, % Diff–Difference in ratings as a percentage of the possible points for that category.

Table 2. Average ratings in written lesson plans for IMB and Control groups.

One source of data to rate the quality of the lesson enactment was the Mathematics Lesson Observation and Analytic Protocol. Table 3 shows the average ratings for each category, out of 5 possible points, of the lessons observed that were taught by participants in the IMB and control groups. Note that for the overall capsule rating observers were asked to give their overall assessment of the quality and likely impact of the lesson, rather that just an average of all previous ratings. The five levels in the overall capsule rating were: 1) ineffective instruction; 2) elements of ineffective instruction; 3) beginning stages of effective instruction; 4) accomplished, effective instruction; and 5) exemplary instruction. The average overall capsule ratings for lessons taught by IMB participants were 2.71 for the first lesson and 2.52 for the second lesson. These ratings are close to a solid 3 or a low 3, which corresponds to beginning strategies of effective instruction. In contrast, the average overall capsule ratings for lessons taught by participants in the control group were 1.83 for the first lesson and 1.92 for the second lesson, which are close to a solid 2, which represents elements of ineffective instruction. Like with the written lesson plans, IMB participants were rated higher for most aspects of the lesson enactment.

	First Lesson Plan					Second Lesson Plan					
	LD	IM	MC	CC	OR	LD	IM	MC	CC	OR	2 L
IMB	2.93	3.21	3.57	2.86	2.71	2.5	3	3.5	2.67	2.33	2.52
Control	2.33	3	3.83	1.83	1.83	1.67	2.67	3.5	1.67	2.0	1.92
Difference	.6	.2	-.3	1	0.9	.8	.3	0	1	2.8	0.6
% Diff	11.9	4.3	-5.2	20.5	17.6	16.7	6.7	0	20	6.7	12.1

Key: LD–Lesson Design, IM–Implementation, MC–Mathematics Content, CC–Classroom Culture, OR–Overall Capsule Rating, 2L–Average rating of two lesson plans, % Diff–Difference in ratings as a percentage of the possible points for that category.

Table 3. Average ratings in Mathematics Lesson Observation and Analytic Protocol for IMB and Control groups.

When examining the differences in ratings in the enactment of the lesson by category, one category, classroom culture, showed a difference or 20% or more in favor of IMB participants. Another category, lesson design, showed a difference of 11% or more, also in favor of the IMB participants. Indicators that the observers attended to when rating lesson design included aspects such as whether the lesson incorporated tasks, roles, and interactions consistent with investigative mathematics; and whether adequate time and structure were provided for "sense-making." Indicators in the category of classroom culture included aspects such as whether active participation of all was encouraged and valued; and whether the climate of the lesson encouraged students to generate ideas, questions, conjectures, and/or propositions.

A third source of data we examined was ratings by University Supervisors and Host Classroom Teachers using the Classroom Teacher and University Supervisor Observation Form. A six-point scale was used to rate performance in each of the five categories, as well as for the overall rating. The mean overall rating reported for IMB participants was 4.91, compared to a mean overall rating of 5.33 for control group participants. The difference represents less that 7% of the possible points and we did not consider it to be significant. So, in the eyes of the host teachers and supervisors, the teaching of all participants was of about the same quality, for the teaching aspects they pay attention to. We are not surprised about this finding as the rubrics and observation forms developed by the project measure the teaching aspects we hope to foster, which may not necessarily be the same that others are looking for.

A final source of information on the lesson planning and enacting of the lessons was the written comments included by observers as evidence to justify their ratings in the lesson observation forms. While written comments included suggestions for all participants to improve their teaching, there was a pattern in the comments about the type of lessons taught by control group participants. The written comments characterized these lessons as traditional, with comments such as:

- Lesson was very scripted with a focus on following a procedure without making a connection to concepts.
- The lesson was again a traditional format. Students reviewed long division. The students did not collaborate with each other. The dialogue was mainly between student teacher and students.
- The instructor began discussing multiplication and reviewed subtraction though a competition at the end of the lesson. This extension was fact practice. Students never talked with other students; little, if any collaboration. No connection to life/applicable context.

Conclusion

The findings described above while modest, show promise of the IMB approach. They suggest that teaching aspects fostered by this approach in the field experience had a positive residual effect that was present during student teaching. Regarding the first research question, we found evidence that lesson planning and lesson enactment by IMB participants exhibited teaching aspects that were fostered during the field experience. In particular, aspects of the written lesson plan that showed strength among IMB participants were the objectives and adaptations sections, the launch part, and the investigate section. As far as the enactment of the lessons, IMB participants did show strength in the aspects of lesson design and classroom culture.

Regarding the second research question, we found evidence that IMB participants were rated higher than control participants in both written lesson plans and lesson observation forms. IMB participants were rated 20% higher in several aspects of their teaching indicating that they strived for student-centered teaching, making connections, and fostering collaboration among students. As far as the third research question, we found evidence to suggest that the teaching by control group participants tended to be more traditional, teacher-centered, with no connections to real life situations.

One limitation of this study is the small number of participants. While the findings reported here are valid for these participants, we will continue following up participants in upcoming cohorts to gather additional evidence about the benefits of the IMB approach. However, we should note that since the participants were volunteers, and they agreed to be observed by strangers, and to provide us with their lesson plans; one may assume that they probably were among the stronger students from each approach.

The Iterative Model Building (IMB) approach to elementary mathematics field experiences focuses on children's thinking and reflective teacher practice. We are encouraged by the initial results that participants are exhibiting during student teaching. For phase three of this study we will follow participants into their first year as teachers in our efforts to develop a research based approach to teacher preparation.

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A TRANSFORMATION OF PRESERVICE TEACHERS' PERCEPTIONS OF THEIR BELIEFS OF MATHEMATICS AND SOCIAL ISSUES

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Preservice teachers often come to college mathematics content courses with beliefs about mathematics that reflect a view of the subject as dichotomous, reserved for certain individuals, and disconnected from a world outside of the classroom. In this study, we examined what happened to a group of preservice teachers' perceptions of their beliefs about mathematics and social issues when they were taught in conjunction with one another. We found that connecting mathematics to social issues transformed these participants' ideas about the nature of learning both subjects.

Although integrating subjects in the classroom has been a concern for educators for over a century (e.g., Dewey, 1902), connecting mathematics to social issues has begun to rise as a major goal for many mathematics educators (i.e., Gutstein & Peterson, 2006). These educators often contend that the purpose of education should be social well-being and all instructors should explicitly address this objective in their classrooms regardless of discipline. However, in order to persuade mathematics teachers to do so, prospective teachers must transform many of their traditional beliefs about mathematics as dualistic, reserved for certain individuals, and disconnected from an outside world (Young, 2002). Therefore, in this study, we examined what happened to a group of preservice teachers' beliefs about mathematics and social issues when the two were integrated in a mathematics content course for elementary teachers.

Theoretical Lenses

Two theoretical lenses are used to explain the findings of the study. The first addresses beliefs that stem from associationist learning theory. This theory of teaching and learning emerges from the philosophy that reality is separated into categorical subjects and that learning about this reality is transmitting a set of facts from a more knowing authority to a less knowing being (Houser, 2006). However, often times this form of traditional pedagogy, which is still the predominate one in western schooling (Houser, 2006), positions the student as passive and inferior in the classroom, waiting to receive knowledge, rather than active, dynamic, and constantly interacting with and reacting to a reality around her. In theory, students who emerge from this kind of educational system can form beliefs about subjects that traditionally mean that knowledge is simplistic (e.g., there is one right way to do mathematics (Young, 2002)) and an authority holds this knowledge.

The second theoretical lens explores the idea of the formation of beliefs that are based on transactional learning theory. Unlike its counterpart, this theory insists that humans are engaged in a transactional give and take relationship with the world around them, and that reality is relational (Houser, 2006). From this perspective, teaching and learning takes on a different meaning. There is no longer an assumption that a prescribed set of facts exists as truth and can be transmitted as such. Subjects such as mathematics and social issues interact with and react to one another, continuously expanding and changing. When mathematics and social issues are taught from this perspective, it becomes the role of all participants in the classroom to negotiate meaning and decide whether or not ideas are valid, just as is done in society. When teaching and learning in this type of classroom, beliefs about the attainment of knowledge begin to transform. The role of the instructor becomes that of someone who develops opportunities for intellectual growth. These opportunities link students to a history and a world outside of the classroom (Dewey, 1938). In theory, mathematics and social issues become more complicated yet more accessible, and students can begin to author their own learning, transforming their beliefs about the nature of subjects.

Methodology

The study took place in a community college classroom in the Southwestern regions of the United States and spanned the duration of a full semester. Nineteen preservice teachers, all the students enrolled in the content course, participated in the study. Seven of the students were nonnative English speakers, and ages ranged from nineteen to forty two. Several students were nontraditional, having returned to school after several years of being away. The instructor of the course was also a participant in the study and a doctoral student at a nearby university.

The course had two major components, a mathematical one and a social one. At the beginning of the semester, the instructor of the course was given a set of topics to “cover.” These included but were not limited to developing an understanding of the real number system and operations within this system (e.g., addition, subtraction, multiplication, and division of integers, rational numbers and irrational numbers). The instructor’s underlying objective was to provide opportunities to develop an understanding of critical social issues such as healthcare reform, poverty, distribution of wealth, and consumerism. Tasks and activities for the course were deliberately selected and created with these objectives in mind; many of these were inspired by and adapted from readings in *Rethinking Mathematics: Teaching Social Justice by the Numbers* (Gutstein & Peterson, 2006) and utilized problem-solving approaches to teaching and learning mathematics (Wheatley & Abshire, 2002; Van de Wall, 2004).

The study was qualitative in nature and employed both case study (Stake, 1995) and practitioner-research (Anderson, Herr, & Nihlen, 1994) designs. Multiple forms of data were collected. They included a reflective journal kept by the instructor, student work, reflective student journal entries, and audio/video tapes of select class periods and informal interviews (referred to as mid-term conferences). Data were analyzed using a data analysis spiral (Cresswell, 2007), in which data were informally analyzed throughout the semester and later formally analyzed through several spirals through the data. Themes were identified during the formal phase of analysis, and examples were extracted through one final spiral through.

Findings

Initial Beliefs

Data revealed that students came to this course with many beliefs about mathematics and social issues that resulted from traditional schooling experiences. Most of the students initially saw the subjects as dichotomous--containing only right and wrong answers--and as only truly accessible to a more knowing other (Young, 2002).

Dichotomous Beliefs

Early in the semester, as students began reflecting on their previous experiences with mathematics, they began expressing the dichotomous view of mathematics that they held. In her reflective journal, one student wrote that mathematics in school is "teaching kids to always have the right answer or just teaching them to pass tests." Others wrote, "[t]here is a wrong way and a right way to do a math problem," and "math was the way the teacher taught it." Many students described mathematics as being a subject that has "one right answer" or is done "one way." In their descriptions of their previous mathematical experiences, almost every student, including those who enjoyed mathematics, referred to the idea that there is only one way to do mathematics. Doing mathematics one way sparked resentment towards the subject for many of the students. In a classroom discussion, the instructor noted one student describing being punished for not doing mathematics in a particular way. She said, "I used to have a teacher that would count off if we didn't do problems exactly like she wanted." Many students echoed this statement, saying they had similar experiences. Several students explained they were not encouraged to understand the reasoning behind the mathematics they studied. One student said her teachers always told her "that's the rule," so she began to refrain from asking.

Similarly, students in this course tended to view social issues as existing within two perspectives, a correct one and an incorrect one, often times based on their faith in others. Students used the words "right" and "wrong" when they discussed social issues. For example, when they began a discussion about healthcare reform, before studying the issue through mathematics, many students simply stated that healthcare reform is wrong, and after being placed in the political party that she opposed for a healthcare debate they had in class, one student said, "I could do this project if I was on the right side." In another instance, as students discussed the idea of socialism, several expressed that they believed socialism is "bad," without considering that we have socialism in the United States (i.e., public schools, fire departments, police officers, etc.). For many of the students, there seemed to be no room for common view points from the two sides or an alternative perspective to the two. There was just the right way and the wrong way.

More Knowing Other

Several students initially viewed mathematics as reserved for certain "other" people. In her reflective journal, on several occasions, the instructor noted one student saying, "I'm just not a math person." Another student wrote in her journal, "I think my brain does not automatically think mathematically it is something I have to work at," indicating that thinking mathematically is a special ability that she does not naturally possess.

Further, students viewed mathematics as a subject that is transmitted as whole and complete from an all knowing teacher to a less knowing student. One student wrote that in her mathematics classes "there was always someone showing the how to's, the right way per say of doing a problem." Although most of the students viewed themselves as not able to know without a teacher, the instructor of this course expected them to discuss and resolve issues on their own. She expected them to justify their thoughts and decide whether or not they agreed with one another. She was instigating the social construction of knowledge rather than using a more typical "show and tell" method (von Glasersfeld, 1995).

Students were initially frustrated with this process. During the first couple of weeks of class, as the instructor rotated from one group to another, various forms of one question kept emerging: "Is this right?" The teacher kept giving the same answers: "I don't know. What do you think? Why?" Students were not always pleased with these inquiries as answers. They seemed frustrated when she refused to answer their questions, as they seemed to exud little faith in themselves as capable of obtaining an answer without the instructor's validation. Various questions and comments such as, "Why can't you just tell us?" or "I don't know. It just seems right" radiated at each table. The instructor continuously asked students to convince other group members, and she probed their thinking by asking questions such as, "What made you think to go about the problem that way? Is there a difference between what you did and what she did? Where?" After the first week, one student wrote in her journal, "I am now nervous because this class seems like it is going to be a lot." Many students used the word "different" when describing the way the class was being conducted. For the first several weeks, the instructor wrote about the frustrations students depicted in the classroom as they were forced to determine solutions and validate ideas without her aid.

Similarly, students' beliefs about social issues seemed to be based on the verification of an "other." In her reflective journal, the instructor wrote about one student saying she had received her information from the news, while another student explained that her family had discussed these issues. Many students' perceived right and wrong perspectives seemed to come from their affiliations with certain political parties or their connections to significant others in their lives. Many students categorized themselves as Democrats or Republicans. One student explained that although she does not always understand issues well, "I do know that when it comes down to it I am a republican." On several occasions, the instructor noted students discussed their spouses, parents, churches, and coworkers' when they described their own ideologies.

The loyalty they felt towards their political parties and significant others was difficult for some students to overcome as they began to view issues from other perspectives. They seemed to feel it was a betrayal if they opposed one of the stances those others perceived as correct. One student explained how difficult it was to rethink some of the issues we studied because of her husband's viewpoint. In class, she explained trying to discuss issues with her husband but encountering resistance when she did so. She said, "He's a staunch Republican and he just doesn't wanna' hear it." Another student, after a discussion about distribution of wealth and a change in opinion said, "I feel like I'm betraying my party if I agree with you," as if she was doing something wrong by considering an alternate perspective.

For several weeks, students seemed uneasy about the notion of constructing ideas individually and verifying them socially. Ironically, in the world outside of the classroom, knowledge emerges from the questions that are asked, answered, and verified through individual curiosity and social interactions, not from an authoritative figure who just knows. In this class, however, students initially had this expectation for the attainment of knowledge.

Transformed Beliefs

As the semester progressed, students began to express their change in beliefs about mathematics and social issues. Although many initially viewed mathematics and social issues as dichotomous and only accessible to certain individuals, by the middle of the semester many students began viewing mathematics differently. Students began rethinking their beliefs about both the duality and accessibility of mathematics and social issues.

Pluralistic Beliefs

As students were exposed to multiple methods and solutions to problems, they began to reconsider their ideas about how mathematics should be done. Many students commented on the effect this course had on their recognition of the varieties of ways of understanding the subject. Students wrote:

- I have learned that math is a subject that causes you to think…I have realized that not everybody thinks the same way. There is always more than one right way…
- There are more than one way to get an answer and it's ok if it's not the way the teacher taught it…
- I really enjoyed this math class. It expanded my horizons and allowe[d] me to understand math a little bit better. I began to realize that there is not just one way to find an answer…

Moreover, after engaging in mathematics lessons that incorporate social issues and concepts, students began to change many of their beliefs about the nature of social issues. They began to reconsider their quick conclusions about which side is "right." One student explained in class that she initially believed she had to "choose a side" when she encountered arguments for or against certain solutions to social issues, but after the healthcare debate, she expressed that she was "torn" and did not know which side was right. Other students seemed similarly confused by their inability to quickly choose a side when considering social issues and concepts. They began to view issues as not so black and white, and they began to revert to mathematics as the connection to their beliefs rather than emotion or an external authority.

Although most of the students in the course continued to view social issues as stemming from two perspectives, they began to have difficulty categorizing each side as right or wrong. Students no longer seemed to be able to quickly label others' points of view as simply correct or incorrect. Many began viewing positives and negatives from both sides. Students wrote in their journals:

- After researching I found that on this topic both sides have great points and I am torn…
- I learned to keep my mind open. There are many good points on each side of the healthcare debate…

Working in groups and realizing that people view mathematics in different ways seemed to contribute greatly to students beginning to see mathematics differently. In their final reflections, almost every student commented on the effect of seeing more than one way to perceive or solve problems. They wrote:

- It was very insightful for our peers to help us with our problems. It allowed for different techniques to be shown…
- It was also a pleasant surprise to be able to see that there are many ways to achieve the right answer and that it was alright for everyone to have different opinions and approaches to logic…the overall acceptance and understanding of viewpoints was comforting…

Further, the transformation of students' beliefs about the nature of social issues seemed to stem in large part from considering other perspectives through mathematical research and working with students' whose perspectives were different than their own. Many students, during mid-term conferences, explained having never thought about diverse outlooks of issues. In their reflections, several students wrote about being affected by other viewpoints. Students wrote:

- The political debate was really interesting, especially since I was on a side that I had never investigated before. Each of the social issue lessons pushed me to think about issues I was choosing to not think about…
- I also found this project to be eye opening to others views…

One student explained that she was open to new ideas because of her interactions with group members. For her, the fact that we were all female and interested in education made a difference. In her journal, she wrote, "I also realized how grateful I was to be debating the subject with just women and teachers…We didn't have any irrational outburst because we were all understanding." In my reflective journal, I noted that the single gender of all the classroom participants might have been a factor in the development of an open, comfortable community in this class. I wrote about a student commenting that one of the reasons she was open to listening and discussing in this class was this factor. During that class period, several students agreed with her. However, I did not systematically assess this factor; therefore, further research is needed to conclude more definitively whether or not the results of the course were affected by the fact that the class was comprised entirely of women.

More Knowing Self

Along with reconsidering their ideas about mathematics and social issues as either right or wrong, students began to change their views of mathematical and social issue knowledge as reserved for certain individuals. During one class session, the instructor wrote about overhearing a student who had previously said that her "brain did not think mathematically," say, "I can do this kind of mathematics." One student who wrote about her bad experiences with mathematics early on in the semester, in a later classroom discussion said:

> Letting us do math this way is also good for students who come from different countries because in their countries, the teacher tells them to do problems one way and when they come here the teacher says do it another way. Like this, they can do it how it makes sense for them.

Other students wrote in their journals, "I'm learning that there are certain parts of math that do click with me," and "I think I would have done much better in math if my teachers let me do it like this." They began to express learning about themselves and their capabilities as mathematics learners and teachers after engaging in this course.

- I learned a lot about myself in this class. This class gave me a boost of confidence mathematically and as a future teacher. I felt so bad about math when I began this class and now I know that I can learn math and I can teach it too!...
- I learned how to teach math and that it's okay to go against the way I was taught as a child…
- So far, I've learned the importance of having a flexible mind when working with math. I need it to be reminded that there are many, many ways to solving math problems…

Students began to question their reliance on significant others in their lives for understanding social issues. They began to trust their own abilities to investigate and understand problems in society. Students wrote:

- I learned how important it is to investigate issues for yourself and not just believe what your friends tell you or what you hear on the news…

- We should not take our leaders information to heart, always question, always do your research…
- It was interesting to investigate both sides and then make my own choice…

Participating in this class opened up space for students to discover verification and construction of ideas from within themselves. Students began to author their own learning and understanding, validating and verifying their ideas without the reliance on others' authority. At the end of the semester, almost every student commented on the effect of realizing that mathematics can shape their beliefs about social issues.

- As a student of the math class, I realized that we are using "MATH" a lot in our real life, not only calculating for our receipt in a store but also reading what happens in our community. Statistic and many kinds of graph can convey a whole story…
- I loved the lessons that covered social issues. These lessons made me think outside of my personal box. Additionally, the lessons showed me just how important math is in our daily lives.
- I liked the lessons dealing with social issues. It brought another dimension to the class and it helped me see how math is used in other ways.

By the end of the semester, students seemed to no longer believe that they needed to rely on a more knowing other to understand mathematics or social issues. They began to view themselves as capable of thinking mathematically and they began to use mathematical information to explain why they accepted or questioned one social viewpoint over another. Connecting mathematics to social issues allowed students to become more autonomous learners and decision makers. They began to expand their beliefs about both mathematics and social issues and were able to extend their thinking about social issues beyond an emotional level, linking it to mathematics and using those connection to actively engage in democratic decision-making.

Implications

Data from this study indicate that most of the preservice teachers that engaged in this course were able to transform their beliefs about both mathematics and social issues. The study implies that combining the two subjects in the way they were in this course can affect beliefs about the nature of understanding mathematics and social issues; however, it is not without its limitations. A few students did not seem able to hurdle their initial beliefs. They were not discussed in this paper. Their resistance may be the result of many years of traditional classroom interactions, or the fact that the lessons in this class created too much dissonance for them, or any number of other factors. Further research is needed to determine what exactly maintained these beliefs and understandings in some students in this course. However, we mention this because it does imply that the interactions of a classroom cannot be completely encompassed within the confines of a paper such as this. Therefore, it is important to note that although meaningful change in beliefs can occur when mathematics and social issues are taught in conjunction with one another, it is not inevitable.

Although not without its limitations, this study seemed to be consistent with other scholars' ideas about the effects of certain learning theories. In accordance with ideas about associationist learning theories (Houser, 2006), most of the students in this course entered with traditional mathematics classroom experiences that yielded certain beliefs about the nature of learning mathematics. They came to the class with assumptions about what knowledge was and who had access to it. After exposure to pedagogical approaches that stemmed from a transactional theory of teaching and learning, those assumptions began to transform. As others have suggested (Houser; Dewey, 1938), teaching mathematics in a way that develops autonomy and connects to a history and a world outside of the classroom shifts ideologies about what it means to learn mathematics and social issues. Beliefs about what is right and who holds that knowledge shift from an outside reality and another individual to an interactive reality, in which subject interaction is complicated, and an autonomous self can know. Therefore, we would advocate creating space in the mathematics classroom where preservice teachers can transform their beliefs about mathematics and social issues when they are intertwined with one another in meaningful ways. Further, we believe that in doing so the importance of linking mathematics to social issues can become more explicit for preservice teachers. Although the preservice teachers in this course seemed compelled by teaching mathematics in this way, this is only one step towards teaching mathematics for social change. Additional research is needed to assess whether or not connecting mathematics and social issues in the way they were here can in fact translate to these teachers future classrooms.

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PRE-SERVICE TEACHER CONNECTIONS: CONNECTING COLLEGE MATHEMATICS TO SCHOOL MATHEMATICS

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This report summarizes findings from the study of 4 prospective secondary school teacher's connections between university geometry curriculum and secondary geometry. Connections are defined as knowledge and beliefs that link one's construct of university curriculum to his or her construct of secondary school geometry curriculum. Transcripts of individual interviews conducted with participants and participant generated maps of secondary geometry curriculum constituted the data for the study. Findings include connections categorized as positive or negative in reference to the promotion of future growth in knowledge and beliefs needed for familiarization of secondary curriculum and effective implementation of standards based mathematics teaching.

In 1985 Cooney called for a discussion of the relationship between teacher content knowledge and teacher beliefs about mathematics, recognizing that a teacher's conception of mathematics has a direct impact on the way in which mathematics is approached in the classroom. Five years later Ball (1990) expanded on the complexity of teacher knowledge stating that teachers needed to understand the nature of knowledge in the discipline, where that knowledge comes from, how it varies with time, how truth is established, and philosophically what it means to know and do mathematics in order to be effective teachers. Studies suggest (Borko et al., 1992; S. Brown, Cooney, & Jones, 1990) that teacher beliefs are influenced by teachers' experiences with mathematics and schooling long before their entry into the mathematics education workforce. These same studies state that without *significant* interventions, these beliefs are very difficult to change (Cooney, Shealy, & Arvold, 1998). Innovative methods courses can address some of the difficulties, but studies show that these methods courses are not enough (Borko, et al., 1992; Ensor, 2001) to overcome the issue that "teacher's conceptions are dependent upon their experiences as learners of mathematics"(Ernest, 1989, p. 13). Prospective teachers gain experience with mathematics in their own preK-12 experience, but also in university mathematics content courses. This site offers another opportunity where the knowledge and beliefs of these prospective teachers about mathematics can be influenced. Therefore the *development* of prospective teacher's knowledge and beliefs about teaching and learning mathematics in these courses should be a goal of their programs of study.

Knowledge, Beliefs, Connections

Knowledge and Beliefs

Both mathematical knowledge and beliefs have been identified as influencing the practices of mathematics teachers. Mathematical knowledge for teaching has been defined and Ball, 1990; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008; Hill, Rowan, & Ball, 2005) described as including more than common content knowledge and involves a complexity specifically of containing "knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum"(Bray, 2011, p. 5). Knowledge is held onto much differently that what is described as beliefs, though it would be a mistake to state that knowledge is unchangeable and static. Beliefs are understandings about the world, situated within the individual, that are *understood* to be true. "What is knowledge for one person, may be beliefs for another, depending upon whether one holds the conception as beyond question" (Philipp, 2007, p. 259). Here we find that varying degrees of conviction result in the categorization of an understanding and existence within a context. Teachers' systems of beliefs, in particular, their conceptions of the nature and meaning of mathematics, may account for variations in practices (Thompson, 1992). However differentiating between knowledge and beliefs may be difficult. Philipp noted that (2007) "distinguishing between knowledge and belief is unimportant for research" (p. 259). Instead, importance is placed on the recognition that preservice teacher *knowledge* and *beliefs* will have effects on their future practice. These "complementary subsets of the sets of things we believe" must have origins specific to the individual (Leatham, 2006, p. 92). Adopting the view that knowledge and beliefs (K&B) are inseparable and university course work may contribute to the development of both, this research seeks to explore the development of connections between K&B about mathematics and K&B about teaching mathematics. In particular, I explore the relationship between K&B generated in a university content course, and how those are projected onto K&B of school curricula. To do so, I define the potential relationship between K&B about mathematics and K&B about teaching as connections.

Connections

Preservice teachers' own preK-12 experience and preservice teachers' exposure to mathematical learning during their undergraduate programs (C. A. Brown & Borko, 1992; S. Brown, et al., 1990; Lortie, 1975; Mewborn & Tyminski, 2006) have been identified as two possible origins of K&B. Relationships between K&B about mathematics and K&B about teaching are fostered when the development of K&B are in an environment that allow prospective teachers to articulate the relationships they see. These links are *connections* between one's construct of university curriculum (mathematics) to his or her construct of secondary school geometry curriculum (teaching). *Positive connections* are defined as connections that inform future growth in K&B needed to learn from the secondary curriculum and effectively implement the curriculum. *Negative connections* are defined as having the potential to impede future growth in K&B needed to learn from secondary curriculum and effectively implement the curriculum. Investigating connections between university curriculum and school curriculum is vital to understanding the success or failure of preservice teacher programs of study in influencing the development of K&B prerequisite for effective mathematics teaching. This study is designed to investigate types of connections linking university geometry course curriculum to the secondary school geometry curriculum as represented in the NCTM documents. Specifically:

> What types of connections are made between the experience in a university level geometry course and participants' understandings of school geometry?

Methods

Participants and Context

During fall 2010, undergraduate mathematics education majors, identified by their advisors, were sent a survey, designed to solicit participation in the study. Those students who had taken Euclidean Geometry, a university starting with high school topics and moving quickly to more developed material, were identified as potential participants. The course is taught by a professor with a professed interest in mathematics education. Anecdotally, the course is known to be helpful and pertinent to preservice secondary mathematics teachers. Potential participants were divided into two subgroups based on their answer to the following question:

> Do you see a connection between college mathematics courses and the curriculum you will be teaching in secondary schools?

Of the 15 potential participants, the two who responded affirmatively and two who responded in the negative were selected for and agreed to participate in the study.

Table 1

Participant Demographics

Pseudonym	Sex	Connection Categorization
Paul	Male	No Connection
Sophie	Female	Connections
Rachael	Female	No Connection
Janet	Female	Connections

Instruments

Two instruments were used to gather evidence of K&B as well as connections: an interview protocol and a mapping activity.

Interview protocol. Participants were interviewed for 30 minutes, after which the audio recordings were transcribed. An interview guide was developed that focused on eliciting evidence of the K&B about university geometry and K&B about school geometry. The exact sequence and wording of questions were decided during the course of the interview by the researcher.

To elicit evidence of connections, participants were asked questions such as:

> As you are taking a post-secondary mathematics course, are you constantly trying to make connections to your vision of your future geometry classroom and what you will be teaching? If so, how do you go about pursuing these connections?

All participants affirming connections were then prompted with a sequence of questions in order to stimulate further elaboration. Such as:

> Are these connections the result of instructor actions? Are connections to school geometry emphasized in a university geometry course?

Participants not affirming connections were exposed to a similar line of questioning in order to stimulate more information about the lack of connections made.

Mapping of geometry curriculum. During the interview participants were asked to develop an image of the secondary geometry curriculum. Specifically, the following prompt was used

> Any way you would like, please draw me your idea of what the curriculum for a typical secondary geometry course would entail.

Participants were asked to explain the facets of their drawings and share insights about lines drawn between words they identified.

Data Analysis

Drawings. The drawings were coded by the number and content of primary subsections and secondary branches. For example in figure 1, the drawing contains 2 primary subsections, specifically “proofs” and “constructions.” Secondary branches of these subsections were categorized and noted. In figure 1 from the primary subsection “constructions,” “perpendicular lines” and “angle bisectors.”

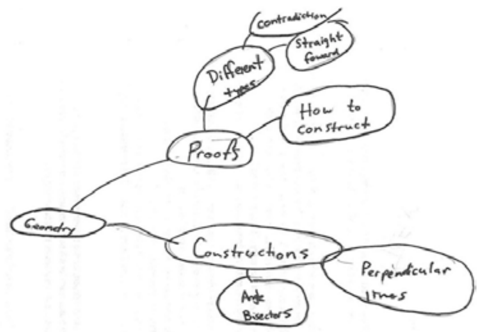

Figure 1. Paul’s drawing of school geometry curriculum

Additionally, the content of the maps were compared to the NCTM standards for school geometry. Similarities and differences in content were noted. For example Paul mentions proofs as primary subsection of school geometry curriculum and the NCTM standards incorporate proof, though much more intricately, as a primary subsection of the idealized school geometry.

Interview protocol. The interview data was coded using categories of positive and negative connections, regardless of claims regarding connections on original survey data used to identify participants. For example the following quote exemplifies a positive connection between the university geometry course and school geometry in explaining that the course curriculum informs and reviews what is entailed in the school curriculum:

> And um, I guess to kinda review geometry concepts in general and to go more in depth and learn more about, you know Euclid and axioms and stuff. (Sophie)

All participants were able to describe positive connections between university geometry and school geometry and shared additional ideas about connections between university mathematics courses and school curriculum. The introduction of comments in reference to other mathematics courses necessitated further coding to investigate connections more broadly. The following excerpt is an example of a connection between university mathematics and school mathematics that was coded as negative.

> None of that [mathematics content] is really ever going to come into play in high school math, as far as like helping me understand why something in algebra works. (Paul)

The interview data was compared with participant’s drawings data to build models of participants’ K&B of university geometry and its connection to their K&B of school geometry. For example, Sophie included a main subsection in the drawing exercise “axioms and common notions”. The participant specifically identified “axioms and common ideas” from the book of Euclid as some of the information drawn from the university geometry course. Thus an inference that there was a positive connection between K&B of university geometry and K&B of school geometry was made.

Results

The discussion will use examples from Paul and Sophie due to limited space, however results apply to all participants.

Across participants, drawings had noticeable difference in the number of primary subsections and secondary branches. Participants originally selected because they viewed university geometry and school geometry as connected had a more elaborate maps as measured by the number of primary subsections and secondary branches. Despite this difference, all four drawings contained very few subsections and branches. In comparison to the NCTM standards, which if drawn in a primary subsection and branching format would reveal four primary subsections corresponding to the overarching standards for grades preK-12: analysis of shapes and shape properties, location and spatial relationships using the Cartesian plane, apply and understand transformations, and use the above facets in problem solving situations. Many subsections branching from each of these primary standards are shared and elaborated in the standards documents (NCTM, 2000, 2009). The comparison between the drawings and NCTM standards revealed very little overlap, suggesting that K&B of school geometry were not informed by curricular content knowledge. For example in Figure 1 "proofs" was identified as a primary subsection, while the standards view of curriculum identifies "establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others" as a branch of "analyze characteristics and properties."(NCTM, 2000).

Participants had difficulty identifying the origins of K&B of school geometry shared in their drawings, but clearly identified their own experiences in high school geometry or mathematics and the university geometry course. Memories of high school geometry were vague as suggested by Paul's comments.

> I have vague remembrances of what I learned, but who's to say I learned the Pythagorean Theorem in geometry? I probably learned it sooner, so I don't know exactly what, if you asked me right now, cause those [pointing] proofs and constructions are different operations you do in geometry. But as far as content goes, specifically like midpoints, angle bisectors, stuff like that I wouldn't be able to tell you the complete list of what you are supposed to teach in geometry. (Paul)

When asked to elaborate, participants expressed concern about their knowledge of school geometry curriculum. For example:

> Um, I guess they [subsections] came from [the university class] or the things that we did in [the university class]… but I would have no idea, I would have no idea what the curriculum [school geometry] is. (Sophie)

Participants reported that thinking about connections while participating in the university geometry course was not a priority to them or in relation to the demands of the course.

> Um, when I was in [university] geometry I definitely wasn't thinking about that at all. It really wasn't even crossing my mind. (Sophie)

Here a contradiction arises in that connections were not fostered in the classroom nor at the forefront of thought, yet linking content in university mathematics courses to the school mathematics curriculum is reported by all participants in being important to their future careers. To investigate this a bit more participants were asked how they might know what to teach in a school geometry class. They identified the school textbook and their own experiences in school geometry as the primary resources for their K&B of the school geometry curriculum:

> I guess I would look at any materials that they [the school] have, like the book or whatever, and I would think back about what, you know, what I struggled the most with geometry. (Sophie)

Connections between K&B of Mathematics and K&B about School Mathematics.

Questioning about the geometry course encouraged participants to elaborate about other university mathematics courses and connections between these courses and their K&B about school mathematics.

> Yeah, I think it is important for them [university mathematics professors] to make a connection between what you are going to teach because, like for example, like abstract algebra I took last semester… I don't see how that connects at all to what I'm teaching. So like I don't understand why I spent my time on it. I don't feel like I'll need it, so I would have rather spent my time learning more about how to teach certain concepts, or maybe like taking a lower level algebra class where you really think about, you know you are not only a student, but a teacher and a student perspective in it. (Paul)

Paul expresses the view that abstract algebra has no value as a tool in his/her development as a teacher. All participants failed to see any connections between the mathematics content they were learning in their university courses and school mathematics. In addition, participants shared evidence of disassociating university mathematics and school mathematics as they identified emotional responses to university mathematics.

> Right now I'm in abstract algebra and… I'm wondering, why am I even in this class? When it doesn't even relate to anything. So, uh, I'm in this math and I did really bad on my first test and it's just making me really frustrated. (Paul)

Beyond the university geometry course, participants expressed no positive connections between university mathematics courses and school mathematics.

Conclusions

This research study set out to investigate connections between K&B about geometry and K&B about school geometry. Positive connections were inferred from drawings and interview data, however the connections were weak as evidenced by the lack of specific links between ideas identified as drawn from the university geometry course and tied to the school geometry curriculum. In elaborating on connections between university mathematics, evidence of negative connections was found. Participants identified emotional responses to university mathematics courses and specifically identified the lack of relevance of the courses as a challenge in preparing for the careers as teachers. Participants viewed their future role as teachers as including the delivery of curriculum and yet they expressed concern regarding their knowledge of that curriculum. In such a vacuum, sources of knowledge were identified as personal experience and the district adopted texts.

Connections to future practice were not emphasized in the participants' content courses, nor were there pedagogical strategies employed that helped inform their K&B of the secondary curriculum. Despite this environment, seemingly devoid of emphasizing connections to future practice, preservice teachers are still building knowledge and belief structures about the mathematics that they will eventually be teaching as evidenced by their responses to mathematics course work they were currently taking.

As posited by Ball (2005) curricular knowledge is a vital component of the overall knowledge necessary to be an effective mathematics teacher. Yet participants were unsure of how to build such knowledge in an environment that focused on common content knowledge. They expressed concern about their limited knowledge of curriculum and the lack of attention to this topic in their university mathematics courses. This lack of attention to the demands of their future profession, led the participants to identify textbooks as resources for what and how they should teach school geometry. This finding suggests that K&B about school geometry will continue to grow in alignment to existing school geometry curriculum as these preservice teachers begin to navigate their teaching career.

Finally, there was a trend of disassociation of the participants that directly affected their knowledge and belief structures in regards to mathematics. Negative connections stem from experiences in theoretical mathematics courses, such as abstract algebra, where students struggled to perform in ways professors valued. Lack of attention in such courses to projection of future practice led participants to focus on performance in the classes rather than building professional K&B for teaching.

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CONTRIBUTING TO SELF-EFFICACY PERCEPTIONS OF ELEMENTARY PRESERVICE TEACHERS THROUGH MATH CONTENT COURSES

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Research continues to show that teacher efficacy is hard to change once it has been established and that it can be positively influenced even before teachers enter the classroom. This article presents the results of a study examining the potential impact of mathematics content courses on teaching self-efficacy perceptions using the Mathematics Teaching Efficacy Belief Instrument. Analysis of the data revealed a statistically significant increase in personal mathematics teaching efficacy of elementary preservice teachers that complete content courses specifically designed for them. Implications for teacher education and directions for future studies are also discussed.

The mathematics preparation of future elementary teachers has picked up momentum in the past two decades. In particular, efforts have been greatly influenced by studies such as Ma (1999); Hill, Shilling, and Ball (2004); and Ball, Thames, and Phelps (2008). These studies have built from the notion of pedagogical content knowledge introduced by Shulman (1986) and have illuminated the importance of developing mathematics knowledge for teaching (MKT). This knowledge for teaching encompasses both content knowledge and a specialized understanding of that content that allows teachers to effectively use their content knowledge to help students learn the subject (Ball, Thames & Phelps, 2008).

As mathematics teacher educators we seek to improve the preparation of our students by developing course experiences that would best prepare them for their teaching careers. We are now particularly interested in providing opportunities for elementary preservice teachers (PSTs) to develop MKT. Ball et al. said, "How well teachers know mathematics is central to their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about presentation, emphasis, and sequencing" (Ball, Hill & Bass, 2005, p. 14). At our institution, the department of mathematics offers two mathematics content courses that have been specifically designed for elementary PSTs. These courses emphasize both content and pedagogy. The first of these courses will be referred to in this article as MATH1.

As teacher education researchers we seek to examine evidence of teacher learning and teacher education that help us find strategies to prepare future teachers more effectively. In particular, identity development has been described as a crucial feature within preservice programs: ‘‘the identities teachers develop shape their dispositions, where they place their effort, whether and how they seek out professional development opportunities, and what obligations they see as intrinsic to their role’’ (Hammerness et al., 2005, p. 384). In this study, we sought to examine influences of mathematics content courses in elementary PSTs teaching self-efficacy as these beliefs relate to the process of identity development.

Theoretical Framework

In recent decades we have seen increased emphasis on students' beliefs in teaching mathematics (Hart, 2002; Leder, Pehkonen & Tőerner, 2002; Spielman & Lloyd, 2004; Watson & De Geest, 2005). One important area of research on students' beliefs is that of teacher self-efficacy (Bandura 1977, 1986, 1997). Bandura (1986) defined self-efficacy as “people’s judgments of their capabilities to arrange and execute courses of action required to attain designated types of performances” (p. xii). It impacts the things we do, our efforts toward them, and how long we persist in working out solutions to problems. In fact, research about teacher efficacy in mathematics and science education has shown that levels of teacher efficacy are related to teacher content knowledge, teacher pedagogical content knowledge, and teacher beliefs and attitudes regarding the content (Gresham, 2008; Huinker & Madison, 1997; Swars, 2005). Thus, self-efficacy and MKT are strongly related. Students’ ability to learn content and pedagogy and their successful application in the classroom depends on their willingness and confidence to do so. Furthermore, research suggests that teacher efficacy is hard to change once it has been established (Hoy, 2000) and that it can be positively influenced even before teachers enter the classroom (Hoy, 2004; Smith, 1996) by providing experiences that promote teaching efficacy.

Researchers have examined the influences that methods courses and student teaching have in self-efficacy levels of preservice teachers (Hart, 2002; McDonnough & Matkins, 2010; Plourde, 2002; Swars, Smith, Smith, & Hart 2009). However, based on our literature review, fewer studies have been conducted to study the effects of content courses on teaching self-efficacy of undergraduate students. Typically, elementary PSTs fulfill their mathematics requirements early in their preparation. Thus, if we can identify and describe the mathematics content coursework that positively affect their sense of teaching self-efficacy, then these levels can be further strengthened in other learning experiences later in their studies.

In this context, Bandura’s (1986) construct of self-efficacy theory framed this study because it is based on the concept that people’s belief or perceived confidence for coordinating and carrying out a specific action influences how well they develop the basic cognitive, self-management, and interpersonal skills that contribute to career development and success. Levels of efficacy affect the amount of effort, persistence and resilience students are willing to undertake when faced with obstacles and failures (Bandura, 2001).

Enochs and Riggs (1990) developed the Science Teaching Efficacy Belief Instrument (STEBI-B) based on Bandura’s (1977) social cognitive theory to measure preservice teachers' sense of science teaching efficacy. Enochs, Smith, and Huinker (2000) have adapted the instrument to assess efficacy in teaching mathematics. The revised instrument is called the Mathematics Teaching Efficacy Belief Instrument (MTEBI) and consists of 21 Likert scale items ranging from “strongly agree” to “strongly disagree”. The MTEBI contains two dimensions: Personal Mathematics Teaching Efficacy (PMTE) and Mathematics Teaching Outcome Expectancy (MTOE). The PMTE contains 13 items and refers to the individual’s personal belief of his/her ability to be an effective mathematics teacher. On the other hand, MTOE contains 8 items and refers to the individual’s belief that effective teaching can enhance mathematics achievement of students regardless of external factors. An example of a PMTE item is, “I will continually find better ways to teach mathematics.” By comparison, an example of a MTOE item is, “Students’ achievement in mathematics is directly related to their teacher’s effectiveness in mathematics teaching.”

Several researchers have employed these instruments to explore issues of mathematics and science self-efficacy in PSTs. An early study by Huinker and Madison (1997) using both STEBI-B and MTEBI showed that content methods courses improved the elementary PSTs efficacy beliefs about teaching mathematics and science. More recently, Bleicher (2002) and Bleicher and Lindgren (2005) used the STEBI-B to help examine changes in science content knowledge and science teaching self-efficacy. They found that the PSTs increased significantly in both areas as they progressed in a methods course. Moseley and Utley (2006) obtained similar results when examining the effects of an integrated mathematics and science course on teaching self-efficacy. MTEBI was also used by Swars, et al. (2009) among other instruments in a longitudinal study examining the effects of a developmental teacher preparation program on multiple constructs. In particular, Swars et al. found that teaching self-efficacy beliefs varied over time as students progressed through two methods courses and student teaching.

In order to better understand how to prepare elementary PSTs for teaching mathematics well, it is important to build from existing research and to investigate not only methods courses, but also, the potential impact of *math content courses* on PMTE and MTOE. In this article we report the results of a study designed to examine changes in the perceptions of mathematics teaching self-efficacy of elementary teacher candidates after participating in the mathematics content course that has been specifically designed for elementary PSTs.

This mathematics content course, MATH1, offered by the department of mathematics, is a semester-long (14 weeks) course that meets twice a week for 75 minutes each time. The material covered in MATH1 is that typically found in textbooks such as Long, DeTemple, and Millman (2011); that is, numeration systems and their characteristics; in-depth look at algorithms for basic operations on whole, integer, and rational numbers, decimals and real numbers; and algebraic and proportional reasoning. Additionally, the department offers a second course subsequent to this one but with a focus on geometry. In contrast to the traditional blackboard-lecture mathematics classes, in these courses knowledge is built via explorations and students' discussions; the instructor's role is that of a facilitator providing guidance to lead students toward developing an advanced perspective on and profound understanding of the mathematical concepts. Students work, individually or in groups, on activities developed to understand the reasons behind familiar mathematical procedures, to explore and discover new concepts, and to analyze children nonstandard approaches. Additionally, students frequently present their ideas and solutions to the entire class. The focus of these tasks is on helping them construct their own knowledge and develop mathematical communication skills while building solid learning skills that will help them approach new mathematics topics in the future.

As part of a larger study investigating influences of math content courses geared toward elementary PSTs (Truxaw, Cardetti & Bushey, 2010), this study focuses on the experiences of the students enrolled in the focus math content course, MATH1. The research questions that guided this study are:

1. Are there significant changes in Personal Mathematics Teaching Efficacy in elementary teacher candidates before and after participation in the mathematics content course?
2. Are there significant changes in Mathematics Teaching Outcome Expectancy in elementary teacher candidates before and after participation in the mathematics content course?

Methods

Design and Participants

For this quantitative study we used a pretest-posttest design. The sample consisted of 24 elementary teacher candidates who were just beginning their teacher preparation coursework at a large public research university in the northeastern United States. These students were predominantly female (21), white (23), and typical ages ranged from 20 to 25 years old. All participants were enrolled in MATH1, the mathematics content course that involves a combination of both pedagogical and content instruction described on the previous section.

These students had completed at least two quantitative courses outside the School of Education to fulfill the school's quantitative requirements. In particular, the majority of the participants (15) had taken the course Mathematics for Business and Economics (calculus-one level). Participants had also taken either the Problem Solving course or the Elementary Discrete Mathematics course (real-world applications of non-routine mathematics).

Procedures

A survey (MTEBI) was administered at the beginning of the semester during the first week of classes to measure the entering level of mathematics teaching efficacy of the participants. The survey was administered again at the end of the semester during the last week of classes to determine whether there were changes in their beliefs before and after completion of the content course.

Instrument

The participants were asked to assess their mathematics teaching self-efficacy using the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) developed by Enochs, Smith, and Huinker (2000). As mentioned earlier, the MTEBI includes two dimensions: Personal Mathematics Teaching Efficacy (PMTE) and Mathematics Teaching Outcome Expectancy (MTOE) with 13 and 8 items, respectively. The instrument uses a Likert scale with five response categories (5=strongly agree, 4=agree, 3=undecided, 2=disagree, and 1=strongly disagree). That is, high survey scores reflected higher levels of self-efficacy.

The PMTE addresses the preservice teacher's belief regarding his or her own ability to teach mathematics effectively. Scores on Personal Efficacy range from 13 to 65. The MTOE addresses the preservice teacher's belief in his or her ability to directly affect students learning outcomes. The scores on Expected Outcome range from 8 to 40. Enochs et al. (2000) found the PMTE and MTOE represented independent constructs based on confirmatory analysis and had Cronbach's alpha coefficients of 0.88 and 0.77, respectively, demonstrating the factors to be internally consistent (i.e., reliable).

Results

All the 24 participants in this study completed both pre- and post- MTEBI surveys; however, we were unable to match pre- and post-survey data for 6 of the participants because of missing identifiers. We analyzed the data in the two ways, described below, using statistical tests to determine whether there were statistically significant changes before (pre-survey) and after (post-survey) the mathematics content course. The statistical significance was established at the 0.05 level.

First, to account for the unmatched data points, the complete data set (N=24) was analyzed using independent-samples *t*-tests, on each dimension separately. The mean of the scores for the PMTE dimension increased from M=47.01 (SD=6.71) before taking the course to M=51 (SD=6.62) after taking the course and the results from the independent-samples *t*-test indicated that this increase was statistically significant; $t(46)=2.07$, $p=.04$. On the other hand, the mean of the scores for the MTOE dimension increased from M=27.67 (SD=2.96) before taking the course to M=28.70 (SD=4.30) after taking the course. This increase was not statistically significant according to the independent-samples *t*-test; $t(46)=0.98$, *ns.*

Additionally, the data from matched pairs (N=18) was analyzed using paired samples *t*-tests, on each dimension separately. Table 1 presents the descriptive statistics for the pre- and post-surveys along with the results of the *t*-tests on both PMTE and MTOE. As can be seen from the table, the means on both dimensions, PMTE and MTOE, increased from pre- to post-survey for the matched pairs as well.

Table 1. MTEBI Paired Samples *t*-tests Results Before and After the Content Course

MTEBI Dimension	Pre-survey		Post-survey		*t*	*df*	Sig. (2-tailed)
N=18	Mean	*SD*	*Mean*	*SD*			
Personal Efficacy (PMTE)	47.58	7.59	52.06	6.37	3.56	16	**0.003***
Expected Outcome (MTOE)	27.41	3.14	28.18	3.81	0.92	16	0.37

* Significant at the 0.05 level

The results from the paired samples *t*-test revealed that the difference in the mean scores on the PMTE dimension was statistically significant. The mean scores at the start of the semester was 47.58 (*SD*=7.59) with an increase to 52.06 (*SD*=6.37) by the end of the semester; $t(16)=-3.56$ (pre-post, thus a negative *t*-value indicates and increase), $p<0.05$. These results suggest that the content course, MATH1, had a positive effect on participants' Personal Self-efficacy levels.

While the mean scores on Outcome Expectancy also showed an increase going from 27.41 (*SD*=3.14) at the start of the semester to 28.18 (*SD*=3.81) at the end of the semester, the increase was not statistically significant; $t(16)=0.92$, *ns.* That is, there was not enough evidence to suggest that MATH1 influences the MTOE dimension.

Discussion

These findings contribute to the body of research on perceptions of mathematics teaching held by elementary teacher candidates. In particular, the findings provide insights into teaching self-efficacy beliefs of elementary PSTs that have implications for teacher education programs.

The findings show that the personal teaching self-efficacy of elementary PSTs, as measured by the MTEBI survey, increased significantly after the mathematics content course, MATH1. It is worth noting that these students had completed at least two traditional mathematics courses, some of which were at the calculus-one level, prior to taking MATH1; yet their sense of personal teaching self-efficacy increased significantly after MATH1. This may be explained by the in-depth treatment of school mathematics concepts in MATH1, and the multiple opportunities given to the students to discuss and explain their mathematical ideas *with* their classmates and *for* their classmates.

Despite the significant increase in PMTE, there was no significant increase in MTOE, even though the mean values increased (pre, *M*=27.41; post, *M*=28.18). This is not surprising since MTOE measures the belief in one's ability to affect learning outcomes regardless of other factors, and these students had not yet experienced teaching; thus, it may be more difficult for them to judge this dimension. These results are consistent with those of others (e.g., Bleicher 2007, Hoy & Woolfolk, 1990; Plourde, 2002) that have found that certain learning experiences can increase PSTs' sense of personal efficacy, but that outcome expectancy beliefs are harder to change at this level.

These results extend those of previous research studies focused on methods courses (Bleicher, 2002; Bleicher & Lindgren, 2005; Huinker & Madison, 1997; Moseley & Utley, 2006; Swars, et al., 2009) by identifying opportunities to positively impact personal teaching self-efficacy at an earlier stage. This positive sense can be further nurtured on future coursework that is more tightly related to their teaching paths (methods courses, field experiences: classroom observations, student teaching, etc.). The results presented here corroborate and complement earlier reported results of the larger study (Truxaw, Cardetti & Bushey, 2010). Those results suggested that students who participate in mathematics content courses, such as MATH1, seem better able to concentrate on the material in the required methods courses and their confidence in teaching mathematics is significantly higher at the end of the methods course than their counterparts. The current study helps us understand further contributions of these mathematics content courses on elementary PSTs preparation. Based on these results, we recommend that teacher preparation programs consider requiring courses, such as the one described here, for all their preservice elementary teachers. Courses such as these, which focus on mathematical content with an eye toward the PSTs' professional goals, may support the development of identities that could shape their current and future professional dispositions (Hammerness et al., 2005).

Further research is needed to support these recommendations. Our findings are subject to at least two limitations. First, these data apply only to one cohort of students in MATH1 that was taught by one particular instructor; and a second limitation is that these data may be particular to the teacher preparation program at our institution. Therefore, we encourage others to build from this research to address these limitations and to continue the important work of understanding the kinds of mathematics coursework that best support elementary preservice teachers.

Endnotes

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2.

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PREPARING MIDDLE SCHOOL MATHEMATICS TEACHERS: RETHINKING ENGAGMENT AND LEARNING

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In this theoretical research paper we describe a collaborative effort between researchers and university faculty to improve teacher preparation in middle school mathematics. Two powerful instructional frameworks, UDL and TPACK are dovetailed within in a web-based dynamic textbook, "Proportional Dynabook" that focuses on proportional reasoning. Theoretical tensions between special and general education teacher preparation programs influence the ongoing design of Proportional Dynabook. Teacher candidates used Proportional Dynabook in a graduate methods class to design instruction for a student struggling with ratio. Teachers developed deeper understanding of ratio and related pedagogical strategies that make ratio concepts accessible to diverse learners.

Classrooms are more diverse than ever due to recent legislation such as No Child Left Behind and the Individuals with Disabilities Education Improvement Act that require all students to have access to the general education curriculum and to be included in the same assessments (Van Garderen, Scheuermann, Jackson, & Hampton, 2009). To meet the increasingly diverse needs of students with learning differences who are being educated in the general education classroom, special education services are being brought to the general education classroom. The lines between general and special education are blurred and teachers are being asked to collaborate to meet the educational needs of an extremely diverse group of learners. These collaborative efforts are based on the idea that each teacher has specific knowledge and expertise that address the instructional needs of the class (Van Garderen et al., 2009). However, collaboration in theory does not address the friction that exists in practice. Philosophical differences in pedagogy and learning theory create tension in the design of instruction. While this tension can be an obstacle to the design of effective instruction, it can also be the vehicle that brings together special and general education researchers to conceive and design innovative instructional tools and practices to meet the needs of *all* students.

Students with high achievement, with learning disabilities and English Language learners are all receiving mathematics instruction from teachers who have been trained as math teachers or special education teachers but not usually both. Special education teachers tend to focus on pedagogy and interventions while general education teachers focus more on the content and structure of mathematics (Van Garderen et al., 2009). Now both teachers are given the formidable task of "raising the floor by expanding achievement for all, and lifting the ceiling of achievement to better prepare future leaders in mathematics…(Friesen, 2008, p. 51). The prevailing special education service delivery model has the special education teacher providing instructional supports within the general education classroom. Hence, the special education teacher remains focused on pedagogy and intervention and the general education teacher remains focused on the math content. Often the special education teacher is attending only to the needs of students with Individual Education Plans (IEPs) and who have formally qualified for special education services thereby underutilizing the additional teaching resource in the classroom. This model may prevail because the special education teacher shies away from the more challenging math content and there is little time in the day for both teachers to do collaborative planning. However, in light of diminishing resources and budget cuts, full utilization of teaching resources must become the norm if we are indeed to raise the floor and ceiling of mathematics academic achievement.

We can begin to make changes in the way we prepare both general and special education teachers to teach mathematics. In teacher preparation and professional development programs, researchers and teacher educators from the fields of general and special education must model collaborative practices and create dynamic learning communities. Silverman and Clay (2010) suggest that teacher candidates need social experiences where they can engage with mathematics in a way that that encourages the development of "deep, connected, unpacked mathematical understandings." Further, the authors suggest that the collaborative use of technology serves to establish the social environment necessary to promote higher levels of thinking and the development of deeper broader understandings of mathematics and pedagogy. Likewise, Shreyar, Zolkower, & Perez (2010) suggest that the 'text' created through dialogue, written response, and diagram, in a social environment such as the classroom, can transcend the actual text in use. In this way, the original text serves a vehicle for collectively making meaning of complex mathematical concepts, problems, and related pedagogy. The instructor in these models works to orchestrate the individual students in the class in such a way that, despite a beginning heterogeneous perception of mathematical concepts and pedagogy, the class ends up with a collective understanding of problems and approaches to solve them.

In this theoretical research paper we describe our innovative approach to engaging pre-service teacher candidates in mathematical and pedagogical thinking. First, we describe our collaborative development of *Proportionality Dynabook*, a web-based dynamic textbook aimed at supporting both special and general education pre-service teachers in their efforts to teach middle school mathematics. Second, we describe the frameworks of Universal Design for Learning (UDL) and Technological, Pedagogical and Content Knowledge (TPACK), which serve as the structural environment of our dynamic text. Third, we explain the curricular activity system designed to create a learning environment where pre-service teachers utilize this Dynabook in a way that encourages social and cognitive engagement in issues surrounding middle school mathematics instruction for diverse learners.

Dynabook

We draw inspiration from Alan Kay's 1972 concept for a tablet-shaped personal computer (see, e.g. Kay & Goldberg, 1977), which he called a Dynabook. Although futuristic visions for the impact of computing stretch back much farther than Kay and have been updated many times since, we see Kay's concept as a distinctive turning point – and the essence of Kay's Dynabook is at the heart of the possibilities now emerging for digital texts in mathematics. Kay's vision, however, requires more than mobile devices and the Web 2.0 capabilities that enable social networking, because Kay also wanted people to more easily and frequently engage with powerful ideas. Like Seymour Papert (1993), Kay saw the dynamic, interactive capabilities of computational media as opening up new ways to playfully engage with powerful ideas – through activities like programming, interacting with visualizations, exploring mathematical models, and playing with simulations.

The present project is a collaboration between SRI International, San Diego State University, San Francisco State University, CAST, Inc., and Inverness. SRI International is a research firm with expertise in math education and innovative uses of technology. San Diego State University and San Francisco State University prepare teachers to serve California students in general and special education, respectively. CAST, Inc. is a leader in educational technology and Universal Design for Learning (UDL). Inverness is a leading research and evaluation firm. Collectively, we have a unique combination of expertise in mathematics, math education, special education, teacher training, technology supported math instruction, UDL pedagogy, and evaluation, which has been productive in designing and building the first iteration of the Proportionality Dynabook.

As a team of researchers and educators, we are exploring and developing Kay's ideas through a mathematics dynabook intended for use by prospective middle school mathematics pre-service teachers and centered on a key middle school concept of proportionality. Just as the strategies of summarization and prediction (e.g., in comprehending reading literature), and questioning and modeling (e.g., in learning from a science text), are essential in comprehension, we believe that there are similar strategies essential in mathematics. These include exploring the same mathematics through multiple equivalent entry points and making connections among different representations. Through our collaborative research with mathematics teacher educators and special education experts, we determined that the goals for this book should be to (a) support and reward the candidates when engaging more deeply in mathematical thinking, (b) encourage candidates to draw connections among related concepts of proportionality, and (c) develop teacher candidates' awareness of potential student misconceptions and instructional options they could choose to support student development of more robust ideas about proportionality.

To support these ends, and cue these strategies, the fundamental structure of the "book" is not a linear narrative but rather a 3x3 matrix that can be navigated through different pathways. The columns of the matrix are concepts in three related strands of middle school mathematics that develop students' proportional thinking: ratio (in the number strand), similarity (in the geometry strand), and linear function (in the algebra strand). Although many middle school mathematics teachers do not recognize this, these three concepts are deeply connected. For example, "slope" is the ratio of two sides (the height and the width) of a slope triangle, and the application of triangles to the slope relies on the concept of similarity. By presenting these three concepts side by side, the Proportionality Dynabook aims to encourage teachers to explore these sorts of connections.

The rows of the Proportionality Dynabook are organized by three different "entry points" for engaging teachers in the mathematics. Teachers can become familiar with the mathematics first by exploring "challenging problems" – mathematics problems that are designed to be challenging to candidates and help them develop their own mathematical thinking. Candidates can further explore the mathematics by watching "video cases" of student thinking as students solve problems with ratio, similarity and linear functions. Finally, candidates can also explore lessons that are specially designed to take advantage of the dynamic medium of the Proportionality Dynabook by presenting mathematical ideas in a visual and interactive format. For example, in the linearity section, candidates develop the idea of linear function as they explore the relationship between timing of thunder and lightning reaching a campsite. This rich matrix of related concepts and related ways to encounter the concepts takes unique advantage of the ability to construct a "book" that does not need to have a strictly linear ordering of pages. In addition, a concept of an expert “tour” is planned, which can overlay a step-by-step trajectory on the book when it is desirable to guide candidates through the book in a linear order.

With our Proportionality Dynabook we seek to further develop the concept of the “book” as a social medium that enhances interaction between an instructor and his or her pre-service teacher candidates. For example, instructors can create “assignments” in the Proportionality Dynabook for their pre-service teachers, support activities of both reading and writing, and allow pre-service teachers to engage deeply with powerful ideas, and extend these benefits to a diversity of learners. By dovetailing the UDL and TPACK framework in the design of the Proportionality Dynabook, we seek to immerse pre-service teachers in use of this resource as an exemplary instance of technology-rich mathematics in three ways: to deepen their own content knowledge, to develop key pedagogical skills, and to support beneficial use of technology in teaching.

Universal Design for Learning (UDL)

Universal Design for Learning supports the needs of diverse learners by providing multiple means of representation, expression, and engagement (Rose & Meyer, 2006; Rose, Meyer, & Hitchcock, 2005). The core idea of UDL is to embed supports in the medium, which learners can optionally activate when they need support to continue their progress. The benefit of UDL as a framework is that it provides a research-based taxonomy of the kinds of supports that learners may need and that technology can provide. Learners need supports for (a) connecting multiple representation of important ideas, (b) interacting with ideas and expressing ideas in new ways, and (c) maintaining a high-level of engagement.

With regard to these three kinds of supports, UDL concisely summarizes a vast literature on the brain, learning, and the role of technology. For example, UDL suggests that providing for multiple representations of a concept not only enables deeper engagement with that concept but also enables access for a broader range of learners (McGuire, Scott, &Shaw, 2006). In the Proportionality Dynabook, UDL is introduced through videos, text, and diagrams as a framework to interpret the digital text. These features are embedded to give teachers ideas about how to support student access to and engagement with challenging mathematics. For example, a glossary is available which defines unfamiliar words and definitions by using a mixture of pictures and words. Multiple means of expression are available to users who can highlight words or sections in the Proportionality Dynabook and take notes in the margins. Further, when answering a question, they can write, draw a picture, explain verbally (into a microphone), or upload a file. To enhance engagement, “Stop and Think” prompts are strategically embedded in the text to encourage candidates to process the text more deeply. These features of UDL are combined with digital resources to increase TPACK in pre-service teachers.

Technological, Pedagogical, and Content Knowledge (TPACK)

Emerging technological advances combined with Schulman’s (1987) work on pedagogical content knowledge have lead to the technological pedagogical and content knowledge (TPACK) framework (Mishra & Koehler, 2006). To increase TPACK in teacher candidates, technology needs to be strategically introduced and utilized by the instructor. It is not sufficient for students to access the Proportionality Dynabook without guidance and thoughtful pedagogy. In addition to careful instruction, content must be rich and challenging for candidates to increase understanding of the complexities of TPACK. Dynabook contains many access points to the three components of the instructional environment: technology, pedagogy and content. It incorporates features of UDL through digital technology and multiple means of expression. Videos of student thinking are assigned to candidates so they have a framework for pedagogy.

The various videos will guide the candidate's developing pedagogy when designing instruction to address student misconceptions. The content aspect of TPACK is distributed through many parts of the Dynabook, especially through the challenging problems for candidates. These are guided problems that include prompts to help them access the underlying mathematics through "How Do I Say It?" and "Get Me Started" features when working through the problems. There are also several dynamic representations involved in showing the solution of the problems when students check their work. The next section focuses on how the Dynabook was presented to teacher candidates who are working towards a special education teacher credential.

Dynabook in a Teacher Education Classroom

The first year of this project was dedicated to planning the design and development of a dynamic web-based text that could improve how pre-service teachers learn middle school mathematics content and related pedagogy. The diverse research backgrounds of team members contributed to lively discussions and ongoing iterations of our evolving Proportional Dynabook. In its current form, we have a completed section on ratio. In an effort to examine how teacher candidates can learn mathematics content, specifically ratio, and related pedagogy with the use of Proportional Dynabook, we introduced our tool in an advanced level methods class for special education teachers at an urban university in Northern California. In California, special education teachers are awarded a credential to teach at the K-12 grade level after successfully completing a series of graduate level courses. Special education interns work as the teachers of record while they attend the university in the evening to complete credential coursework. Most courses center around assessment, pedagogy, legalities and literacy with only one course devoted to mathematics. Since teacher educators in classes such as advanced methods often teach specific strategies, they choose content areas that tend to be more accessible to new teachers, leaving mathematics methods to the mathematics professors in the general education department where special education teachers take their one math class. Mathematics is an area that is frequently overlooked because it is considered difficult and less interesting than other topic areas. It is our hope that the Proportional Dynabook will provide a fun and interesting way for teacher educators, who are not experts in proportionality, to engage their pre-service teachers more fully in the concepts and pedagogy related to teaching proportional reasoning to diverse learners.

Thirteen pre-service and intern special education teacher candidates participated in two classes dedicated to interacting with the ratio section of Proportional Dynabook. Participants had varying levels of mathematics proficiency and teaching experience. During the first session, we introduced Dynabook with a scavenger hunt activity through various sections of Ratio (i.e., Introduction, UDL, Challenging Problems, and Video Cases). In addition, our teacher candidates watched two instructional videos related to the shifts in proportional reasoning outlined by Lobato, Ellis, Charles and Zbiek (2010).

During the first observation of the Dynabook training and proportionality development session, all the researcher observers agreed that the teachers seemed a bit overwhelmed by all the new material and technology. In addition to learning a new technological tool, and a new mathematics framework (Lobato et al., 2010), they were also ask to think about middle school ratio problems that they found difficult to solve. Many might call these problems simple, but the students still had difficulty thinking through these problems and coming up with answers and novel ways to approach the problems. During a researcher debriefing after the first session, we realized that the teacher candidates were uncertain about their role in relation to the Dynabook. Questions like, "Who is this for?," and "Would I use this with my students?" made us realize that we did not provide enough background information about the development and purpose of Dynabook. In addition, we had not provided a rational for these teachers to truly engage with Dynabook in the way that we had hoped they would. It is our contention that if a special education teacher candidate does not have a reason to engage with challenging mathematics content, he or she will not and instead attend to the many other responsibilities required by a job or classwork. For the second session, we designed a specific curricular activity that would provide purpose and rationale for our teacher candidates to engage with Proportional Dynabook.

In the beginning of the second three-hour class session, the teacher candidates finished trying to solve the ratio problems from last session and post answers to the communal whiteboard in Dynabook. While one of the researchers used the large screen computer in the lab, each teacher candidate followed along in their own Dynabook on individual screens. The anonymous postings on the whiteboard initiated a lively discussion of ratio concepts and common misconceptions shared by several of the teacher candidates in the class. We described the rationale and purpose of this session by setting up a scenario where they were going to have to teach ratio to middle school students starting tomorrow. Specifically, they were going to prepare to address the learning needs of a student struggling with the concept of ratio as depicted in one of the videos in Dynabook. Each pair viewed the video of Kayla, and began to discuss her misunderstandings along with their own confusions and partial understanding of ratio. As we observed, pairs moved in and out of the different sections of Dynabook utilizing various features to improve their own understanding so that they could develop a script to address Kayla's level of understanding. Each pair remained engaged in discussion until the end of class.

In the video, Kayla, a middle school student, attempts to solve a ratio problem by incorrectly utilizing a procedural algorithm. Despite her answer not making sense, Kayla defends her use of the algorithm, representing a fairly typical procedural approach to proportionality in the absence of understanding. After carefully watching the video, the teacher candidates worked in pairs to create a lesson that addresses Kayla's conceptual misunderstandings. Each pair created a script for the lesson and used that script to create a movie of the lesson using *Xtranormal*, a free, web-based software for creating animated movies. We explained that by creating a script and movie, each pair must think about what words and language they will use to explain ratio. Then, by watching the movies, we have a way to share and discuss changes in the candidates' thinking about ratio and the pedagogy they used. As the groups were working on the movie, we told them they were free to take a break, but most groups did not stop. By the end of class, we observed that not only were the teacher candidates able to talk more precisely about ratio, they were also able to talk about how to assess student understanding of ratio. They discussed how to define what level of thinking the student is operating at according to the Lobato et al. framework and ways to get them to the next level. This is the type of engagement we had hoped to initiate with the introduction of Dynabook.

One of the researcher observers commented "that we have a way to really engage pre-service students in UDL and TPCK -- it was completely evident to me that the candidates were really in that space with their thinking throughout the evening. We need to remember to describe both the tool --- the Dynabook -- and the curricular activity system in which it exists. Both are necessary to achieve the reasoning we saw …. It is not our story that the Dynabook causes great pre-service classroom experiences' -- instead that the Dynabook is an enabler for a curricular activity system that increases candidate's social and cognitive engagement in these important issues. The Curricular Activity System language may be unfamiliar. We should capture both the tool's contribution and the activity design contribution -- that's the heart of it."

As we continue to develop and test the Proportional Dynabook, we hope to share our ideas and collect feedback from those interested in improving how we prepare teachers to think about and teach mathematics. Our next steps include formal testing and data collection to more thoroughly examine how the Proportional Dynabook can expand meaningful engagement and develop teachers' and students' sense of connections across the middle school mathematics curriculum.

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FOSTERING BELIEFS AND VIEWS OF PRE-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS ABOUT TECHNOLOGY

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The literature shows that beliefs and views of teachers have an important influence on their instructional decisions. Many pre-service teachers seem to hold negative views about the potential of graphing calculators as tools for concept development. In this research, we studied a section of mathematics teacher education students as they encountered activities we designed for use with TI-Nspire. This paper reports the extent to which portions of three 2 hour, 45minute class sessions were effective in pointing pre-service teachers toward more positive and cognitively richer views of calculators in mathematics instruction.

The use of graphing calculators in school mathematics classrooms is now in its third decade. Despite this fact, mathematics teacher educators continue to encounter numbers of pre- and in-service teachers who either eschew the use of calculators during instruction or voice support for their use, but "… only after students master things by hand." Among many pre-service teachers, there does not seem to be much awareness of the potential of graphing calculators as tools for concept development. Therefore, the goal of this study was to learn whether exposure to graphing calculator-based activities could foster pre-service teachers' awareness of and beliefs/views about the potential of graphing calculators as tools for concept development.

Conceptual Frameworks

We employed two conceptual frameworks to guide our study. The first is based on a broad model of the process of teacher change. The second deals with the roles that teachers ascribe to calculators.

A Model of Teacher Change

In order to better understand the process of teacher change, we used the model developed by Edwards (1996). This model was based on a constructivist perspective on learning. If one views the active construction of knowledge as a cycle involving interaction with something new, some sort of perturbation resulting from the interaction, reflection, and change (von Glasersfeld, 1983; Cobb, 1988, 1989, 1994; Confrey, 1991), it is a natural step to view the process of teacher change as an instance of such active, constructive learning. However, as shown in Figure 1,

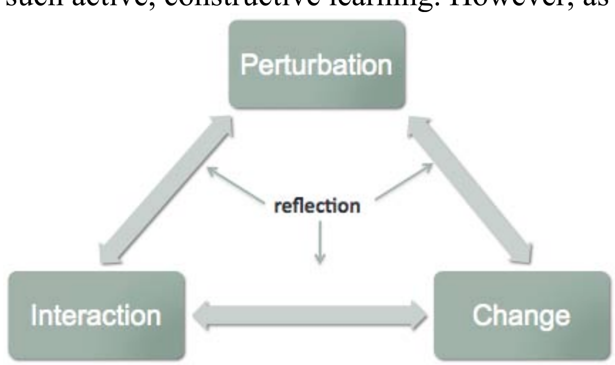

Figure 1: Edwards's Cyclic Model of Teacher Change

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

Edwards's cyclic model is a three-cycle, rather than a four-cycle, as others have proposed (see for example Underhill, 1991). As a three-cycle, interaction, perturbation, and change are the three points of the cycle, and reflection forms the connections between those points.

The Roles of Calculators

The second conceptual framework of this study centers on the classification of Doerr and Zangor (2000) for the roles of graphing calculators in the construction of mathematical meaning. Following 21 weeks of classroom-based observations, they were able to classify the patterns and modes of using graphing calculators in two pre-calculus classrooms. The modes of using graphing calculators were as a:

- computational tool to evaluate numerical and perhaps even algebraic expressions,
- transformational tool "whereby tedious computational tasks were transformed into interpretative tasks" (pp. 152-153),
- data collection and analysis tool when using Calculator Based Laboratory devices such as motion detectors or temperature probes,
- visualizing tool "to develop visual parameter matching strategies to find equations that fit data sets, to find appropriate views of the graph and determine the nature of the underlying structure of the function, to link the visual representation to the physical phenomena, and to solve equations" (pp. 154-155), and
- checking tool "to check conjectures made by students as they engaged with the problem investigations" (p. 156).

Since Doerr and Zangor explained that their checking mode emerged when students check their conjectures, we renamed this role as exploratory tool. Doerr and Zangor also concluded that the "nature of the mathematical tasks and the role, knowledge and beliefs of the teacher influenced the emergence of such rich usage of the graphing calculator" (p. 143). As Doerr and Zangor highlighted that teachers' beliefs are as important as their knowledge, the activity sequence in this study was aimed to bring presevice teachers face to face with their own beliefs through reflective activities.

Method

Twenty-four elementary education students who were seeking a mathematics endorsement for grades 7 and 8 took part in the study. All of the participants were students in an algebra course for middle school teachers. They were a mix of undergraduate and graduate students. Most were pre-service teachers, but a few were in-service teachers. The class met weekly for 2 hours, 45 minutes. The study took place over three consecutive weeks.

In the first two weeks, we engaged the participants in two TI-Nspire based lessons, one dealing with linear regression (Özgün-Koca & Edwards, 2010), and the other dealing with quadratic functions (Edwards & Özgün-Koca, 2009; Özgün-Koca & Edwards, 2008). In the third week, participants viewed video of 7th and 8th grade students completing the same TI-Nspire activities.

Our sources of data were four reflective writings that asked participants to respond to open-ended questions about the use of calculators to support mathematics teaching and learning. Reflection 1 was a pre-survey in which they discussed their views of the use of calculators in teaching and learning mathematics. The first reflective writing was completed one week prior to the first TI-Nspire lesson. Reflections 2 and 3 were completed immediately following each of the TI-Nspire lessons. They consisted of writing prompts aligned with the Technological Pedagogical and Content Knowledge (TPACK) model (Niess et al., 2009). Reflection 4 was a post-survey aimed at eliciting their beliefs about the use of calculators in mathematics teaching after their experiences in these three weeks. It was completed immediately after the participants viewed the classroom videos.

The raw data were transcribed as Word files and analyzed using Weft QDA software. This facilitated keyword searches and allowed us to analyze participants' reflective writings at the level of individual phrases or sentences. Data and investigator triangulation were used to ensure the trustworthiness of this study.

Results

Intentions to Use Calculators

In both the first and fourth reflective writings, we asked participants directly to discuss their intention to use calculators in their future teaching, or not. Fifteen of the 17 participants in the first reflective writing and all twenty of those who participated in the fourth reflective writing said yes to using calculators. Because there was not much difference in the proportion of participants who said yes to using calculators in these two reflective writings, we decided to look at the ways of using calculators that they mentioned in those writings.

We deemed using the calculator solely to check answers computed by hand or solely as a fast and accurate computational tool to be trivial uses of calculators. Two examples of participants' earlier writing about how they would use calculators in their future teaching that we considered trivial uses follow:

I would use calculators to engage students in learning to check their answers.

I would … show them things such as square roots and other operations that can't be computed by hand.

We then looked at the proportion of participants who mentioned any non-trivial use of the calculator in the first and fourth reflective writings. Only 7 of the 15 participants (47%) who said yes to using calculators in the first reflective writing mentioned a non-trivial use. However, in the fourth reflective writing, 17 of the 20 (85%) participants who said yes to using calculators mentioned a non-trivial use. Two examples of what we considered to be non-trivial uses of calculators mentioned in the fourth reflective writings can be seen in the following quotes:

Having a calculator to generate graphs that are easy to manipulate gives students the opportunity to recognize patterns both quickly (in terms of generating graphs) and independently.

I think the Ti-NSpire helps the kids visualize what they are doing, and this helps them understand better what is going on.

A Chi-square test revealed this difference to be statistically significant ($p<0.015$).

Roles of Calculators

Throughout the four reflective writings, participants mentioned various roles that they saw for calculators in the mathematics classroom. We decided to compare the roles mentioned in the first two reflective writings with those mentioned in the last two, because at the time of the first two reflective writings they had minimal exposure to TI-Nspire, while at the time of the last two they had experienced both TI-Nspire lessons. These comparisons were across two different reflective writings. Therefore, sometimes the number of times a role was mentioned is greater than 24, because one or more of the participants mentioned the role in both writings. We will focus on four of the calculator roles that emerged from their reflective writings, because these four exhibited a dramatic shift in focus.

In the first two reflective writings, participants mentioned calculators as a checking tool 10 times and as a computational tool 11 times. In the last two writings, there were only 2 mentions of the calculator as a checking tool and 3 as a computational tool. In contrast, during the first two reflective writings, the calculator as an exploratory tool was mentioned only twice, and the calculator as a visualization tool was mentioned 9 times. However by the time of the last two writings, the number of mentions of the calculator as an exploratory tool had risen to 18, and those of the calculator as a visualization tool had increased to 27. We believe this conceptual shift away from the more trivial roles of the calculator as a checking or computational tool toward more substantial roles as an exploratory or visualization tool is of practical significance, particularly in light of the relatively short duration of the participants' TI-Nspire experiences during the study.

How and When to Use Calculators

When we look at the things that participants wrote about how or when to use calculators, a sense of their Technological Pedagogical Knowledge development emerges. We again compared four types of responses, after mastering skills, not for basic facts or abuse, not for developing concepts, and to develop concepts, across the first two and last two reflective writings. In each case, there was very little difference in the number of responses when the first two writings were compared to the last two. In a number of instances, there was also not much difference in the things that they wrote. For example, on the first reflective writing, a participant wrote:

> *I think calculators are a great teaching tool as long as students understand the concept (know how to solve the problems first).*

Similarly, on the fourth reflective writing, a participant wrote that (s)he would:

> *not offer the students the opportunity to work with the calculators until I am confident that they understand the concept we are working with.*

We think these results show, not surprisingly, that dislodging beliefs about how and when to use calculators, or even if they ought to be used, will take more than the brief interaction that we provided our participants.

Issues Related to Student's Learning

Three issues related to student learning appeared across the four reflective writings. Some participants mentioned their belief that calculators are helpful for learning or understanding and are fun or motivating for students. However, some in their writing also revealed a fear of students becoming calculator dependent.

In the first two reflective writings, participants said that calculators are helpful for learning or understanding 11 times. By the last two writings, that number had grown to 22. Similarly, in the first two reflective writings, the fun or motivational aspects of using calculators were mentioned 3 times, but by the last two writings that number had increased to 9. While the numbers are small in both cases, there was also an increase in the number of times that a fear of calculator dependence was mentioned from 3 in the first two writings to 5 in the last two.

We interpret the first two increases as indicators of some developmental changes in participant's TPACK levels. As to the third increase, fear of calculator dependence may well be yet another pernicious belief that can only be altered via more time and experience with appropriate classroom uses of calculators.

Technological Knowledge and Confidence

Sixty-nine percent of the participants in this study stated that they were confident using technology in the mathematics classroom. The technology was not limited to TI-Nspire. Some participants stated that they would use TI-89, computers, or Smartboards.

I feel confident with technology in the classroom. I love computer & calculators. I found the TI-Nspire a little more difficult than calculators or computers.

Even though some mentioned that TI-Nspire was confusing, as this pre-service teacher did, this could be overcome by time and experience. Another participant mentioned that

I am not too confident at using technology in the classroom, but I intend to use it despite this. Once I gain experience and practice, especially with teacher specific technology, I'm sure my confidence will improve.

Here we see the importance of the combination of technological knowledge with disposition. When technological knowledge or confidence are non-existent, believing in the value of using technology or believing in or knowing the positive influence on the teaching and learning environment acts as the driving force to increase ones technological knowledge and confidence. In this case, technological pedagogical knowledge initiates the desire for an increased technological knowledge.

These pre-service teachers are still transitioning from being a student in a mathematics class to being a teacher. When they were talking about technological knowledge, some parts of their reflections focused on their technological knowledge as students. Perhaps that is why nine reflections mentioned the rationale for using technology was to serve a technology-driven student population.

With younger, technology driven students involving accurate use of calculators, and graphing calculators in older grades, will become more & more essential to solid understanding of mathematical concepts.

However, 16 reflections stated that their future students would need to know how to use technology for their own good and for their future mathematics classes.

I think it will be good for younger students to get some hands-on experience with the technology they will be using later in school as they get older.

Technological Capabilities of TI-Nspire

When we coded participants' reflective writings that followed their exposure to activities with the TI-Nspire, two codes emerged which referenced two capabilities of the TI-Nspire—linked representations and dynamic features. Twenty-three reflections in the third and fourth writings mentioned the importance of linked representations. They discussed that TI-Nspires became a better visualization and exploratory tool with the help of the linked representations.

Seeing is believing, being able to see the change in the graph as it correlates with points indicated.

We mainly used the calculator to plug in equations & move them around to create answers to equations about the technology and how it can be used to explore answers about graphing. It was a great visual to see how things happen on the axis when a parabola was moved.

The participants liked the dynamic feature of the TI-Nspire and commented on its effects on the teaching and learning environment.

Having a calculator to generate graphs and that are easy to manipulate gives students the opportunity to recognize patterns both quickly (in terms of generating graphs) and independently.

They can easily manipulate graphs and have a quicker time to understand how graphs can change with little work involved.

When empowered with linked representations, many participants opined that the dynamic feature created a teaching and learning environment for understanding.

I really like how the calculator can show how graphs shift. Also, how it changes the equation as you shift the graph. This lets kids see how it changes the equation. When the graph moves, I feel that this will help kids understand graphs and how they shift much better. It will no longer require kids to memorize all this info. With activities like this, they will learn it without even knowing it.

The fact that participants were not able to mention these features in the first two reflective writings is not surprising. They had little or no experience with linked representations or dynamic features on a graphing calculator at that point. However, statements such as the one above seize our attention, because they clearly highlight the importance of technological knowledge for mathematics education in the 21st Century.

Discussion

Reflections on the Teacher Change Model

We believe that the teacher change model presented earlier is a good fit for some of the phenomena we observed in this study. We designed the study so as to provide participants with a number of interactions. They interacted with TI-Nspire activities, they interacted with the reflective writing prompts, they interacted with the instructor, and they interacted with each other.

It was our hope that, through reflecting on these interactions, participants would encounter something to perturb their thinking, and such perturbation was evident in some of their reflective writings. Moreover, we provided participants with reflective writing prompts that we hoped would resonate with what was perturbing them. Then, we hoped to see evidence of a conceptual shift in their consideration of the potential of graphing calculators as tools for concept development. Again, we saw evidence of such change in participants' reflective writings. For example, one participant directly described just such a cycle when (s)he wrote:

I am <u>very</u> for using calculators when teaching math, because it will generate more constructivist based conceptual learning. Initially, I had thought calculators would be misused, because students would stop thinking when using calculators. After being able to use TI-Nspire myself with student activities, I realized how wrong I was. … Calculators are ultimately changing math learning for the better, following the trend of more conceptually, constructivist, activity-based teaching as opposed to lecture style, direct teaching.

We think this example suggests the power of the approach we used. If we were able to provoke such a shift in one participant's thinking based only upon such a short exposure to TI-Nspire and a few reflective writings, the potential of a broader-based longer-term exposure to the appropriate, effective use of graphing calculators, as well as other technological tools, seems clear.

Reflections on the Roles of Calculators Framework

Seeing the departure from the trivial uses of calculators towards the nontrivial uses of graphing calculators was encouraging. This could be related to the participants' experiences as students learning or doing mathematics with calculators. If their prior experiences were limited to what we called trivial uses, it would be expected that they only mention such uses when asked to think about the use of technology in the mathematics classroom. This result highlights the significance of the combination of technological knowledge with technological pedagogical knowledge. If a teacher knows the ins and outs of technology; that is, (s)he has well-rounded technological knowledge, this would not be enough to plan or implement an effective mathematics lesson using technology. The teacher needs to have the complete technological, pedagogical, and content knowledge in order to deliver a lesson that makes effective and appropriate use of technology. In teacher education programs, we can help our pre-service and also in-service teachers first face their beliefs, views, and knowledge, and then we can guide them to extend their knowledge toward a more complete TPACK.

When we reflect back to the roles of graphing calculators conceptual framework by Doerr and Zangor (2000), we see that the framework was very efficient in the analysis of our data. Other than the role as a data collection and analysis tool, we saw all the roles described by Doerr and Zangor emerge in the reflective writings of our participant teachers. This result again highlights the importance of the past experiences with graphing calculators (or lack of, in this case) of the participants. Moreover, when coding our data, we saw graphing calculators as a transformational tool also come up when participants described uses of the graphing calculator as a visualization or computational tool. Therefore, we decided to restrict our coding to clear-cut functional roles of the calculators in this study. Furthermore, some of the writings of the participants indicated that they were assigning different roles to the calculators that we could not categorize with our existing codes. For instance, one pre-service teacher wrote that

> *I am currently teaching and use calculators to show relationships between different problems and in multistep problems where I am trying to see if my student knows what steps need to be taken in order to solve a problem instead of focusing on their computational skills.*

Clearly, this could be coded as a transformational tool, but then we would lose the problem solving aspect. Another participant stated that "I think it is a good way to scaffold students to learn what used to be concepts that were out of their reach cognitively." Other keywords that they used that we could not code within our existing framework were graphing calculators to reinforce skills or concepts, as a supplement, as an aid, and as a resource.

At this point we believe that an extension of our conceptual framework is necessary. In addition to the clear-cut functional roles of graphing calculators as computational tools, checking tools, exploratory tools, and visualization tools, we can add another layer of pedagogical roles, as

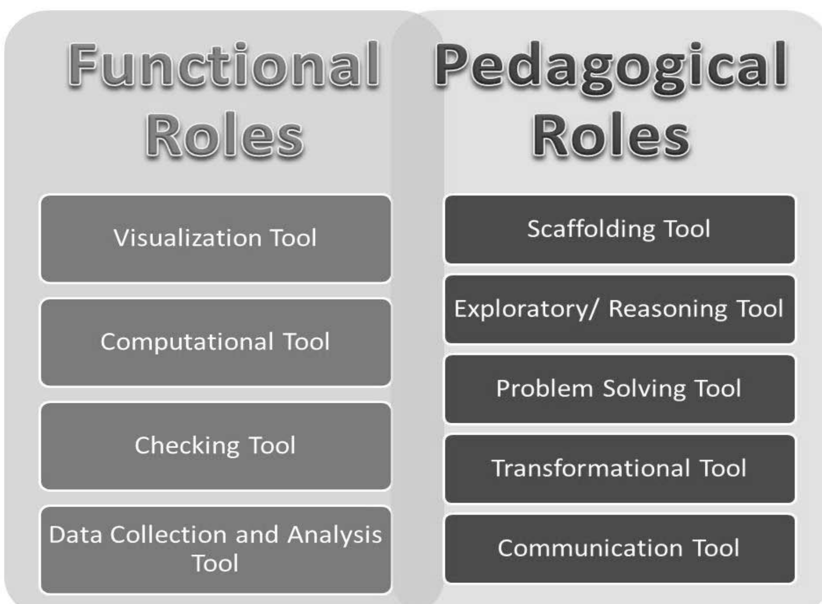

Figure 2. Revised Framework for the Roles of Calculators

in the NCTM (2000) process standards. One of the process standards suggests "reasoning tool" as a role, but we see some overlap between the exploratory tool role and a reasoning tool role. Therefore, we will add using graphing calculators as reasoning/exploratory tools, transformational tools, problem solving tools, scaffolding tools, and communication tools to the framework, as depicted in Figure 2. Still we expect that some of boundaries among these roles will be ambiguous at first.

In time, we, as teacher educators, and hopefully also the pre-service and in-service teachers we work with, will focus not only on what graphing calculators can do with their technological capabilities, thus highlighting the functional roles, but also how they can affect and enrich the teaching and learning environment, thus highlighting the pedagogical roles. Then all of our pre-service and in-service teachers might realize, as one of our participants did, that "engaging the students with technology will also be essential to effective teaching in the future."

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A CASE STUDY OF PRE-SERVICE TEACHERS WRITING MATHEMATICS FOR TEACHING IN A SECOND LANGUAGE

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This is a case study on the development of Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008) among three pre-service teachers who are English language learners (ELLs). We investigate the potential of thinking and learning reflection logs (TLR) to promote the development of MKT. Participants were enrolled in a required course for teachers seeking middle grades certification. Data were gathered from informal interviews and participant-observation in addition to logs. Analysis focused on demonstrated use of MKT. Findings suggest that writing TLRs is beneficial for ELLs's development of MKT, among other reasons, because it necessitates the use of multisemiosis and it allows time to communicate understanding.

The assessment of mathematical knowledge among English Language Learners (ELLs) has become an important priority for mathematics educators concerned with equitable access for all students. The research literature has pointed out how language may impact equitable access. For instance, Abedi & Lord (1998) show that students' language background impacts performance on word problems in standardized tests. "English language learners scored significantly lower than proficient speakers of English" (p. 231).

However, alternative assessment, such as student logs in which students write out their thinking and propose solutions to mathematics problems, may pose challenges for ELLs as well. A study by Brown (2005) indicates that alternative assessments such as student logs demand high literacy skills, which might especially impact ELLs of beginning proficiency.

We argue that writing TLR logs potentially promotes pre-service ELL teachers development of MKT discourse. In this study, we focus on the challenges and opportunities that writing logs posed for pre-service ELL teachers at a university near the US/ Mexico border. Many pre-service ELL teachers at this particular institution have developed much linguistic and mathematical knowledge in Spanish (Lesser & Winsor, 2009) but are being trained in courses in English. We present the experiences of three pre-service teachers who are seeking to become certified to teach in the middle grades. Enrolled in a course on conceptual algebra, participants were required to write Thinking and Learning (TLR) logs.

Current approaches to teacher education outline Mathematical Knowledge for Teaching (MKT) that mathematics educators should have (Ball et al., 2008). Researchers are beginning to ask how this knowledge base can be built in teacher education programs (McLeman & Cavell, 2009). In this paper, we aim to understand how pre-service teachers' writing TLR logs contributes to their development of MKT.

Theoretical Framework

The discourse of mathematics is challenging—even for fluent English users. Mathematics discourse practices are characterized by argumentative modes including precision, brevity, and logical coherence as well as cognitive and linguistic abilities such as "abstracting, generalizing, searching for certainty, and being precise, explicit, brief, and logical" (Moschkovich, 2007, p.

10). According to linguist Kay O'Halloran (2005), mathematics discourse is characterized by the use of multiple semiotic resources for constructing mathematical meanings. Mathematical meanings are constructed three semiotic resources: symbolic representation, visual representation, and language.

In this work, we draw on the framework on Domains of Mathematical Knowledge for Teaching (Ball et al., 2008) to describe the mathematical knowledge that pre-service teachers must master. The following is compilation of definitions for categories discussed by Ball. Subject Matter Knowledge includes (1) Common Content Knowledge (CCK), which is non-specific knowledge of mathematics (and skills) that is common for many different areas where mathematics applications are successfully used; (2) Horizon Content Knowledge (HCK), which is knowledge of the "big picture" and awareness of interconnectedness of all mathematical topics; and, (3) Specialized Content Knowledge (SCK), which is mathematical knowledge unique to teaching. Pedagogical Content Knowledge (PCK) includes (1) Knowledge of Content and Students (KCS), which is mathematical knowledge connected to the knowledge of students' thinking (conceptions and misconceptions, motivation); (2) Knowledge of Content and Teaching (KCT), which is mathematical knowledge connected to teaching, including knowledge of design and instruction, creating sequence of relevant and adequate, challenging tasks, making decisions about selection of manipulatives and resources for the task; (3) Knowledge of Content and Curriculum.

In our approach, we used the MKT framework as an analytical tool to understand the development of mathematical knowledge for teaching. A mathematician compresses and "packs" the knowledge and data (e.g., one short formula may represent huge amount of data that is literally "packed" in this formula) (Ball et al., 2008; Ball & Bass, 2000). SCK knowledge indicates specific "unpacking" of mathematical content knowledge in such a way that it is specifically suitable for teaching mathematics. Writing in TLR logs is an opportunity to develop mathematical discourse for teaching. With proper guidance and appropriate prompts, pre-service teachers could "unpack" mathematical content knowledge specifically for teaching mathematics. We present the experiences of pre-service ELL teachers writing TLR logs, and their development of MKT as evidenced in the logs.

Methodology

For the present study, we drew on data gathered from a larger qualitative study on pre-service ELL teachers' development of mathematical discourse[1]. The larger qualitative study draws on qualitative research data collection strategies, including participant-observation, informal interviews and analysis of student writing. Participant-observation was carried out over the span of one semester in a section of a course on conceptual algebra; the first author attended 70% of class meetings. According to course materials, the course is an inquiry-based course covering mathematics concepts including fractions, ratio, proportion, functions, algebra and geometry with an emphasis on reasoning and mathematical thinking. Some of the habits which the course aims to develop in students are a) attention to the meaning of symbols and numbers, b) analysis of problem situations, and c) hypothesizing and justifying mathematics solutions. The course was organized so that students would become socialized to the discourse of mathematics. One of the main goals of the course is for students to develop mathematics concepts to such an extent that it is possible for them to converse about it. Problem-based activities that require mathematical solutions were the core of discussions. Students' mathematical reasoning and mathematical communication are promoted through problems requiring small group discussion, whole class discussion, and writing in TLR logs.

Participants

Three students were invited to participate in the research based on the observed use of Spanish in the classroom. All three received formal education in Mexico, and for all three writing TLR logs was a completely new endeavor. All three reported having taken ESL courses in high school or college. The pseudonyms used to identify them are Betty, (a Mathematics and Science major), Yolanda (a Special Education major) and Laura (a Bilingual Education major).

Data Collection

For the present study we used data from TLR logs to assess MKT. Students were assigned semiweekly TLR logs. The TLR logs were designed to allow pre-service teachers to extend their thinking beyond what was covered in class and for students to offer new insights about mathematical concepts. TLR assignment required students to use multiple meaning-making resources (symbolic representation, visual representation and language), which is an opportunity to write mathematical discourse.

The unit of analysis is the TLR logs, a weekly writing assignment in which students had the opportunity to use the discourse of mathematics for teaching. Each weekly log is analyzed for the demonstrated use of MKT (Ball et al, 2008).

Data Selection

Data were selected based on topic and comparability across student writers. Data were selected from the first six weeks of the course in which the topic of fractions, a key concept for these pre-service teachers, was covered. Moreover, data were selected from assignments that could be compared. Not all students completed every log assigned. We selected logs for week 2 (log 1), week 4 (logs 1 and 2), week 5 (log 2) and week 6 (log 1).

Data Analysis

Data analysis took place in two phases. In the first phase, four raters independently rated each selected log. Raters are all Spanish/ English bilingual doctoral students with an expertise in mathematics education. For each log, raters examined each of the following MKT domains CCK, SCK, and PCK. Each domain was rated based on the demonstrated knowledge of the three domains. Raters, who were instructed to not discuss their ratings among each other, used the rating scale in the figure below. Individual ratings were averaged and standard deviations were calculated.

Rate on a scale of 1 to 5
1= **little or no evidence** that student has/ is using this knowledge
2= **some evidence** that student has/ is using this knowledge
3= **evidence** that student has/ is using this knowledge, with room for improvement
4= **significant evidence** that the student has/ is using this knowledge
5= **abundant evidence** that student has/ is using this knowledge

A second phase was designed, after standard deviation was consulted. In this phase, which is ongoing, raters selected one log to discuss the features of the log that each rater was observing to determine writer's demonstrated knowledge of CCK, SCK and PCK. In this phase, raters reach a consensus on the aspects that demonstrate knowledge of each domain[2].

Findings

Regarding different Domains of Mathematical Knowledge for Teaching Fractions we specifically focus on how these pre-service teachers understood different representations of fractions, and how different models were used to represent a multifaceted concept of fraction.

Overall, Betty (the Mathematics and Science major) performs differently on log writing compared to Laura and Yolanda (the Bilingual Education and Special Education majors, respectively) do. Betty's efforts in writing TLR logs are usually low, while Laura and Yolanda are consistently high. Ratings for each student's performance in five focus logs; ratings are reported for each domain evaluated. The table in Appendix A compiles all the ratings, including average ratings. Next, we use average ratings to report each participants' demonstrated knowledge in each domain.

We assumed that Betty (Math and Science major) would receive the highest average in CCK and Laura (who reported a lack of confidence in both mathematics and English) to get the lowest CCK average. However, the results surprised us. Betty appeared with the lowest average in every log but one, while Yolanda showed the highest average in every log but one, as we can see in Chart 1.

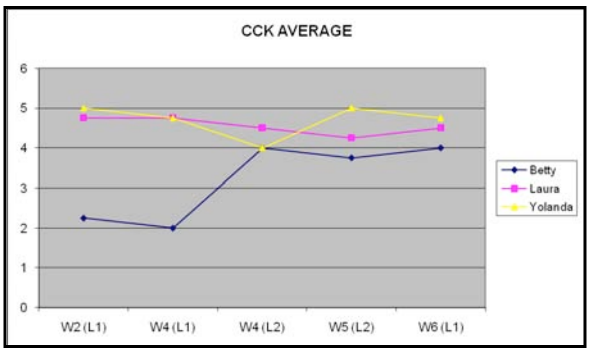

Chart 1. CCK average

Betty was very low in the first two logs but improved substantially in the following three. Moreover, Laura was the one who performed consistently, having almost the same rating in each activity log. Week 4 activity log number two "W4 (L2)" and week 6 activity log number 1 "W6 (L1)" showed the highest average overall by the three participants. Qualitatively, Betty copied brief definitions from the textbook that week, placing an emphasis on correctness. However, she did not use diagrams to further explain her meanings. This performance was generally characteristic of Betty's work in TLR logs. In an informal interview, she noted that she did not see the point of writing TLR logs. She fell behind in submitting her weekly TLR logs, but did manage to turn in a semester-end portfolio. She struggled to write logs in which she "unpacked" the mathematical knowledge. It seems that she found it hard to explain it in such a way that it would help children to understand.

Next, an analysis of SCK averages shows that Betty demonstrated the lowest evidence level on each week. Betty performed very poorly in the SCK domain. Meanwhile, Yolanda had an increasing performance every single week. Laura has almost the same grades in each week but week 4 activity log 1 "W4 (L1)". Yolanda had the highest average overall. However, the grades obtained by every participant were lower than the grades obtained by them in the CCK domain and hence, the SCK average was lower than the CCK average.

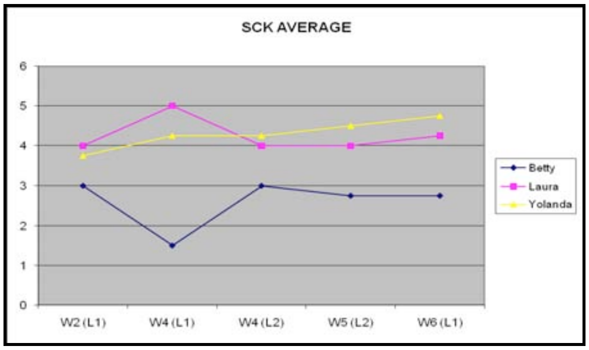

Chart 2. SCK average

Thirdly, the PCK domain gave the lowest grades overall as we can see in chart 3. Laura was the only one who decreased her performance every week but the second. On the other hand, Yolanda was the only one who increased her performance every week except the second one. Furthermore, Betty was the one with the lowest grades overall. Betty had the lowest averages in the first two activity logs and then she improved substantially.

Yolanda had the highest PCK average overall. To illustrate the point, taking an example from one of Yolanda's logs, she used her own words to explain two different meanings of fractions. When talking about division, she seemed to be confused about dividend and divisor, e.g., "3/6 in part-whole indicates three one-sixths and only need one whole. When we refer to division meaning, we are talking about how many 3s are in 6. That means that three wholes will be divided into six equal parts." She also provided pictorial representations using the area model. The representations seemed to help her understand the correct meaning of fraction as a division. She also discussed difference between continuous and discrete models. After that, she discussed whole numbers and fractions as part of a whole. Qualitatively, Yolanda's attitude about writing TLR logs was the most positive of the group. She regarded as a learning experience for both mathematics and English. She was conscious of the positive impact that writing logs had on her learning. She wrote about the logs that they help "to clear my understanding and while I was trying to explain at the same time I discover new concepts and ways to explain that will help as a future teacher." Furthermore, Yolanda demonstrated progress in the development of CCK when she says that she learned a significant amount of mathematical content by attempting to explain concepts instead of "rushing into a procedure." Revealingly, when she was asked if she would ask her future students write logs, she was enthusiastic about the benefits of writing them.

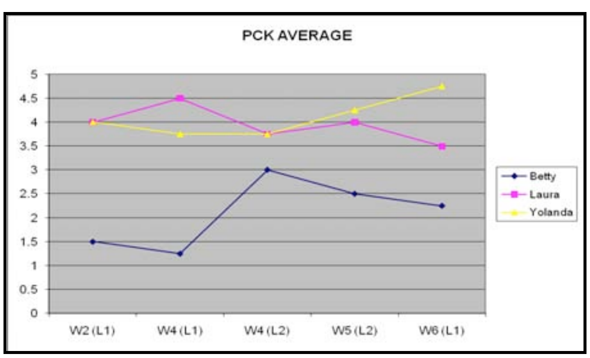

Chart 3. PCK average

In comparison to Yolanda, Laura said in an interview that her general mathematics knowledge was inadequate. She also noted that her English proficiency was still developing, and that she struggled to write the TLR logs. Laura was notably quiet in class; she never volunteered to present her solutions in front of the rest of the class—the only participant to not do so. Although Laura struggled, she was a very persistent student. In her semester-ending reflection, she wrote that, compared to other courses, in which she "was just memorizing formulas and procedures in order to pass the class" the class allowed her to learn the "meaningful part of math." Laura also stressed that she learned to understand many mathematical topics during the course. Laura is an example of a student for whom writing TLR logs may have been very fruitful. Writing them actually helped her to understand mathematics better.

Finally, in order to measure the reliability of the ratings, we calculated standard deviation. The four raters had the activity logs separately without discussing it among each other. Chart 4 shows the standard deviations resulting from the rating of each domain (CCK, SCK and PCK) in every weekly log by the four evaluators. The lowest standard deviation (0.66) obtained by the researchers corresponded to the week 2 activity log number 1 "W2 (L1)" in the SCK domain. Unfortunately, only three standard deviations showed in the chart were below 1.00. Additionally, CCK appears as the domain with the lowest standard deviation in every single activity log but the first one. PCK had the highest standard deviation average overall. Moreover, the average of the standard deviations of the three domains in all activity logs was 1.2.

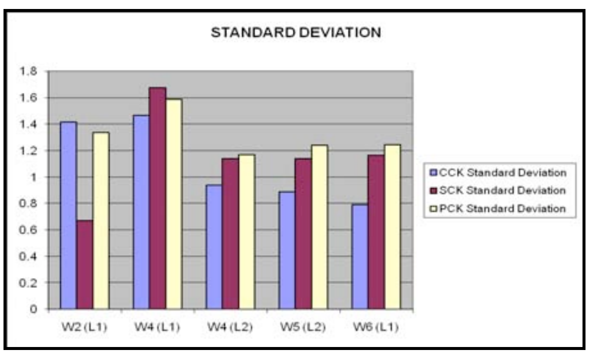

Chart 4. Standard deviations

Discussion

From the standard deviation data, we can conclude that it was necessary for raters to discuss and arrive at a consensus about what constitutes CCK, SCK and PCK. The process will make it possible to rate the rest of the log data. We have begun a second phase of research in which raters discuss their individual ratings with other raters. Raters pay attention to clarity and precision in symbolic representation, the use of examples, the use of visuals and a step by step presentation of a solution.

Still, we are able to comment on what our analysis so far can mean for training pre-service ELL teachers. Often this group may place an emphasis on teaching rules (procedural understanding) vs. teaching concepts. This aspect became relevant in the writing log where pre-service teachers discussed algorithms for operations on fractions. Still, we found that participants do develop MKT, but their learning trajectory is different.

On the one hand, writing TLR logs throughout the semester may allow students like Yolanda the experience and courage to develop mathematics discourse for teaching, particularly SCK and

PCK, which is very important for future teachers (Ball et al., 2008). Laura and Yolanda demonstrated consistently submissions medium to high levels of SCK and PCK. Their effort and level of detail were notable. However, sometimes their levels of CCK lead to some mistakes (e.g., omitted parentheses) that in turn lead to incorrect explanations. Writing TLR logs was especially beneficial for Yolanda. By the end of the semester, she was confident enough to go to the front of the class to propose a solution to a mathematics problem in English. While Laura's confidence never reached that point, in her reflection she notes that she learned from writing TLR logs.

On the other hand, the case of Betty is an interesting example because as a Mathematics and Science major she was expected to have high levels of CCK. Also, she was proficient enough in the second language to make verbal explanations in English in front of the class. However, her attitude about writing TLR logs was mostly negative. She regarded it as an unnecessary task in mathematics class. In her logs, she shows a preference for very concise, short descriptions. She prefers definitions from the book because they correctly represent the answers to the questions and, in her understanding are clearly explaining the meaning of the concepts. As she indicated in oral discussion, she doesn't understand the need for additional explanations or drawings. Thus, she submitted short logs, textbook definitions or paraphrased writing. She usually uses only traditional algorithms. She seemed to believe that no detailed explanations are necessary for these elementary concepts, and that everything is self-explanatory. When faced with some inconsistent example, she merely demonstrated that the result obtained was incorrect. Betty, as compared to Laura and Yolanda, may seem to receive no significant benefits from writing TRL logs. Moreover, it is evident that her SCK and PCK development is delayed because it is consistently evaluated at a low level. For students like Betty, it is important to make explicit the importance of unpacking mathematics for the purpose of teaching for her to develop SCK and PCK.

All mathematics students must become socialized in these mathematical discourse practices (Moschkovich, 2002; Gee, 1996; New London Group, 1996). We suggest that, to promote ELLs' equitable access to the discourse of mathematics for teaching, mathematics educators implement assessment strategies such as TLR logs. Writing logs has the potential to allow ELLs, such as the participants in this study, to develop MKT through writing. Writing mathematics discourse in TLR logs allows them time to plan and suit the message to the purpose of "unpacking" mathematics for teaching. Short & Fitzsimmons (2007) note that additional time for learning has been shown to benefit ELL students. Writing logs can be an opportunity to learn mathematics discourse, and, with proper guidance, mathematics student can learn to learn mathematics discourse. ELLs may require additional guidance that focuses on language choices (Schleppegrell, 2007). For instance, teachers can offer guidance focusing on 1) presenting a mathematical problem to be solved, 2) arranging mathematical symbols in patterned ways so that they encode a limited range of meanings, 3) accompanying the symbols with language to clarify the symbols using language to explain the problem-solving sequence, and 4) discussing the implications of the proposed solution. Finally, ELL writers can learn how to include visual images such as statistical graphs, geometric diagrams and other kinds of drawn or computer-generated visual displays, which illustrate the relationship in a space-time format (O'Halloran, 2005). Bringing attention to the discourse practices of mathematics can help support ELLs' development of mathematics.

Endnotes

1. Because pre-service ELL teachers were students in a college teacher preparation course, we use the term pre-service ELL teachers and 'students' to refer to the same group.
2. In this article we report on the first phase.

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Appendix A

Betty	CCK				SCK				PCK				Average				Average			Average
	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D	CCK	SCK	PCK	
W2 (L1)	3	2	3	5	2	3	3	4	1	2	2	1	2.00	2.33	2.67	3.33	3.25	3.00	1.50	2.58
W4 (L1)	1	2	2	3	1	2	2	1	1	1	2	1	1.00	1.67	2.00	1.67	2.00	1.50	1.25	1.58
W4 (L2)	3	3	5	5	1	3	4	4	2	2	5	3	2.00	2.67	4.67	4.00	4.00	3.00	3.00	3.33
W5 (L2)	3	3	4	5	2	3	3	3	2	4	2	2	2.33	3.33	3.00	3.33	3.75	2.75	2.50	3.00
W6 (L1)	5	3	3	5	3	4	3	1	2	3	3	1	3.33	3.33	3.00	2.33	4.00	2.75	2.25	3.00
Average	3.00	2.60	3.40	4.60	1.80	3.00	3.00	2.60	1.60	2.40	2.80	1.60	2.13	2.67	3.07	2.93	3.40	2.60	2.10	2.70
	3.40				2.60				2.10				2.70				2.70			2.70

Laura	CCK				SCK				PCK				Average				Average			Average
	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D	**CCK**	**SCK**	**PCK**	
W2 (L1)	5	5	4	5	4	4	4	4	4	4	4	4	4.33	4.33	4.00	4.33	4.75	4.00	4.00	4.25
W4 (L1)	5	4	5	5	5	5	5	5	5	4	4	5	5.00	4.33	4.67	5.00	4.75	5.00	4.50	4.75
W4 (L2)	5	5	3	5	4	4	3	5	4	5	2	4	4.33	4.67	2.67	4.67	4.50	4.00	3.75	4.08
W5 (L2)	5	4	3	5	5	4	2	5	5	3	3	5	5.00	3.67	2.67	5.00	4.25	4.00	4.00	4.08
W6 (L1)	5	4	4	5	5	4	4	4	4	3	4	3	4.67	3.67	4.00	4.00	4.50	4.25	3.50	4.08
Average	5.00	4.40	3.80	5.00	4.60	4.20	3.60	4.60	4.40	3.80	3.40	4.20	4.67	4.13	3.60	4.60	4.55	4.25	3.95	4.25
	4.55				4.25				3.95				4.25				4.25			4.25

Yolanda	CCK				SCK				PCK				Average				Average			Average
	A	B	C	D	A	B	C	D	A	B	C	D	A	B	C	D	**CCK**	**SCK**	**PCK**	
W2 (L1)	5	5	5	5	3	4	4	4	3	5	4	4	3.67	4.67	4.33	4.33	5.00	3.75	4.00	4.25
W4 (L1)	5	4	5	5	5	4	5	3	4	3	5	3	4.67	3.67	5.00	3.67	4.75	4.25	3.75	4.25
W4 (L2)	5	4	3	4	5	5	3	4	4	5	3	3	4.67	4.67	3.00	3.67	4.00	4.25	3.75	4.00
W5 (L2)	5	5	5	5	5	4	4	5	3	5	5	4	4.33	4.67	4.67	4.67	5.00	4.50	4.25	4.58
W6 (L1)	5	4	5	5	5	4	5	5	5	5	4	5	5.00	4.33	4.67	5.00	4.75	4.75	4.75	4.75
Average	5.00	4.40	4.60	4.80	4.60	4.20	4.20	4.20	3.80	4.60	4.20	3.80	4.47	4.40	4.33	4.27	4.70	4.30	4.10	4.37
	4.70				4.30				4.10				4.37				4.37			4.37

CROSS-SECTOR LEARNING: TRANSFORMING AN ELEMENTARY METHODS COURSE THROUGH COLLABORATION AND STANDARDS-BASED GRADING

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In this manuscript, we describe an unusual outcome of school-university collaboration use of standards-based grading (SBG) to transform the curriculum and assessment in an elementary mathematics methods course. This discussion includes a high school teacher's view of the value of SBG, a university professor's course changes, and students' response to the changes.

School-university collaboration has been a frequent area of study in teacher education research (e.g., Burton & Greher, 2007; Jeffrey & Polleck, 2010; McLaughlin & Black-Hawkins, 2007). However, few studies discuss the effect that this type of collaboration can have on the work of university faculty. In this paper, we discuss a collaboration of high school and college faculty focused on mathematics education that resulted in fundamental changes in an elementary preservice teacher (PST) mathematics methods course. The nature of the changes stemmed from the use of an assessment approach referred to as *standards-based grading* (SBG). In addition to bringing about change in assessment, the use of SBG had a transformative effect on course objectives and curriculum. The purpose of this paper is to demonstrate how the collaboration impacted the instructor's thinking about the course as well as to suggest the value of SBG in college-level courses.

Throughout the time that Janet taught her methods course, she was also a planner and facilitator of a professional development project that combined high school and college mathematics faculty (Frost, Coomes, & Lindeblad, 2009). During the project, a local large urban high school had decided to institute SBG as part of its district's consideration of future SBG adoption. Some of the teachers in the project and their colleagues were early initiators of this approach. Janet's conversations with a project participant and his colleague, Charlie, both early users, exposed her to the concept and rationale behind it. We begin this paper with Charlie's perspective on SBG as a theoretical framework before going on to Janet's discussion of the research method and transformation of her elementary mathematics methods course.

Theoretical Framework

After teaching high school for more than 25 years, I concluded that traditional grading had so little correlation with student understanding that I had to change. The old system of points, tests, quizzes, homework, and penalties created a game that some students learned to play even though their knowledge of math was limited. I'm fairly certain that my students who earned perfect scores knew some math, and those who earned zeros knew little math, but everyone in between remained a mystery. Searching for a more meaningful grading system led me to SBG.

Finding a consistent definition for SBG is a difficult task. The U.S. Department of Education, many individual states, and several prominent authors each have definitions of their own (e.g., Marzano, 2000; O'Connor, 2002; Popham, 2008; Tomlinson & McTighe, 2006; U.S. Department of Education, n.d.). In this section, I outline the parts of definitions consistent with my belief about what grades should mean, citing the sources that helped me to develop these beliefs.

Two main principles appear in the literature concerning effective grading practices (Marzano, 2000; O'Connor, 2002; Popham, 2008; Tomlinson & McTighe, 2006). The first principle is that achievement must be the only determining factor in assigning grades. The second is that grading practices should facilitate learning. From these two principles many characteristics emerge which help identify what I mean by SBG.

In order for grades to reflect achievement, most parts of traditional grading must be eliminated. Assignments, penalties for late work, extra credit, and punishment for inappropriate behavior have no place in effective grading practices (O'Connor, 2002). Summative assessments must be clear indicators of achievement towards pre-established standards (Marzano, 2000; O'Connor, 2002; Tomlinson & McTighe, 2006). Because grades must be an indicator of academic achievement, clear standards need to be established to determine how achievement will be measured. Changing to a SBG system requires teachers to develop the standards and the means to assess those standards.

This leads directly to the second principle. Having standards and the means to measure progress towards them allows teachers to assess throughout the unit. These formative assessments inform both the teacher and the learner on how to proceed. In other words, a teacher looking at the results of the formative assessments can gauge both the class and individuals' progress towards the standard. This powerful source of information helps guide differentiated instruction (Black, Harrison, Lee, Marshall, & Wiliam, 2003; Tomlinson & McTighe, 2006). Students can learn to use these formative assessments to plan for their own success as well.

With traditional grading, early quizzes often discourage students because these points always remain a part of their grades. Early failure leads to disillusionment and belief that the student is incapable of learning. When grades are used as a guide to instruction, no penalties arise from early misunderstandings. Instead, only the most recent assessments are used to determine a student's final grade.

Context and Methods

Context

The context for this qualitative study was formed by three components: the nature of the cross-sector collaboration, discussed above, that occurred while Janet was reconsidering her methods course design, the nature of the program in which the methods course occurred, and some specific aspects of the previous course design and grading approach that prompted a desire for change. Janet explains the latter two components below.

PST program. My methods course was part of a Masters in Teaching [MIT] program designed for PSTs who had earned a non-education baccalaureate degree and wanted to complete a short and intense program that would result in a teaching credential and a master's degree. In this program, PSTs did not take a mathematics content course. During the time they took the mathematics methods course, they were also taking methods courses in science, social studies, and literacy. The group of courses was taught in an 8-week session, with students attending each course two times per week.

Dissatisfaction with course design and grading approaches. I had several concerns about the PST course design. Many of the PSTs came to the course with low content understanding and mathematical confidence. This fact seemed to hinder them from understanding the methods being taught – they focused on the content, but complained that they were not learning enough about methods. As I tried to bolster PSTs' mathematics content understanding alongside the methods study I felt that I was trying to cover too much material in a short time, thereby only offering shallow coverage of the ideas.

In the course, I used traditional grading policies that allotted certain percentages to each topic and to student participation. However, I was concerned that students were ill prepared to master all of the ideas covered, due to the short course length and their heavy course load. For this reason, my grading was often lenient, resulting in PSTs receiving inflated grades that indicated higher levels of understanding than they had actually demonstrated. Yet, as Charlie discussed, the reverse could also be true. If a PST was late in submitting several assignment of high quality, the policy of taking points off for late submissions could result in grades that were lower than the level of mastery shown.

Method and Mode of Inquiry

This study was conducted as a combination of case study and action research. I regarded the PSTs and myself as participants. Data collected included documentations of the changes initiated over time, as indicated in course materials such as syllabi, assignments, and rubrics, and PSTs' response to these changes, as shown through course work and evaluations conducted twice during each course. My reflections were used as an additional data source on the process of making the changes.

Results

In this section, I explain several aspects of the changes made in my methods course, relating these changes to the cross-sector collaboration and the use of SBG. Course content changes and student comments in 2008 and 2009 are described in the first section, followed by the changes in grading practices during the same time period. These sections are accompanied by descriptions of the cross-sector collaboration that influenced my perspective. The last section addresses the course content and grading changes made in 2010.

2008-2009 Course Content and Schedule Changes

Course content. In order to address about PSTs' lack of mathematics content understanding, I increased time for content study and assessment. I attempted to cover the following mathematical content both years:

- Whole numbers, including number sense, place value, operations, facts, computation, and estimation
- Fractions/decimals/percents/proportional reasoning, including number sense, operations and computation, estimation, and conversions
- Geometry
- Algebra and algebraic thinking
- Data analysis/statistics, probability
- Integers, exponents, real numbers and other number types

It was not unusual for at least one of the topics to be dropped due to additional time needed for topics that students found difficult, such as fractions and geometry.

Methods topics focused on the NCTM (2000) and state process standards, including problem solving, reasoning and proof, communication, connections, and multiple representations, with brief attention given to technology and assessment. Students were asked to complete two lesson plans, teach "minilessons" in class, analyze a textbook unit in comparison to the process standards, and take tests on mathematics content.

Schedule changes. In 2008 and 2009, I increased the length of the course in order to cover the mathematics content and methods. On course evaluations, both positive comments and recommendations for improvement seemed to indicate that PSTs believed the mathematics content (assessed with tests), rather than the methods (assessed with lesson plans and curriculum analysis), was the most important aspect of the class. Some comments seem to indicate that the PSTs believed the tests were the only important assessment.

Sample positive comments:

- This course is great for deepening my understanding of basic mathematical concepts. I feel challenged in areas where I thought I was confident and the way I think about math has completely expanded.
- Open note tests were useful – took the pressure off and made me study more than I would have otherwise.

Sample recommendations for improvement

- More time for tests, focus only on what we are being assessed on because we don't have time to spend on other things… we have spent time on other things that were not related to what we will be tested on.
- I would suggest maybe allowing for more time on fractions and number concepts if possible.

Some of the recommendations also indicated that the PSTs felt the coverage of material was too fast.

- I feel like we rushed through a lot of the material to fit it all in and disregard whether we actually have a good understanding of it before we see it on the test. I think we need to take more time on some things and get rid of some things (like videos) that don't necessarily help us become better teachers.
- We go so fast, I do not feel I can fully master what we are learning.

I studied these comments, recognizing that the increased time was still insufficient to do a complete study of the mathematics content that PSTs would need to know. In conversations with my colleagues, I also learned that the additional course time could not continue, since the program could not include extended time for all the subject areas. Furthermore, more than half of the students earned lower scores on lesson plans and other opportunities to demonstrate understanding of the methods than they earned on math content assessments. This outcome showed the need for more focus on methods.

2009 Standards-Based Grading (SBG) Initiation

As described above, I had been concerned about grade inflation in my course but was unable to find ways to maintain a supportive atmosphere that also included high expectations and assessments that would indicate students' mastery levels. In 2008, student difficulties brought the issue of fair assessment and grading to the fore. As described above, a student's failure to submit work on time, due to family and work responsibilities, would have resulted in a grade far lower than his actual level of understanding. I regarded the situation as a dilemma – how could assessment be devised that would reflect the student's true level of mastery while still holding him accountable for prompt submission of assignments?

Conversations with high school teachers had led me to awareness of SBG. When the dilemma presented itself, I initiated conversations about the issues in my course and the use of SBG with one of the high school teachers in the professional development project and Charlie, a colleague of that teacher. I made the decision to implement SBG, focused solely on achievement of the learning targets for the course. Student disposition issues would be addressed through individual meetings with students to discuss issues that arose and strategies for addressing these issues.

The first use of SBG in the methods course occurred in 2009. The syllabus description of grading practices is shown below.

2009 Grading

Description	Score
Student demonstrates full and independent mastery of the objective.	4
Student demonstrates full mastery but has minor errors or requires some support or assistance to do so.	3
Student has significant gaps in mastery of the standard but shows mastery of most main ideas.	2
Student has significant gaps in mastery of the standard, including mastery of the main ideas.	1
Student did not submit work.	0
Your final grade will correspond with your final average across all of the competency areas: 4 = A, 3 = B, 2 = C, 1 = D. Please note assignment descriptions regarding fluidity of scores – in most cases, you will have more than one opportunity to demonstrate mastery. Your final score in each area is the score that will be used to determine your grade.	

Table 2. 2009 grading description

As described above, students were given multiple opportunities to demonstrate mastery of each learning target. If their assessed level of understanding increased, the older score was dropped in favor of the new score. If the new score was lower than a prior score on that learning target, it was averaged with the higher score. These scores were converted to grades only at the end of the course, so students could work to improve their scores without penalty until that time.

The use of SBG also required changes in rubrics used to score students. Samples of 2008 and 2009 rubrics are shown to illustrate the changes made in the first year of SBG.

2008 Lesson Plan Rubric

Lesson Plan #1 Points: Components and organization (6 points); detail and clarity of explanation (6 points); lesson plan represents activity accurately or summarizes created activity (6 points); presentation (2 points)

Lesson Plan #2 Points: Appropriate lesson *type*: consistent with NCTM principles and standards, current learning theory, appropriate mathematics methods as discussed in text and presentations, appropriate for identified grade level and good fit with unit plan (9 points); all lesson plan components included and organized appropriately (9 points); explanation of components is detailed and clear, all aspects of each component addressed (9 points); presentation (3 points)

2009 Lesson Plan Rubric

Lesson Plan Rubric	Surpasses Expectations	Meets Expectations	Below Expectations
Learning Targets (5 points)	All learning targets addressed in detail and with clear alignment. Learning targets are closely connected to previous learning experiences.	Instructional plan purpose, state learning targets, and content objectives are aligned (4 pts); previous experience explained (1 pt).	Aspects of learning targets not identified, unclear, or not aligned with each other
Assessment Strategies (5 points)	Multiple formative and/or summative assessment strategies and procedures for gathering evidence explained in detail and aligned with learning targets and experiences. Multiple learning modalities addressed in assessment plans. Ways of reducing bias explained.	Formative and/or summative assessment strategies are aligned with learning targets and experiences (2.5 pts). Procedure for gathering evidence for each student or for representative sample included (2.5 pts).	Assessment strategies incomplete or unclear OR lack of alignment between assessment strategy and learning targets OR inappropriate or no procedure for gathering evidence
Learning Experiences and Student Grouping (30 points)	Learning experiences match and complement the learning targets, include detailed plans for differentiation, remediation, and extension OR complex instruction with group-worthy tasks. Learning activities include detailed before, during, and after components that incorporate all 5 process standards. Includes culturally responsive, multicultural, interdisciplinary, and/or technology aspects.	Learning experiences aligned with learning targets (5 pts). Learning experiences include appropriate before, during, after components (10 points). Learning experiences utilize 3-4 process standards (problem solving, connections, communication, representation, and reasoning/proof) (15 points).	Lack of alignment between learning experiences and targets OR only 1-2 of before, during, and after components OR learning experiences utilize 2 or fewer process standards.
Student Voice (5 points)	Lesson plan provides multiple opportunities for student voice and student self-evaluation as well as opportunities to incorporate student voice into existing lesson plan.	Lesson plan includes at least one of student voice component.	Lesson plan includes no use of student voice techniques or Assessment For Learning strategies.
Organization and Conventions of Writing (5 points)	Proper structure and organization utilized. Writing conventions are appropriately addressed and all descriptions are clear. Fewer than 3 errors in writing mechanics.	Follows instructional plan format or very similar format (1 pt). Explanations clear (2 pts). Fewer than 8 errors in writing mechanics (2 pts).	Errors in following instructional plan format or missing important details from checklist. More than 8 errors in writing mechanics.

Table 3. 2009 lesson plan rubric

As I used the 2009 rubric, it was clear that additional changes needed to be made so that students could be assessed on each learning target individually, rather than on groups of targets.

In response to the use of SBG, some students offered positive comments in their evaluations, such as “Emphasized growth, not grades.” Others felt that the flexible scores and lack of concrete grades left them wondering where they stood. “I’m not a fan of the scoring of the course, because it makes me feel confused about where I stand.” This latter type of comment seemed to indicate that students were having difficulty making a cultural shift from a focus on getting an A to the new focus on demonstrating mastery of the learning targets.

2010 Changes

In this section, I describe the effect of introducing SBG and my subsequent reflection on the course content. This effect is demonstrated in the course content change and the 2010 lesson plan rubric. Student comments follow the description of the changes.

Course content and assessment changes. The changes enacted in the 2009 course and the ongoing detailed discussions with Charlie about his use of SBG prompted further improvements in 2010. As I recognized the need to delineate the individual learning targets and considered students’ scores and the impossibility of covering all of the mathematics content in the time available, it became clear that the focus of the course needed to shift. The curriculum needed to be much more narrow and focused with clearly defined learning targets solely related to learning methods, and the rubrics used to score the assignments needed to be more detailed and delineated.

The course content was changed to focus on the NCTM (2000) and state process standards, with additional discussion of levels of questioning and cognitive demand (Stein, Smith, Henningsen, & Silver, 2000), analysis of lessons that did or did not address the standards, revision of the lessons that did not, development and teaching of minilessons, and analysis of textbook units. PSTs were asked to identify the presence or absence of the process standards in commercial videos, their peers' minilessons, and the lesson plans they analyzed. Mathematics content was reduced to whole and rational number sense and operations, with a brief consideration of algebraic reasoning and geometry. This mathematics content was used as a context for considering the use of the process standards, rather than as an end in itself.

In the 2010 course, the 2009 grading description of the 0-4 scores was used. However, the lesson plan rubric was changed to separate out each component discussed in class.

2010 Lesson Plan Rubric
1. Process Standards Applications In lesson plan, student demonstrates at least three of the five process standards: a. Mathematical problem solving b. Reasoning and proof c. Mathematical communication d. Mathematical connections e. Mathematical representation
2. Standards and Learning Targets a. Learning targets, state standards, and learning activities are grade-appropriate. b. State standards, learning targets, objectives, and learning activities are all aligned. c. Class is informed about objectives/learning targets, including state standards.
3. Meaningful Learning a. Lesson is likely to be engaging for students. b. Learning activities are accessible to multiple skill levels c. Learning activities build meaningful in-depth conceptual mathematical knowledge and critical thinking ability, with appropriate cognitive demand. d. Learning activities address needs of all students, including those who struggle with the lesson or who are ready for more challenge. e. Lesson includes appropriate before, during, and after activities.

Table 4. 2010 lesson plan rubric

The change in the course content and rubrics had two types of positive effects. First, students could be assessed on delineated learning targets. By having specific information about their areas of strength and weakness, they could focus on making changes or asking for help in order to strengthen their demonstration of learning on the elusive targets. Second, I could immediately identify the areas of difficulty and adjust my plans to meet those needs.

2010 Student Comments

Student comments on the 2010 course indicated that the changes had many of the desired effects as well as pointing the way for further changes. Positive comments were more focused on the process standard of teaching through problem solving, which was a new idea for most students. They also recognized that the daily plans were often adjusted to meet the needs demonstrated in the assessments.

Sample positive comments:

- I liked the process grading. I think it gave us great opportunity to LEARN from our mistakes…. Janet was extremely helpful and understanding with our student needs.
- Going into depth about concepts really was useful. Observing and experiencing math lessons everyday gave concrete experiences and understanding of how to present lessons.

Recommendations for improvement also indicated PSTs' focus on methods and the desire to have more resources for and practice of lessons that utilized the methods taught.

- I would like more concrete examples of effective teaching. Janet was great at modeling practices but I would appreciate more reference materials.
- More in-class lessons without assessment so we can practice.

As a result of the student suggestions and my ongoing reflection, I plan to develop more opportunities for students to work collaboratively and individually designing and teaching lessons that reflect the process standards and high cognitive demand, as well as more in-depth study of teacher videos that model different ways of incorporating these ideas. Two additional areas of focus emerged from the assessment of PSTs' understanding. The first addition will be increased discussion of ways to engage students, since the PSTs did not express the same level of understanding of that idea as they did of the process standards and cognitive demand. The second addition will be analysis of student work, since PSTs have had little experience studying students' thinking and designing instruction to fit it.

Discussion

In an address to mathematics teacher educators, Jim Hiebert (2010, January) claimed that U. S. teacher education programs were failing to have significant effects on classroom teaching and learning. He suggested that varied and short teacher education programs cannot provide the critical mass of learning required to overcome PSTs' culturally developed beliefs about teaching. Therefore, teacher education programs should focus on helping PSTs become learners within the teaching context, rather than assuming the program can fully develop their teaching skills before they enter the classroom.

In this paper, we provide an example of how cross-sector collaboration and SBG led to new ideas about addressing issues in an elementary mathematics methods course, and ultimately the transformation of that course in a way that may address Hiebert's suggestion. The collaboration provided new perspectives: although often used in K-12, SBG was not used or discussed at the university level prior to Janet's work with two high school teachers and subsequent introduction of SBG in her course. This new perspective provided a possible solution to issues she had identified: use of SBG helped her to narrow and deepen her course content and it allowed more support for the process of learning, which PSTs noticed and appreciated. Thus, the changes made in this course indicate one aspect of course design – the use of SBG – that benefited Janet and her students. Janet gained insight and innovative ideas and the students gained a more effective curriculum and assessment system. When traditional grading practices are used, students – in this case PSTs – may not be treated as learners but as "knowers" by receiving summative grades that are factored into their final grade, rather than flexible scores that change as they demonstrate increasing mastery. SBG offered this kind of formative kind of assessment as well as providing input of how course content could be changed in response to PSTs work. At the same time, the principles and data from SBG informed Janet's instruction throughout the course, so she remained a learner as well. In this way, SBG had and will continue to transform the thinking and learning of Janet and her students, just as it did for Charlie in his high school teaching.

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TEACHING TRIGONOMETRY: A DIRECTED LENGTH APPROACH

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This empirical study investigated the effects of an instructional sequence on pre-service secondary teachers' understanding of trigonometric functions. Guided by the APOS framework, we designed activities to promote a directed length interpretation of the six trigonometric functions using a dynamic geometry environment. Results of a paired t-test (two-tailed) showed a statistically significant difference in performance. Qualitative analysis of student work revealed a shift from ratio based strategies to directed length based strategies. Findings reveal the potential this method holds for transforming the teaching and learning of trigonometry.

Trigonometry is a subject that is highly valued. It has a long history as a component of the mathematics curriculum with methods reaching far back before the 20^{th} century. Trigonometry has appeared in both the Curriculum and Evaluation Standards (NCTM, 1989) and Principles and Standards for School Mathematics (NCTM, 2000). At the same time, it is also a subject that has been regarded as an important in domains outside of mathematics particularly those in science, technology, and engineering. Careers in physics, astronomy, computer graphics, optics, and many branches of engineering require an understanding of trigonometric functions and their application. Moreover, principles of trigonometry have a practical application in the work of carpenters, surveyors, architects, and navigators (Hoachlander, 1997). Despite its relevance, trigonometry continues to be a subject with which many students struggle (Blackett & Tall, 1991; Kendal & Stacey, 1998; Weber, 2005).

The modern approach to the subject, which considers both triangles and circles, is taught over a period of years as students progress from middle school to high school. Historically triangle trigonometry and circle trigonometry emerged from separate traditions and were developed for different purposes (Bressoud, 2010; Van Brummelen, 2009). Triangle trigonometry, which focuses on ratios between side lengths in a right triangle, has its origins in the surveying and measurement techniques used by the Babylonians and Egyptians. Circle trigonometry, which focuses on chords and their associated arcs, was developed by the Greeks in their study of the heavens. Anyone familiar with modern secondary mathematics will immediately recognize that these two approaches are visible in the predominant teaching methods, the ratio method and the unit circle method. There has been disagreement and debate over the use of these two methods since the “New Math” movement (Kendal & Stacey, 1998). Rather than choosing a side, we posit that both methods play an important role, laying the groundwork essential for building an understanding of trigonometric functions.

Framing the Problem

The study of trigonometry typically begins with a focus on the ratios formed by the sides of right triangles. This ratio method is effective for the purpose of solving right triangles and thus should hold its place in the curriculum (Kendal & Stacey, 1998). However, this method relies heavily on memorization (e.g., the mnemonic SOH-CAH-TOA) and computation. Using this method requires students to create nontrivial connections that pose many obstacles as they attempt to connect diagrams to numerical relationships and manipulate different ratios to determine side lengths (Blackett & Tall, 1991). In our opinion, the ratio method fosters a static interpretation of the trigonometric functions in which the value of a function for a given angle is related to a single right triangle rather than a family of triangles. Further, this method is limited to angles measuring less than 90° with particular attention on "special" right triangles (30°-60°-90° and 45°-45°-90°). Although this focus on specific angles is helpful in establishing benchmarks for comparison, it does not easily connect to an understanding of function.

The unit circle method, which defines sine and cosine as the coordinates of a point on the unit circle or as directed lengths, resolves a number of the issues with the ratio method. The domain in the unit circle method expands to include all real numbers. In addition, the method provides students with a straightforward way to compare the values of different angles by focusing on changes in directed lengths. This type of reasoning can also be used to identify the sign of the sine and cosine functions in each of the four quadrants without relying purely on memory. Unfortunately, this directed length interpretation is typically not extended beyond sine or cosine. Instead, there is often a focus on special angles (i.e., multiples of 30° or 45°) and reference triangles in order to help students see the connection between the quadrants. For many students and teachers, the remaining four trigonometric functions are interpreted as algebraic manipulations of sine and cosine. Thus, working with these four functions requires either reasoning about how they are related to the sine and cosine or recalling facts from memory.

Our research aims to help students build a robust and connected understanding of trigonometric functions and explores the role that dynamic geometry environments (DGEs) can play in its development. DGEs are one of the most widely used technologies in mathematics education. Many mathematics teachers and mathematics education researchers have focused their attention on the use of DGEs within the classroom. Designers of DGEs have stressed the central role that the software can play in linking geometry to other topics (Hohenwarter & Lavicza, 2007) by offering users a way to manipulate algebraic objects dynamically. We believe that DGEs may be one way to introduce a directed length interpretation and emphasize its connection to trigonometric functions.

Purpose

The purpose of this study was to develop an instructional sequence of trigonometric activities using a dynamic geometry environment and to analyze its influence on pre-service secondary teachers' understanding of trigonometric functions. These activities were designed to promote a directed length interpretation of the six trigonometric functions as functions that map one measurable quantity, angle, to another measurable quantity, directed length.

Research Questions

1) How can a dynamic geometry environment be used to develop pre-service secondary teachers' understanding of trigonometric functions?
2) How does an instructional sequence focused on a directed length interpretation of trigonometric functions influence the strategies used by pre-service teachers' when solving trigonometry problems?

Theoretical Lens

We used the APOS (action, process, object, schema) framework as a theoretical lens (Asiala et al., 1996). This theoretical perspective was built upon Piagetian ideas and designed for studying mathematical learning at the undergraduate level. According to this perspective,

> An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations. (p. 5)

Below we briefly discuss the four components – action, process, object, schema – and describe how we envision student responses on a single trigonometry comparison task from our assessment (Figure 1).

In the figure (below) line BC is tangent to circle A. How does the value of sin θ compare to the value of sec θ? (i.e., >, <, =) Explain your answer.

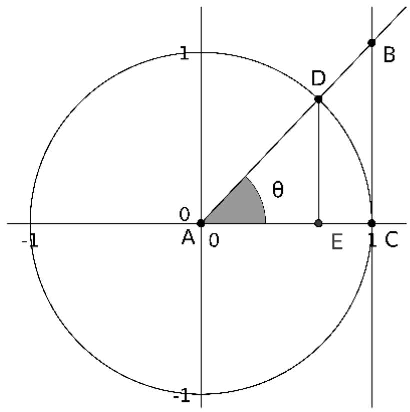

Figure 1. Comparison of sin θ and sec θ task

A student with an action conception of trigonometric functions would require external cues to solve this problem. These cues could include things such as using sine = opposite/hypotenuse and secant = 1/cosine. A student with this conception might also use an estimation strategy based on a benchmark (e.g., theta appears close to 60°) or a construction (e.g., drawing in the segments that correspond to sine and cosine). In any case, the student would need to write this information down and reason through the problem externally.

Following the APOS framework, the shift from action to process conception occurs when a student internalizes what were previously external cues. Thus, we anticipate that a student with a process conception of trigonometric functions would have a solution strategy similar to that of a student with an action conception. However, the student with a process conception would be able to work through the problem internally without needing external cues.

We believe that a student with an object conception of these two trigonometric functions would be able to approach this question from multiple perspectives. They would be able to reason about the secant without needing to compute a value or imagine the process of division. A student with an object conception might interpret the sine and the secant as directed lengths and conclude that secant is greater than the sine based on a mental image or use their knowledge of the range of the trigonometric functions to determine secant is larger (e.g., $\sec\theta > \sin\theta$ because $\sin\theta \leq 1$ and $\sec\theta \geq 1$). It is important to note that distinguishing between a process conception and an object conception is difficult without asking the student to explain his or her reasoning.

A schema is formed by the organization of actions, processes, objects, or a combination of these ideas. Based on this definition, schemas can be built by individuals holding any of the different conceptions. For example, a student might construct a schema using their action conception of sine and secant. They could then use this schema to estimate the values of sine and secant for the task in Figure 1. A student who had constructed a schema using an object conception would be able to respond quickly and efficiently to the sample task. They would have several methods for working through the problem and be able to identify the connections between them. For example, they might view the sine and secant as directed lengths while imagining the graphs of the functions. We see our current work as helping students move from action conceptions of trigonometric functions to process conceptions and aiding them in the development of rich and connected schema.

Methodology

The participants for this study (n=23) were enrolled at a medium sized Midwestern university in the United States. All participants were enrolled in a course that focused on the use of technology tools for the secondary school level. One student was a practicing teacher while the rest were pre-service secondary mathematics teachers. The course met once a week for three hours and was co-taught by the researchers. Three lessons dealt specifically with trigonometry; however, the course dealt with a variety of different mathematical topics. All students received the same experimental instruction.

Design of the Assessment Instrument

Following the APOS framework, we first analyzed what it meant to understand the concept of a trigonometric function. This was based primarily on our own understanding of trigonometric functions as well as relevant research. We then designed a paper-pencil instrument to assess students' knowledge of a directed length interpretation. The instrument included: a) one trigonometric identity question; b) three questions that compared the value of different trigonometric functions for a given angle (e.g., Figure 1); c) one question analyzing how increasing an angle changed the value of a trigonometric function; and, d) two questions that compared the magnitude of two trigonometric functions in the first quadrant. The overall aim of the assessment instrument was to determine if there were any changes in student reasoning about trigonometric functions as a result of engagement in the experimental instruction.

Design of Experimental Instruction

Our analysis of the concept of a trigonometric function also guided the design of our experimental instruction. We designed a sequence of instruction consisting of several different explorations within a DGE. The sessions were sequenced to first introduce the directed length interpretation and then connect this interpretation to the graphs of the trigonometric functions. Our goal was to engage students in activities that explored interpretations of the six trigonometric functions as functions that mapped angle measures to directed lengths in the coordinate plane. We briefly discuss the first session below.

The first session began with an activity exploring the “chord function.” This activity served two purposes. First, to connect to the historical origins of the word sine as the mistranslation of the word “chord” (Bressoud, 2010). Second, to demonstrate how the concept of function could be used to explore the relationship between angle measure and directed length. We first constructed a unit circle with central angle BAC (with point B fixed) and chord CD as shown in Figure 2. Students then considered how changing the size of angle BAC affected the directed length of segment CD. We then had the students reason about what it would look like if they were to map the angle measure to a point, D', on the x-axis and construct a copy of CD on D' with point D mapped to D'. Connecting the copies resulted in a rough sketch of the function y =

2 sin *x* (Figure 2). This lead to a discussion of the historic connections between the words "chord" and "sine." Focus then shifted to the standard unit circle representation of sine where half of the directed length of the chord CD is the value of the sine of angle BAC.

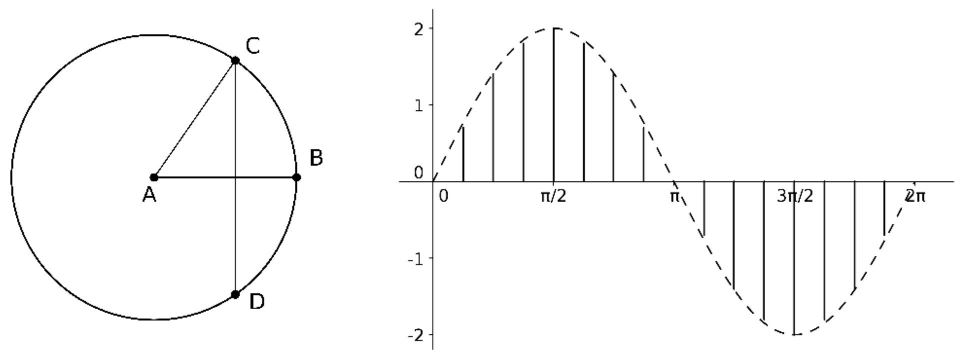

Figure 2. The chord function

Subsequent sessions explored similar constructions and mappings with the other trigonometric functions. Sessions began in the classroom and then transitioned to the computer lab where each student constructed his or her own sketch of the problem(s) using dynamic geometry software. Group work and collaboration were encouraged.

Procedure

Students completed the pre-test prior to the first session on trigonometry. The instructional sequence then took place over three consecutive class periods. Two weeks after the third session, the post-test was administered. Neither test was announced in advance. Students were given 25 minutes to complete each test and were not allowed a calculator. All questions were assigned a value of one point and the number of correct answers was totaled to obtain a score for each student. The highest possible score was seven. A two-tailed dependent t-test was used to test for significant change.

Written work was analyzed using a constant comparative method (Merriam, 1998) to identify and categorize solution strategies. One question was omitted from this analysis because it was multiple choice and did not include space for written work. Responses were analyzed and clustered into progressively more inclusive categories. If a response included parts of a strategy (either incorrect or correct) it was coded as having shown that strategy. Consequently, some responses received multiple codes because they exhibited more than one strategy (this occurred only on the post-test questions). This coding process resulted in the identification of three different solution strategies - ratio, directed length, graph - and a collection of answers that had no identifiable strategy.

Ratio based strategies (Figure 3) involved the use of ratios and mnemonic devices (SOH-CAH-TOA). These responses typically had specific ratios in their solution or cited facts (e.g., sine is positive in first quadrant). Also common were case based arguments in which a specific example was used to justify a more generalized statement.

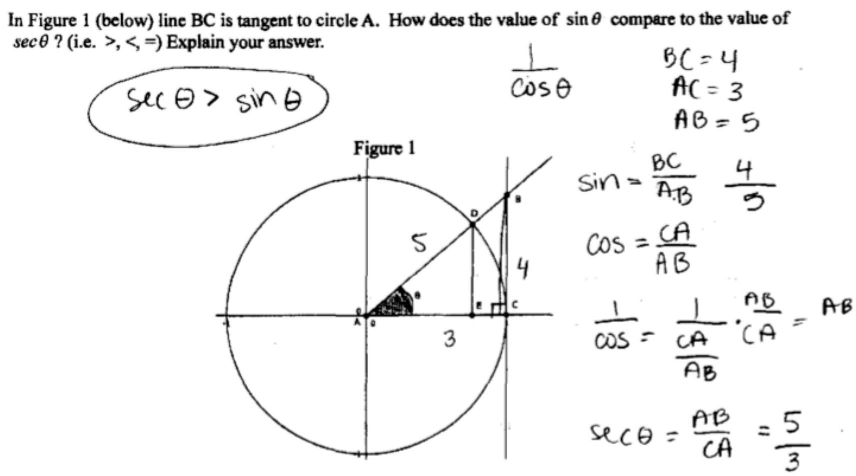

Figure 3. Sample ratio strategy

Figure 4 displays an example of a directed length strategy. These responses typically included the construction and labeling of segments corresponding to the value of a trigonometric function for a given angle. In some cases, discussion of the length of the segments followed.

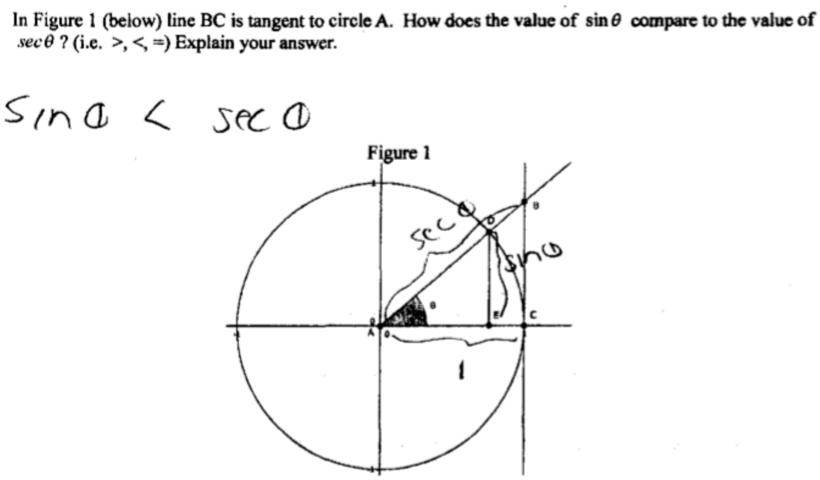

Figure 4. Sample directed length strategy

Graph based strategies included a graph of a trigonometric function in the response. Responses were coded as having no identifiable strategy when one could not be determined from the written work or no work was presented other than the solution.

Results

The mean of the pre-test scores was 2.26 with a standard deviation of 1.74. The mean of the post-test scores was 4.17 with a standard deviation of 1.92. Results of the two-tailed dependent t-test indicated that there was a statistically significant change in scores from pre-test to post-test $t(22) = 7.19$, $p < .0001$. The calculated effect size was $d = 1.04$. Results of the qualitative analysis of solution strategies are summarized in Table 1.

Strategy	Number of Instances (Number of Times Correct)						Total	% Correct
Pre-Test	Q1	Q2	Q3	Q4	Q5	Q6		
Ratio	11(7)	21(8)	21(10)	16(4)	15(4)	15(3)	99	36.36%
Directed Length	0	1(1)	1(0)	1(1)	0	0	3	66.67%
Graph	0	0	0	0	0	0	0	
No Strategy	12(5)	1(0)	1(0)	6(0)	8(0)	8(0)	36	13.89%
Post-Test								
Ratio	11(9)	6(3)	5(1)	6(4)	6(2)	6(2)	40	52.5%
Directed Length	9(9)	19(17)	18(11)	18(12)	6(5)	7(7)	77	79.22%
Graph	0	0	0	1(1)	2(1)	2(0)	5	40%
No Strategy	3(2)	0	2(0)	2(0)	8(2)	8(1)	23	21.74%

Table 1. Summary of pre-test and post-test strategies

Discussion

We attribute the statistically significant growth from pre-test to post-test to both the directed length interpretation as well as the use of a dynamic geometry environment. The DGE provides unique access to the directed length interpretation by allowing students to construct figures, identify specific attributes, quantify these attributes, and analyze the relationship between these quantities by graphing them in a coordinate plane. Thus, the connection between directed length and the graph of a trigonometric function can be easily made within a DGE. We believe this approach is useful in addressing many of the difficulties students have with trigonometry concepts (Blackett & Tall, 1991; Kendal & Stacey, 1998; Weber, 2005) by providing them with a flexible model for reasoning about trigonometric functions. Hence, this study demonstrates how DGEs can be used to foster student learning and shift the focus of trigonometric instruction.

Our qualitative analysis, presented in Table 1, resulted in two findings: 1) substantial changes in student strategy use and 2) increases in effectiveness across the three identified strategies. The ratio strategy represented 99 out of 138 (72%) of all coded responses on the pre-test and 40 out of 145 (28%) on the post-test. The direct length strategy, in comparison, represented 3 out of 138 (2%) of all coded solutions on the pre-test and 77 out of 145 (53%) on the post-test. These results show that many students who used ratio strategies on the pre-test transitioned to more flexible strategies using ratios and directed lengths on the post-test. In addition, the percentage of correct responses across all three strategies (ratio, directed length, graph) increased from pre-test to post-test. These numbers indicate that students improved in their ability to use strategies effectively. Together these results show that students not only understood the directed length concept, but also integrated it into their trigonometric schema.

Based on our analysis of the written work we are confident that the majority of the students have, at least, an action conception of trigonometric functions. Anything beyond this is speculation due to the nature of our assessment. What is clear, however, is that the instructional sequence aided students in developing ways to reason about these trigonometric functions on the post-test that were different from the ways they reasoned about them on the pre-test. This, in turn, lead to a significant improvement in performance.

Our goal is to assist pre-service teachers in developing rich and connected schema that contain multiple representations and interpretations of trigonometric functions. We believe that the action conception exhibited by the majority of pre-service teachers can develop into a process or object conception given further time and interaction with the directed length interpretation. In our opinion, this interpretation lends itself to a process (and later object) conception of trigonometric functions by providing students with tools to reason about the functions as *things* themselves instead of the *result* of memorized facts or algebraic manipulations. Future research should examine the benefits and drawbacks of this interpretation, determine its utility in the teaching and learning of trigonometry to different students at different levels, and explore the role that DGEs can play in this process. This will require the development of assessment tools to probe student thinking to establish if either a process or object conception is present. We believe the directed length interpretation has the potential to profoundly impact both the learning and teaching of trigonometry – embodying the idea of transformative mathematics.

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IDENTITY DEVELOPMENT OF PRESERVICE ELEMENTARY MATHEMATICS TEACHERS IN STANDARDS-BASED REFORM PEDAGOGY AND ITS CONTRIBUTING FACTORS

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The current study is part of an ongoing, longitudinal study of prospective teachers' identity development as a mathematics teacher. Drawing on case study of five teacher candidates, this paper investigated how their professional identity has developed over time and what the contributing factors of such changes. Findings indicated that most of student teachers entered the program with traditional perception of math teachers and it evolved endorsing more of a reform model. Prospective teachers' prior experiences didn't seem to be a great factor that contributed to change. The university instructor appeared to be the greatest influence and their mentor's teaching practices has a greater potential for modifying and re-constructing student teachers' professional identity and their teaching practices.

Preservice teachers enter teacher education programs with early experiences in mathematics and they have perceptions of themselves as mathematics learners and teachers (Drake, 2006). While participating in their university methods program, under the mathematics reform movement, prospective teachers tend to be exposed to reform-oriented teaching models. Prospective teachers need to understand what it is like to participate in the figured world of reform pedagogy, learn models of identities of reform pedagogy, and negotiate new constructions of mathematics (Horn, Nolen, Ward, & Campbell, 2008). They are constantly transitioning between their current model of what a teacher is, and practicing what their identity as a teacher may be through student teaching, as well as learning how teaching is done through modeling by a mentor. When they eventually transition into their own career, teachers are constantly constructing and reconstructing their identities as math teachers through the negotiating within the community of practice.

A recent study (Pressini et al, 2009) found student teachers' norms and expectations about mathematics teaching practices were fundamentally different depending up on whether they saw themselves as teachers or students. Other studies emphasize that having a clear self-image as a teacher is critical to translate what they've learned from their teacher education program into real classroom practice (Bullough, 1992; Kagan 1992; Mewborn,1999). These studies document that prospective teachers struggle with their teaching practices when they don't have a secure self-image of themselves as teachers. Whether they see themselves as a traditional mathematics teacher (image from early experiences) or if they endorse a modern model of teaching (provided by teacher education programs) their teaching practice will look fundamentally different.

Despite the importance of a teacher candidate's identity, we know little from the literatures about how teacher candidates construct their professional identity within complex contexts. Specifically, little emphasis was placed how mentoring potential teachers contributes to their construction of their teaching identity. In addition, there are few longitudinal studies that examine the effects of preparation of teachers over time. Although a teacher's transition is a continuous trajectory of learning and growing, the research is rather fragmented and isolated.

In summary, my study explores how novice teachers construct their professional identity of themselves as mathematics teachers as they move from their earlier perceptions of themselves as teachers to newer models of teachers within a longitudinal framework. Specifically, their struggle to develop their self-image as a mathematics teacher between two competing models is explored.

Theoretical perspective

I adopted a situative perspective because this perspective allows me to focus on a contextualized individual interacting in particular settings that have been shaped by social variables. Relying on Eynedy (2006) and Wenger (1998), I interpreted the identity as identity is an on-going process and a persons' identity is shaped and negotiated through everyday activities including their personal history. It means novice teacher's prior beliefs, knowledge, and experiences are important factors of identity construction. In specific, this study focused on two constructs that are particularly relevant to teachers' instructional practice: professional identity and the identity developing process. The construction of professional identity here means conception of how one defines oneself as a mathematics teacher in a negotiation with other participants in the same community. How relationships in this community develop is important because student teachers experience various relationships as they move along their path of learning how to teach.

In order to understand identity construction, novice teachers' personal history involving their prior experiences, relationship with cooperating teachers, and mentor's teaching practice were explored. All of these factors have been considered to contribute to a teacher's identity development. The following questions will be addressed,

1. What are the changes experienced by a prospective teacher during his or her identity development while transitioning from a university method program to their student teaching?
2. What are the critical experiences, knowledge and skills that contribute to the change?
3. How does the context of a teacher education program impact the identities and teaching practices?

Methods

The current study is part of an ongoing, longitudinal study of prospective teachers' identity development as a mathematics teacher. Drawing on the strengths of qualitative research (Merriam, 1998; Stake, 1995; Yin, 1984) case study method is utilized. The case study method will highlight the complex social context that attributes to development of teacher identity in a real-world setting.

Participants include five students enrolled in an undergraduate mathematics course that was offered by a state university. One of the participants was male and the remaining four were female. The investigator has followed beginning elementary school teachers from their last year of their mathematics method courses into their student teaching period, and will continue to follow them into their first year of full-time teaching. The method course designed to have students experience principles and ideas from NCTM standards. Within that framework, instructor specifically focused on students' thinking on learning and teaching mathematics.

Data collection for this case study consisted of interviews with student-teachers and their mentors, as well as their journals and direct classroom observations. During the method course, I observed all the classes for 16 weeks between August and December 2009, and wrote an observation field note each time to record how teacher candidates participated in the classroom. During the student teaching period between January and May 2010, I observed five participants' mathematics classrooms once a week. I interviewed all five participants regarding their prior math related experiences and experiences about their current teacher education program. All the interviews were audio-recorded and transcribed for analysis.

Analysis

Pattern coding from Miles and Huberman (1994) was utilized. Pattern matching of the collected evidence from field note observations and interview transcripts was conducted. Common themes were categorized, such as, prior school experiences, experiences of method courses, mentor relationships, impact of field experiences, their teaching philosophy and their current understanding of teaching practices (see table 1).

Table 1 : Analysis theme of interview and observation

Collected data	How does this relate to identity construction
<Interview> ►Prior experience -mathematical background -teacher education program	• Story telling is a part of identity (Drake, 2006), • personal history is important on identity construction (Wenger,1998) • How this program influence participant's prior belief of teaching mathematics
►Current social context - relationship with mentor and mentor's teaching practice	• How their identity is practiced by the master teacher (Wenger, 1985) • How they interpret/evaluate master's teaching practice? How is this model of teaching close to their image of mathematics teacher they want to be? (Brown et al, 1999) • degree of autonomy and participation is important (Wenger, 1985) • what they learn during this experiences
<Classroom Observation> ►Teaching Practice	• How does student teachers' teaching mathematics look compared to their mentor's teaching? • What is the basis of the major resources that student teachers use to teach mathematics? • How is their teaching practice close to their image of mathematics teacher they want to be?

The Survey of Professional Identity as Mathematics Teacher Survey

In order to grasp the overall view of five pre-service teachers' professional identity as mathematics teachers, participants were asked to do a quick survey of how they see themselves between two different teaching images: 1) traditional and 2) reform on a scale from 1to 10 where 1indicates endorsing more of a traditional model and 10 indicates endorsing more of a reform model. Before asking this self-reported survey interviewer discussed what the participants think about traditional and reform based image of teaching practice. This scale informed how these teacher candidates interpreted two different teaching images of modeled teachers (the same university instructor and different mentors) and how they viewed themselves between these models. The scale is reviewed in the following section.

Teachers from own experiences in general
Method course instructor
Yourself (at the end of field experience)
Current mentor teacher
Ideal goal (when full-time teaching)

1 2 3 4 5 6 7 8 9 10
Traditional Reform

Mentor Relationship

Under the mentor relationship, two areas were focused; student teacher's autonomy level and opportunity to teach mathematics. I considered autonomy level is important because even though they wanted to adopt newly learned method from teacher education program they are may not allowed using them and instead, they are supposed to follow mentor's teaching practice whether they like it or not. Student teachers' autonomy level was categorized as high, medium and low based on how much responsibility they had to prepare their lesson. The definition of each category is the following.

• High autonomy: As long as it covers standards mentor teacher let student teacher have full responsibility of preparing and teaching materials. Mentor's feedback is supplementary. Student teachers take over the math class from the beginning of student teaching and more than 90 % of observation time, student teachers teach the whole math class.

• Medium autonomy: Mentor and student teacher plan lesson together and teaching practices are integrated. Mentor teacher lead the math lesson and student teachers' participation is gradually increased.

• Low autonomy: Mentor teachers have full responsibility of planning and curriculum and students teachers adopt them. Student teachers take over the math class at the end of semester and their role in teaching math is supplementary. Less than 20 % of observation times, student teachers teach the whole math lesson.

Classroom Observation (Student Teacher's Teaching Practice)

During the classroom observation period, the major focus was placed on mentor's teaching practice and student teaching practice. Observation teaching practice is critical in this study because it provides what type of teaching practice mentor modeled to their student teachers and how student teachers adopt such model of teaching. In addition, it illustrates understanding of how their identity as a math teacher is practiced in a real classroom situation, student teacher's decision making, what resources they utilize for their teaching, their teaching pedagogy between two models (reform and traditional) compared to their beliefs about teaching mathematics.

Result

Table 2 shows the survey result of overall view of five pre-service teachers' professional identity as mathematics teachers.

Table 2 *Results From a Survey of Teacher Image*

Survey questions	*Participant ratings				
	Meg	Mike	Vicky	Kerry	Jackie
Prior math experiences 1) Successful in math? 2) Confident teaching math	1) Yes 1) Yes	1)Yes 2)Yes	1) No 2) No	1) Yes 2) Yes	1)No 2)No
How do you see math teachers from your own school experiences in general	5	3	2	6	3
How do you see the instructor of university method course?	10	8	9	6	10
How do you position your mentor as a mathematics teacher?	3	7	5	9	8
How do you position yourself as a mathematics teacher?	7	4	4	8	5
What is your ideal goal to reach when you move into your own career?	9	6	6	9	10

Note: All of the participant names are pseudonym and they are using the rating scale from above.

In essence, the results of this survey indicate that most of the participants challenged their initial beliefs about being mathematics teacher throughout their teacher education program, and their teacher identity evolved. However, one participant didn't endorse the reform pedagogy model as much as the other participants did.

Similar to a study conducted by Ball (1990), the majority of the participants in this study reported traditional experiences concerning their own school's mathematics teacher (M=3.8). All of participants agreed in the interview that the university instructor's teaching practice challenged their earlier image of mathematics teacher. When they moved to field experiences, 3 student teachers reported that their mentor's teaching was closer to reform pedagogy and the remaining two considered their mentor's teaching practices as more traditional. Compared to their mentor's teaching practices, all but Meg, considered themselves as being less reform pedagogy type teachers than their mentors. Meg said during the interview that she considered her mentor's teaching as more traditional than herself. When asked their ideal goal of what their self-image as a teacher might look like, most of the teacher candidates (80%) but Meg wanted to endorse more of a reform pedagogy style than what their mentors do in the classroom.

To better understand teacher candidates' identity as a mathematics teacher it is necessary to investigate relationship with mentor's teaching, how they teach mathematics and student teachers' autonomy level. The following table is the brief summary of participant's overall experiences of student teaching.

Table 3: Overall summary of student teacher's experiences during student teaching

	Meg	Mike	Vicky	Kerry	Jackie
Confidence in math	Yes	Yes	No	Yes	No
Success in math	Yes	Yes	No	Yes	No
Mentor type	traditional	traditional	reform	reform	reform
Self image of themselves	reform	traditional	traditional	reform	between
Grade level	2^{nd}	5^{th}	5^{th}	5^{th}	5^{th}
Autonomy level	High	High	Low	Medium	Low
Opportunity of Teaching mathematics	Take over math class from the beginning	Take over math class from the beginning	Take over at the end, relatively less chance compared to other participants	Gradually take over	Take over at the end, relatively less chance compared to other participants

In sum, here is a summary of each student's professional identity development.

<u>Meg</u>: she was confident in mathematics and she wanted to become a very reform based teacher (scale 9). Her mentor teacher was holding a traditional teaching pedagogy. But, as her autonomy level was high she was able to practice her identity as a mathematics teacher as she wanted to be. It is very often observed that the student teacher utilized materials from university program, which she evaluated as very reform, and focused on group work, discussions, games and sharing.

<u>Mike</u>: He, similar to Katie, was confident in mathematics and he evaluated that his mentor teacher hold a traditional view of teaching mathematics. However, from his own experiences, he adopted more traditional way of teaching than reform pedagogy. Thus, rather than taking reform pedagogy his goal of teaching practice was in the middle of traditional and reform. During the observation time, his teaching mathematics seems to closer to the traditional way such as working on worksheets all the time, time task, procedure oriented, and teacher directed.

<u>Vicky</u>: Unlike Meg or Mike, Vicky was not confident and successful in math during her own school experiences. During student teaching experiences, she was practiced by master whose teaching was evaluated as in between. Compared to other participants, she had little experience of teaching mathematics to her students. In the interview, Vicky said she loved what she learned from methods class and what her mentor is teaching in the class.

However, it seems that her lack of confidence made her uncomfortable of trying something not familiar with her.

Kerry: Her prior experience with mathematics was positive and her teaching pedagogy was not either traditional or reform. Her mentor was experienced and reform based teachers who took a considerable amount of Cognitive Guided Instruction (CGI) training from the district. Kerry and her mentor planned the lesson together and Kerry started teaching a part of math at the beginning and expanded her role to the whole lesson. Kerry told during the interview that she absolutely valued her mentor's teaching practice so she adopted most of her mentor's teaching practice.

Jackie: Similar to Vicky, she was not confident and successful in math during her own school experiences and she was practiced by master whose teaching was evaluated as reform.. Vicky said she loved what she learned from methods class and what her mentor is teaching in the class. She was the one who expressed the highest desire to become a reform based teacher. However, Jackie also had little experience of teaching mathematics as Vicky did. In the interview, Jackie said math would be the last subject to take over because she was not confident teaching it. Her lack of confidence of teaching mathematics appears to be an obstacle of practicing her identity as a reform teacher.

Contributing Factors

Prior Experiences

In this study, pre-service teachers' prior experiences didn't seem to be a great factor that contributed to construction of identity. Four participants reported their initial perceptions of mathematics teacher (more traditional) were changed into more reform, except one participant, Mike. For this participant, mathematics was his favorite subject and his best math teacher was the one who taught math in a traditional way. Drawing on his positive perspective to seeing a traditional teaching practice from his high school, he stated in his interview that he wanted to adopt a more traditional teaching practice. Specifically, he stated:

> My best math teacher was in high school, Russian man. He always tells us you shouldn't care why you do, just do it what I say and you would pass the test. That is what I did… This is how I see it and how I succeeded it and what I am trying to pass on.

Teacher Education Program

Among various variables that relate to identity construction, the university instructor appeared to be the greatest influence on their identity change. All of the participants replied in the interview that their university instructor brought different and innovative perspectives to the teaching experiences in mathematics. As shown in the table, students considered her as a very reform mathematics teacher, indicated by rating of the scale above (M=9). However, Mike, who adopted the least from teacher education program, stated during the interview that even though he liked the teaching methods provided by the instructor, he thought it was not grade appropriate for him. He considered using manipulatives are more suitable for primary grade such as K-3. When teaching 5th grade he thought that such methods wouldn't be appropriate for his students.

Relationships with Mentor

Drawing on apprenticeship metaphors from Lave and Wenger (1991), I particularly focused on student teacher's authority level, their negotiation process, and how mentor's modeled teaching influenced to novices' teaching practices. Regardless of level of autonomy, all of the student teachers stated that they had a certain level of autonomy, but they wanted to respect their mentor's teaching style. Also they wanted to rely on mentor's current teaching materials as well as adopted their teaching strategies. This result indicated that mentor's teaching practices has a greater potential for modifying and re-constructing student teachers 'professional identity and teaching practices.

Conclusion and Discussion

This analysis has shown that prospective teachers developed identities of what it means to be a what so called 'traditional' or 'reform' mathematics teacher through the teacher education program. Rather than prospective teacher's prior beliefs, the reform pedagogy of university instructor has influenced the most on their professional identity. When novices entered field experiences leaving from reform world, their masters' teaching model was various. The result indicated that no matter of masters' teaching model, all of my participant teachers stated that they considered mentors' teaching practices as major recourses. In addition, student teachers who had high autonomy level of teaching seem to have more opportunity to practice what they believed and what they wanted to try. It implies that how mentors support their student teachers is also a factor that influences novice teacher's initial teaching practices. It brings an attention that we need to pay more attention on mentoring program and also consider placement of mentors as a critical potentials in order for them to construct professional identity in ways consistent with the standards reform movement.

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TRANSFORMATIVE TASKS IN ALGEBRA TEACHER PREPARATION

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This paper reports on a study using algebra-tile tasks in the mathematics methods course to develop prospective secondary mathematics teachers' algebra knowledge for teaching.

Introduction

What is the nature and role of tasks in mathematics teacher education? The tenth-anniversary volume of the *Journal of Mathematics Teacher Education (JMTE,* December, 2007) targeted this under-represented area of research asking: What impact do the tasks of mathematics teacher education have on prospective teachers' mathematical and pedagogical understanding? Guest editors of the volume, Zaslavsky, Watson, and Mason, proposed a lexicon of tasks to frame this emerging area of research. Among the general task types highlighted were mathematical explorations, cognitive conflict, classification and analogical reasoning, examples and counter-examples, alternative and idiosyncratic algorithms, symbols and multiple representations, alternative problem-solving strategies, mathematical connections, use of technological tools, and inductive and deductive reasoning. Algebra researchers in this volume further examined the efficacy of concept-map (Schoenfeld et al, 2007) and big-ideas (Hsu et al, 2007) tasks to measure change in prospective teachers' understanding of algebra. Results of the research on big-ideas tasks unearthed the difficulties prospective teachers often experience when called upon to organize the ideas of a subject (in this case, algebra) into a network of interconnected notions so that, having taught one big idea (e.g., distributive law), other ideas (e.g., combining like terms) can be logically deduced from the initial "big idea." The concept-map studies show these tasks to be effective on two counts: first, as pre-and-post assessments of prospective teachers' mathematical understanding, and second, as cognitive tools for pre-service teachers to organize their thinking about a topic when planning a lesson. Interestingly, then, the research on concept-map tasks resolves, at least in part, the learning-to-teach dilemma that research on "big ideas" tasks unearths in a single volume of JMTE.

As these examples suggest, tasks research is an important source of insight into pre-service teacher understanding of the mathematics they are preparing to teach. The goal, therefore, in advancing this line of inquiry is to lay the framework for the creation of instructional materials and strategies that can help pre-service teachers improve their understanding and teaching of mathematics. In this paper, I focus specifically on task research in algebra teacher preparation by investigating the impact of one manipulative-modeling task, *The Imbalance Task,* on prospective secondary teachers' understanding of school algebra. Aiming to contribute both empirically and theoretically to algebra teacher preparation, this work is guided by the following two research questions: (1) What tasks promote algebra concept development in a secondary mathematics methods class? (2) How can we measure their development? In what follows, I explain the analytic framework that will be used to measure the quality of pre-service teachers' algebra knowledge; describe the study participants, data collection, and coding methods; illustrate *The Imbalance Task* which is the focus of the study; and analyze the enhanced algebraic insight that prospective teachers report from doing the imbalance task. The paper concludes with reflections on algebra teacher preparation tasks.

Theoretical Framework

Measures of Algebra Understanding

Guided by the framework of Perkins and Simmons (1988), I characterize deep knowledge (or understanding) of any algebra topic in this study as consisting of five interlocked types of knowledge (content, concept, problem solving, epistemic, and inquiry). The *content-level* of understanding refers to the vocabulary, skills, and isolated bits of information typically connected with a discipline or school subject. Characteristic performances that indicate this type of understanding are rote memorization and recall of previously learned material. The content-level is a superficial kind of understanding akin to Skemp's *instrumental understanding* (i.e., knowing the rules and procedures of mathematics without understanding the reasons for them).

The remaining three, non-content levels of understanding are akin to what Skemp called *relational understanding,* but differentiated by the focus of knowledge (concepts, problem solving, epistemic, or inquiry). In particular, the concept-level of understanding refers to a contextual understanding of what is typically called vocabulary. Concepts, according to Schwab (1967), are the key orienting ideas of a discipline (e.g., variable, function, equality, or rate of change in mathematics). The problem-solving level of understanding refers to (1) the heuristic ability to monitor and reflect on one's own thinking and (2) the procedural ability to execute on demand specific problem-solving strategies such as find-a-pattern, make-a-table, or induction. Epistemic-level understanding relates to the evidence and warrants for claims in a discipline. Algebra students exhibit understanding at the epistemic-level when they construct arguments and provide mathematical reasons to support their thinking. Inquiry-level understanding refers to independent exploration beyond the problem-solving level. At the inquiry level, students pose problems and implement strategies to resolve them with little assistance from a teacher or guide. What is resolved at the inquiry level is more complex than puzzles resolved at the problem-solving level.

Using this framework, I will analyze the changes in algebraic understanding that prospective teachers report as resulting from the imbalance task (Kinach, 2002).

Participants, Data, and Methodology

Data for this report were collected from a five-year longitudinal study of prospective teacher learning in the author's secondary mathematics methods course. Taught annually by the author, the course typically enrolled three to five students. Prospective teachers' written coursework, video of methods course discussions, and pre-service teachers' post-course responses to the following prompt constitute the database for the current study.

> Throughout the course, we explored ways to make mathematics meaningful for middle-grades and high-school students. *Teaching with manipulatives* (e.g., algebra tiles, attribute blocks, geoboards, etc.) was one of the teaching strategies we investigated. In the course of learning to teach with these materials, many of you commented that your own mathematical understanding was deepened. Describe (if possible) five instances in the course when your own mathematical insight grew as a result of your modeling work with manipulatives.

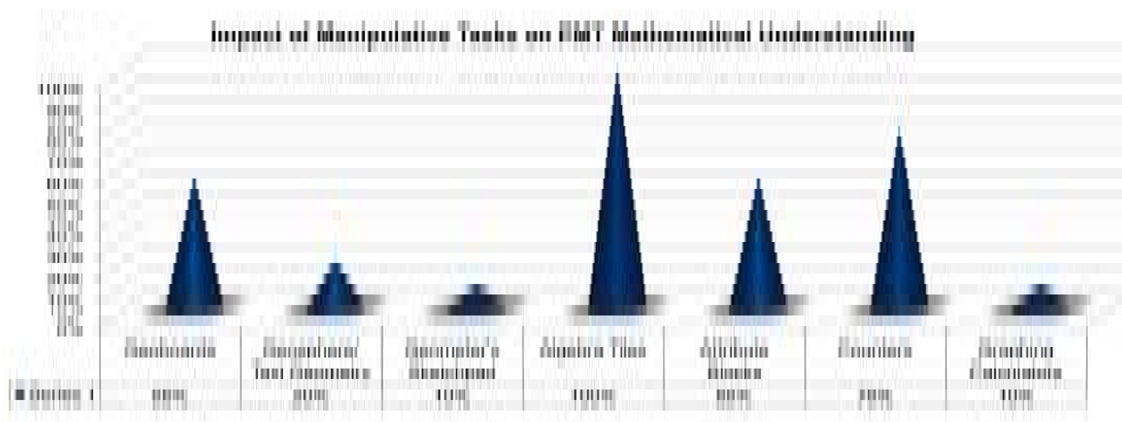
Using a qualitative design, data were first coded by manipulative type and task, and then sub-coded using the levels-of-understanding framework. Before describing and analyzing results for "The Imbalance Task" specifically, I summarize results from the larger study of manipulative modeling tasks and their impact on the mathematical understanding of methods students.

Results

Impact of Manipulative Modeling Tasks in the Larger Study

Twenty prospective mathematics teachers (PMT) responded to the above prompt resulting in 100 instances of self-reported growth in mathematical understanding from participating in ten manipulative-modeling method course tasks (Table 1). Collectively, the tasks employ seven manipulatives (algebra tiles, geoboard, clay/geosphere/taxi-geometry grid, counters, attribute blocks, graphing calculators, and dynamic geometry software called *The Geometer's Sketchpad)*.

Table 1



Algebra-tile modeling tasks scored the most instances of reported gains in mathematical understanding (Table 2). Of twenty prospective teachers, 100% report enhanced understanding at the concept level of the multiple meanings of the symbols "-" and "+". Next in rank was the imbalance task. 90% of the pre-service teachers reported knowledge gains at the concept level related to the equal sign, equation, variable, and at the epistemic level related to justification for the processes of equation solving, transposition, and algebraic factoring.

Table 2: End-of-Course Reported Gains in Algebra Understanding from Methods Course Manipulative-Modeling Activities

1) Algebra Tile Manipulative Modeling Tasks			
a. *The Imbalance Task*	**Form of Knowledge**	**Freq**	**Percent** (n=20)
Enhanced understanding of . . .			
equal sign	concept	18	90
equation	concept	18	90
variable	concept	18	90
equation solving	epistemic	18	90
justification for transposition	epistemic	18	90
b. *Two-Color Chip Task*			
Enhanced understanding of . . .			
Operation vs. signed number	concept	20	100
subtraction vs. addition	concept	20	100
addition vs. positive	concept	20	100
subtract, negative, opposite	concept	20	100
c. *Area Model of Factoring/Multiplication*			
Enhanced understanding of . . .			
Algebraic factoring	concept/problem solving	11	55
Area	concept	11	55
Area formulas	epistemic	11	55

The area model of multiplication tasks with algebra tiles produced new insights at the concept level into the connection between numerical and algebraic factoring (55%). Mathematical reasoning tasks with the ESS attribute blocks generated both concept and epistemic level gains in understanding about logic, Venn diagrams, and critical thinking in the mathematics classroom (55%). For the geoboard tasks, pre-service teachers reported concept-level insight into area and epistemic-level insight into the origin of area formulas and the Pythagorean Theorem (55%). Less frequently mentioned were epistemic-level insights into the relationship between the shape of space and definitions of different geometric objects (4%) from modeling tasks with the geosphere, clay, and taxi geometry grid.

Perhaps the most surprising result was the gain in concept- and epistemic-level understanding that these pre-service secondary mathematics teachers reported from using counters to represent whole-number division. Prospective teachers reported surprising new insights into the meaning of division (concept level) and justifications for division by zero (epistemic level). Fifteen prospective teachers (75%) commented that they had never encountered the mathematical rationale for division by zero during their mathematical education. The biggest revelation for prospective teachers was the epistemic insight that different mathematical reasons (or principles) justify why 5/0 and 0/0 are undefined.

Description of The Imbalance Task

Figure 1: The Imbalance Task

The Imbalance Task presents the classic "balance beam" analogy to solving equations in a new light as an inequality. The aim of the task is to stimulate prospective teacher thinking about solving equations in order to better understand the cognitive puzzles and challenges of young algebra students. The task relates a new idea of how the concept of solving equations algebraically can be solved geometrically in terms of an inequality and physically constructed visual of an unbalanced scale.

The Imbalance Task (Figure 1) asks prospective teachers to determine the number of small squares that the strip would have to equal for the scale to balance. Methods students construct the diagram in Figure 1 using the small squares (dimensions 1 by 1) and rectangular strips (dimensions 1 by *x*) from the student-set of algebra tiles (www.etacuisenaire.com). Note that the dimensions of the small square and strip are manufactured deliberately so that the strip does not equal an integral number of squares (i.e., $x \notin$ integers). The intention is for the strip to represent a variable capable of taking on different values despite the fact that the plastic model has a fixed length and width. Students using the tiles are, thus, to imagine the strip (*x*) as capable of varying in value. The value of the small square is 1.

By engaging familiar material in new ways, prospective teachers report deeper concept- and epistemic-level understanding of basic algebraic concepts and processes such as equation, equation solving, variable, and the equal sign. Next I analyze the changes in understanding that 90% of the prospective teachers reported from doing the imbalance task.

Analysis of Knowledge-Change Reports for the Imbalance Task

Insight 1: Meaning of the equal sign. About the equal sign, one methods student's comment is representative:

In working with the balances and algebra tiles, I had new insights into what the equal sign means – that both sides have the same weight or have equal quantities. I previously conceived it (the equal sign) as a do something symbol, even if it meant the "do-something" was to manipulate it to solve for a variable. Also this gave new ideas on what it was to solve an equation (choice of balance or imbalance).

Prospective teacher reflections like these suggest that despite advanced undergraduate mathematics study, foundational misconceptions persist at the concept level about the equal sign. Building upon Kieran's (1981) findings about how students misconstrue the equal sign in arithmetic, this result suggests further investigation of algebra students' concept-level understanding of the equality symbol and its meaning in equations. Is it common for advanced mathematics students at the college level to conceptualize the equal symbol in equations as a signal to solve for the variable? This bears further investigation as 90% of study participants held this view.

Insight 2: Equation and equation solving. The idiosyncratic representation of an equation as a scale out of balance generated many rich reflections about linear equations and the process of solving equations. Several representative reports illustrate the ways in which prospective teachers symbolic understanding of equations and equation solving were deepened through the idiosyncratic approach to thinking about these basic algebraic concepts and processes. One methods student wrote:

The balance model for equations introduced another concept that I had never thought twice about. I had always assumed that an equal sign meant equal, always equal. This model could treat equations in this way or it could present an equation as unbalanced or unequal, and it needed help to get balanced. I thought of an equation as a statement not a problem. This made me think of how many other ideas/concepts in math could have double meanings for my students. I realized that I needed to always try to understand and communicate better with them.

Another teacher candidate wrote:

A fourth insight occurred with the discussion of the meaning of an equation in terms of the balance model. I had always assumed that an equation was "balanced" to begin with, unless proved otherwise. It was a new insight for me to see how it could be interpreted as being unbalanced, with the process of solving equivalent to balancing the scale. This brought out the point that in mathematics, there is not necessarily one right way of looking at something. Sometimes insights are made just by approaching a problem from a different point of view. As a teacher, I want to encourage divergent thinking. I don't want my students to believe that there is only one way to reach an answer. This is not true to the actual "doing" of mathematics.

When prospective teachers operate on the scale by adding or subtracting equal quantities from both sides of the pan balance, they treat the equation as an entity or object in itself and exhibit a structural conception of equation (Sfard, 1991). The alternative procedural conception of equation would be present if students solved the equation by substitution. By inserting different numerical values of the variable into each side of the equation, numerical equality is eventually achieved (or not). This approach to solving linear equations treats each side of the equation as an algebraic expression whose numerical value can be determined by calculating the numerical value of the algebraic expression for different values of the variable. While all prospective teachers in this study exhibited concept-level understanding of equation solving of the structural kind, not all possessed epistemic-level understanding their equation-solving procedures. As the following example dramatically illustrates, justification of equation solving procedures was problematic:

When I learned equations, I learned that you want to get all similar things on one side. To do this, I was told that the equal sign is the moon and that to go from one side to the other side of the equal sign, you jump over the moon, changing the sign of the number or variable as you jump. I know now that what I learned was a trick. The balance idea for solving equations helped me see that the trick I learned was actually a substitute for showing me that what I should do is subtract, add, multiply, or divide equal amounts. I learned that if you treat the equal sign as a balance and solve the problem keeping the balance equal, then at the end you know what a strip will represent in terms of squares because they will be in balance.

Clearly, this student had learned to manipulate linear equations to find the solution without knowing the mathematical (epistemic-level) principles behind the symbolic manipulations. What she describes is the process historically called *al-jabbr,* or transposition. This is an equation-solving method that utilizes the notion of inverse operations to solve equations. Another methods student describes the epistemic-level understanding she gained to explain why transposition procedures work:

The balance model of an equation showed me a great way to understand why we "take" a number's opposite to the other side. Really, we are either adding or subtracting the same thing from both sides.

Insight 3: Different conceptions of the variable. By tilting the scale, this model of a linear equation emphasizes the point that the two functions under consideration are equal only for certain values of the variable. Prior to this activity, prospective teachers report that they missed the (obvious) concept-level idea that a variable *varies* in value. On this point, one teacher wrote:

What does the strip have to represent to balance the scale? This question showed me a deeper insight into the meaning of variables. Before I thought them to merely be representations of specific values. This exercise drove home the point that variables can take on many values, each having an effect on the balance.

Implications for Algebra Teacher Preparation

In this study, The Imbalance Task was transformative. It expanded prospective teachers' content-level understanding of basic school algebra concepts and processes (equal sign, equation, variable, equation solving) to include epistemic and concept level understanding of these ideas. Although the data set is small, the strong results warrant further research on this task and its effectiveness for enhancing pre-service teachers' algebra concept development in the methods course. The analytic framework used in this study emphasized the focus of knowledge (i.e., concepts, problem solving strategies, justificatory rationales) and not the form of knowledge (e.g., strategies or beliefs). This emphasis was useful for identifying the gaps in understanding that these advanced mathematics students brought to the process of learning to teach in the methods course. Aspects of this task design that deserve the attention of future researchers include the juxtaposition of symbolic (algebraic) and spatial (geometric) representations of mathematical concepts, processes, or rationales. Also worthy of more attention by designers of tasks for algebra teacher preparation are the task's employment of cognitive dissonance, analogical reasoning, and idiosyncratic approaches to familiar mathematical material.

Prior to the imbalance task, prospective teachers universally conceptualized an equation is a *statement* of equality (which it is). The unbalanced scale, however, suggested an alternative interpretation of an equation (e.g., $5x + 3 = 4x + 8$) as a *question* about two expressions ($5x + 3$ and $4x + 8$) or about two functions ($y= 5x + 3$ and $y = 4x+ 8$) and the conditions under which they will be equivalent. The idiosyncratic approach generated unexpected insights into basic algebraic notions. Future research on the affordances of such tasks especially in situations where years of proficient procedural practice hides from prospective teachers the underlying conceptual meaning and epistemic rationale for algebraic concepts and procedures learned long ago.

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THE IMPACT OF MATHEMATICS TEACHER EDUCATION RESEARCH ON PRE-SERVICE TEACHER EDUCATION

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The "research-to-practice" relationship in mathematics education and in education more generally has been examined from a variety of perspectives. This research examined the research-to-practice relationship of mathematics teacher educators. Using the Mathematics Teacher Educator Instrument, developed for this project that addresses 11 dominant areas of research in the field, course syllabi were examined from Australia, Canada, England, Ireland, Italy, Malaysia, New Zealand, and the US. Our results suggest that major bodies of research are only moderately represented on course syllabi. Recognizably, the research areas identified on the instrument may nevertheless be enacted in mathematics teacher education classroom. Potential implications for mathematics teacher educators are discussed.

Introduction

Of great interest over the last decade have been the relationship, and often the gap, between research and practice in education (NCTM Research Committee, 2007). Numerous and robust research has emerged within the field of mathematics teacher education and mathematics education. Dominant research themes include: reflective practice, mathematical tasks, lesson study, assessment and evaluation, theory and practice connections, policy and politics of mathematics education, technology, beliefs and attitudes, content knowledge, pedagogical knowledge, and equity and diversity.

Our interest in this research is to examine the extent to which research is informing the practices of mathematics teacher educators, and thus the way in which pre-service teachers learn to teach mathematics. To examine research-to-practice in mathematics teacher education we analyzed pre-service mathematics teacher education course syllabi, provided to us from mathematics teacher educators from around the world. The question framing this research is: *How is research reflected in mathematics teacher educators' practices, as evidenced in course syllabi?* The aim of this paper is to take a purposive sample from the syllabi collected in order to evaluate the instrument developed for this research.

Theoretical Framework

According to the Merriam-Webster's dictionary, *syllabus* is defined as a summary, a course of study, or an outline. Matejka and Kurke (1994) proposed that there are four key functions of a syllabus: a legal agreement between the instructor and the student, the student and the university, and the instructor and the university; a communication device regarding the learning outcomes and goals of a program of study; a plan or description of the events to occur within the course; a cognitive map, outlining the way in which knowledge will be shaped by the content of the course.

Baker (2001) suggested that "the syllabus may have begun as a modern administrative overlay, but it has become a powerful teaching tool, … used to communicate important aspects of the course to students" (p. 391). We recognized that what is presented on the syllabi may be vastly different from what is enacted in the classroom. In many regions, the course syllabus, is as Matejka and Kurke (1994) suggested – a legal contract, a plan, a communication tool, and a cognitive map. As such, the details presented in a syllabus are at the very least a good starting point for examining the extent to which research is informing the practice of mathematics teacher educators – but may not present the full picture.

Very few studies exist that examine mathematics teacher education syllabi from a global or regional perspective. The studies that do exist largely focused on common elements (Taylor & Ronau, 2006), programmatic goals reflected in syllabi (Harrington & Enochs, 2009), and also tended to be regionally situated. We found no studies that explored the relationship between research and practice from the mathematics teacher educator's perspective. This research aims to fill this gap.

Relevant Literature

Reflection

Prospective teachers need opportunities to learn how and to engage in thinking about their teaching. Artzt and Armour-Thomas (2002) contended that for teachers to develop in their practice, they must engage in reflection before, during, and after implementing a lesson. In the same way, prospective teachers must participate in the various forms of reflection throughout their teacher preparation program (Chapman, 2008). Three different types of reflection have been proposed: a priori reflection, where a pre-service teacher thinks in advance about a topic; initeri, where a pre-service teacher engaging in reflection in the moment of a task or teaching; and a posteri, where reflection upon an action, lesson, or task occurs after the fact (Clark, Kotsopoulos, & Morselli, 2009).

Mathematical Tasks

As Watson and Sullivan (2008) noted, we may distinguish between "classroom tasks", that are "questions, situations and instructions teachers might use when teaching students", and "tasks for teachers", that are "the mathematical prompts, many of which are classroom tasks, that are used as part of teacher learning" (p. 109). The authors observed that tasks for teachers also serve different purposes. Consequently, the understanding promoted by mathematical tasks in teacher education is a blend of the pedagogical and mathematical, with an emphasis on one dimension or the other, depending on the way the teacher educator implements the tasks.

Lesson Study

Adopted from practices of Japanese teachers, lesson study in pre-service teacher education has been has been widely embraced internationally (Fernandez, 2005). Lesson study is a process aimed at improving teaching practices by involving groups of teachers collaborating and reflecting upon lesson plans developed jointly, and then taught by one or more teachers while the other teachers observe. Studies have shown that lesson study enhances the pedagogical practices of teachers, the development of pedagogical practices amongst pre-service teachers, and also improves student learning outcomes (Post & Varoz, 2008).

Assessment

Generally accepted in mathematics education is the need for pre-service teachers to engage in assessment practices through the act of analyzing pupil level mathematization. Studies in the area of assessment practices for pre-service teachers address various issues including the preconceived notions that affect pre-service teachers' evaluative skills (Morris, 2007) and the affects of pre-service teachers' traditional conceptions of mathematics assessment opposed to newer assessment reform ideas. The act of analyzing pupil level mathematization gives preservice teachers the opportunity to experience authentic assessment practices that encourage and support further learning.

Theory and Practice

Numerous scholars have identified the extreme challenges and great necessity in connecting theory and practice for pre-service teachers and practicing teachers (Jaworski, 2006). The need for connecting theory or research to practice stems from the concern that practices of teachers are not stimulated or perhaps supported by research, and are increasingly required to engage in evidenced-based discourses in their practice. Research in this area considers the ways in which pre-service teachers make connections between research and practice when learning to teach (McDonnough & Matkins, 2010), the consumption of research in the pedagogical approaches of practicing teachers (Mathern & Hansen, 2007), and theoretical perspectives (i.e., challenges with making connections; ways to occasion connections) (English, 2003; Heid et al., 2006).

Policy and Politics

Policy documents pervade mathematics education, particularly in the form of curriculum and teaching standards and guidelines for teacher preparation programs (NCTM, 2000; NCATE, 2003). Although these documents are US-specific, parallel documents drive what is taught and by whom throughout the world. In this light, mathematics education is considered a highly political act, to the point that the dominant population holds the power and "policy and most research about diversity, cultural pluralism and a Eurocentric curriculum evolve around this conception of power" (Popkewitz, 2004, p. 254).

Equity and Diversity

Widely documented in mathematics education is the underachievement and marginalization of ethnically, racially, linguistically, academically, or sexually diverse students (Lee, 2002; Richardson, 2009). There have been numerous factors identified as contributing to the underachievement and marginalization of certain populations of students that include: pedagogical factors (Esmonde, 2009), teacher biases and teacher preparation (Sleeter, 2001). Such research points to the fact that there may be serious deficiencies in the ways in which some students are permitted to learn. Consequently, the way in which teachers are prepared during pre-service teacher education is a central concern.

Affect

Beginning with studies on problem solving and meta-cognition, research on affect has developed further in mathematics education, as evidenced by the growing amount of literature (Leder, Pehkonen, & Törner, 2002; Maasz & Schlöglmann, 2009). Traditionally, three key components of affect are studied: emotions, beliefs, and attitudes (McLeod, 1992). Goldin (2002) also introduced the concept of meta-affect. For our aim, a specific sub-theme is worthy of consideration: teachers' affect, with a particular reference to teachers' beliefs about mathematics and its teaching and learning, seen as a driving force influencing teachers' instructional practice (Philipp, 2007).

Content Knowledge

In the current decade, many scholars have attempted to describe the aspects of a teacher's content knowledge that contributes to their teaching effectiveness. A US report called for mathematics teachers to possess deep understanding of mathematical content at least three years beyond the level they teach and delineates topics within specific content strands (e.g., number and operations; algebra and functions; geometry and measurement) that prospective teachers must know well (CBMS, 2001). Other scholars describe content knowledge for teaching mathematics in less content-driven ways. Ball, Thames, and Phelps (2008) categorized four domains of "mathematical knowledge for teaching" (p. 399), including common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching.

Pedagogical Content Knowledge

Shulman (1986) described the different components of teachers' knowledge. Among them, there is pedagogical content knowledge, a special blend of mathematical and pedagogical and knowledge, described as "an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and background bring with them to the learning of those most frequently taught topics and lessons" (p. 9). As Ball (1990) observed, it is important to help prospective teachers to "learn to do something different from – and better than – what they experienced in mathematics classes" (p. 11). More recently, Even and Tirosh (2008) explored the issue of teacher knowledge and understanding of students' mathematical learning and thinking, as a significant part of pedagogical content knowledge.

Technology

Some of the issues and implications addressed through technology research include: professor demonstrated or modeled use of mathematics technology (Sturdivant, Dunham, & R., 2009), opportunities for pre-service teachers to engage in investigation of mathematics technology (da Ponte, Oliveira, & Varandas, 2002), and authentic implementation of mathematics technology by pre-service teachers (Lin, 2008). Studies have shown that when these three teaching strategies are enacted in pre-service classrooms, the result is a pre-service teacher with maximized and current knowledge and understanding of technology integration in a mathematics classroom (Blubaugh, 2009; Niess, 2001)

Methods

Using email, we solicited the submission of syllabi from colleagues around the world using primarily the listserv mailing lists from the Psychology of Mathematics Education. In total, 146 syllabi were collected which ranged from K-12 to graduate level. We were unable to obtain representation from every continent. This pilot research focuses on a sample of 10 syllabi (one or two each) in total from: Australia, Canada (2 analyzed), England, Ireland, Italy, Malaysia, New Zealand, and the US (two analyzed).

The Mathematics Teacher Education (MTEd) Instrument was developed and used to examine the syllabi (see Appendix A). The MTEd Instrument is a rubric that explores the level in which the research areas mentioned in the introduction are evidenced within a syllabus (see Appendix A for a brief version of the instrument). The level four criteria was an outgrowth of the existing research on optimal recommended learning opportunities for pre-service teachers learning how to teach mathematics (Arbaugh & Taylor, 2008). Low, moderate, and high tags were assigned to a syllabus based upon the cumulative level obtained from the syllabus, determined by adding up the assigned levels from each of the individual research areas identified. Descriptive statistics were computed to summarize overall levels across the 11 research areas analyzed.

Results and Discussion

Mathematical tasks, assessment, belief and attitudes, lesson study, and equity and diversity were the lowest represented on the syllabi (see Table 1). Additionally, assessment had the greatest amount of variability. All but one syllabus achieved an overall level of "moderate" in terms of representation of the research areas on the instrument (see Appendix A for a description of the overall level ranges). One syllabus achieved a "high" level in terms of representation of the research areas. None of the samples analyzed for the pilot achieved a low level in terms of research areas. The mean overall level assigned to the syllabi was 27.7 and the standard deviation was 3.53. The overall level assigned to the syllabi ranged from 22.0 to 34.0.

There is preliminary evidence to suggest that some areas may be particularly overlooked. We note again, that while our pilot analysis of course syllabi showed moderate evidence of research-to-practice connections for mathematics teacher educators, connections may nevertheless exist in classroom practices. However, the absence of evidence has led us to question the importance of course content transparency to students. Numerous scholars have argued that this sort of transparency is essential and indeed may be why some institutions have policy statements regarding course syllabi (Adler, 1999; Baker, 2001; Matejka & Kurke, 1994). It is also important to note that items may be stated on a syllabus and not enacted in the classroom.

Table 1. *Mean levels* (n = 8)

	Mean	Min	Max	STD Dev
Reflection	3.50	2.00	4.00	0.71
Mathematical tasks	1.90	1.00	4.00	0.99
Lesson study	2.20	1.00	4.00	1.32
Assessment	2.00	1.00	4.00	1.25
Theory and practice	2.50	1.00	4.00	1.08
Policy and politics	2.60	2.00	4.00	0.84
Equity and diversity	2.10	1.00	4.00	1.20
Beliefs and attitudes	2.00	1.00	4.00	1.05
Content knowledge	2.80	1.00	4.00	1.22
Pedagogical knowledge	3.00	1.00	4.00	0.94
Technology	2.80	1.00	4.00	1.14

Conclusions

In the pilot sample, research-to-practice was only moderately evident in the syllabi analyzed for our pilot study. The MTEd Instrument appears to be a useful tool for this analysis. Our goal is to now engage in the analysis of the full data set. There are several limitations to this study which include: language barriers existed for some colleagues who may have wanted to participate but could not translate their documents or understand which document we were interested in analyzing, the syllabi provided to us likely does not represent what is actually enacted in the classroom, this research only considered the research-to-practice take-up by mathematics teacher educators. It would be of great interest to explore the way in which the pre-service teachers incorporate (or not) research in their subsequent practice, distinguishing between pre-service teachers who participated in courses that exhibited low, medium, and high research-to-practice evidence within the syllabi. Finally, the MTEd Instrument may need elaboration or revision. We may have missed other major bodies of research that contribute to the field of mathematics teacher education and mathematics education. We also recognize that a wide range of culturally-relevant pedagogies exist and may be unique to a particular mathematics teacher education program. The MTEd Instrument is intentionally not designed to address culturally-relevant pedagogies but could benefit from such elaborations in certain settings.

Appendix A: MTEd Instrument (brief version)

Low evidence of research (less than 22)	**Moderate** evidence of research (total assigned levels from 22 -32)	**High** evidence of research (total assigned levels from 33 -44)

Categories	Level 1	Level 2	Level 3	Level 4
Reflection	**NO** or limited opportunities to engage in a priori, initeri, a posteri reflection.	Opportunities to engage in only **one** type of reflection.	Opportunities to engage in only **two** types of reflection.	Opportunities to engage in a priori, initeri, a posteri reflection.
Mathematical Tasks	No direct engagement with mathematical tasks.	Opportunity to engage only in **either** pupil or pre-service level tasks.	**Some** opportunity to engage in both types of tasks.	**Extensive** opportunity to engage in both types of tasks.
Lesson Study	No lesson planning.	Developing **individual or collaborative** lesson plans that are **not** enacted.	Developing lesson plans **individually or collaboratively** that is presented to the class.	Developing lesson plans **collaboratively**; presented to the class; **and reflected** upon collaboratively or individually.
Assessment	No opportunities to engage in assessment.	**Limited** opportunities to analyze pupil level mathematization.	**Some** opportunities to analyze pupil level mathematization.	**Extensive** opportunities to analyze pupil level mathematization.
Theory and Practice Connections	**No** opportunities to engage with research.	**Limited** opportunities to engage with research through course readings and discussions because of language barriers.	**Some** opportunities to engage with research through course readings and discussions (course is **somewha**t grounded in research and research is evident in the course content).	**Extensive and authentic** opportunities to engage in and with research (course **is** grounded in research and research is evident in the course content).
Policy and Politics of Mathematics Teaching	**No** evidence of any exploration of the political aspects of mathematics education.	**Limited** evidence of exploration of the political aspects of mathematics education.	**Some** evidence of exploration of the political aspects of mathematics education.	**Extensive** evidence of exploration of the political aspects of mathematics education.
Equity and Diversity	**No** evidence of any exploration of the equity and diversity considerations in mathematics education.	**Limited** evidence of such exploration.	**Some** evidence of such exploration.	**Extensive** evidence of exploration of the equity and diversity considerations in mathematics education.
Affect	**No evidence** of addressing the implications of affect on the teaching of mathematics.	Evidence of **addressing** affect in the teaching of mathematics.	Evidence of **addressing and challenging** affect.	Evidence of **addressing, challenging, and potentially changing** affect.
Content Knowledge	**No evidence** of exploration of content knowledge at any level.	Engaging in a **selective component** of content knowledge at the instructional level of the students.	Engaging in content knowledge **at the level of** instruction of the students.	Engaging in broader ranges of content knowledge **beyond the level** of instruction of the students.
Pedagogical Content Knowledge	Didactic methods of introducing pedagogical knowledge (little opportunity for critical analysis).	Interrogating pedagogical strategies (limited opportunity for critical analysis).	Interrogating and developing, pedagogical strategies (some opportunity for critical analysis).	Interrogating, developing, enacting, and analyzing pedagogical strategies (extensive opportunity for critical analysis).
Technology	No evidence of technology integration.	Didactic methods of exploring technology integration.	Evidence of pre-service teacher investigation and implementation.	Extensive evidence and demonstration of technology integration, modeling, and pre-service teacher implementation.

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THEMES FOR VIDEO CASE INTEGRATION IN MATHEMATICS AND SCIENCE

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Video cases have frequently been developed for use in the education of preservice teachers in mathematics and science. While attending to the viewers' predilections, specific content preservice teachers desire in mathematics and science should influence how these cases are created and integrated. This article investigates the content elementary teachers prefer in a video case specific to mathematics and science. Using Personal Construct Theory, constructs were identified and then analyzed using hierarchical cluster analysis. Three primary clusters were discovered with a very small fourth cluster. These results can be used to guide the future creation and integration of mathematics and science video cases.

Introduction

According to Walen and Williams (2000, p. 4) case methodology "refers to a pedagogical method in which learners [such as teachers] examine situations that deal with the practice of that profession." Oftentimes, cases are used in teacher education (Cannings & Talley, 2002) Participants evaluate cases in a whole class university setting to try to anticipate and address issues that may occur and develop a repertoire to address these potential situations (Manouchehri & Enderson, 2003). Case study methodology has been shown to be a powerful component to improving teacher education in mathematics. Cases have helped teachers evaluate and address their classroom concerns, increase knowledge about practice, increase motivation about what encompasses effective teaching strategies and raise awareness of student learning (Manouchehri & Enderson, 2003; Walen & Williams, 2000; Wang & Hartley, 2008). Video cases are a common form of case, and they often designed to help prepare preservice for potential classroom situations while also providing a context for the university theory being addressed (Brophy, 2004).

Video cases have been shown to be advantageous in helping preservice teachers develop into more fruitful observers. In a study by Alsawaie and Alghazo (2010), video cases were shown to improve preservice teachers' ability to analyze mathematics teaching. In their study, an experimental group analyzed 10 video cases in mathematics throughout the semester; the control group did not. It was found that the preservice teachers who analyzed the videos were able to improve their ability to analyze mathematical lesson on two levels. The preservice teachers were able to focus on students' learning more than the control group. Additionally, they were able to identify and comment on noteworthy events within the video case. There was a large gap between the control group and the experimental group in these two areas. In addition, video cases have been shown to improve students' achievement on knowledge based tests (Kim, Sharp & Thompson, 1998; Sariscany & Pettigrew, 1997). And in a study by Stockero (2008), preservice teachers showed more depth to their learning when instructed using video cases. Those who had been instructed using video cases showed more complex thinking patterns in relation to reflective analysis. More specifically, the preservice teachers using video cases began to focus more intently on student thinking and evidence used to justify reasoning.

It is important to have a purpose when integrating video cases. "Although video can provide a concrete and vivid resource for professional study, unfocused viewing is like reading a text without a purpose." (Ball & Even, 2009, p. 256). In order to utilize video cases in the classroom, a purpose must be defined and used to guide its integration. The article addresses this concern by evaluating what preservice teachers want in a video case designed to support mathematics and science instruction at the elementary level. Mathematics and science are examined as these are two areas that preservice teachers have often have problems. Bischoff, Hatch, and Watford (1999) found that preservice teachers were unable to write lesson plans appropriately aligned to national guidelines in mathematics and science; only 1 out of 10 preservice teachers were able to do so. Stevens and Wenner (1996) found that most preservice teachers had very weak understandings of how to teach mathematics and science. The preservice teachers need more content and methodological guidance.

The predilections of elementary preservice teachers in the area of video case use and development in mathematics and science were evaluated. Specifically, if training preservice teachers using video cases while focusing on mathematics and science instruction, what content do they voice as pertinent and relevant to their training? What do they want to see and what supportive information do they want from these cases? Using Personal Construct Theory (Kelly, 1955) themes for the use of video cases were identified. These themes can be used to guide the future creation and integration of video cases in mathematics and science.

Theoretical Framework

Personal Construct Theory (Kelly, 1955) was used to guide the data collection and analysis. Within this theory, each person is viewed as a scientist and he/she makes active distinctions among objects. Elements and constructs are used for analysis. Elements are the items of analysis and constructs are a way of defining similarities and differences among the elements. Constructs are systems of thoughts and they are linked and interrelated to one another. Distinctions are shown mathematically as a multidimensional space where the distances demonstrate how closely related or unrelated constructs are to one another (Walker & Winter, 2007). In this study one element was used, video cases. Through questioning and comparing, preservice teachers identified the constructs. After the constructs were created, they were analyzed in terms of mathematics and science. Because this research is focusing on the use of video cases in mathematics and science, all of the constructs that were statistically significant at the 0.01 level were evaluated using hierarchal clusters to identify requisite themes.

Methodology

Participants

Ninety-three (93) preservice elementary school teachers were the participants of this study. They were from a culturally diverse university in the western United States. Sixty percent of the participants were students attaining a teaching credential in a fifth-year program after completing their Bachelor's degrees. The remaining forty percent were working on the fifth year program while also taking courses for their Bachelor's degree. Eighty-three percent were women and while this is a high percentage of women it parallels the demographics of preservice elementary teachers (Zumwalt & Craig, 2005).

Procedures and Materials

To gather the constructs for analysis, the stages as outlined by Beail (1985) were used to conduct and analyze the repertory grids in this study. In stage one, the elicitation of elements occurred; video cases were the only defined element. In stage two, the elicitation of constructs took place. The preservice teachers answered five different questions that focused on the use of video cases to: prepare for teaching, extend learning beyond what can be done traditionally, contrast traditional teaching, include topics you would like to see and to support preservice teachers' weak areas of preparation. Each of these questions provided feedback for the creation of the repertory grid constructs. Constructs can be supplied by the researcher(s) when using Personal Construct Theory. However for this study, participant-supplied constructs would be more appropriate as they would address the individual predilections of the novice teachers with less researcher bias. As well, elicited constructs provide better data for analysis (Fransella, Bell & Bannister, 2004).

Stage three involved the creation and administration of the repertory grid. This procedure requires the participants to classify and evaluate the given element on a numerical scale based on personal beliefs (Borell, Espwall, Pryce, & Brenner, 2003). Evaluating the preservice teachers' responses in stage two, the constructs were discovered and listed across one dimension of the repertory grid (Walker &Winter, 2007). Then the preservice teachers were asked to rank each construct on a five-point scale (cases demonstrating the given topic would not be helpful (1) to would be extremely helpful (5)). Any preservice teachers who left one or more of the constructs blank were not included in the analysis.

Stage four involved the analysis of the repertory grid, with a specific emphasis on mathematics and science instruction. All constructs were analyzed while emphasizing a statistically significant correlation to both mathematics and science education. Constructs correlating with mathematics or science that were other specific content areas (such as social science or language arts) were not included. The statistically significant construct "the integration of various subjects into a lesson (integrated curriculum)" represented these content areas. The significant constructs were analyzed using hierarchical cluster analysis. The fifth stage was the interpretation of clusters, to be discussed in the next section.

Results and Discussion

Design and Data Analysis

The first analysis looked for correlations between mathematics and science and the other 105 constructs identified. The relationships were determined using Phi correlation coefficients. Overall, there were 32 significant Phi correlations at the 0.01 level or below in reference to mathematics education and science education. These 32 constructs were then categorized into clusters to illustrate themes.

Ward's method was used to analyze the repertory grids with significant correlations of the preservice teachers using hierarchical cluster analysis (Ward, 1963). A Euclidean metric for determining cluster methods is one of the components of Ward's method. It differs from other methods that use a divisive algorithm to find clusters (Fraboni & Cooper, 1989; Fraley & Raftery, 1998). In utilizing Ward's method, "clusters are progressively formed on the principle of minimization of variance" (Morey, Blashfield & Skinner, 1983, p. 314). It is more appropriate for this study as it identifies similar ways of thinking and showing construct connections. Additionally, Ward's method is one of the better methods to use for clustering techniques as it shows dimension that is sensitive to profile elevation (Morey, Blashfield & Skinner, 1983). Inverse scree tests were used to find the number of significant clusters (Lathrop & Williams, 1989; 1990).

There were four distinct clusters discovered. Three of the clusters were complete clusters while the fourth cluster only contained one construct. Table 1 displays the constructs in their entirety along with the cluster membership. The mean average for each construct and an overall mean of each cluster is also presented. The means provide a guide to the degree in which video cases of this type are perceived to address preservice teachers' trepidations in mathematics and science. Constructs considered similar and in the same cluster were then described as a theme (Kelly, 1955).

Table 1
Video case descriptive statistics for each construct

Video cases demonstrating…	**Cluster Membership**	**Mean**	**SD**
Well-run classrooms	1	4.55	.77
Real teachers working with real students	1	4.48	.97
Classroom management	1	4.43	.85
Lessons to improve my weaknesses	1	4.42	.96
A variety of teaching techniques	1	4.42	.84
How to interact with struggling students	1	4.40	.84
What works in the classroom	1	4.30	.99
How the teacher adapts the lesson based on student understanding	1	4.28	.85
The integration of various subjects into a lesson (integrated curriculum)	1	4.10	.94
Cluster 1 mean		**4.38**	
Lessons with older students (grades 4-6)	2	3.98	1.18
Step-by-step procedures	2	3.90	1.24
The use of manipulatives	2	3.88	1.11
Student motivation issues	2	3.80	1.07
Lessons to guide my strengths	2	3.79	1.21
Attention getters	2	3.78	1.31
How to use visuals in the classroom	2	3.76	1.20
The use of educational tools with students	2	3.74	1.03
Classroom routines	2	3.70	1.25
Complex topics	2	3.65	1.21
Classroom organization	2	3.57	1.13
Projects with students	2	3.53	1.13
Group work	2	3.39	1.04
Cluster 2 mean		**3.73**	
Time management	3	3.76	1.24
Mainstreamed students	3	3.74	1.11
Trouble-spotting analysis	3	3.57	1.22
How to prep activities	3	3.48	1.24
Diverse student populations	3	3.47	1.20
Expert analysis of the presented lesson	3	3.47	1.32
Reciprocal teaching	3	3.42	1.28
Feedback given by the teacher	3	3.39	1.24
Time fillers	3	3.15	1.36
Cluster 3 mean		**3.49**	
How to collect and pass out materials	4	2.37	1.26

Cluster 1

Authentic, highly effective classrooms focusing on helping students understand was the first cluster. This cluster had the highest overall cluster mean at 4.38. Some of the constructs contained in this cluster included authentic well-run classrooms that show a variety of teaching techniques with multiple subject areas. In addition, teachers want to see lessons of what works in the classroom while factoring in preservice teachers' weaknesses. When developing mathematics and science video cases, preservice teachers want to learn how to run effectively managed classrooms with quality curriculum being modeled.

In terms of authenticity, it seems very important to preservice teachers that video cases show actual teaching without the lesson being edited down to perfection (Gainsburg, 2009; Sherin, Linsenmeier & van Es, 2009). The construct mirroring this idea was "real teachers working with real students." Lessons need to show actual, ordinary students with typical responses and actions to help preservice teachers bridge the video case to the actual classroom setting. Authenticity is a key element (Gainsburg, 2009); preservice teachers must view the case as genuine and unmodified to showcase what they will actually experience in the classroom as a new teacher.

Cluster 2

The second cluster was labeled *intermediate tool-based lessons with procedures showcasing cooperative learning*. The cluster mean was 3.73, the second highest mean of all the clusters. This cluster focused more on routines and procedures; it seemed to focus more on viewing mathematics and science lessons with an emphasis on how to use tools and visuals to help support cooperative learning in the classroom. Evaluating the constructs contained within the cluster, it signifies that preservice teachers need help with the procedures veteran teachers often take advantage of knowing. While procedures and routines may seem innate, the preservice teachers in this study point out that they need these skills to be modeled and showcased through the use of video cases.

When designing and/or implementing video cases, it is important to allow preservice teachers to see tools used in a cooperative setting. Inquiry-based instructional models generally encourage teachers to allow students to speak and analyze in a cooperative learning setting. This cluster shows that preservice teachers need to see this model in practice through the use of video cases. Video cases that demonstrate student thinking and talking in groups have the potential to help preservice teachers see and appreciate student thinking and understanding in a cooperative learning context (Ding, Li, Piccolo & Klum, 2007). Additionally, preservice teachers would like to see the use of tools at the intermediate elementary level to help them teach appropriately. According to Star and Strickland (2008) preservice teachers do have difficulty noticing what is mathematically significant when analyzing a video case. They need guidance to understand the mathematical content and how it supports student learning. Video cases developed and implemented with the use of tools, procedures and cooperative learning can be used to cultivate this concept.

Cluster 3

Preparing, timing and analyzing lessons with diverse students was the third cluster identified. This cluster's mean was 3.49. The primary theme for the cluster was the need to understand what the preservice teachers are viewing in a mathematics and science video case. They need feedback to better understand what they are seeing and how it relates to effective (or ineffective) teaching.

In addition, diverse student populations should be integrated into the video case with specific analysis of the students' varying academic and social needs.

Expert analysis of the video case is one of the components of this cluster. According to the constructs "Expert analysis of the presented lesson" and "Feedback given by the teacher," preservice teachers need experts to guide their viewing of the case. The experts can be available online or can be the university instructor. Using experts to guide video case observation is possible (Herrington & Kervin, 2007) and expert analysis can help the viewers understand what they are looking at and why it is (or is not) quality instruction (Beck, King & Marshall, 2002). Sherin and van Es (2005) describe video cases as helping preservice teachers notice and understand what they can and should be looking for in classroom situations, particularly what encompasses quality instruction in the classroom setting. It is important to also note that the video cases can also be analyzed with the university instructor serving as the expert while using support from the preservice teachers. With this kind of analysis, it bridges the authenticity of the video case to the preservice teachers allowing them to contribute to the analysis (Gainsburg, 2009).

Cluster 4

This fourth cluster has just one construct "How to collect and pass out materials" and was labeled *Materials distribution and collection*. The mean for this construct was 2.37. Because this cluster contains only one construct, it cannot be considered a coherent super-ordinate construct. Nonetheless, it shows that when teaching mathematics and science, preservice teachers need guidance and assistance from video cases outlining the distribution and collection of the necessary mathematics and science materials and supplies. Preservice teachers are novices and procedures such as the distribution/collection of materials may be perceived as confusing or overwhelming. Seasoned teachers have established a routine that seems natural and instinctive. So while collecting and passing out materials may seem straightforward and uncomplicated to experts, it can pose a daunting challenge to those just entering the field, particularly in mathematics and science where tools are often used.

Conclusion

The results of this study provide guidance regarding the integration of video cases in mathematics and science elementary preservice teacher education. Video cases have been shown to benefit preservice teachers' instruction; they can improve how preservice teachers' ability to analyze lessons greatly improves with their integration (Alsawaie & Alghazo, 2010; Star & Strickland, 2008). Stockero (2008) found that video cases can enhance how preservice teachers contemplate student thinking and reflective analysis. However, it is important to also consider not just video cases per se, but how video cases should be used in the classroom to support preservice teachers' learning and development. How effective video case integration is can be greatly influenced by how they are used in the classroom (Darling-Hammond & Hammerness, 2002).

Preservice teachers cannot be expected to know everything they need to be highly effective teachers, as they are novices. It is up to the experts (professors) to use the findings in this study to help guide the integration of video cases in university coursework. While preservice teachers may articulate that they need help with using tools to support instruction in mathematics and science, it is up to the professor to determine what tools can best meet his/her students' needs while integrating constructive video cases examples.

If one is going to implement cases, it is important to also understand preservice teachers' predilections in terms of video cases integration. Deciding how to develop or use video cases in the mathematics and science classroom is a multifaceted process (Wang & Hartley, 2003). The goals of the lesson, the finished instructional product, along with curricular ideas somewhere in the middle of the goals and finished product must all be considered. This research provides some guidance in relation to the requisite features as voiced by preservice teachers. These features can be used to support and enhance the teaching of mathematics and science by providing a framework for video case integration.

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PRESERVICE TEACHERS USING LESSON STUDY: RESEARCHING THE NATURE OF LESSON STUDY DISCUSSIONS

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In this study we researched elementary preservice teachers who are using lesson study in a mathematics field experience. More specifically, we document the nature of discussions in the post-lesson meetings by examining the topics discussed in these meetings and exploring if there is a change in the nature of the discussions. Findings suggest that lesson study, which is becoming a common way to foster communities of practice among inservice teachers, can be a useful tool in early field experiences to support preservice teachers to be reflective on their teaching and to learn authentic teaching knowledge.

Teacher education research suggests that teachers learn a lot from the professional community of practice they belong to, and that meaningful teacher learning occurs when teachers have opportunities to work in professional communities in order to solve problems that are relevant to their teaching practice (Ball & Cohen, 1999; Hiebert, Morris, & Glass, 2003). When teachers participate in communities of practice, they can learn mathematical content knowledge, pedagogical content knowledge, and how to respect other's knowledge (Nickerson & Moriarty, 2005), which helps teachers reflect productively on their profession.

A community of practice can be created when using lesson study. Lesson study is a model for school-based professional development that emphasizes reflection on practice. Lesson study is frequently used in preservice and inservice teacher education in Japan as a strategy for improving instruction (Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999). In lesson study, a group of teachers work together to improve their lessons and produce knowledge for successful teaching by going through cycles of planning, teaching, observing and reflecting, and revising lessons.

However, in the U.S., the lesson study method has not been widely used in preservice teacher education, even though it has increasingly been applied to professional development for inservice teachers. Lesson study could be useful for preservice teacher education in that it can help produce good quality teachers, who have appropriate mathematical content knowledge, the desire to understand student thinking, and the ability to use resources effectively in order to promote students' meaningful learning (Hart, Alston, & Murata, 2011; Lewis, Perry, & Hurd, 2009; Parks 2008). Also, lesson study can encourage preservice teachers to have a commitment to lifelong professional development because teachers who have experienced the advantages of lesson study may have a positive attitude towards communities of practice, and this can be a driving force to continue collaborating with peers after becoming a teacher.

In this study we researched elementary preservice teachers who are using lesson study in a mathematics field experience. More specifically, we document the nature of discussions in the post-lesson meetings by examining the topics discussed in these meetings and exploring if there is a change in the nature of the discussions.

To achieve this purpose, the following research questions were investigated:

1. What topics are covered in a lesson study meeting?
2. What is the nature of preservice teachers discussions during lesson study meetings?
3. Are there aspects of the nature of discussions that can be quantified?

Literature Review

Theoretical Perspective on Lesson Study

Lesson study is based on two major theoretical perspectives: cognitive theories of teacher learning and situated learning theories (Lewis, Perry, & Hurd, 2009). Cognitive theories explain learning as changes in an individual's mental schema. From this perspective teachers construct their knowledge for teaching from their prior perceptions; that is, the knowledge development is accounted for as a transformation and a reorganization of teachers' prior knowledge (von Glasersfeld, 1995; Hashweh, 2003). In lesson study, participants confront conflicts between their own ideas and the ideas they receive from their colleagues, research, students, and curriculum and try to solve the cognitive imbalance through the practical research lesson and discussion. Situated learning theories explain learning as the result of participation in a community of practice. Participation and communication forms the identity of members, which affects their future actions and commitments (Lave & Wenger, 1991). Effective teacher learning occurs when knowledge for teaching, beliefs, and practice are shared in a community (Borko, Davinroy, Bliem, & Cumbo, 2000). In lesson study, preservice teachers can develop a professional community and establish those practices which are desirable for instructional improvement.

A Model of Lesson Study

Lesson study is a professional learning approach using a collaborative community for living instruction. Teachers using lesson study go through the following steps: (1) Setting goals; (2) Planning the lesson; (3) Teaching, observing, and debriefing the lesson; (4) Revising and re-teaching; and (5) Reflecting and sharing results (Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999; Stepanek, Appel, Leong, Mangan, & Mitchell, 2007). Setting goals means identifying a research theme by considering students' current knowledge, the key concepts, and the existing curriculum. Planning the lesson involves developing a classroom lesson and writing a lesson plan. For the teaching, observing, and debriefing the lesson step a teacher from the lesson study group conducts a lesson while other group members observe the lesson to collect data about the effectiveness of the lesson. After the lesson, the teachers have debriefing time to discuss the data from the lesson. The revising and re-teaching step involves revising the original lesson plan based on feedback from colleagues and then teaching the same topic again to a new group of students with the revised lesson plan. Finally, in the reflecting and sharing results step, the lesson study group reports on what they learned from the research lesson and group discussion.

However, in this study, we will use the term lesson study in a more narrow sense. That is, we use it to refer to an approach which uses the steps mentioned above except for part of the fourth step. This is because participants in this study were not asked to re-teach a revised lesson plan due to constraints of the field experience. Therefore, this modified version of lesson study includes planning, teaching, observing, debriefing about the research lesson, and writing a revised lesson plan. Debriefing about the research lesson (lesson study discussions) consists of three stages: 1) the instructor's self-reflections; 2) a discussion of the lesson based on observations from peers; and 3) plan of the next lesson. In the first stage, a pair of PSTs who taught the lesson shares their observations about how the lesson went. In the second stage, the other participants share their observations about how the lesson can be improved. In the final stage, participants brainstorm ideas about how the next lesson should be taught.

Advantages of Lesson Study

Recently, the National Research Council (2005) noted that elementary children have to be taught mathematics by knowledgeable teachers, and that practice-based learning is important in order to produce knowledgeable teachers in mathematics education. The NRC emphasized the potential of lesson study because it could increase teachers' content knowledge, their understanding of students' thinking, and could help teachers pursue reform-oriented teaching. Marble (2007) researched a lesson study approach in an early field experience. He found that there was striking progress in lesson design, classroom management, the quality of students' engagements through meaningful activity, and the quality of student assessments. Furthermore, Lewis, Perry, and Hurd (2009) reported that lesson study improves instruction in several ways, namely, lesson study changes teachers' knowledge, beliefs, professional communities, and teaching-learning resources; and these changes are related to instructional improvement. Additionally, Lewis (2009) studied the nature of knowledge development in lesson study based on a lesson study case from a Japanese elementary school. She reports three kinds of benefits from lesson study: development of knowledge, interpersonal relationships, and personal qualities and dispositions. Research studies such as the ones mentioned above have shown that lesson study has been an effective tool in teacher learning. In this study we explored if elementary preservice teachers would also benefit from an adapted version of lesson study by researching the nature of post-lesson discussions in an early field experience.

Methodology

The Participants and the IMB Project

A lesson study group consisting of six preservice teachers, a field experience instructor, and an inservice teacher (host teacher), were the focus of this study. The six elementary preservice teachers (PSTs) enrolled in mathematics and science methods courses concurrently with the field experience course. Preservice teachers attended a 75-minute mathematics methods course twice a week and went to an elementary school once a week to teach children one lesson. The PSTs taught mathematics during the first half of the semester (6 weeks), and taught science during the second half (6 weeks). PSTs were juniors and most of them had considerable motivation to learn how to teach mathematics because they had to teach their students mathematics based on what they had learned in their mathematics methods course.

In the field experience, the six PSTs are placed in an elementary classroom to implement Iterative Model Building (IMB) cycles. IMB cycles consist of four stages: formative assessment interviews; model building sessions; teaching a lesson; and the lesson study meeting. Prior to the teaching of the lesson, pairs of PSTs conduct formative assessment interviews (FAIs) and then share information about children's thinking in a model building session. The goal of FAIs is to understand students' thinking about the topics of the lessons and use this information when planning future lessons. Then a pair of PSTs teaches a whole-class lesson that had been planned the previous week with the lesson study group, while the others in the lesson study group observe that lesson. After teaching the lesson, the lesson study group deliberates about the lesson, in order to give feedback, and discuss how to improve it for future teaching. Then another pair of PSTs presents a draft of the next lesson to the lesson study group and the lesson study members suggested ideas for improving the lesson. The PSTs take turns teaching the whole-class lesson each week, and each pair of PSTs teach two mathematics and two science lessons during the semester.

Data Collection

We watched videos of the lesson study discussions and analyzed their transcripts in order to investigate the interactions between the participants in a lesson study group. Only the second and the fifth lesson study meetings out of six mathematics lesson study meetings were used for this analysis. During the first lesson study meeting, PSTs were learning the lesson study method, while the sixth lesson study meeting included content for the preparation of a science lesson that would be taught the following week. Furthermore, the two PSTs who taught the lesson were the same in both the second and the fifth session, even though the roles of the main teacher and the assistant teacher were changed. Thus, we decided it was more meaningful to compare the second and the fifth lesson study sessions for the purpose of this study.

Data Analysis

To analyze data we used the content analysis method (Krippendorff, 2004) and the reconstructive coding method (Carspecken, 1996). Content analysis is used not only to find out the meanings of data but also to articulate some working categories of codes by counting the occurrences of those codes. Content analysis is conducted following three steps: (a) Locate the data that should be representative of some describable universe of content; (b) Do the coding; and (c) Do quantifiable analysis of the data. After collecting the data from the lesson study meetings, we started coding the data. For specific coding, we used the reconstructive coding method based on the principle of inference and the principle of pragmatics. We started with *emic* coding, which involves developing coding based on an internal understanding of the meaning of the data.

First, we began the data analysis by reading through the transcripts of the post-lesson discussions, focusing on the level of meaning that is evidenced in a *meaning field* or a *meaning reconstruction*. Then, we assigned a short-form label for meaning to sentences and then linked that back to data through levels of inference. We named the codes according to three levels of inference (high inference, medium inference low inference) – based on how close the codes were to the data. After labeling, this coding got layered in two ways: collecting the first level codes into families or super codes; and describing the relationships among the super codes as themes. Finally, these codes, families, and super codes were combined to produce thematic categories or statements by uniting and distinguishing between the various levels of codes. After completing the *emic* coding, we tried *etic* coding, which involves developing codes grounded in an external understanding of the meaning of the data. *Etic* coding was conducted by referring to the relevant literature in order to refine the *emic* coding as a pathway for quantifiable analysis of data (Lewis, 2009; Marble, 2007; Parks, 2008; Lewis, Perry, & Hurd, 2009). That is, we examined some of the categories used in those studies and looked for categories that overlapped with the ones we found in order to narrow down our codes into fewer categories by combining those categories. Through this process, we arrived at the refined categories of *lesson plan*, *student*, *collaboration*, *beliefs*, and *miscellaneous*. After the coding work, we counted the frequency of contributions of words related to each code in order to investigate if the nature of the preservice teachers' discussions could be described quantitatively.

Analysis and Findings

In this section, following the order of the research questions, we first describe the kind of topics covered in the lesson study meetings and then examine the nature of lesson study discussions and finally present our attempts to quantify the change in the quality of discussions.

Topics Covered in Post-Lesson Discussions at Lesson Study Meetings

Through the coding of the transcripts, we found that the topics covered in the lesson study meetings fell into five categories: *lesson plan*, *student*, *collaboration*, *beliefs*, and *miscellaneous*. The category of *lesson plan* includes lesson goals, gear up and gear down activity, launch, investigation, summarize, time, task, manipulatives, and group organization. The category of *student* includes students' thinking (strategies), difficulty, attitude, prior knowledge, students' needs, and response. The category of *collaboration* includes sharing experience, suggestion, reflection by lead teachers, feedback from observer teachers, and elaboration. The category of *beliefs* includes the beliefs about students, beliefs about teaching, and beliefs about mathematics. The category of *miscellaneous* includes things such as chatting and announcements.

Here are some examples of quotes in each category. A comment like "we met our objectives, which were finishing and sitting together, covering an area, counting sets of objects up to 60" was coded in the category of *lesson plan* because the comment mentioned the lesson goals. A comment like "They [students] all seemed more confident with the shapes than they are with like numbers" was coded in the category of *student* because it referred to the students' attitude. A comment like "I am thinking if you had done that you could have said what was the least number of shapes you could make it with or what was the largest number [of them]" was coded in the category of *collaboration* because it is related to a suggestion to improve teaching. A comment like "it's hard to get everything within that short time, [and] I think that's why I got frustrated because I wasn't trying to fit things in" was coded in the category of *beliefs* because the remark is related to teaching difficulty, namely beliefs about teaching.

The Nature of Preservice Teachers Discussions in Lesson Study

After investigating the topics discussed in lesson study meetings we researched the nature of those discussions. Three categories did show remarkable change in the frequency and quality of contributions from participants: *lesson plan*, *collaboration*, and *student*. We now describe the nature and quality of discussions for these three categories.

First, in the category of *lesson plan*, PSTs in the second lesson study were more focused on classroom management and group organization for implementing the activity. However, in the fifth lesson study, the preservice teachers focused on content knowledge as well as on pedagogical knowledge by emphasizing the connection between the previous task and the next task. Also, in the second lesson study, the facilitator, Susan, and the host teacher, Ms. Tyler, had the main roles leading the discussion about planning the next lesson; while in the fifth lesson study, the preservice teachers led the discussion about lesson planning.

Next, in the category of *student*, in the second lesson study PSTs focused mainly on the students' superficial attitudes rather than on their mathematical thinking. However, in the fifth lesson study, the preservice teachers reflected more on the students' thinking, with detailed sample evidence. Also, in the fifth lesson study, PSTs often quoted the result of probing the students' thinking, in the form of an interview, in order to back up their observations.

Finally, in the category of *collaboration*, in the second lesson study, the facilitator and host teacher chiefly led the discussion about suggestions and reflections on lesson. However, in the fifth lesson study, the role of the facilitator and the host teacher (who was absent on that day) changed, from a person who actively suggested opinions to a person who listened to preservice teachers' opinions and gave only simple feedback remarks such as "good," "I like that idea," and "Yeah." Also, in the fifth lesson study, preservice teachers reflected on the lesson in terms of launch, investigate, and summarize parts, referring to the structure of the lesson plan.

Quantifying Changes in Lesson Study Discussions

We attempted to quantify the change in the quality of lesson study discussions in two ways: the change in the number of contributions to each category of topics and the change in the frequency of participation in the lesson study discussions. Given that lesson study meetings are opportunities to reflect on practice and to deliberate on how to improve teaching, we thought that one indicator of the quality of such meetings would be that all participants contribute to the discussions. We also thought that the frequency of contributions of all participants would tend to increase throughout the weeks and get to a point when it would be about equal for all members. As a result of keeping track of the frequency of participants' contributions for each category, we found that the overall quality of lesson study meetings improved (see Table 1).

Category	Second lesson study	Fifth lesson study	Increase
Lesson plan	11	42	31
Student	40	56	16
Collaboration	85	106	21
Beliefs	8	9	1
Miscellaneous	12	18	6

Table 1. Change in the frequency of contributions made in post-lesson discussions

In the category of *lesson plan*, the number of contributions increased from 11 to 42. In the category of *student*, the number of contributions increased from 40 to 56; and in the category of *collaboration*, the number of contributions increased from 85 to 106. In the categories of *beliefs* and *miscellaneous*, the number of contributions increased from 8 to 9 and from 12 to 18 respectively. Based on this result, the change in the category of *lesson plan* was the most remarkable, with an increase of 31; *collaboration* was second, with an increase of 21; and the change for the category *student* was a noteworthy third, with an increase of 16. The increase in the category of *lesson plan* is attributable to the preservice teachers' active participation in planning the next lesson and suggesting appropriate activities. The increase in the category of *collaboration* is attributable to the active interaction between lesson study members, in things such as feedback and suggestions. The increase in the category of *student* is derived from the participation of various preservice teachers when reflecting on students' thinking and responses.

PST	Second lesson study	Fifth lesson study	Increase
Amy	8	10	2
Christina	26	19	-7
Erin	17	37	20
Hera	6	32	26
Jane	17	30	13
Mary	0	22	22
Susan	50	81	31
Ms. Tyler	34	Absent	

Table 2. Change in the number of participations in post-lesson discussions

In reference to the change in the number of participations in lesson study discussions, five out of six preservice teachers showed an increase in their contributions, while one preservice teacher showed a slight decrease in the number of her contributions (see Table 2). Also, the participation rate during the fifth lesson study shows more equal frequency of contributions than that of the second lesson study. Particularly, Mary who was the only African-American preservice teacher in the lesson study group and Hera who was shy about participating in the second lesson study discussion showed a remarkable change in the number of times they participated.

Discussion and Conclusion

In this study we investigated the topics covered in lesson study meetings and the nature of preservice teachers' discussions during their participation in lesson study meetings. We examined the nature of these discussions qualitatively and quantitatively. We found that in lesson study meetings, discussion topics fell into five categories: *lesson plan*, *student*, *collaboration*, *beliefs*, and *miscellaneous*. Based on the frequency of contributions in these five categories, we also found that the level of preservice teachers' discussions improved quantitatively and qualitatively through the use of lesson study meetings. In particular, the changes in the categories of lesson plan, collaboration, and student – in that order – were remarkable. When examining vignettes from the transcripts, we noticed that the improvement in the category of *lesson plan* came from preservice teachers' active participation in planning the next lesson. The incremental change in the category of *student* was derived from the preservice teachers' participation in reflecting on the students' thinking and responses. The rise in the category of *collaboration* was attributed to the active interaction between the lesson study members through feedback and suggestions.

From these findings, we can see that lesson study in their early field experience was helpful in supporting these preservice teachers to be reflective on their teaching and to learn authentic teaching knowledge. These results are consistent with those of other studies showing benefits of lesson study (Lewis, 2009; Marble, 2007). However, in this study, the change of beliefs was not as much as that found by Lewis (2009) in her study. This might be because, in this study, the preservice teachers' post-lesson discussions were guided by a facilitator in order for them to focus more on topic-centered discussions rather than in sharing their beliefs, and also because of the way in which we quantified this change, using the frequency of their contributions during the post-lesson discussion phase of lesson study.

This study has implications for preservice teachers and teacher educators. The first implication is that preservice teachers would benefit from participating in a professional community continuously into the future. Through this early field experience, based on the IMB project, preservice teachers exhibited the benefits of lesson study. However, to maintain the benefits gained from lesson study, preservice teachers will need to create or join in an inquiry community with their colleagues in the future, and discuss their teaching problems or questions and make reflection on practice a part of their professional practice. If this disposition takes firm root in their teaching practice, preservice teachers will be willing to apply this approach to their teaching after they become inservice teachers.

The second implication is that preservice teachers can benefit from using the lesson study approach in mathematics methods courses, as well as in field experiences. Lesson study is becoming more popular in professional development and many are embracing it as an effective way to improve teaching practice. In this study, we saw how these preservice teachers also benefited from lesson study. Our findings give some support to the importance and value of using this method with preservice teachers.

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CONSTRUCTING AVATARS TO CHALLENGE PRESERVICE TEACHERS IN TEACHING ALGEBRA FOR EQUITY: AN EXPLORATORY STUDY

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As an initial effort to challenge preservice teachers in teaching algebra for equity, we investigated constructing avatars in Second Life to simulate a diverse middle-grade class. Participants created avatars that had many of their own social-cultural characteristics along with algebraic knowledge. Specific research-based guidelines are necessary to assure authentic and viable avatars are developed for use in classroom simulations.

Introduction

As a multi-user interactive virtual environment, Second Life (SL) has great potential for providing interactive, effective and engaging teaching and learning for the field of education, including preservice teacher education (Bransford & Gawel, 2006; Cunningham & Harrison, 2010; Goodband, Bhakta, & Lawson, n.d.). Teaching and learning in the SL environment has been documented as extremely positive, enjoyable and facilitated (Gao, Noh, & Koehler, 2009; Ritzema & Harris, 2008).

As animated characters with human capabilities and communication tools, avatars can be utilized to facilitate teaching and learning with numerous potential benefits (Blake & Moseley, 2010). Despite the recent increasing research interests in avatar issues, constructing student avatars to facilitate teaching and learning in K-12 classrooms has yet to be fully studied (Annetta & Holmes, 2006; Hew & Cheung, 2008).

In this study, through constructing avatars resembling a diverse student population in a real-life middle-grade class, we intended to take the initial effort to provide virtual teaching experiences for preservice teachers, specifically in the area of mathematics education. We investigated, from the perspective of a project team, the process of conceiving and developing a group of avatars which will portray middle-grade students with various socio-cultural and mathematical backgrounds in the simulated virtual environment for teacher preparation. We expect to gain insights into how to design the avatars in the best way to help address equity in algebra teaching for preservice teachers.

This study describes a particular component of the work in the first phase of a 5-year NSF-funded project. The project aims to design, develop, and test technology-enriched teacher preparation strategies to address equity in algebra teaching and learning for all students. The project endeavors to discover methods to assist in closing the achievement gap between White students and students of color in mathematics. A hypothesis of the project is that technologies such as SL can support teacher preparation for a thorough understanding of diverse learners and apply deep and specific mathematical knowledge in teaching.

This study is aligned with the objective of the project to design technology-enriched teacher preparation strategies to address equity in algebra learning for all students. Virtual technologies will be employed to provide preservice teachers with initial teaching simulations and experience in diagnosing, presenting, and evaluating activities of middle-grade student avatars in the content area of algebra. We hope that through the interactions between the constructed middle-grade student avatars and preservice teachers in SL, the preservice teachers can be better prepared for teaching algebra for equity.

The following research questions guided our study:

1. What are the social-cultural and mathematical characteristics of middle-grade student avatars as constructed by participants in our study?
2. What are the perceptions about how avatar construction challenges preservice teaching in teaching algebra for equity?

Theoretical Framework

As illustrated by Sleeter (2001), most preservice teachers lack experience in confronting diversity issues in multicultural classes, failing to meet the expectations of working with diverse students. Other researchers also pointed out that ''very few teacher education programs have successfully tackled the challenging task of preparing teachers to meet the needs of diverse populations'' (Watson, Charner-Lind, Kirkpatrick, Szczesiul & Gordon, 2006, p. 396). Ladson-Billings (1994) argued, however, culturally relevant teaching will benefit students of color since they have a distinct history and culture they bring into the classroom as prerequisites that should be nurtured accordingly. According to Lewis (2009), prospective teachers should understand that many students of color have the following belief (Kunjufu, 2001):

I don't become what I think I can,
I don't become what you think I can,
I become what I think you think I can.

Providing knowledge and experience in the classroom for prospective teachers is a significant component of teacher education programs. Moreno and Ortegano-Layne (2007) asserted that we can learn through either directly or indirectly observing people in real or fictitious situations. SL offers exciting opportunities for simulated teaching and learning in teacher education programs among other major affordances provided by SL for education such as communication, experience (Hew & Cheung, 2008), scaffolds and professional development (Cunningham & Harrison, 2010), To be specific, SL "provide(s) access to authentic simulations of…situations that would be otherwise impossible to experience" (Cunningham & Harrison, p.98).

Regarding the content, mathematics educators and researchers have increasingly recognized the gatekeeper role of algebra in preK-12 schooling (Carraher & Schliemann, 2007). The concepts of change, variable, and operations have been identified as core algebra concepts in the middle grades mathematics curriculum (AAAS, 2000; Kulm & Capraro, 2008). In this study, we intended to restrict the range in mathematic ability of algebra, thus avatars were developed with these three mathematical concepts, and the related prior mathematical knowledge that leads to these concepts.

Methods

Seven graduate assistants participated in this study. All participants are pursuing doctoral degrees in Curriculum and Instruction at a large, Southwestern research intensive university. All but one participant has a background in mathematics education. The participants are actively involved in the NSF-funded project, fully aware of its aims and hypotheses. Before collecting the data for this study, researchers informed the participants that they were faced with preservice teachers who are poorly prepared for working with diverse students. Several meetings were held for the project involving graduate assistants. During the meetings, researchers clearly indicated that graduate assistants were expected to construct middle-grade student avatars in SL to represent a sample of diverse students and thus better prepare preservice teachers for issues relating to diversity. Researchers also informed graduate assistants that the goal of constructing avatars in SL was to start considering possible interventions for preservice teachers by simulating a diverse student population in a real-life middle-grade class.

Given the above prompts, the participants were asked to describe the process of constructing a middle-grade student avatar by filling in a survey. The survey consisted of several open-ended questions regarding the social-cultural and mathematical characteristics of the avatar they conceived. The participants were told that they were going to play the role of their constructed avatar in the virtual SL environment and interact with preservice teachers in later phases of the NSF-funded project (see Figure 1).

Through substantive analyses of the open-ended questions in the survey, researchers initialized a set of response categories. The first two researchers read the responses to the survey individually and exchanged their ideas about the responses. After identifying the key themes of the responses to the survey questions, the researchers modified the initial categories to accommodate the responses. Responses to the survey were coded on three dimensions: (1) social-cultural characteristics of the avatars and underlying reasons for those characteristics, (2) algebraic knowledge of the avatars, and (3) perceptions about how avatar construction can challenge preservice teachers in teaching algebra for equity.

Figure 1. Constructed middle-grade student avatars interacted with preservice teachers in the front of a SL classroom specially designed for the project.

Results

Social-cultural Characteristics of Avatars and Underlying Reasons

Participants were asked to describe their avatar's name, gender, ethnicity, grade, qualification for a free lunch at school, family life, first language, personality, and hobbies and interests. In terms of gender and ethnicity, most avatars were identical to the participants who constructed them (See Figure 2 and Table 1). Most participants explained that by choosing the same gender and ethnicity for their avatar as their own, they will be able to relate better to the avatar. Only two participants changed the ethnicity for their avatars from their own and one participant changed the gender for his avatar. The participant who constructed an avatar of a different ethnicity, African American in this case, stated, “The math performance of African American students is unsatisfactory. The achievement gap in math between African American students and White students has been well documented. Also, students of color are underrepresented in gifted classes”.

Figure 2. Examples of constructed student avatars.

Table 1
Summary of Avatar Demographic Characteristics

Avatar Name	Grade	Avatar Gender	Different From Participant?	Avatar Ethnicity	Different From Participant?	Free Lunch
Glenn Polandia	6	F	Yes	Hispanic	Yes	No
Browniana	NR	F	No	Hispanic	No	No
Yong Wang	8	M	No	Asian	No	No
NattyJo	8	F	No	African American	No	Yes
Chih-Yu	8	M	No	Asian	No	No
Tonya	8	F	No	African American	No	Yes
Angel	7	F	No	African American	Yes	Yes

The ethnicities of the constructed avatars included African American, Hispanic, and Asian. However, none of the participants developed a White student avatar. Results of the socio-economic status dimension of the avatars showed that only African American avatars receive a free lunch at school. Participants explained that according to their research and teaching experiences, many African American students are categorized in this socio-economic status.

Algebraic Knowledge of Avatars

In regard to avatars’ prior algebraic knowledge, the most frequently occurring theme was the concept of mathematical operation. All participants reported that their avatars have some prior knowledge of operation. Selected responses include, the avatar “was very good at multiplication tables when she was in elementary”, has knowledge of “fractions, ratios, decimals, operations with fraction numbers”, and generates “equivalent forms of integers, fractions, decimals, and percent". The least frequently occurred theme was the concept of change. Only three participants regarded knowledge of algebraic change as a part of their avatars' prior knowledge.

The results also showed that participants attached great importance to avatars' prior knowledge and lack of a solid foundation for algebra. For instance, when considering the connections between arithmetic and algebra, a participant wrote, "Student does not learn well in sixth grade and has hard time to connect to the math knowledge in the following grades", and "the knowledge base of my other peers is not very deep".

When describing their avatars' knowledge depth, most participants highlighted lacking the knowledge of underlying reasons and applications of formulas and theorems in problem solving as indicators of shallow knowledge depth. To be specific, "Though I [the avatar] may not always know why I am using the formulas and theorems, I know how to use them. My knowledge is probably more procedural. I have not yet gotten the full picture of the theoretical reasons for using some math knowledge"; "She [the avatar] understands only the superficial formulas in recognition. She cannot apply them", and "She [the avatar] can recall some algorithms and represent situations with symbolic representations. However, she has difficulties in posing and solving real-world mathematical problems using equations. She also has a difficulty in understanding the underlying reasons and applications of algebra concepts and algorithms."

The theme of variable emerged frequently, together with operations, in the category of misconceptions of algebra. Since the concept of variable is a higher-level concept than operation, most participants chose some misconceptions related with variable for their avatars. For example,

> *Because in his (my avatar's) previous arithmetic learning experience, he treats '=' as a procedural representation. For example, his habit of computation is:*
> *5+2=7+3=10-5=5*3=15*
> *In algebra, his misconception is*
> *5+X=2+7*
> *X=2+7+5*

The participants further illustrated that their choices for the prior knowledge, knowledge depth and misconceptions of their avatars were based on their personal experiences, classroom observations, and common misconceptions identified by the literature. For instance, participants reported that "Problem solving was a major weakness for me when I was in school"; "Some of the math problems encountered by my avatar are the same with which I was faced with when I was an adolescent"; "directly related to myself in middle school"; and "I will relate the avatar to my previous misconceptions in mathematics."

Perceptions about How Avatar Construction Challenges Preservice Teachers in Teaching Algebra for Equity

Some participants implied or directly stated how avatar construction can challenge preservice teachers in teaching algebra for equity. One participant provided a sharply focused description about why the avatar was constructed as a female student: "My experience has shown that young men often display more discipline issues than young women. I wanted my avatar to focus their troubles more on the subject (algebra)." It seems that the participant expected that the preservice teachers can primarily focus on how to teach algebra in the classroom rather than contend with discipline or behavior issues.

Other participants shared views of how their avatar construction can purposefully simulate a middle-grade student in a racially diverse class to challenge preservice teachers to teach algebra for all students. For example, some participants posed specific challenges or opportunities of teaching algebra for equity for preservice teachers through constructing avatars: "I want to see how teachers negotiate this balance of education, this balance of 'intelligences'"; "I want to see how teachers address and assess the weaknesses of their students"; "I feel that my teachers have to go slower because the knowledge base of my other peers is not very deep"; "My avatar would like to interact with her teacher in SL. She hopes that her teacher can know more about her culture, and teach her accordingly, integrating her culture and the content of algebra"; and "I hope that through playing a role of an African American student in SL, I can help prospective teachers get some experience about how to work with students of color. I hope that through interacting with African American middle grade avatars, prospective teachers can gain some instructional practices for meeting the challenge of teaching African American students, providing equitable learning opportunities for all students".

Discussion

Through this exploratory study, we have examined constructing middle-grade student avatars as an initial effort to address teaching algebra for equity. Our analysis suggests that without specific guidance in conceiving avatars, participants will tend to create avatars that have many of their own characteristics. In our case, participants only thought of minority students when considering a diverse classroom, as if White students should not be included in the model for a diverse student population. Disparate views of equity and diversity were prominent in our study and need to be addressed by further study. Our participants displayed a narrow view of African American students, especially in socio-economic aspects. We were left with the nagging question, "Why are only the African American students considered to have low socio-economic status (receiving a free lunch at school)?"

The findings of our study suggest that constructing a diverse group of avatars as an initial effort to help address teaching algebra for equity lead to positive outcomes. The avatar construction may facilitate preservice teachers' learning to teach algebra for equity in a virtual environment since the avatars bear diverse characteristics resembling middle-grade students. However, in the process of construction, participants need to follow specific research-based guidelines concerning characteristics of real middle-grade students. This finding is consistent with the previous study that good preparation and management is required in role-playing scenarios such as goal setting, context and role defining, and character researching and preparing (Teed, 2009).

Theoretically, this exploratory study offers a unique contribution by drawing on constructing middle-grade student avatars as a means to promote preservice teachers' learning to teach for equity. Although an important starting point, this study provides a foundation to further explore how virtual teaching experiences support preservice teachers' learning in the later phases of the NSF-funded project. This study implies that effective design principles and criteria should be fully considered for avatar construction (Blake & Moseley, 2010). This study also clearly indicates the need for additional research focusing on addressing preservice or in-service teachers' challenge with equity.

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TEACHERS' MODES OF INTERACTING: EXAMINING CHILDREN'S WORK

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We examined the interactions of 10 prospective secondary mathematics teachers with video-based episodes of high school students' work on mathematical tasks. We analyzed and catalogued their comments on each case episode to determine their particular orientations along a continuum from naïve to mature in mathematical and content specific pedagogical domains.

Introduction

The use of illustrations of children' mathematical work in designing tasks for teachers has become increasingly popular in mathematics education (Smith & Friel, 2008; Seago, 2004; Goldsmith & Seago, 2008; Brophy 2004; Borko, Jacobs, Eiteljorg, & Pittman, 2008). Grounded in a situated cognition perspective on learning this approach assumes that by providing contexts in which teachers study concrete and tangible examples of children's mathematical practices meaningful opportunities for advancing their knowledge for teaching might become more immediately accessible (Brophy, 2004; Star & Strickland, 2008).). These assumptions have given rise to development of different genres of resources (videos, written cases, animations of practice) to be used in teacher education. Although the number of reported empirical studies that examine the utility of such resources on teacher learning is limited, a close inspection of what exists reveals several major gaps in discussions regarding the criteria used in both design and evaluation phases of the work. First, these reports fail to list the specific learning objectives (mathematical, pedagogical) that had guided the development of resources used in studies (Santagata, Gallimore, & Stigler, 2005; Goldsmith & Seago, 2008) or ways in which acquisition of those objectives were monitored. Second, little is shared about the quality of interactions of the teachers with such resources so to shed light on elements that might be considered when designing and/or implementing similar learning media. Third, even in the presence of claims to the effectiveness of such resources for teacher learning information regarding aspects of teacher knowledge that might have been enhanced from their use is not shared. These discussions are critical to the genre since they can greatly impact how these resources might need to be gauged against the particular strengths and orientations of the individuals using them (Homphrey & Boaler 2005).

In this exploratory research project our aim was to gain an understanding of factors that might need to be considered when designing and using video-based case tasks in teacher education in order to optimize their impact as learning tools. Using three video-based illustrations of children's mathematical work, we examined the quality of interactions of a group of 10 prospective secondary teachers with the video-episodes and their reactions to each. Two specific research questions guided our data collection and analysis:

1. What do teachers' extract from viewing video-based data of children's work on mathematical problems?
2. What factors seem to influence how teachers analyze and assess episodes they view?

Background and Context

The current study is a part of a much larger longitudinal research and development project in which we address three interrelated goals. The first goal of the project is to trace the development of mathematical thinking, problem solving skills and attitudes of a cohort of 70 children from 18 different urban communities as they progress from 7th through 10th grade in their respective schools. Children are interviewed twice a year, either individually or in small collaborative groups, experience technology-based explorations and work on a variety of problems from secondary school curriculum. The second goal of the project is to draw from what is learned about (and from children) to design educational resources to be used with secondary mathematics teachers so to enhance their mathematical knowledge for teaching. The third goal of the project is to study the impact of these materials on different teacher audiences, including prospective and practicing teachers at 6-12 grade levels. Of particular interest to use is determining ways in which different contexts might be designed and used to optimize among teachers.

A major design issue for us is determining what mathematical concepts to include in each case. We contemplate also the amount of information that needs to be utilized in the media. Additionally, we study the type of instructional materials that need to accompany the case based contexts to assure specific learning objectives are accomplished. Our primary objectives for using the vide-based case studies are to increase teachers' knowledge of mathematics and content specific pedagogy. The research reported here attempted to provide guide on these issues namely, what specific prompts might need to be included in the video based case tasks in order to facilitate development of mathematical knowledge of teachers. Our data collection and analysis focused on factors that enhanced or impeded meeting these objectives using three specific contexts.

Methodology

Participants

The participants consisted of 10 prospective secondary mathematics teachers enrolled in a graduate degree program at a large public institution. At the time of data collection all participants had completed three courses in methods of mathematics teaching. They had fulfilled all mathematics coursework required for a BS degree program in Mathematics. All participants had similar academic backgrounds (had taken the same number of mathematics courses, maintained GPAs above 3.4). Six of the participants were female.

Each participant was interviewed individually. During each interview the participants reviewed and analyzed 3 video-based cases. Each interview session lasted between 2 to 4 hours. All interviews were video and audio-taped. The video camera captured both the computer screen on which the videos were shown and the physical movements of the participants and their gestures as they interacted with the videos. This taping format allowed us to identify specific segments of the videos that seemingly gained the most attention by the participants.

Instrumentation

Data for the study was collected using three different video illustrations of children's work. A brief description of each video-episode is offered below.

1. (Similarity and congruence- 2 minutes and 40 seconds) illustrated a debate between two students as they tried to resolve meanings associated with (and consequences of) two different definitions found on the internet for similar figures. The teacher had asked students to first find the definition of similar figures on the internet and to then determine if specific shapes given on a worksheet were similar. The children had found two different versions of the definition (exclusive and inclusive). The children's argument concerned whether congruent shapes were also similar. The mathematical issue of the episode centered around the role of inclusive and exclusive definitions in mathematics.

2. (The rope problem-approximately 5 minutes) depicted a whole group discussion moderated by the teacher as four different students presented four different responses to the same task involving addition of fractions. Two of the methods offered close approximations techniques often used in real life. One method offered an incomplete process of finding common factors. One student had relied on the use of unit drawing. The mathematical issue of concern in this episode was the connections among these various methods, particularly the merit of approximation methods children had used in numerical analysis methods used in Calculus.

3. The third video-case (Horses and Humans problem-approximately 12 minutes) illustrated segments from clinical interviews with three children as they solved the same problem using different representational schemes. The problem concerned the use of system of linear equations. Each child had used a particular representational system for solving the problem (iconic, tabular and graphic) and without algebra. The mathematical concern of the episode related to importance of each representation for solving particular classes of problems in Algebra, Geometry and Calculus.

The participants were asked to respond to six questions at the end of each case:

1. What might be considered the major instructional issue in this segment?
2. What is your assessment of children's mathematical arguments? (Which of the responses is correct?)
3. What might have contributed to the children's thinking?
4. What mathematical concepts could relate to the topic of students' discussions?
5. Which of these concepts should the teacher pursue with children at this point in their discussion?
6. What other information might you need to offer a more accurate analysis of what you viewed or to suggest for how instruction might need to progress?

The first five questions were used specifically to capture the participants' knowledge in three specific domains: Knowledge of the discipline and its rules and principles; knowledge of connections among different mathematical concepts; pedagogical reasoning and decision making as it pertains to the knowledge of trajectory of concepts. The last question was used to document whether participants considered alternative approaches to teaching in light of different interpretations of mathematics that children shared. The participants were reassured that they could watch each of the videos multiple times, if they felt a need to do so.

Data Analysis

We documented the amount of time each of the participants spent on the analysis of each of the videos, the type of questions they asked the researcher during the interview, their interpretation of individual and group mathematical production and merit of what was proposed, ways in which they framed their analysis of each episode and their rationale for pedagogical solutions they offered. We recorded the number of times each of the participants reviewed each of the tapes and noted specific portions of the videos they choose to review multiple times.

Additionally, we coded the participants' interactions with each case along four broad-categories: Reflections on pedagogy and student work; reflections on mathematics of children and their connections to structure and form; explanation of mathematical topics that could be explored with children; and pedagogical actions which included suggestions for how a teacher could proceed with her lesson in the presence of children's ideas.

A network was generated for each of the three episodes first. These individual networks were combined later to build an interactional profile for each of the participants. At the conclusion of each individual analysis we compared the entire cohort's responses to each of the three cases in order to construct a model of interaction with video-data that took into account their mathematical and pedagogical orientations. These are described more fully below.

Results

Cataloging Interactions

Since a primary goal of our research was to identify the type of instructional materials we needed to develop in order to engage teachers in focused mathematical and pedagogical analysis, we coded their comments and analysis as they pertained to these two particular domains. As such, drawing from data the following indicators were generated and used to code their discourse:

Mathematical analysis

- Could identify critical points in each of the arguments that children had used or referenced in the episode
- Could explain why children's ideas were (or were not) mathematically sound
- Could identify the sources of children's difficulties from a mathematical standpoint
- Could identify trajectory of the content under study and how different mathematical topics could be addressed using children's particular suggestions and conceptions

Pedagogical (content specific) analysis

- Could explain why certain pedagogical moves were (or were not) adequate to pursue with children
- Could justify why certain topics should or should not be pursed with children
- Could identify the advantages and disadvantages of different instructional tools when trying to build on children's ideas to design follow up instruction
- Could identify specific and relevant approaches for pursuing discussions
- Could identify how the different pedagogical choices should be implemented individually, small group, large group discussions)

In analyzing the range of the teachers' modes of interactions with the videos and their discussions of the three cases, three specific categories of knowledge/orientation emerged: Mathematically Conceptually Mature, Pedagogically Mature (MC_mP_m); Mathematically Algorithmically Mature and Pedagogically Immature (MA_mP_i), Mathematically Immature and Pedagogically Mature (M_iP_m). Specific behaviors that characterized each group are summarized in table 2.

Category	Actions	Focus when reviewing topic
Mathematically Conceptually Mature, Pedagogically Mature (MC_mP_m) (n=2)	• Could articulate the basis for mathematical decisions children had made, offer other relevant mathematical concepts that could be pursued with them. • Suggested other instances of similar conceptual structure from the discipline, offer specific guides for how to extend children's intellectual work-space and a rationale for choice. • Referenced personal encounters with similar issues.	• Interpreted children's work • Identified strengths and relevance of children's ideas from the mathematical standpoint • Commented on connections among representations • Identified strengths and weaknesses of each approach drawing from a long term perspective on content development • Referenced the impact of curriculum and instruction on children's behaviors and mathematical performance.
Mathematically Algorithmically Mature, Pedagogically Immature (MA_mP_i) (n=2)	• Could explain the children's work and sources contributing to their development. • Could offer explanations of why certain mathematical actions were sophisticated. Could not identify trajectory of the concepts. • Managed to see the connections between different representations that children had offered. • Choice for practice was insensitive to context and particularities presented of the case.	• Identified children's errors or misuses of language • Commented on the amount of time it took children to complete tasks • Commented on how children might be guided in the right direction • Identified other instances where children may not be able to perform adequately
Mathematically Immature, Pedagogically Mature (M_iP_m) (n=3)	• Could not adequately unpack the mathematical work of children or their connections to other concepts. Could identify some related concepts that addressed the mathematics presented in the case. • Capitalized on the general practices (affective) suggested as effective for use (group work). • Could set some mathematical goals for children and instruction. • Could identify the impact of school curriculum and instruction on children's work and thinking. • Managed to draw from theoretical views on learning to offer perspectives on why certain pedagogical choices would be suitable.	• Attempted to interpret children's work • Commented on children's apparent talents and verbal ability • Asked questions about what would be appropriate to do • Asked questions about what children might mean • Asked questions about mathematical ideas • Asked questions about tasks that could follow each of the discussions
Mathematically Immature, Pedagogically Immature (M_iP_i) (n=3)	• Could not adequately unpack the mathematical work of the children or their relevance. • Endorsed decision making to students. • Did not formulate specific goals for the children's mathematical learning. • Not committed to learning.	• Avoided commenting on mathematics • Commented on the general attitudes that children had exhibited as they solved or discussed mathematical problems

Table 2: Behaviors and foci of attention of different groups

The different layers of knowledge and maturity of the participants directly impacted what they extracted from illustrations they viewed and how they interacted with them. Figure 1 illustrates a roadmap of anaytical pathways taken by different groups.

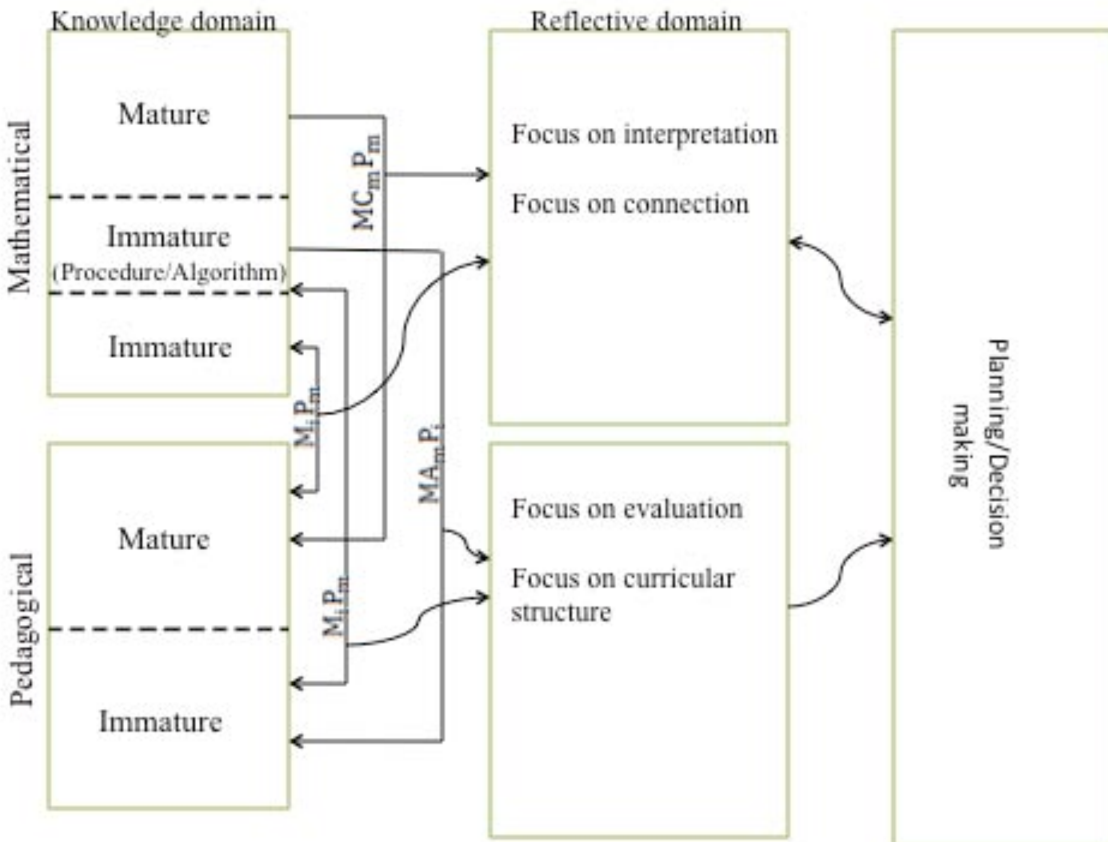

Figure 1: Interactional roadmap

The M_iP_i and MA_mP_i groups consistently spent the least amount of time on discussing the video data (Mean-4 minutes one each case). The MA_mP_i group quickly judged children's work as right or wrong and rarely addressed the connections among various ideas that children had shared. This group also identified what they perceived as gaps in children's understandings and resorted to conventional approaches for leading follow up discussions in class including (showing additional examples, describing procedures). Neither one of these groups expressed a need for additional information. Neither one of these groups showed a desire to consider alternative ways of interpreting children's ideas, unless they were asked to do so specifically..

The M_iP_i participants spent over a half of their analysis time discussing how the manipulative materials could be used to show children different concepts. These discussions did not address the particular ideas presented in the case. In analyzing children's work they focused primarily on affective dimensions associated with classroom learning (i.e. a teacher should acknowledge all answers as correct and praise children for their good work). The general tendency of the group was to avoid judging ideas and refrain from offering specific guides for how a teacher might proceed with instruction. In suggesting pedagogical actions they were willing to endorse mathematical authority to either the children or outside sources. These individuals did not ask questions during the interviews, felt secure in their knowledge about children and/or the pedagogical methods they offered. They were less committed to making sense of the children's ideas or testing their merit since they seldom referenced rationale for children's particular choices or practices.

In contrast, the M_iP_m group consistently spent the largest amount of time analyzing children's behaviors. They reviewed each of the episodes more frequently(n=3). They were concerned about whether they had fully understood the arguments that children presented or accurately assessed the competencies children had shown at each stage. These individuals successfully offered the largest number of mathematical concepts that could be explored in response to the children's ideas in each of the cases they examined. Lastly, these individuals routinely attempted to elicit feedback from the interviewer on accuracy of their analysis or ideas. Indeed, these individuals, when asked to generate a list of topics and areas about which they needed to growth, offered a more accurate assessment of their own educational needs.

Pedagogically and mathematically mature group members showed the tendency to fold back to reviewing the children's thinking at various points in their analysis. They used an "if-then" mode of reasoning when they offered perspectives on what a teacher could pursue in class.

Discussion

The findings of our exploratory study support four conjectures. First, the use of illustrations of children's work and thinking offer fruitful contexts for engaging teachers in generative mathematical and pedagogical inquiry. These discussions, if faciltiated robustly, can significantly enhance the quality of teachers' mathematical knowelge for teaching. Second, in order for resources of this type to have transformational power careful attention needs to be devoted to designing differentiated contexts for discussion and exploration by teachers in order to accomadate for particular strenghts or gaps that might exisit in their knowledge.

Our data indicated that teachers viewed and interpreted tasks differently. Naturally, in the diverse setting of a classroom or professional development program what teachers gain from activities depends largely on their particular learning trajectories. Therefore, in order to optimize the educational impact of case based tasks different structures might need to be put in place. This point drives our third conjecture which highlights the need for development of detailed instructional resources to augment case based tasks. Avilable case based resources, for the most part, rely on a set of guiding questions for individual investigations or group discussions. However, rarely do these materials offer mathematical guide for teachers. This gap is particularly paramount at the secondary teacher education level. A significant finding of this work concerns the sample of teacehrs who despite their immature understanding of mathematics tended to show great sensitivity and care when making decisions about how to proceed with anaylsis of children work or pedagogical decision making. This group seemed most receptive to learning and prepared to seek and justify the need for mathematical knowledge.

Lastly, our data indicate that video-based illustrations of children's work do provide an ideal outlet for motivating detailed analysis since they embody mutiple sources of data for teachers to consider when analyzing children's thinking. In commenting on children's work the teachers frequently referenced particular gestures children had exhibited as evidence of particular congitive actions in which they may have been engaged. Both verbal and non-verbal explanations of children created a space for discussions regarding affective factors influencing mathematics learning.

A particularly powerful outcome of the use of video-based illustrations was evidence of their efficacy for asessing mathematical knowledge for teaching secondary school mathematics. This was an un-expected outcome. Certainly, teachers' analysis allowed us to make and test conjectures regarding their current state of understanding of mathematics, and to formulate plans for the type of follow up activities that might productively advance their thinking. Such an outlet is particularly important for augmenting the content of courses designed for teachers and assessment tools used in teacher education.

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MOVING FORWARD AT AN EXPONENTIAL RATE: CROSSING DISCIPLINE BOUNDARIES TO PREPARE TEACHERS FOR 21ST CENTURY STUDENTS

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Technological progress follows an exponential curve. In order to prepare teachers for middle school students who are accustomed to having digital technologies on hand, teacher educators must help teacher candidates (and in-service teachers) develop technological "habits of mind". This project focuses on the development, implementation, revision, and extension of an instructional sequence designed to support prospective teachers' development of increasingly flexible proportional reasoning skills. Results suggest that prospective teachers may learn how to integrate technology into teaching most successfully when the engage in these practices as part of their own learning rather than when taking technology-specific classes.

Prognosticators suggest that technological innovation will continue to grow at an exponential rate. Given that their social lives are enmeshed in social technologies, it is not surprising the subjects of our study, 38 undergraduates with an average age of 21 years old, are well aware of the ways in which accelerated technological change has altered their lives. What is surprising is that the majority of these prospective teachers feel that they are already "behind the curve" when it comes to figuring out ways to leverage innovations such as e-books and interactive technologies to teach mathematics. The goal of our work has been to explore the nature of this ambivalence by unpacking its constituent parts: prospective teachers' knowledge of --and confidence with—mathematics (proportional reasoning), instructional strategies, and technology in general. According to Mishra and Kohler (2006), teachers who integrate technology effectively in their practice have developed knowledge that lies at the intersection of Technological, Pedagogical and Content Knowledge (TPACK).

Theoretical Framework

In order to support prospective teachers' development of TPACK, we have created an online textbook called the "Dynabook" for use in University-based teacher preparation courses at either the undergraduate (prospective) or graduate (credential) levels. The Dynabook resource incorporates a "mash-up" of research findings and interactive tools from two here-to-fore bounded disciplines: mathematics education and special education. In particular, we have developed challenging proportional reasoning problems based on the work of Susan Lamon (1999) and content described in the Common Core State Standards (ref). The teaching frameworks include a) the Universal Design for Learning (ref), b) the proportional reasoning classification framework described by Khoury (2004), and c) the description of effective teaching shifts in proportionality by Lobato., Ellis, Charles, & Zbiek, R. (2010). Finally, the technological tools include applets for moving blocks on a balance, graphing points and lines, moving gears, creating tables, and links to scripting videos. Figure 1 shows how these pieces fit within Mishra and Kohler's TPACK diagram.

The idea of using TPACK as a framework for organizing modern prospective teacher education programs is not new. For example, Wilson, Lee, & Hollebrands (2011) describe how prospective teachers developed TPACK by examining each of these knowledge components via a video-case. First, the prospective teachers solved the *mathematical problems* using the software featured in the case. Second, they read the lesson plans in the case and discussed various *pedagogical practices* that the teacher in the case had demonstrated. Finally, they attended to the ways in which the students' activities were mediated by their "conversations" with each other and the *technology*.

Zbiek and Hollebrands (2008) discuss two ways that university instructors can support the emergence of technology-enhanced classroom norms in their methods classes. First, they can ask more cognitively demanding questions that can be explored using modeling software. For example, in the current project, we asked a series of questions regarding how weights can be distributed to make a see-saw balance. These questions were explored using a balance applet. Second, instructors can explicitly discuss their efforts to establish norms for argumentation within the classroom culture. For example, instructors may want their preservice teachers to become the ultimate arbiters of whether others' explanations are clear and accurate.

When instructors who use technology effectively can also describe the pedagogical practices they are enacting, the prospective teachers have opportunities for apprenticeship observation (Lortie,1975). In other words, the instructors are able to describe learning as a socially-situated process in which norms for discussion and participation are negotiated on an on-going basis and profoundly affected by practices in which they engage and the tools at hand (cf., Sfard & McClain, 2002; Cobb & Yackel, 1996; Sfard, 2008). Given the view that learning is profoundly affected by the tools we use and the discursive norms that emerge around these tools, how did one promote the aim of preparing teachers to teach mathematics in 21st century classrooms?

The Study

This study was conducted in one mathematics class for prospective middle school mathematics teachers. Activities from the Dynabook were enacted during 12 consecutive ninety minute class sessions. This particular report will focus on the three day sequence involved in using the balance metaphor to model a variety of proportional reasoning examples.

Subjects and Data Collected

The class contains 38 prospective elementary teachers who are taking extra mathematics classes to earn a "mathematics specialization". The particular course focuses on supporting the mathematics learned in middle school from a conceptual point of view. The data set includes, digital images of student work, videotaped classroom lessons, video-based interviews with students working on the tasks, results from a final "far transfer" balance question posed on the final exam, and pre-post results of proportional reasoning items on the *Learning Mathematics for Teaching* test (Hill, Schilling, & Ball, 2004).

Intervention: The Balance Sequence

In choosing contexts and tools that would be a part of the bricolage (Papert, 1993) we bring to bear to support our prospective teachers in proportional reasoning, we were mindful of both their experience as mathematics students and their preparation for teaching. We believed the balance model, or lever, could be a representation that would help us accomplish several things. First, we knew that balance, or more broadly lever, problems are prevalent in upper elementary and secondary texts. While one might argue that "textbook word problems" are not necessarily realistic, they *are* realistic in the realm of prospective teachers who will be helping students solve these types of problems.

Second, we wanted a representation or model on which they could build connections across problem contexts. Students often see proportional reasoning in algebra, geometry, physics, and other quantitative disciplines as distinctly different. Students can gain a conceptual understanding of levers, averages, mixture problems and the geometric mean by connecting them through a familiar model (Flores, 1995). The balance representation had the potential to help our prospective teachers recognize proportional reasoning in different settings and in so doing develop flexibility as proportional reasoners.

Third, we believe the balance or seesaw is an experientially-real context (Gravemeijer, 1999) from which students could potentially draw on their physical experience to immediately engage in personally-meaningful activity. In the absence of physical childhood experiences with seesaws, students could experiment with a physical or virtual model. Finally, our choice was further influenced by another of the design heuristics of Realistic Mathematics Education (RME), that is: students develop models of their informal activity, which become models for more formal mathematical reasoning (Gravemeijer, 1999; 2004; Stephan, Bowers, Cobb, & Gravemeijer, 2003). We believed the balance could serve as a powerful model, such as the empty number line, array model of multiplication, or ratio table. In reifying the notion of equal quantities in the symbolism, they could extend reasoning to contexts beyond those encountered directly with the model.

Results

Results from the paired t-test conducted on pre-post proportional reasoning items of the LMT indicated that the increase in average score was significant at the $p<.001$ level. This indicates that the class did learn how to identify and solve proportional reasoning situations, as well as some pedagogical knowledge associated with teaching such topics. The goal of the qualitative analysis was to explore *how* these pre-service teachers' discursive engagement changed over the course of the study. The outline for this "mini sequence" and the associated mathematical practices that emerged is shown in Figure 1.

	Day 1	Day 2	Day 3	Day 4
Activities	• View intro videos • Read transcript • Solve 3 see-saw problems	• Modeling solutions with applet 1 • Solving extension problems	• Using decimal distance measures	• Introduction of indirect variation
Associated Practices in which students engaged	Students used the drawing tools, video transcript, and experiential knowledge to solve problems. No way to check viability	Assigning any quantity to be represented by weights. Equivalence via multiple representations Notation emerges that reifies action with balance	Modified balance allows for more flexibility in weights, distances, and monomials	Approximately half the class was able to extend the metaphor of the balance to model indirect variation using notion of mathematical equality

Discussion

As mathematics educators interested in the development of prospective teachers' TPACK, we have attempted to create a challenging and cognitively demanding mathematical sequence and associated exploratory applets that engage them in activities that model the effective use of technology in classrooms. Our research has shown that the students did in fact engage in exploratory investigations wherein the balance applet became a strong visual mediator of communication. Moreover, as we had hoped, the use of the applet because reified through students' notation and hence their use of the applet –while initially integral as a model of their problem solving efforts—became reified through their notations as a model for more extensions to the balance metaphor both in terms of "short term" transfer items such as related problems as well as "long term" balance problems such as those involving indirect variation.

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A SELF-STUDY EXPLORING THE PRACTICES AND TECHNOLOGIES OF A FACULTY ADVISOR IN SECONDARY MATHEMATICS TEACHER EDUCATION

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Pre-service teachers' negotiations of theory-practice transitions call for an exploration of innovative models for faculty mentoring and professional development. This paper describes a self-study research project to explore a blended real and virtual model for faculty advising (e-advisor) during secondary mathematics teacher education field experiences. One goal of the study is to disrupt traditional notions of teacher education programs as places to 'train' and 'prepare' teachers, with field experience being viewed as the supervised enactment of these preparation techniques. Drawing on concepts of Foucault, the research analyzes the traditionally performed roles of student teacher, teacher educator, and faculty advisor.

Description and Purpose of Research Study

In the field of teacher education, pre-service teachers often reminisce fondly on their experiences as students, holding tightly to their preconceived notions (formed through many years of the 'apprenticeship of observation') of what it means to teach mathematics and, in general, to be a teacher (Begg, Davis, & Bramald; 2003; Britzman, 2003; Lerman, 2001; Lortie, 1975; Nolan, 2010a). Disrupting and reconsidering these notions can prove challenging for both pre-service teachers and teacher educators alike. Pre-service teachers' time in a teacher education program represents a critical period of transition from student to student teacher to teacher. While there has been valuable research to date on the nature of the transitions required for becoming teachers (Garcia, Sanchez, Escudero, & Llinares, 2006; Jaworski & Gellert, 2003; Klein, 2004; Nolan, 2008; Ritchie & Wilson, 2000), the specific areas of teacher education field experiences and faculty supervision models are still relatively under-documented and under-explored in the research literature. This is especially true in the case of teacher educator self-study research that seeks to foster a continuum between the university and the field while, at the same time, disrupting the traditionally performed roles of student teacher, teacher educator, and faculty advisor.

This paper discusses one aspect of a larger project, referred to as the e-advisor project. The research is a three-year self-study of my own practice as a faculty advisor, working with pre-service teachers (interns) during their four-month field experience (internship). The purpose of the evolving e-advisor project is two-fold: (1) to create and sustain a professional development relationship between myself, as teacher educator and faculty advisor, and secondary mathematics interns through the use of multiple technologies (such as desktop video conferencing, video flip-cameras, online chat and discussion forums), and (2) to disrupt traditional notions of teacher education programs as places to 'train' and 'prepare' teachers, with field experience being viewed as the 'supervised' enactment of these preparation techniques. In other words, the overall intent of the self-study is to better understand and deconstruct the role of a faculty advisor in reconceptualizing secondary mathematics teacher education programs and field experiences. This paper focuses predominantly on the second purpose, using Foucault's concepts of discourse, dividing practices, and technologies of the self to analyze two particular storylines from the research data.

CONTEXT, METHODOLOGY, AND THEORETICAL PERSPECTIVES

As a mathematics teacher educator and faculty advisor in an undergraduate teacher education program, one of my greatest challenges is encouraging pre-service teachers to reflect and act on inquiry-based strategies for instruction and assessment in secondary school mathematics. Research indicates that, in spite of introducing new strategies during curriculum courses in teacher education programs, traditional textbook and teacher-directed approaches still prevail in many secondary mathematics classrooms (Lerman, 2001; Nolan, 2006). Pre-service teachers' negotiations of theory-practice transitions from university mathematics curriculum courses to field experience in secondary school classrooms call for an exploration of innovative models for mentoring and professional development. To respond to this call, I initiated a research project to explore the possibilities of a blended real and virtual model for faculty advising during field experience.

At my university (University of Regina), the culminating field experience in a four-year undergraduate teacher education program is a four-month internship experience in schools. Each intern is paired with a cooperating (mentor) teacher and assigned a university faculty advisor. As part of the faculty-intern mentorship relationship and professional development process, faculty advisors are expected to visit, observe and conference with their interns three-five times during this four-month internship. A mentorship based on only three-five visits over four months, however, is not adequate to bring about change in pedagogy nor to disrupt and challenge the view that teacher education programs merely train and prepare pre-services for the *real* experience of school classrooms. As a faculty advisor, my role in the internship feels superfluous, since the cooperating teacher is ultimately responsible for overseeing and evaluating the day-to-day activities of the intern. I have often reflected on my faculty advisor role in the context of feeling like a third wheel—one that does not have clearly defined boundaries or expectations associated with it, and one generally not valued on a journey mapped out for two (intern and cooperating teacher). Through these reflections, I became motivated to better understand my positioning (as a teacher educator and faculty advisor) and the positioning of the pre-service teachers with whom I work during this four month internship.

According to research (Towers, 2008; Van Zoest & Bohl, 2002), the school intern placements—the 'real' classrooms—are frequently not well-aligned with the inquiry-based philosophy and pedagogy being advocated in teacher education programs. In reconceptualizing mathematics teacher education, one must consider the importance of either selecting placement schools with an environment that is well-aligned with reform ideas or working toward "a redefinition and expansion of the role of teacher educator" (Van Zoest & Bohl, 2002, p. 285). In my case—in pondering the shortcomings of the traditional faculty advisor model at my university, including dissatisfaction with my superfluous role in the entire process— I was drawn to the opportunity for redefinition and expansion. I decided to develop a faculty advisor model that would supplement the three-five 'real' classroom visits with additional 'virtual' visits. Thus, the self-study project was initiated in order to understand my role as a faculty advisor and how I could make the role more meaningful to me and, hopefully as a result, also make it more valuable and meaningful to the interns and cooperating teachers.

As a methodology, self-study can be defined as the intentional and systematic inquiry into one's own practice. In teacher education, self-study is powerful because of the potential to influence pre-service teachers, as well as impact one's practice as a teacher educator. Furthermore, self-study can have a broader impact by informing other teacher educators' practice (Loughran, 2007) and thus other teacher education programs. According to LaBoskey (2004), self-study research practice should produce two kinds of knowledge: the personal embodied knowledge of the researcher that often leads to his/her own transformed and reformed practices, and a form of public knowledge that (could) contribute to the transformed and reformed practice of others (in this case, other teacher educators and pre-service mathematics teachers). Both these forms of knowledge are significant in the context of my own Faculty of Education, as well as beyond this context into other teacher education programs across the country and internationally.

Framing self-study in Foucauldian discourse analysis privileges teachers' stories (text and spoken discourse) while providing insights into the ideologies and power relations involved in the discursive practices of mathematics teacher education and field experience. This research draws on poststructural and socio-cultural theories to challenge and disrupt the traditional discourses of mathematics teacher education and field experience and to integrate more reflexive, critical approaches to learning to teach, and teaching to learn, mathematics (Skovsmose & Borba, 2004; Vithal, 2004). Foucault provides a framework for exploring the normalized practices and discourses of schooling as strong forces in shaping teacher identity and agency. Drawing on Foucault's concepts of discourse, power, dividing practices, and technologies of self (Foucault, 1977, 1988; Walshaw, 2007, 2010), this paper explores the practices and technologies of a faculty advisor in secondary mathematics teacher education.

Methods and Data Sources

To transform and reform my own practice as a teacher educator and faculty advisor, my research involves listening to the stories of pre-service teachers. Thus, data collection for this self-study includes interviews and focus groups with the three (3) pre-service teachers (interns) with whom I worked as a faculty advisor during two internship semesters (2009, 2010). The interviews and focus groups were conducted both in person and through video conferencing software. I also kept a self-study reflective artefact (in the form of a Weblog) to understand the role that the e-advisor model might play in the education of secondary mathematics teachers. Through the use of multiple technologies (such as desktop video conferencing, video flip-cameras, online chat and discussion forums, and collaborative authoring/editing through Wiki spaces), I sought to create and sustain a professional development relationship between myself as a faculty advisor and my pre-service teachers (the first purpose of this research study). The second purpose of the study—that of disrupting traditional notions of teacher education as 'training' and 'preparing' teachers—is the focus of data analysis for this paper.

With year two of the three-year self-study research project now complete, key findings have made an impact on our teacher education programs (see, for example, Badali & Nolan, 2010; Nolan, 2010b), in addition to being used to inform and shape the third year of my research study. During each of the first two years of this self-study, a Professional Learning Community (PLC) was sustained 'virtually' through the use of desktop video conferencing (*Skype* and *Adobe Acrobat Connect Pro*), as well as the recording and analysis of videos (*Flip cameras*), online chat and discussion forums, and collaborative authoring/editing through Wiki spaces. These multiple technologies enabled me to supplement the 'real' (traditional school) visits with 'virtual' visits, such that the faculty-student conferencing process evolved into a mentorship relationship that was ongoing, synchronous, and without geographical boundaries. In this brief paper, it is not possible to conduct a thorough analysis of the data, so instead I focus on two particular storylines in the data. The first relates to my attempts to disrupt traditional notions of faculty supervision and classroom observations through the use of flip video cameras; the second storyline conveys my efforts to understand the interns' perceptions of my role as a faculty advisor in their professional development during the internship semester.

Discussions and Interpretations of Data

In the first storyline of the data, I sought to deconstruct traditional models of faculty supervision based in surveillance and 'super'vision techniques. Typically, as a faculty advisor, I would position myself in the back of the classroom to 'observe' the interns, taking notes on anything and everything that the intern might want to know about her practice. While the intern may identify a few targets for me to focus on, it is generally expected that I comment on the lesson structure (set, development, closure, etc.) as well as student and classroom dynamics, from the start of the lesson until the end. My goal in this research, however, was to disrupt such techniques of observation and surveillance. One way I approached this was to ask each intern to use a flip video camera to record three (3) short segments of a lesson, which they uploaded to a website for me to access and view. It was challenging for the interns to understand how this process might provide me with enough visual and auditory 'evidence' of their teaching so as to provide them with appropriate feedback for their professional development. Once they were able to move beyond the notion of professional feedback coming in only one form and obtained by me being physically in the classroom from beginning to end of the lesson, I think that they actually learned a great deal from the process. Instead of the faculty advisor taking on the role of identifying what aspects of a lesson were worth focusing on and analyzing, I was requiring the interns to initiate this decision and reflection on their own, thus disrupting systems of power that "both produce and sustain meanings that people make of themselves" (Walshaw, 2010, p. 111).

After viewing the videos, I engaged in a *Skype* conference with each intern individually. I was able to begin this "post-lesson conference" in a manner quite different from the usual conferences that take place after I have been physically in the classroom for the entire lesson. This time I began the post-conference by asking each intern why she selected these particular aspects of the lesson to capture on video for me to see. As part of my research Weblog, I reflected on how these short video segments changed in focus as the semester progressed. At the start of the semester, each intern captured *herself* on video, generally positioned at the front of the room beside the whiteboard (or SmartBoard), displaying the traditional teacher-centred image of "teaching." As the semester progressed, however, the interns began to feature their *students*, including their conversations with each other and with the (pre-service) teacher, much more prominently in their "teaching videos." In negotiating their identities within the social and political networks of the discursive practices of schools, it seems that the interns began to engage in a reconstruction of traditional images of teaching and the disciplinary practice of the "panoptical gaze" (Foucault, 1977) that is offered by the university through its faculty supervision techniques.

The second storyline conveys my efforts to understand my interns' perceptions of my role as a faculty advisor in their professional development. During the final focus group session with the three interns (using *Adobe Connect*), I questioned the three of them on my role as faculty advisor and its overall value to them in terms of their professional development during the internship semester. The following excerpt is taken from the transcript of that focus group session:

> Intern1: If our coop is doing their job right they really *should* be doing that professional development process with us, so having you there is just kind of extra, I guess. I don't know if it's completely necessary. But if you *were* to do it, I would probably still prefer that you come out and see me.
>
> Faculty Advisor: What [Intern1] brings up is interesting… do you see my role from the university being different—or potentially should be different— than what the cooperating teacher has to offer you in your professional development?

> Intern2: Well, *I* have because [my cooperating teacher] was very new to this… I know that you two have had *some* contact, I don't know how much, but it was nice for her to be able to have you as a liaison kind of, to ask questions to and bounce ideas off of on what her role is because, really, she had no idea of what she was supposed to do.
> Intern1: And I know with [my cooperating teacher] it really *wasn't* different because he's done this before and he's had experience and… like, if I had had problems with him then I would want you there, I would *need* someone else, but since we got along then the roles kind of seem the same to me.
> Intern3: [My coop] and I have had that same conversation because [she] *has* had some interns in the past that didn't go over so well. I don't know, things seemed to go really well with us but if they *didn't*, like [Intern1] said, it would have been nice to have you out…

I didn't exactly leave the conversation here; in fact, I went on to 'dig a little deeper' into their responses to my question. Even though they eventually added that I may have provided them with a few things that their cooperating teachers did not (for example, a few ideas and methods, a once-per-month growth perspective that the coop could not give, etc.), the underlying message of their responses was essentially the same: I was "just kind of extra."

My efforts to disrupt the benign and distant role of the faculty advisor had been constituted by the interns as *extra* and *much the same as the cooperating teacher had to offer*. In their eyes, I had not expanded and redefined my role in the manner I set out to; instead, the interns constructed an identity for me as liaison, mediator, umpire, even peacemaker, but definitely *not* advisor or mentor. They specified that, as long as there are no "problems" with the cooperating teacher, I am not needed. However, the interns' constructed 'truth' of my value as a faculty advisor is not so simply stated if one draws on Foucault's concepts of dividing practices and technologies of the self as a means to unpacking it.

Once in the schools for their field experience, pre-service teachers are "confronted with the task of learning the discursive codes of practice" (Walshaw, 2007, p. 124) in the secondary mathematics classroom, and no longer in my own university classroom. It makes sense that interns would identify their cooperating teachers as being much better positioned to initiate them into these practices. Embedded within the discursive codes of classroom practices are dividing practices—those practices that function to categorize and distinguish people/contexts, and which can be seen to operate across various social situations. Systems of dividing practices have a strong impact on identity formation and "[d]ividing practices that are at odds with each other are most keenly felt by pre-service teachers as they move from one disciplinary institutional site to another" (Walshaw, 2007, p. 100). Since different practices across different contexts contribute to the constitution of one's identity, there can be a sense of internal conflict in attempting to reconcile competing practices. Foucault (1988) proposes that people come to understand (or know the 'truth' about) themselves through various techniques, or technologies. He describes four such technologies, which are interrelated and together form a specific body of reasoning techniques. For the purposes of this paper and brief analysis, Foucault's concept of "technology of the self" is most useful in understanding the truth games being played by the interns.

According to Foucault (1988), technologies of the self "permit individuals to effect by their own means or with the help of others a certain number of operations on their own bodies and souls, thoughts, conduct, and way of being, so as to transform themselves in order to attain a certain state of happiness, purity, wisdom, perfection, or immortality" (p. 18). Interpreting the interns' above response to my question on the value of my role as faculty advisor in the context of Foucault's concepts of dividing practices and technology of the self, I gain a better understanding of the interns' constructed truth stories. As Walshaw (2007) puts it, "the transfer from the university course into the school brought specific dividing practices sharply into focus" (p. 126). Since interns must strategically negotiate the social and political networks of power in the cooperating teacher's classroom, this probably means making a choice: between engaging the technologies and practices which are part of the cooperating teacher's mathematics classroom or engaging (valuing) the set of technologies and practices that I ask them to import, so to speak, from the university classroom. As long as everyone is "getting along" (which, in the context of the transcript excerpt, I interpret to mean that there are no obvious personality tensions or conflicts at work between intern and cooperating teacher) then my technologies and dividing practices are not needed; in fact, they are viewed as a third wheel on the journey, and not one that is easily reconciled by interns as they seek to "transform themselves in order to attain a certain state of happiness, purity, wisdom, perfection…" (Foucault, 1988, p. 18).

CONCLUDING THOUGHTS

Firstly, it should be noted (almost as an aside to the storylines of this paper) that introducing *virtual* visits with the interns has resulted in *real* examples of how the use of multiple technologies can create a faculty-intern relationship that establishes more of a continuum between university courses and internship field experience (Mulholland, Nolan, & Salm, 2010; Nolan & Exner, 2009). The significance of this relationship as interpreted by interns, cooperating teachers and faculty advisors is, however, an issue to unpack. I initiated this self-study research project in order to understand how I could make my role as a faculty advisor more meaningful and valuable to me as well as to the interns and cooperating teachers. Through the lens of Foucault, one sees how multiple 'truths' are constructed for what it means to be meaningful and valuable. I must not confuse *my* truth of what is meaningful and valuable with the *interns*' constructed truth for the same. At the very least, I should not expect my practices and technologies to have a visible (and immediate) impact on my interns' truths at this point in their teacher identity constructions and negotiations. As I delve more deeply into Foucault's concepts for the purposes of further analyzing this research, I appreciate that I am being asked "to always be mindful of the technologies and truth games that simultaneously mold and discipline ways of thinking and being" (Hassett, 2010, p. 462).

Nonetheless, by pursuing what I initially thought would be a meaningful way to fulfill my role as a faculty advisor I was surprised by several emergent aspects of the research. For example, I was able to engage pre-service teachers in dialogue on the role of field experience in shaping student-teacher transitions (even if I was not all that pleased with what I heard in the dialogue!); to disrupt notions of the roles of teacher education programs and field experience; and to challenge traditional images of faculty observation and supervision. Teacher education programs are currently steeped in a technical rational model, reflected in the normalized use of language such as teacher 'training' and 'preparation'. To challenge the dominant image of teacher education as the 'place' where theory makes the transition to practice through teaching tips and techniques, this research takes critical steps toward reconceptualizing secondary mathematics teacher education and associated field experiences.

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INTEGRATING CONTENT AND PEDAGOGY IN SECONDARY TEACHER PREPARATION: COLLABORATING ACROSS DISCIPLINARY BOUNDARIES

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Traditionally, there has been a sharp separation between mathematics content courses and methods courses for secondary school math education majors. Content courses, many times housed in mathematics departments, and methods courses, housed in colleges of education, provide two different but important perspectives on teaching mathematics. However, often these courses end up working at cross-purposes. This study examines the territory of cross-disciplinary collaboration between mathematicians and mathematics educators in order to understand the ways in which these courses can better integrate mathematical content and pedagogical issues. Observing these collaborations provides a window into the shared and conflicting philosophies of these disciplines.

Undergraduate mathematics teacher education in the U.S. typically consists of content preparation through mathematics courses taught by mathematics department faculty, general education courses taught by college of education faculty, and mathematics pedagogy courses taught by mathematics educators in either colleges of education or mathematics departments (depending on where the secondary teacher preparation program is housed). Recent modifications of this structure include (1) the addition of a “capstone” course focusing on connecting the mathematics students study in college with the topics they will teach, (2) the institution of separate sections of selected upper division mathematics courses for teachers, and (3) the integration of students’ classroom-based experiences with pedagogy courses and the expansion of pedagogy courses over multiple semesters. However, despite these changes, separation of the study of mathematics from the study of mathematics curriculum and pedagogy continues across much of students’ preparation for teaching. The separation persists, in large part, because effective methods for cross-disciplinary collaboration in service of teacher education are not well understood.

The separation of subject matter knowledge from teaching methods knowledge is unnatural and ineffective – teachers must use their mathematical knowledge *in practice* and continually build new knowledge *through practice*. Teacher preparation will be more effective when students are supported in building integrated knowledge; that is, knowledge of mathematics that is tied to issues and concerns of teaching, and knowledge of teaching that is embedded in mathematical understandings. How best to link content knowledge development to development of knowledge of pedagogy is a central question. It is a problem that demands the combined attention of mathematicians and mathematics educators. However, jurisdiction over teacher preparation is generally divided across separate and distinct academic units, and administrative and cultural barriers impede communication and joint work. Clearly, if teacher preparation programs are to be organized so as to support the development of integrated mathematical knowledge for teaching, then traditional disciplinary lines must be blurred and collaborations among mathematicians and mathematics teacher educators must become the rule rather than the exception. Without robust and generalizable knowledge about the processes and products of collaboration, productive collaborations will continue to occur in a hit-or-miss fashion, if they occur at all.

The research reported on here attempts to address this need through an investigation of five math/math-ed co-teaching collaborations. Each collaboration consisted of one university mathematician and one university mathematics teacher educator who co-taught both methods and mathematics courses for future middle and high school teachers. The following research questions guided the investigation.

- How do collaborations develop and evolve? What is beneficial and what is challenging about collaborating?
- What intellectual resources (knowledge, tools, practices) are brought to collaborations and developed through collaborations?
- How does the collaborative process influence the collaborators (e.g., how they think about content/methods, how they think about their practice)?

Theoretical framework

Underlying the investigation is a sociocultural perspective on learning, in which learning is understood to be a fundamentally social phenomenon inseparable from its historical, cultural, and institutional context (Vygotsky, 1981; Wertsch, 1985; 1998). Sociocultural theory assumes that access to the world is mediated by psychological and cultural tools (e.g., languages or other symbol systems). These tools perform supportive, constraining, and transformational functions. Sociocultural researchers examine interactions among human agents as mediated by psychological and cultural tools, and how these interactions are situated within local and broader contexts (e.g., classroom communities, disciplines, and institutions).

Within a sociocultural framework, an academic discipline is viewed as a set of theories, texts, and methods, but also as a community of individuals with particular social practices and styles of discourse (Lattuca, 2002). The methods and practices of disciplines frame the thinking of those who work within them; members of a discipline tend to "see" things in similar ways. At the same time, it is possible to find great variations among members of a single discipline with respect to methodologies, pedagogies, and epistemologies (Lattuca, 2001). Sociocultural researchers interested in cross-disciplinary collaborations must consider how individuals are situated within their respective disciplines, how the disciplines are situated within institutions, and how the disciplines are situated with respect to one another. They must also consider the historical contexts of these interrelationships. Studying any of these aspects of cross-disciplinary collaboration presupposes an understanding of each discipline in terms of (1) its historical, cultural, and institutional context, (2) its tools and methods, and (3) its social practices and discourses. Although a full, or even partial, description of the disciplines of mathematics and mathematics education is not possible in the space of this report, the following is a very brief summary of each discipline, focusing on historical context, methods, and practices.

Mathematics education developed as an area of study at universities during the early 1900s, with research in mathematics education growing out of the fields of mathematics and psychology (Kilpatrick, 1992). Nesher (1997) describes "investigations that could improve the practice of learning mathematics" and "defining (and redefining) the boundaries of mathematics as a subject matter to be learned in school" (p. 34) as two central activities within the discipline of mathematics education. To this list a third activity may be added: developing the pedagogical and pedagogical content knowledge (Shulman, 1986) of future and current teachers. Until quite recently, investigations within the discipline utilized mainly behaviorist theories and an objectivist research paradigm, with quantitative research methods borrowed from the "hard" sciences. In the 1980s, neo-Piagetian, constructivist epistemologies, and associated research methodologies, began to be adopted and utilized by mathematics educators and researchers. Such theories underlie much of the current research on mathematics learning, as well as the practices utilized by (and advocated by) mathematics educators in their instruction of pre-service and in-service teachers.

Mathematics education research took a "social turn" in the late 1980s, however, with the adoption of "theories that [view] meaning, thinking, and reasoning as products of social activity (Lerman, 2000, p. 23). Over the past decade and a half, the analytical lens has shifted from the individual to the group, and many researchers now focus on interactions around mathematics within small groups of students and across large groups of students and teachers (Tsatsaroni, Lerman, & Xu, 2003). Following this trend in mathematics education research, methods for facilitating the development of "mathematical learning communities" have become a focus of teacher education. Currently, both psychological and sociological perspectives influence the content and process of both research and teacher education. With respect to knowledge for teaching mathematics, current influences on the field include the idea of "principled knowledge" as described in *The Mathematical Education of Teachers* (Conference Board of the Mathematical Sciences (CBMS), 2001), the concept of pedagogical content knowledge (Shulman, 1986) and its key components as described by Grossman (1990), and descriptions of the mathematics teachers do in practice (Ball, 2003) and mathematical knowledge for teaching (Ball, Thames, & Phelps, 2005).

Mathematics is typically seen by its practitioners not as a cultural product, but rather as the study of external objects whose properties are not in doubt once discerned and established. "Every problem of mathematics gets solved sooner or later. Once solved, a mathematical problem is forever finished: no later event will disprove a correct solution" (Rota, 2006). This Platonic stance was challenged by discoveries in geometry and analysis in the 19th century, and by problems with the foundations of mathematics in the early 20th century. These events led to various solutions such as logicism, formalism, and intuitionism, all of which, however, retained as a central value the underlying certainty of mathematical ideas (Hersh, 1979). The logical structure of mathematics and the relative permanence of its ideas results in a culture that enjoys an extraordinary degree of consensus compared to other scientific disciplines. Mathematicians tend to view their subject as having a rigid structure, a cumulative nature, and an extraordinary compactness. Paradoxically, this allows for a combative intellectual culture—when there is little doubt that the truth or falsity of a proposition will ultimately be settled by appeal to external norms, there is no need for social practices of negotiation and compromise. Recently, however, some mathematicians have advocated grounding the philosophy of mathematics in human practice, viewing mathematics as a subject whose ideas and methods are still external to its practitioners but nonetheless embedded in human culture (Gowers, 2006; Lakatos, 1976).

Mathematicians can be characterized as working within a "realist" mode of inquiry, while mathematics educators are frequently "relativists." Becher (1987) suggests that an important division exists between these modes of inquiry. For example, scholars working in the humanities (relativists) tend to explore particulars or complexity and interpretation, while pure or natural scientists (realists) focus on universals and simplification (Frost & Jean, 2003). The research reported on here is grounded in an awareness of the ways in which disciplines may differ, but also a wariness of simplistic or naïve assumptions about the nature of disciplines. *Both* *differences in the disciplines themselves and beliefs about differences influence the cross-disciplinary interactions of individuals.*

Collaboration across the disciplines of mathematics and mathematics education is the focus of this research report. Although many definitions of collaboration exist, most include the following elements: (1) commitment to a common goal and the use of a coordinated effort to achieve it, (2) shared authority over the processes and products of collaboration, and (3) sustained engagement in interactions that promote collective and integrative cognition (Amey & Brown, 2004; Bruffee, 1993; John-Steiner, Weber, & Minnis, 1998). From a sociocultural perspective, a fourth element must be added – the development and utilization of a shared discourse through the appropriation and transformation of psychological and cultural tools. In our research, we seek to understand the how this shared discourse develops, how difference in disciplinary cultures are negotiated, and how mathematics teacher education can be enriched by the creation of interdisciplinary knowledge and practice.

A central goal of the collaborations reported on here was to interweave of content and pedagogy in courses in which these two elements have traditionally been separated. The method for achieving this goal was co-teaching, including joint planning, implementation, reflection, and evaluation. Of course, collaborators varied in the nature and development of their collaborations, including the ways in which they engaged together with students in the classroom. Analysis of the nature and processes of co-teaching encompass four broad areas – planning, content integration, teaching, and assessment (Davis, 1995). With respect to content integration, analysis is guided by the concept of "principled knowledge" as described by CBMS (2001), and categories and descriptions of "mathematics knowledge for teaching" suggested by Ball, Thames, and Phelps (2005), which include (1) common content knowledge, (2) specialized content knowledge, (3) knowledge of content and students, and (4) knowledge of content and teaching. The use of this framework allows for analysis of the extent to which collaborations between mathematicians and mathematics educators support the development of the specific types of mathematical knowledge that prospective teachers will need in practice. With respect to the analysis of the teaching component of the collaborations, we utilize the classification of the classroom roles suggested by Haynes (2002), which include model learner, observer, co-lecturer, discussion leader, case facilitator, and resource, and (2) the dimensions of collaboration described by McDaniel & Colarulli (1997), including the degree of integration of ideas and perspectives, the degree of interaction of faculty members with students, the degree of student engagement, and the degree of faculty interdependence.

Methods

Multiple data sources are being used to investigate the work of each collaborative team. All meetings of collaborative teams at their home institutions were audio-taped by the team members themselves, "records of practice" were created by each team for one unit of instruction per course taught, and team members were interviewed individually at regular intervals (approximately twice per month) by research team members during the semesters during which the collaborative teams co-taught. The interviews are being used to gain insight into underlying perspectives of the collaborators and the influence of their perspectives on the collaborations.

The unit of analysis in the study is the interactions between team members, as mediated by psychological and culturally specific tools, and as situated within sociocultural settings. For our purposes, an "interaction" is defined as a bi-directional instance of communication around a common topic. The beginning of an interaction is marked by entry into the topic by one collaborator, followed by an on-topic response by another collaborator. The end of an interaction is marked by disengagement from the topic by one collaborator. Analysis of movement between topics, i.e. “boundaries of interactions”, is crucial, as it is in these spaces where power relations are most transparent. Analysis of strings of interactions over time allow for the identification of patterns, with respect both to the development of the intellectual terrain and the nature and process of collaboration. Rather than depending on self-reports, data includes transcriptions of all meetings of collaborative pairs at their home institutions. The within-collaboration purpose of these meetings is to plan and reflect upon course activities, but from a research perspective the meetings are an irreplaceable source of unfiltered information about the nature of out-of-class collaborative work. Within the courses, videotape was collected for one unit of instruction (minimum of 3 hours of instruction).

Analysis includes the identification of tools and the ways they are utilized and transformed by team members as they interact with each other and with students, and patterns of interaction within teams (as mediated by tools), e.g., the development of a shared discourse, and how it evolves over the course of the collaboration. Discourse analysis techniques are being used to examine the evolution of speech patterns within teams, including the nature of speaking turns, and the use of contrastive pronouns (Edwards & Lampert, 1993; Gee, 2005). Consideration is paid to the way tools limit and constrain interactions as well as the ways they afford and transform it, with the goal of identifying both generative and limiting collaborative practices.

Analysis of the nature and processes of collaboration encompass four broad areas – planning, content integration, teaching, and assessment (Davis, 1995). We use qualitative data analysis software (Transana) to organize segments of interaction into conceptual categories, and analytical frames derived from the theoretical framework as starting points for analysis. However, we also develop meanings and relationships through the analysis process. With respect to teaching, analysis is guided by (1) the classification of the classroom roles of team teachers suggested by Haynes (2002), which include model learner, observer, co-lecturer, discussion leader, case facilitator, and resource, and (2) the dimensions of collaboration described by McDaniel and Colarulli (1997), including the degree of integration of ideas and perspectives, the degree of interaction of faculty members with students, the degree of student engagement, and the degree of faculty interdependence. Attention is also paid to issues identified in the literature on collaborative practice, including differences in roles, time demands, competing obligations, expectations, and differences in work methods (John-Steiner, Weber, & Minnis, 1998).

The trustworthiness and generalizability of findings are established by (1) the independent analysis of a representative samples of data by multiple individuals trained in the use of the particular theoretical and analytic frames adopted in this study, (2) the systematic search for alternative categorizations and meanings, and disconfirming evidence, throughout the data analysis process, (3) “member checks” in which findings are discussed with individual participants, and (4) the use of multiple sources of evidence (i.e., triangulation). With respect to triangulation, findings are based on the following data sources: (1) within-team meetings that occurred throughout the semesters of co-teaching, (2) bi-monthly interviews with each team member, (3) “records of practice,” including videotape, for of one unit of instruction per course taught, (4) interviews of colleagues at participating institutions, and (5) discussions of findings with team members.

Results

Preliminary results have yielded a promising window into the rewards and challenges of collaboration across fields. Collaborators participating in the study not only created meaningful strategies for the integration of content and pedagogy in their classrooms, but also uncovered their own basic assumptions about the effectiveness of particular teaching methods. Issues that arose across the teams included the lack of adequate preparation time, the challenges of decision-making with a partner (both within and without the classroom), and the demands of clear communication with students. However, every participating team felt they had a greater understanding of the norms and concerns of their partner's field of study as a result of the collaboration. Mathematics teacher educators spoke of gaining a greater sense of the importance of mathematical rigor and why mathematicians emphasized it so strongly in class. Mathematicians were exposed to alternate ways of teaching mathematics and, in some cases, carried these new methods into courses that were not co-taught(or even necessarily made up of preservice teachers. Results from two of the four sites, encompassing three collaborations, will be discussed in further detail here.

Southeast

This site was home to two collaborating teams who co-taught a total of five courses. Elizabeth Durand, the mathematician, and Cynthia Roberts, the mathematics educator, were initially wary of the other's knowledge in their own primary field of work. However, they found their teaching philosophies to be more aligned than they had previously assumed. Dr. Durand jumped eagerly into unfamiliar teaching actions, like facilitating class discussions, even as she struggled with being so far out of her comfort zone. Dr. Roberts, meanwhile, found more mathematical confidence from working with Dr. Durand. In contrast, the other team from this site, Angela Laurent and Dejan Merakovski, started their collaboration with extremely different teaching styles and underlying philosophies. This led to extensive conversations on the use of technology in the classroom and the development of appropriate tools for assessment. In fact, one discussion that spanned two semesters of their Geometry course concerned the creation and utilization of a rubric for grading homework. A career mathematician, Dr. Merakovski expected formal mathematical proofs from his students, and had previously relied on a primarily intuitive method of evaluating student work. He "knew" when a proof was sufficiently rigorous. However, since he and Dr. Laurent shared responsibility for grading homework, it quickly became necessary to create a system that ensured consistency in assessment. Dr. Laurent's solution to the problem was to create a formal rubric that clearly let students know what was necessary for a mathematically rigorous proof. For Dr. Merakovski, however, the rubric was too codified and seemed to take some of the elegance out of a well-constructed proof. This difference of opinion was central to understanding the different approaches to teaching and mathematics that these collaborators embraced. While Dr. Laurent proritized communication and clarity over other considerations, Dr. Merakovski felt that part of his job as a mathematics professor was to help students grasp some of the more ephemeral concerns of aesthetics in mathematics.

Developing good mathematical communication in students was a central objective in both collaborations. To this end, Drs. Durant and Roberts, frustrated by the quality of work they were receiving from their students, began to ask their students to critique responses that had been produced by their own classmates. Responses were culled from quizzes they gave and activities were built around assessing the responses for correctness, clarity and completeness. When working on these activities in groups, not only did students usually manage to appropriately assess their peers' work, they also gained the pedagogical experience of evaluating student work.

The pedagogical aspects of these activities were also assessed in the form of a test question that asked the preservice teachers to critique hypothetical student answers to mathematical problems.

Midwest

At this site, the collaborators had co-taught a course before they participated in the study, so they spent less time negotiating their way through the preliminaries of collaboration and more time explicitly thinking about strategies for integrating content and pedagogy. Moreover, due to the structure of the mathematics education program at this site, the mathematician, Ibrahim Haddad, had already garnered some experience with preservice teachers. Given the chance to be involved with the methods course, however, gave Dr. Haddad a chance to observe student teachers in action and experience life in secondary school classrooms. This exposure led him to reevaluate some of the assumptions he had made about the efficacy of certain elements of the curriculum for preservice teachers.

Since Dr. Haddad and his partner Chris Adams began the study as a functioning team, they began planning the courses with a shared vision: there would be a clear mathematical goal and a clear pedagogical goal for each class meeting of each course. They developed instructional cycles for both the content class and the methods class with the intent of constantly contextualizing content within classroom realities (content course) and continually modeling teaching methods by teaching content (methods course). Both instructional cycles included a "classroom connection," and both courses were designed with "mathematical habits of mind" as an articulated student learning goal. As both of these courses ran concurrently and there was a large intersection between the class rosters, Drs. Adams and Haddad found themselves occasionally losing track of which class was which. In fact, both classes were taught in the same classroom, on alternate days of the week. In planning meetings, the collaborators found themselves moving from discussions of one course into discussions of the other fluidly. Interestingly, the extreme amount of overlap seemed to make it easier for them to integrate content and pedagogy, since the confusion may have broken down preconceived notions of what subject matter belonged in each course.

Data analysis is ongoing, however initial analysis suggests that significant boundaries exist between the mathematics and education communities with respect to (1) appropriate use of in-class time, (2) methods and value of utilizing technology in the service of instruction, and (3) goals and processes of student assessment (discussed earlier). With regard to in-class time, we found the mathematics educators to be more likely than the mathematicians to think of class time as time for "doing mathematics" while the mathematicians saw greater value in demonstrating ideas to the whole class. Importantly, both types of individuals seem to believe that active engagement was essential for learning; the difference was in the role of in-class time in this process. All collaborators felt the fundamental tension of desiring both depth of understanding and coverage of content. With respect to utilization of technology, the mathematics educators developed Powerpoint slides for organizational purposes, to highlight key ideas, and to introduce problems. They also used technological drawing tools (e.g., GeoGebra). The purposes and value of such technology was questioned by the mathematicians, some of whom wondered whether the time and energy spent in learning and using the technology could not be better spent elsewhere.

Discussion

The purpose of the study described here was to investigate the processes and outcomes of mathematics/mathematics education co-teaching collaborations. Participants constituted five collaborative teams and have, to date, co-taught ten courses (mathematics and methods) for future middle and high school teachers. Although significant data analysis is currently in process, we are able to report some findings at this time. The collaborative experience was quite intensive - both intellectually and with respect to the sheer amount of time involved – but all collaborators agree that is has been a worthwhile endeavor. It is important to note here that co-teaching comes with a significant financial cost, since two professors receive credit for teaching one course. However, we would argue that resulting knowledge in the form of connections between mathematical and pedagogical learning for students, and knowledge of what is involved in working across the academic cultures of mathematics and mathematics education are valuable goals deserving of substantial investment.

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MATHEMATICS EDUCATION AS AN EXTENSION OF DEWEYAN REFLECTIVE MIND DEVELOPMENT

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I extend the theoretical work on reflective thinking by John Dewey—in particular, that education involves a certain type of mind development—into the supra-individual realm to argue that mathematics education can involve the development of mathematizing intellectual entities that stretch across student partners, small groups, whole classes, and even teacher professional development collectives. Beginning with the reflective development of their own minds in collaborative practice, teachers can learn the skills necessary to help individual students and collections of students become reflective mathematics problem-solvers.

Introduction

Although mathematics education has used a variety of psychological perspectives on the teaching and learning process to inform its work, these perspectives have, with rare exceptions, focused on how *individual students* make sense of mathematics. Intensive work with constructivist epistemologies, for example, involves only the individual pupil's cognitive constructions. Even social constructivist researchers, although acknowledging the social component of mathematics learning, still maintain the individual as the "locus of learning" (Davis & Simmt, 2003, p. 153) when they examine the social implications for learning; classes still are viewed as "groups-of-individuals" (Lave, 1996, p. 149).

When it comes to teaching mathematics, the story is much the same: Contemporary mathematics instruction is designed for augmenting the learning of individuals. Differentiated instruction is a clear example of carrying classroom mathematics learning to the individual-is-the-focus extreme, where some researchers suggest teachers should adapt their instruction to meet the diverse needs of unique individuals (Darling-Hammond, Ancess, & Ort, 2002). Other researchers have called for smaller class sizes (Bloom, 1984) to facilitate the teaching of the individual. Homogenizing the mathematics class through tracking or grade-level advancement is an attempt to "individualize the group," perhaps glossing over the richness inherent in classroom environments. The fact that students are assembled in a classroom to learn mathematics results from efforts to make schooling efficient and affordable, and not from a plan to create and harness community action for learning. In addition, the competitive nature of classes, the assignment of individual grades, and the assessment of the isolated individual's capacities through closed-book, written examinations confirms that although students are physically together in classrooms, mathematics learning has focus on the individual. This paper extends the theoretical work on reflective thinking by John Dewey—in particular that education involves the development of mind—into the supra-individual realm, to argue that mathematics teaching can involve the development of mathematizing intellectual entities that stretch across collectives such as student groups, classes, and even teacher professional development groups.

Theoretical Perspective and Method

Various researchers have suggested such dominant individualistic perspectives could be balanced with a perspective on the social (Boaler, 1999; Davis & Simmt, 2003; Davis & Sumara, 2001; Schoenfeld, 1994). Such a view would be beneficial, especially as many believe mathematics, as a domain, transcends any individualistic perspective (Ernest, 1990; Romberg, 1994). From this perspective, mathematics emerges through communicative correspondence, socially posed questions, and group deliberations; all mathematical discovery, all mathematical activity, all mathematical explanation can be seen to take place in a complex social context. As Boaler (1999) stated, “The behaviors and practices of students in mathematical situations are not solely mathematical, nor individual, but are emergent as part of the relationships formed between learners and the people and systems of their environments” (p. 260). Even Newton recognized this social dependence: “If I have seen farther than others,” he said, “it is because I stood on the shoulders of giants.”

The field of mathematics education has been developing novel approaches to the study of collective classroom learning (Cobb & Yackel, 1995; Davis & Simmt, 2003). This paper extends this work through a theoretical exploration of the ideas of John Dewey regarding education and reflective thought, as applied to supra-individual phenomena. In particular, I first synthesize his ideas about education, reflective thinking, and mind development, and then I apply these ideas beyond the individual; this theoretical coupling is an instrumental study that generates theory for broader application (Stake, 1994).

Results

According to John Dewey (1981/1933), the act of teaching mathematics is intimately involved with the development of mind, namely, of enhancing the mind’s capacity to think: “We state emphatically that, *upon its intellectual side education consists in the formation of wide-awake, careful, thorough habits of thinking”* (p. 177, emphasis in original). Dewey also stated: “LEARNING IS LEARNING TO THINK” (p. 176, emphasis in original); he believed that the teaching/learning process helped students develop their mental capacities. Dewey also proposed that minds can refine their mode of thinking through practice—this implies that wherever there is mind, there can also be “mind refinement.” He also maintained that minds could operate on different levels, i.e., not all types of thought were equal:

> Some of these ways [of thinking] are better than others; the reasons why they are better can be set forth. The person who understands what the better ways of thinking are and why they are better can, if he [or she] will, change his [or her] own personal ways until they become more effective… The better way of thinking … is called reflective thinking. (p. 133)

John Dewey considered reflective thinking as a special thought process. Out of the experience of life arise perplexities or problems, and “*demand for the solution of a perplexity is the steadying and guiding factor in the entire process of reflection”* (Dewey, 1981/1933, p. 122, emphasis in original). Reflective thinking involves careful scrutiny of available data, the formation of possible hypotheses, and their testing through action. There is “a question to be answered, an ambiguity to be resolved, [which] sets up an end and holds the current of ideas to a definite channel” (p. 121). It is a highly developed, refined, and disciplined mode of thinking, not exercised by all minds. Dewey (1981/1933) stated:

> One can think reflectively only when one is willing to endure suspense and to undergo the trouble of searching. To many persons both suspense of judgment and intellectual

> search are disagreeable; they want to get them ended as soon as possible. They cultivate an over-positive and dogmatic habit of mind, or feel perhaps that a condition of doubt will be regarded as evidence of mental inferiority. (p. 124)

A reflective thinker is able to overcome the urge to make a quick, unjustified, spontaneous decision about a perplexity and begin a process of inquiry that leads to knowledge generation and an understanding of the dilemma:

> It is at the point where examination and test enter into investigation that the difference between reflective thought and bad thinking comes in. To be genuinely thoughtful, we must be willing to sustain and protract that state of doubt which is the stimulus to thorough inquiry, so as not to accept an idea or make positive ascertation [sic] of a belief until justifying reasons have been found. (Dewey, 1981/1933, p. 124)

Through the process of reflective thinking, an individual is able to provide grounded evidence for beliefs held, and act in reasonable, justifiable ways. Refined thinking leads to refined actions, higher forms of thought leading to higher forms of action:

> [Reflective thinking] emancipates us from merely impulsive and merely routine activity…. it enables us to act in deliberate and intentional fashion to attain future objects or to come into command of what is now distant and lacking. By putting the consequences of different ways and lines of action before the mind, it enables us to *know what we are about* when we act. *It converts action that is merely appetitive, blind, and impulsive into intelligent action*. (Dewey, 1981/1933, p. 125, emphasis in original)

In addition, the reflective mind learns through all its actions: "A great advantage of possession of the habit of reflective activity is that failure is not mere failure. It is instructive. The person who really thinks learns quite as much from his [or her] failures as from his [or her] successes…. Nothing shows the trained thinker better than the use he [or she] makes of his [or her] errors and mistakes" (Dewey, 1981/1933, p. 206).

The formation of the mathematics teacher's *individual* mind is at the heart of the teaching process: "The only way to increase the learning of pupils is to augment the quantity and quality of real teaching (Dewey, 1981/1933, p. 140). If the teacher's mind can be adequately formed, reaching sufficient levels of reflective thought, then it is in a position to provide for the formation of other reflective minds such as the students (and as I will argue later, supra-individual minds like the class or professional development groups). Teachers who have developed and refined their minds carefully control their actions, basing their actions upon beliefs that are rigorously grounded on a foundation of evidence. Teachers should be consciously aware of why they act, and should be able to support their decisions and judgments with evidence. As Shulman (1987) stated:

> The goal of teacher education… is not to indoctrinate or train teachers to behave in prescribed ways, but to educate teachers to reason soundly on their teaching as well as to perform skillfully. Sound reasoning requires both a process of thinking about what they are doing and an adequate basis of facts, principles, and experiences from which to reason. Teachers must learn to use their knowledge base to provide the grounds for choices and actions. (p. 15)

Reflective teaching turns teachers into powerful agents that can influence strongly the students in their classrooms. These teachers carefully scrutinize their experience against their knowledge to reach justifiable conclusions and act accordingly. Other's opinions form ideas to be considered rather than mantras to be followed: "The material supplied from the experience of others is testimony: that is to say, evidence submitted by others that is to be employed by one's own judgment in reaching a conclusion" (Dewey, 1981/1933, p. 323). These teachers are powerful because of their powerful actions; powerful actions resulting from powerful modes of thought.

When teachers are reflective, they have a greater capability of helping their students to become reflective. Children naturally exhibit reflective behavior (Dewey, 1981/1933, p. 181); the key is only to capitalize on this natural instinct. In addition, "any subject… is intellectual… in its function—in its power to start and direct significant inquiry and reflection" (Dewey, 1981/1933, p. 149). It seems surprising then that many teachers are unable to combine naturally reflective children with subject matter that possesses reflective capabilities and generate reflective discussion. This is because many teachers are not yet reflected thinkers themselves. "Many a teacher is misled into supposing that he [or she] is engaged in disciplining the mind of pupils, when in reality he [or she] is creating an aversion to study and a belief that using the mind is a disagreeable, instead of a delightful, operation" (Dewey, 1981/1933, p. 183). As John Dewey mentioned:

> Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his [or her] own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect the mental integrity and force of children. (1904, p. 22)

Preservice, beginning, or experienced teachers should strive to become better reflective thinkers so as to better form the minds in their midst; collaboration is a principal means to achieve this.

Working in communities would help preservice, beginning, and experienced teachers develop their reflective capacities. The community provides an environment for the "steady infusion of new ideas" (Hiebert & Stigler, 2000, p. 15) and as a testing element for inquiry. Rodgers (2002) stated:

> No teacher outgrows the need for others' perspectives, experience and support— not if they are interested in being what Dewey calls life-long students of teaching. The community… serves as a testing ground for an individual's understanding as it moves from the realm of the personal to the public. A reflective community also provides a forum wherein the individual can put form to what it is he or she was thinking—or feeling—in the first place. (p. 857)

Schön (1983) explained the importance of community for reflection: "The teacher's isolation in [his or] her classroom works against reflection-in-action. [He or she] needs to communicate [his or] her private puzzles and insights, to test them against the views of [his or] her peers" (p. 333). Communities help teachers "to realize how difficult and yet how essential it is to be able to consider phenomena from various perspectives and to experience others' constructions of meaning in the development of one's own knowledge" (Feldt, 1993, p. 401). Rodgers (2002) also considers other benefits of community for reflective development:

> I have identified at least three factors that highlight the benefits of collaborative reflection: 1) affirmation of the value of one's experience: In isolation what matters can be too easily dismissed as unimportant; 2) seeing things "newly": Others offer alternative meanings, broadening the field of understanding; 3) support to engage in the process of inquiry: The self-discipline required for the kind of reflection that Dewey advocates, especially given the overwhelming demands of a teacher's day, is difficult to sustain alone. (p. 857)

The community provides both the support and training needed for reflective development. Teachers can be exposed to others' ways of thinking, and experiment and test ideas in a non-threatening environment (Stigler & Hiebert, 1999, p. 152).

By working in communities, teachers can develop and refine their minds' capacities for reflective thought. Feldt (1993) stated that Piaget himself "emphasized that … peer interaction, that is, discussion and intellectual exchange, facilitated the knowledge-acquisition process by serving as a catalyst for reflective thought" (p. 401). Shulman has said (as quoted by Frykholm, 1998, p. 318):

> This teaching [*Standards*-based] can only be achieved if we also buy a Deweyan model of teacher development where the end of teacher development is not some fixed and final set of teaching knowledge, understandings, skills, [or] competencies… If we recognize teacher learning is an ongoing process with a form of inquiry as its goal, then that type of inquiry must go on in institutions and settings beginning with teacher education and continuing throughout their careers. [These settings must be] characterized by teachers actively contemplating their own teaching and their students' learning, having the opportunity to reflect on these activities… and to do that with teachers who are engaged in similar activities, in a culture that makes space and time and rewards consistent with the notion that if teachers aren't learning, student learning will be a transient phenomenon. That is the Deweyan vision. (Shulman, 1995)

Working effectively with colleagues is an essential component in the reflective development of any teacher. In a way, a teacher's professional development group forms a cognitve entity that tutors the teacher.

Following Dewey's belief that minds can operate at different levels of effectiveness, I extend this idea to include that *mind*, wherever it may be found, either individual or collectively-formed, can function at different levels of cognitive thought—and reflective thinking forms a better mode of operation for *any* mind. Like the individual human mind, this implies supra-individual minds can operate at various cognitive levels as well, the highest and most developed being reflective and collective cognition. Sounds far-fetched? It shouldn't be: Researchers have been touting the brilliance of collectively-formed minds from lowly bacteria to the wisdom of human crowds for decades (see Shapiro, 1998 or Surowiecki, 2004 for concise reviews of this literature).

The mathematics classroom environment provides an ideal locale to bring together minds. In school, minds are brought together in one location, but their proximity in space does not necessitate their being brought *together cognitively*. Teachers have the opportunity of taking the individual minds present in their classroom and weaving them *together* to form a higher cognitive identity, galvanizing the collection of single student minds into a larger collective thinking unit—a class mind or a supra-student mind, if you will. Thus, mathematics teachers can be involved in mind formation at various levels: both the creation of larger minds and the refinement of existing minds toward reflective possibilities.

Discussion

A handful of researchers have begun to investigate mathematical learning in such collectives using the newly-developed theory of *complexity theory* (Davis & Simmt, 2003), which investigates the cognition of learning systems: Mathematics is a collective action by thinking entities that engage in communal mathematizing—as a whole—and provide substantial opportunities for individual contributions. Because "complex systems transcend their components" (Davis & Sumara, 2001, p. 88), creating novel phenomena unpredictable from the components' behavior, a new kind of research is needed to understand the behavior of these systems:

> At each level of complexity entirely new properties appear, and the understanding of the new behaviors requires research which I think is as fundamental in its nature as any other.… At each stage entirely new laws, concepts, and generalizations are necessary, requiring inspiration and creativity to just as great a degree as in the previous one. Psychology is not applied biology, nor is biology applied chemistry. (Anderson, 1972, p. 393)

Complexity theory allows mathematics classes functioning jointly as mathematizing superminds to be envisioned and studied from a scientific perspective. It opens up a vista heretofore unseen by previous individualistic perspectives.

These complex systems are composite entities formed from interacting, interrelated components: "For reasons that are not fully understood, under certain circumstances agents can spontaneously cohere into functional collectives—that is, they can come together into unities that have … potential realities that are not represented by the individual agents themselves" (Davis & Simmt, 2003, p. 141). These smaller entities interact synergistically to form a whole whose potentialities are larger than the sum of the parts. The whole becomes an object with power that did not exist previously in any of the components, and the whole often exhibits holistic learning capabilities at the system level (Delic & Dum, 2005). Examples of complex systems abound, from ant colonies to economies to nations. The human body is made up of trillions of individual organisms—cells—that are independent living creatures (they can even be separated from the larger host and kept alive, as in blood transfusions, organ transplants, skin grafts, etc.), but when brought together these tiny creatures interact in such a way as to form a larger whole that is much more than the sum of its parts. The cells exist together not only in one location, they also exist together functionally. And just as individual cells can form a larger person, so too can individuals in a classroom merge to form a larger learning entity—a mathematically functioning classroom "organism."

Complexity theory may help researchers understand classroom dynamics because students' actions are affecting the system they constitute while simultaneously being affected by that system (Davis & Simmt, 2003). The mathematical development occurring is an entire class phenomenon—the result of "joint productive activity" (Stein & Brown, 1997, p. 175). Knowledge becomes "stretched over" (Lave, 1988, p. 1) the entire class, not the domain or possession of any one individual; the mathematics is situated, social, and distributed (Putnam & Borko, 2000). Whereas individual and social constructivist paradigms focus on the individual as the "locus of learning," complexity theory sheds light on how the class as a whole develops mathematics. Individual knowledge in such a situation cannot be understood, complexity theorists claim, by slicing up the classroom and ignoring the larger collective entity of which the individual is an active part.

Conclusion

This paper has explored the writing of John Dewey on reflective thinking and mind development to extend these concepts into the realm supra-individual mind development. Just as a student can reflectively engage in mathematics problem-solving, so too can a group of students, or even the entire class. This perspective suggests that individual student mathematical learning can be augmented if teachers attend closely to cognizing classroom collectives (Davis & Sumara, 2001); these entities, as "mathematizing communit[ies]" (Sfard, 2003, p. 381), parallel the type of communities mathematicians learn and work in (Romberg, 1994). In particular, mathematics educators can apply the principles of good teaching espoused by John Dewey into a pedagogy that respects not only individual student or teacher learning, but various supra-individual learning such as partner, small group, whole-class, or teacher professional development groups. I have extended the theoretical work on reflective thinking by Dewey—in particular, that education involves a certain type of mind development—into the supra-individual realm to argue that mathematics education can involve the development of mathematizing intellectual entities that stretch across student partners, small groups, a whole class, and even teacher professional development collectives. Beginning with the reflective development of their own minds in collaborative practice, teachers can learn the skills necessary to help individual students and collections of students become reflective mathematics problem-solvers.

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USING A LESSON EXPERIMENT TO INVESTIGATE THE PROMISE OF DIFFERENTIATED INSTRUCTION FOR PROSPECTIVE ELEMENTARY TEACHERS: AN AREA AND VOLUME LESSON

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A lesson experiment was used to investigate how differentiated instruction impacted prospective elementary teachers' conceptual understandings of area and volume. Results include how the differentiation supported learning as well as how to enhance future instruction. Information will be provided about differentiated instruction, prospective teachers' understandings and learning in measurement, and lesson experiment methodology.

The mathematical preparation of teachers has recently been a renewed topic of national attention. *Mathematics Teaching Today* (National Council of Teachers of Mathematics [NCTM], 2007) states teachers need to be "fluent in the language of mathematics and have broad and deep knowledge of mathematical content, processes, and contexts" (p. 119). Similarly, several sources are calling for teachers to possess "mathematical knowledge for teaching" (e.g., Delaney, Ball, Hill, Schilling, & Zopf, 2008; National Mathematics Advisory Panel, 2008). As such, there is little disagreement that teachers need a deep and robust knowledge of mathematics for teaching. The more challenging aspect however is "In what ways do the content-learning experiences in your teacher-preparation . . . program help develop the robust and connected mathematical understandings needed for teaching?" (NCTM, 2007, p. 197). This dilemma is magnified by the varying understandings, interests, backgrounds, and learning preferences that prospective teachers bring to their undergraduate mathematics courses.

Such student variance is evidenced in nearly all classrooms. To address such diversity in grades K-12, teachers have been using *differentiated instruction*, a process of proactively modifying instruction based on students' current understandings, interests, and learning preferences. Effective characteristics of differentiated instruction include clear learning goals, ongoing and diagnostic assessments that modify instruction, and challenging tasks for all students. The approach is supported by literature on learning and has resulted in the improvement of students' learning. Yet, very little research literature reports on the use of differentiated instruction at the undergraduate level, especially with prospective teachers in mathematics courses. Chamberlin and Powers (2010) demonstrated the potential of differentiated instruction at the undergraduate level by finding that prospective elementary teachers receiving differentiated instruction experienced greater gains in their mathematical understandings. Questions remain however about specifically *how* differentiated instruction impacts the learning of prospective teachers. The objective of this study is to investigate how differentiated instruction impacts the mathematical understandings of prospective elementary teachers enrolled in a mathematics content course.

Background and Theoretical Framework

Differentiated instruction is a process of proactively modifying curricula, teaching methods, learning activities, and assessments to meet the diverse needs of students and thereby to maximize access to, motivation for, and efficiency of learning (Tomlinson, 1999). These changes

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are based on students' readiness (current understandings), personal interests, and learning profiles (learning styles, culture, and gender) (Tomlinson et al., 2003). Differentiated instruction is supported by theoretical literature on learning (Subban, 2006; Tomlinson et al., 2003; Tomlinson & McTighe, 2006) as well as empirical research. Empirical studies across several grade levels have found that differentiated instruction leads to achievement gains on standardized tests, including mathematics assessments (Hodge, 1997). While achievement gains have been found for all students and across all racial and socioeconomic groups (Brighton, Hertberg, Moon, Tomlinson, & Callahan, 2005; Tomlinson, 2005), some studies have found particular improvement for historically low-performing students, students with exceptional needs, and students with gifted abilities (Batts & Lewis, 2005; Brimijoin, 2002; McAdamis, 2001).

Several core principles guide differentiated instruction (Tomlinson, 1999; Tomlinson et al., 2003; Tomlinson & Eidson, 2003). First, assessment is ongoing, continuously informs instruction, and includes the assessment of students' understandings, their personal interests, and their learning profiles. Second, teachers attend to student differences, accepting students as they are but expecting them to become and understand all that they can. Third, all students participate in respectful work at a level attainable for them. Lessons for all students emphasize critical or creative thinking that promote individual growth. Fourth, the teacher and students collaborate in learning, maintaining a balance between teacher-assigned and student-selected tasks and working arrangements. Fifth, teachers are flexible in their use of groups, whole class discussion, time, materials, and classroom space. Finally, differentiated instruction is proactive rather than reactive. The teacher plans lessons that address learner variance from the outset rather than relying on adjusting instruction during real-time. Differentiated instruction is not synonymous with individualized instruction. Rather, it draws on flexible groupings, space, time, and materials to target instruction for *subsets* of the entire class, which fosters accommodation without the overwhelming effort of individualization. Furthermore, the teacher does not differentiate instruction during every class. Whole class instruction is still utilized but done so purposefully, informed by on-going assessment of students' needs. Teachers may choose to differentiate their instruction with regard to content, learning process, learning product, or learning environment.

Differentiated instruction is grounded in a sociocultural theory of learning, which also serves as the theory of learning for this study. Sociocultural theory emphasizes the role of social interaction and culture in the development of the learner (Vygotsky, 1987; Wertsch, 1991). Higher order functions develop out of social interaction and culturally organized activities. In addition, social and individual human action is mediated by tools and signs. These tools facilitate the co-construction of knowledge and aid in future independent problem solving activity. An instructional implication of sociocultural theory is that learning should be matched in some manner with the child's level of development. As such, Vygotsky proposed the zone of proximal development (ZPD), known as the difference in a child's performance when she attempts a problem on her own compared with when an adult or an older or more advanced peer provides assistance. Differentiated instruction, with its emphasis on continuous assessment and targeting instruction for students' ZPD, aligns with the basic tenets of socioculturalism.

Methods

For the present study, 19 prospective teachers enrolled in Math 2120 Geometry and Measurement for Elementary Teachers in spring 2011. The course utilized the book *Reconceptualizing Mathematics for Elementary School Teachers* (Sowder, Sowder, & Nickerson, 2009) and had three primary emphases: two- and three-dimensional shapes, symmetry and congruence, and processes of measurement. Megan was the instructor for the course, while both Megan and Michelle served as researchers for the study. The course included two in-class exams, a cumulative final exam, formative assessments, homework assignments, quizzes, a group project, and various writing prompts. All students in the course were elementary education majors with 63% sophomores, 26% juniors, and 11% post-baccalaureates or seniors. Students self-identified their mathematical backgrounds as ranging from high school algebra to calculus, and all students had taken the first two courses in the elementary education mathematics sequence. Students also varied in their self-identified like or dislike of mathematics, which they were asked to rate on a scale ranging from 1 (low) to 10 (high). Students' answers ranged from 1 to 10, with most students falling between 5-7 and the class average being 6.1.

Instruction was differentiated in multiple ways. Several pre- and formative assessments were used to gauge students' readiness, interests, and learning preferences. Varying activities and flexible use of small groups (sometimes heterogeneous, other times homogeneous) were implemented to meet students' different needs. Multiple modalities, in particular visual, audio, and kinesthetic mediums, were incorporated throughout the course. Students completed the same quizzes and exams, but homework assignments were differentiated. To differentiate the homework, students' progress was tracked using learning logs that outlined the learning goals for each unit. Students were given choices on writing prompts and the group project, as well as being able to select the medium through which material was presented for the project, e.g., website, skit, brochure.

To examine the impact of differentiated instruction on prospective teachers' mathematical understandings, a *lesson experiment* was utilized. The purpose of a lesson experiment is to engage in cycles of creating and testing hypotheses about cause-effect relationships between teaching and learning during classroom lessons (Hiebert, Morris, Berk, & Jansen, 2007; Hiebert, Morris, & Glass, 2003). It is a deliberate and systematic process that focuses on the key question, "What did students learn, and how and why did instruction influence such learning?" (p. 48). The process is composed of four steps that take on a character somewhat akin to teacher as researcher, reflective practice, and disciplined inquiry. The first step is explicating the learning goals for the students, which allows one to investigate whether and how the instruction helped the students achieve the desired goals. The second step is assessing whether and to what extent the learning goals are achieved during the lesson by gathering data on students' thinking from videos, transcripts, or students' written work. The third step consists of developing hypotheses for why the lesson did or did not achieve the learning goals. The fourth step entails using the hypotheses to revise the current lesson or to provide additional ideas about supporting learning that can be utilized and tested in future lessons. In conducting a lesson experiment, such steps are not necessarily performed in a distinct sequential fashion, although the order of the steps provides a general framework for progression. This approach provides a shift of focus from teaching in the moment to including preparation and reflection outside the classroom. In addition, the approach is a promising response to the gap between research and practice. While Hiebert and his colleagues foremost recommend lesson experiments as a way to help teachers learn from teaching, they also recommend the approach for teacher educators.

The lesson experiment took place the third week of the semester and involved a differentiation of activities based on students' conceptual understandings of area and volume. The purpose of the lesson experiment was to address how the differentiated area and volume activities impacted the students' learning. Table 1 provides the learning goals for the differentiated lesson (Wiggins & McTighe, 2005).

Enduring Understandings (EU)	Knowledge (K)	Skills (S)
1. A measurement system allows one to measure attributes with a unit appropriate for the particular context, including the ability to measure an attribute with different magnitudes. 2. In learning to measure, elementary students are best supported by a progression of a.) learning to perceive an attribute, b.) comparing objects with the same attribute, c.) measuring with a unit (nonstandard and then standard), and d.) working within a standard measurement system.	1. Area is the number of square units that cover a region. 2. Volume is the number of cubic units (solid or liquid) that fill a space. 3. Surface area and volume are independent. 4. Without a consistent unit, an attribute may have different measures.	1. Compare, order, and measure area and volume using nonstandard and standard units (including selecting an appropriate unit or tool). 2. Develop familiarity with standard units for area and volume (metric and English).

Table 1. Learning objectives for differentiated area and volume lesson

At the beginning of the course, students completed a measurement pre-assessment. The pre-assessment asked students to draw and offer an explanation of five attributes (length, area, volume, mass, and weight), to order groups of objects for each attribute, to match up various objects with an appropriate measure and unit, and to rate their comfort level with English and metric units. The pre-assessments revealed that some students had only developing understandings of area or volume (e.g., confusing the two, confusing volume with weight or surface area), that other students defined area or volume in terms of the measurement formula for a rectangle or a rectangular prism, and that most students were not familiar with the metric system. Thus, for the differentiated lesson and due to time constraints, students completed activities in either area or volume as per their performance on the pre-assessment. The activities led students through the learning progression for measurement (see EU2 above). Specifically, for the volume group, the first activity was to compare the volume of three cylinders, each formed from a standard sheet of paper but taped together in different ways. Second, they measured a cylinder with a non-standard unit and then compared their measurement with another group (who, unbeknownst to either group, was measuring the same sized cylinder, but with a different sized non-standard unit.) This activity highlighted the need for a consistent unit. The third activity was to measure the volume of a cylinder using base ten pieces to emphasize the advantages of a standard cubic unit. Finally, students filled a hollow base ten cube with water to realize that one liter is equivalent to the volume of a base ten cube and thereby to realize that one milliliter is equivalent to the volume of one cubic centimeter. The area group went through a similar progression, first directly comparing shapes of different area, then using a non-standard unit to measure area, then working within the metric system by measuring a variety of items using the faces of base ten blocks.

After the students completed their respective activities, each volume group presented their findings to an area group and vice-versa. For the presentation, students talked about how their attribute is measured, what metric units are used to measure their attribute, and related their activities to the measurement learning progression. Finally, in order to assess students' understandings on both attributes, a post-assessment was given in which students compared the area of two shapes and then justified their decision, considered the best type of unit for measuring volume (either a cylinder, cube, or rectangular prism), justified the measurement formula for the volume of a rectangular prism, and listed any remaining questions they had.

During the lesson experiment, various sources of data were collected from nine students, a subset of the students who agreed to participate in the research. The data sources included students' work on the measurement pre-assessment, students' work on the area and volume group activities, audio recordings of one area and one volume group, video recordings of Megan's instruction, and students' work on the post-assessment. Data analysis is in progress. Analysis to date has included examining the students' written work, coding their understandings and developing interpretations about area and volume in regard to the learning objectives. The associated audio recordings have also been listened to and students' thinking and discussion paraphrased to form initial ideas about how the social interaction and measurement objects are mediating students' learning. Future analyses planned include transcribing the audio and video recordings and using grounded theory and constant comparative approaches (Strauss & Corbin, 1998) to refine the interpretations of the students' understandings and how the differentiated activities impacted their social interactions and use of measurement tools.

Results

Preliminary results describing which learning goals were met refer to the objectives given in Table 1. With regard to K1 and K2, the lesson appeared to enhance students' understandings of area as covering a two-dimensional shape and volume as filling a three-dimensional shape, especially for students that demonstrated misconceptions on the pre-assessment. For example, on the post-assessment when asked about justifying that a shape had a larger area, all of the students either commented about placing the shapes on top of each other or counting the number of square units required to cover each shape. However, not many of the students described counting the number of square units. While the students appeared to develop an appreciation for area as coverage and volume as filling, they did not often describe quantifying these attributes by counting the number of square or cubic units. With regard to K3, students seemed to understand well that measuring an attribute with different-sized units leads to different measures, as students in both groups described this experience in their presentations. In addition, during interactions with Megan, it was apparent that the students were implicitly making sense of the inverse relationship between the size of the unit and the associated measure. For K4, the volume group was shocked by the realization that cylinders with equivalent surface areas may have different volumes. Also, due to their interaction with Megan, some students may have gained an appreciation for why the circumference and resulting radius formed by the sheet of paper has a greater impact on the volume than the height.

During the presentations and on the post-assessment, students demonstrated some strong understandings of S1, selecting appropriate units for measuring area or volume. Students clearly understood that circles or cylinders do not work well as units because when placed next to each other gaps remain. Nearly all students also deferred on selecting a unit for measuring area or volume that would include dimensions of different lengths, e.g., a rectangle or a rectangular prism. However, as evidenced on the post-assessment, it was not apparent whether students could articulate the complexities involved with using such units, e.g., the need to pay careful attention to which dimensions of the unit are used to measure which dimensions of the associated two- or three-dimensional object. With regard to S2, students appeared to gain an understanding that one milliliter is equivalent to one cubic centimeter. Due to a lack of commentary by the students, they did not seem to learn about any other common referents for standard units of area or volume. From listening to the area group audiotape, it appears that these students may have developed a general sense of the magnitude of the different metric units for area (EU1). For example, when completing an activity entitled ‘Guess the Unit’, they discussed the relative magnitude of a kilometer to a decameter to a meter. It is uncertain whether the volume group gained a similar sense of such magnitudes for volume as their activities seemed to deal less with this issue. Finally, with regard to EU2, all of the students seemed to understand the need to help elementary students understand an attribute, e.g., the idea of area as covering and volume as filling. However, it is not clear that the students understood all of the steps of the learning progression, as they did not talk about the later steps.

After analyzing the lesson, various hypotheses as to how the instruction helped students attain the learning objectives were determined. First, as recommended for K-12 students learning about measurement (NCTM, 2000), the activities provided the students with concrete and active experiences in finding, comparing, and exploring the attributes of area and volume. In fact, one of the students commented on the audiotape that he particularly appreciated the fact that these activities involved the activity of measuring rather than just listening to a teacher and performing computations on paper. This active process contributed to students’ understanding of area as coverage and volume as filling. Second, activities were purposely designed such that the students would proceed through the measurement learning progression (Inskeep, 1976; Van de Walle & Lovin, 2006) due to our belief that this progression is helpful for all learners regardless of age. In this particular lesson, the students also had misconceptions about the attributes of area or volume; therefore students were differentiated by being placed in the appropriate group and leading them through the learning progression. This differentiation also allowed the students’ interactions with the instructor to focus on their developing understandings. For example, while interacting with Megan, the volume group was able to further explore the formula for the volume of a cylinder. The lesson structure also allowed the students to learn from each other about the other attribute. From the presentations and students’ work on the post-assessment, it appears that as a result of the area presentation, the volume students learned about the option of placing shapes on top of each other to compare area and about a circle not working as a unit of area because of the resulting gaps. As a result of the volume presentation, the area students came to understand volume as filling, that cylinders do not work as a unit of volume, and that one milliliter is equivalent to the volume of one cubic centimeter.

Analyzing the data of students’ thinking also revealed some differences between the volume and area group. In general, the volume group was more active in verifying, justifying, and critiquing their own work. For example, on multiple occasions after determining the volume of a cylinder by using the formula $\pi r^2 h$, the students also verified their calculation by filling the cylinder and counting the number of units used. On another occasion, the students were very deliberate about wanting to know why the volumes were not equal for cylinders with equal surface areas. In contrast, the area group did not verify their measurement calculations, used estimation rather than actually measuring certain objects (such as the surface area of a door), and seemed satisfied with answers that appeared “fairly close” (when upon further investigation would have been shown to be somewhat inaccurate). With their reliance on estimation, the area students often thought of quantities in terms of English units and then made rough approximations to metric units. This approach was in contrast to the intent for students to become familiar with metric units. In hypothesizing about what instructional aspects led to the differences between the two groups, several possibilities were considered. First, the students in the volume group have been known through other class sessions to frequently pursue the reasoning behind their mathematical ideas and concepts. This raises the question of how to encourage such behavior on behalf of other students and how social dynamics and culture may play a role. Second, the difference could be attributed to the lack of measuring activity on the part of the area group. Because of this, in future offerings of this lesson and for other measurement activities in this course, the instructor may need to take an active role in making sure the students engage in the activity of measuring. Third, the volume and area activities themselves may have contributed to this difference. For the volume activities, the students were finding the volume of an object other than a rectangular prism. Thus, the students were using a volume formula with which they were less accustomed. Perhaps this led them to be more active in verifying their calculations. In contrast, for the area activities, the students were primarily finding the area of rectangles. Perhaps their familiarity with the area of such shapes decreased the need or desire to verify such calculations. Thus, upon utilizing these activities in the future, more irregular two-dimensional shapes in the area activities would be included.

In considering how the current lesson might be revised or formulating ideas about supporting future learning, two unexpected mathematical complexities arose. First, when selecting nonstandard units for the measurement activities, unforeseen complexities from using units with varying dimensions emerged. As a result, a successive lesson was taught in which students cut out nets of cubes and of rectangular prisms and discussed in small groups which of the two would serve as a better unit for measuring volume. Second, in covering shapes with sticky notes, the area group alluded that shapes with equal area may not necessarily be congruent. Thus, an additional lesson was taught using geo-boards to investigate the area of triangles and parallelograms which had the same base and altitude (and thus the same area) but looked very different. Finally, some learning objectives that did not appear to be met during this lesson were addressed in future lessons: the concept that a measurement is the result of counting the number of units that match the given attribute, the various magnitudes of metric units of volume, and developing familiarity with the metric units, including learning of common referents.

Discussion

Lesson experiments provide a means for connecting research and practice, as we found to be the case for us. In this lesson experiment, we were engaging in two roles: researchers in teacher education and teachers of mathematics content. We found this process to benefit both roles and to provide feedback between the two. Due to the collection of research data, we enhanced our interpretations of the students' thinking which thereby impacted our future instruction. Similarly, our instructional roles provided insight into the students' activities and experiences during class, enhancing our ability to hypothesize how the instruction was impacting the students' understandings. We also found continued promise in the use of differentiated instruction at the undergraduate level with pre-service elementary teachers. Differentiating by area and volume in this lesson enhanced the students' understandings, allowed us to maximize the use of class time, and possibly provided a model of differentiated instruction for the students. Indeed, we have been open with the students about the differentiation process and are undertaking other research projects to investigate the potential impact of this on the pre-service teachers' future pedagogical practices. Finally, the lesson experiment process has begun to reveal hypotheses about how differentiated instruction is impacting the students' learning and mathematical understandings – the primary purpose of this project.

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MATHEMATICS REFORM FROM INTERACTIVE VIDEO EXPERIENCES (MR FIVE): EXAMINING PRE-SERVICE TEACHERS' NOTICING AND KNOWLEDGE

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We present the results of an exploratory study utilizing interactive video examples of reform teaching in an elementary mathematics methods course and their impact on pre-service teachers' pedagogical content knowledge (PCK) and ability to identify and interpret reform-based pedagogy. The study utilized a quasi-experimental research design in which all participants completed both a pre- post-pedagogical pedagogical content knowledge assessment and a pre- post-case analysis assessment. We present the results of this intervention; interpret the results and present implications for further research.

Introduction

In the United States, the current mathematics education reform movement works with teachers to develop pedagogy, which values a deep conceptual understanding of mathematics in students rather than a rote procedural understanding. The reform movement's emphasis on mathematical problem solving, reasoning, communication, representation and connections (NCTM, 2000) answer the call of the 21st century social and intellectual skills. Teachers must be prepared to teach in a manner that incorporates and integrate four knowledge domains, knowledge of: content for teaching, pedagogy, learning, and evaluation (Lappin, 2000). Despite the efforts of many talented researchers in partnership with inservice and preservice teachers, the impact on classroom pedagogy has remained minor. Thus, mathematics teacher educators and researchers continue to explore new and different ways to provide the momentum to enable reform-based pedagogy to thrive in working classrooms.

Shulman (1983) stated that, "…the teacher must remain the key. The literature on effective schools is meaningless; debates over educational policy are moot, if the primary agents of instruction are incapable of performing their functions well…" (p. 504). The traditional American teacher preparation program has three main components to develop preservice teachers abilities to perform their function: coursework that develops pre-service teachers' (PSTs) knowledge of both content and pedagogy and various field experience components. In order to prepare teachers to teach using a reform-based approach, mathematics teacher educators and researchers have taken an empirical look at various approaches within each of these three phases to help prospective teachers develop reform-based pedagogical approaches (e.g. Mewborn, 1999, 2000; Sowder, 2007; Ball & Wilson, 1996). This paper examines the results of an exploratory study utilizing interactive video examples of reform teaching in an elementary mathematics methods course and their impact on PSTs' pedagogical content knowledge and ability to identify and interpret reform-based pedagogy.

Perspective

Developing Reform-Based Teachers

The constructivist philosophy of learning originates from the research of Vygotsky and Piaget. Here, mathematical learning occurs from the active participation in one's own navigation of the curriculum. Knowledge cannot be objectified and then transferred, but must be discovered through the processes of perturbation, accommodation, and assimilation into existing constructs. Students use these tenets to learn mathematics by making sense of their own experiences and then also the collaborative sharing of these experiences amongst their peers. This type of mathematical teaching is demanding on the practitioner, and requires a specific skill set. Realizing its complexity, Ball (1990) suggested that teachers need to understand the nature of knowledge in the discipline, where that knowledge comes from, how it varies with time, how truth is established, and philosophically what it means to know and do mathematics in order to be effective teachers. The complexity and versatility of knowledge for teaching mathematics is important for the implementation of mathematics reform and also the future design of methods courses designed to train PSTs.

The release of the National Council of Teachers of Mathematics (NCTM) (2000) *Principle and Standards for School Mathematics* document describes an idealized future for mathematics classrooms in the US and echoes the international community in its call for access for all students to rigorous, high quality mathematics instruction. To make the transfer between ideal and reality is the task of the classroom teacher.

What students learn in their K-12 mathematics experiences is a reflection of the procedures in those classrooms. The construction of learning environments will invariably reflect the classroom practitioners' views on mathematics and mathematics instruction. An absolutist teaching of mathematics supports the behaviorist thought that mathematics is a rigorous, masculine discipline of which mathematics is out there, awaiting discovery, and can be learned by rigorous procedural applications. Research indicates "that classroom instruction, which tends to focus almost exclusively on the knowledge base, deprives students of problem-solving knowledge"(Schoenfeld, 2004, p. 263). As a product, this falls short of the goals set forth by mathematics education professionals.

The belief of mathematics and mathematics instruction being a process, or a series of ideas and refutations provides a different fundamental platform for learning. Discovery, improvisation, inquiry, student centered, these adjectives speak of the nature of this type of mathematical teaching and learning. This builds towards the high standards set forth by the international research community. This type of mathematical learning also reveals complexities of what is required of practitioners that are urged to teach this way (Hill, Sleep, Lewis, & Ball, 2007).

Studies suggest (Borko et al., 1992; Brown, Cooney, & Jones, 1990) that teacher beliefs are influenced a great deal by teachers' experiences with mathematics and schooling long before their entry into the mathematics education workforce, and without *significant* interventions, these beliefs are very difficult to change (Cooney, Shealy, & Arvold, 1998). Innovative methods courses can address these difficulties inclusive with exposing these practitioners to more clinical experiences. Field experiences can provide difficulties in that there is a lack of control for the beliefs of the inservice teacher. Therefore, the task may fall on technology to provide video cases imbedded in hypermedia in order to provide a quasi-field experience for preservice teacher training. This controls for contrasting beliefs, while simultaneously providing quality role models for preservice teachers to project methods onto their own vision of their future classroom and classroom practices.

Video Cases

Teacher education programs have long been criticized that its didactic instruction and limited field experience cannot equip pre-service teachers with the insight and experience dealing with the messy and complex teaching practice in the real world. In the last several decades, case-based method has increasingly been advocated as an effective method of preparing pre-service teachers for the real classroom context (Shulman, 1986; Merseth, 1996;

Harrington, 1991). Case teaching has a long and productive history in several fields of professional education such as law, business and medicine, but it did not gain attention in the field of teacher education until mid 1980s. The use of video cases in teacher preparation has grown substantially within the last few years.

Compared with traditional narrative cases, video cases offer further advantages. Primarily it satisfies the need for pre-service teachers to observe how teachers handle daily problems in real classrooms (Georgi, Redmond, Talley, & Cannings, 2002). Video cases also increase PSTs' opportunities to observe authentic teaching situations; each of which can be viewed multiple times. The opportunity to view and reflect on an episode multiple times allows for deeper understanding of the methods and environments that are being viewed. (Georgi et al., 2000; Sherin, & van Es, 2002).

In the preparation of preservice mathematics teachers, it has become apparent that importance must be placed on facilitating skills such as listening to, hearing, and watching students. Many programs such as CGI (Phillip, Armstrong, & Bezuk, 1993), IMAP (Phillip et al., 2007), and DMI (Schifter, Russell, & Bastable, 1999) have found success in the emphasis on studying children's thinking. Various studies have demonstrated the advantages of using video cases over written cases in promoting such skills. In a study that compares the reflective abilities of two groups of pre-service teachers in a mathematics methods course, the group that enrolled after the introduction of a video case curriculum tends to focus more on individual student's mathematical thinking (Stockero, 2008).The video cases applied in the curriculum are unedited recordings of classroom activities during middle school math classes. Compared with written cases, the unedited video series provide pre-service teachers with the opportunity to locate the important issues and concerns from the recorded classroom interactions. This finding is in accordance with Wang and Hartley's (2003) conclusion that, video cases are more efficient than written cases in facilitating the development of the knowledge and skills of observation. Sherin (2004) also pointed out that such development enhances the ability of identifying the issues that are critical to the profession, such as discovering and understanding students' mathematical thinking patterns.

In addition to helping teachers to realize the students' mathematical thinking levels, video-based pedagogy has also been demonstrated to facilitate teacher's reconstruction of their beliefs about how children learn mathematics and shape a student-centered rather than the teacher-centered perspective (Friel & Carboni, 2000). The use of video cases in teacher education is also found to be helpful in developing the pre-service teachers' ability to examine student ideas (Richardson, 1999; Sherin, 2004) and to clarify students' conceptual understanding (Sherin & Han, 2004).

Examining Pedagogical Content Knowledge

In his landmark article, Shulman (1986) raised the following questions, "Where do teacher explanations come from? How do teachers decide what to teach, how to represent it, how to question students about it and how to deal with problems of misunderstanding?" (p. 8). The type of knowledge teachers required to address these questions is what Shulman referred to as pedagogical content knowledge (PCK). Our position is that interactive video examples of reform-based pedagogy can be an accessible source of knowledge for PSTs to develop PCK. Thus we have employed Shulman's description of PCK to use as a lens through which to view what PSTs know and notice about reform-based teaching. Thus, as we analyzed our data for what PSTs knew and noticed about reform-based pedagogy, we examined it for evidence of PSTs' knowledge of students and student learning, appropriate and powerful mathematical representations, and PSTs' ability to interpret and make sense of students' mathematical work and thinking. In order to further frame our examination of PSTs knowledge and noticing of reform-based pedagogy, we employed the NCTM *Principles and*

Standards for School Mathematics (2000) document. Specifically germane to our assessments are he teaching and learning principles, the number and operations content standard, and the five process standards of: problem solving, reasoning and proof, communication, connections, and representations.

The purpose of current study was to examine the influence of video cases on pre-service teachers' pedagogical content knowledge. Specifically, this study addressed following research question, What is the influence of video cases on pre-service teachers' pedagogical content knowledge as it pertains to reform-based mathematics instruction?

Methods

Participants

Forty-two students enrolled in two sections of elementary mathematics methods at a large Midwestern university course participated in the study. The third author served as the instructor for the two sections. There were 38 Females and 4 Males. All participants were seniors enrolled in the methods course the semester prior to their student teaching experience. In conjunction with course, participants also spent 32 hours observing and teaching mathematics and science within an elementary classroom.

Measures

Pre-post pedagogical content knowledge assessment. Participants completed a pedagogical content knowledge assessment in a pre-post format. The open-ended questionnaire included four questions that assessed pre-service teachers' understanding of number and operations and how they might approach teaching this content. Specifically, the questions were: 1. What mathematics makes up number and operations?; 2. How would you teach topics in number and operation (i.e., what are some approaches, methods, etc.)?; 3. How do you know a student is struggling with ideas in number and operation?; 4. How do you know a student has a good understanding or mastery of ideas in number and operation?

Pre-post case analysis assessment. Participants also completed a video case analysis assessment, which was utilized to assess their ability to observe and notice reform-based mathematics teaching practices. Participants watched a video clip (6 minutes and 21 seconds) that showcased a sixth grade classroom engaged in learning fractions and percentages. After watching the video, participants completed a questionnaire with three questions that assessed their ability to notice teaching practices. Specifically, the questions were: 1. What do you notice about the teacher? 2. What do you notice about the students? 3. What do you notice about the mathematics?

Procedure

The study utilized a quasi-experimental research design where students enrolled in one section of math methods course acted as experimental and another section served as control. Both sections were taught by the same instructor, which removed any potential instructor effect. Participants in the experimental section had access to an online video case library that showcased reform-based mathematics instruction where as control section did not have access to the video case library. The video case library included 4 teachers teaching various lessons within the topic of numbers and operation with a total number of 37 video clips. The participants could search the video case library using a theme, keyword, or teacher. The themes included more broad categories, such as task launch, task exploration, student explanations, and task summarize, where as keywords were more specific instances of mathematics teaching practices. Participants from the experimental section were given seven bi-monthly assignments to watch a series of video clips and respond in an electronic reflection journal to three prompts: "What do you notice about the teacher?", "What do you notice about the students?", and "What do you notice about the mathematics?". The control participants, on the other hand, did not watch the video cases, and were instead required to respond in their electronic journals to a series of journal prompts selected from the end of chapter discussion questions in their course textbook. All participants completed both the pre-post-pedagogical content knowledge assessment and the pre- post-case analysis assessment described above.

Data Analysis

Researchers developed a coding rubric to analyze the pedagogical content knowledge and case analysis assessments. The rubric was developed using the assigned readings for the course (Van de Walle, 2009) and the NCTM *Principles and Standards for School Mathematics* document (2000). Specifically, the rubric consisted of acceptable responses for each of the questions and participants were assigned a 1-point for each correct response and could score multiple points for each question based of the total number of acceptable responses for the given question. The researchers coded each participant's assessments individually and then met to discuss their assigned scores and to reach consensus for the final score for each participant. One example of coding scheme for each assessment is in Table 1.

Pedagogical Content Knowledge Assessment
Question 1: What mathematics makes up number and operations?
Code: 0-2 (continuous); 0 points for "I don't know" or other incorrect answers; 99 for blank
Scale – provide 1 point for supplying each answer or similar worded idea (up to 2): • Identifies the four main operations (addition, subtraction, multiplication, division) • Identifies various number systems or classes of numbers (whole numbers, fractions, decimals, percents, ratio)
Case Analysis Assessment
Question 1 – "What do you notice about the teacher?"
Code: 0-9 (continuous); 0 points for "I don't know" or other incorrect answers; 99 for blank
Scale – provide 1 point for supplying each answer or similar worded idea (up to 9): • Teacher ask for/encourages different ways of explaining • Teacher asks students for students' attention • Teacher shows interest in student explanations/thinking • Teachers values student explanations/thinking • Teacher revoices/restates student thinking for class • Teacher encourages student-student questions communication • Teacher asks questions to encourage students to clarify/refine their thinking • Teacher moves around the room and talks to students during exploration time • Teacher provides opportunities to students to work on the problem on their own

Table 1: Assessment rubric samples

Results

Pre-post Pedagogical Content Knowledge Assessment

The results suggest that overall video cases increased students' pedagogical content knowledge of numbers and operation (See Table 2 for means and standard deviations). However, ANCOVA results suggest that condition was not a significant factor for participants understanding of what mathematics makes up number and operations (F(1, 41)=3.70, p=0.06); how to teach topics in number and operation (F(1, 41)=0.09, p=0.77); identifying student struggling with ideas in number and operation (F(1, 41)=0.48, p=0.49); identifying student mastery of ideas in number and operation (F(1, 41)=0.01, p=0.92).

Condition	Q1	Q2	Q3	Q4
Traditional	0.21 (0.42)	1.84 (0.83)	0.95 (0.62)	1.16 (0.69)
Video Cases	0.65 (0.77)	1.91 (0.90)	1.04 (0.64)	1.09 (0.67)

Table 2: Means and standard deviation of pedagogical content knowledge assessment

Pre-post Case Analysis Assessment

Results showed that pre-service teachers' ability to notice reform-based mathematics teaching practices increased when video cases were used (See Table 3 for means and standard deviations). ANCOVA results exhibited that video cases significantly influenced pre-service teachers' ability to notice students (from the video vignette) using variety of tools, student-student mathematical interactions and students giving procedural and conceptual explanations of equivalence, ($F(1, 40)=4.72$, $p=0.04$). However, ANCOVA results exhibited that condition was not a significant factor for pre-service teachers' ability to notice reform-based teaching practices ($F(1, 40)=0.28$, $p=0.60$); and identifying mathematical content ($F(1, 40)=2.58$, $p=0.12$).

Condition	Q1	Q2	Q3
Traditional	0.21 (0.42)	1.84 (0.83)	0.95 (0.62)
Video Cases	0.65 (0.77)	1.91 (0.90)	1.04 (0.64)

Table 3: Means and standard deviation of case analysis assessment

Discussion and Conclusions

Past research has indicated the use of video case examples can impact teachers' knowledge and abilities. Aside from the one significant result, the findings from this exploratory study do not seem to support these results. We provide a possible explanation for the lack of effect and present some directions for our further investigations.

A valuable aspect of using video cases is PSTs opportunity to have discussions with their peers and instructor of the episodes, what they noticed, and how they reflected on and made sense of their observations. Due to time constraints within the structure of the methods course, PSTs only gave written responses to the video cases and were not afforded this opportunity. We believe relying solely on written reflection limited the PSTs' opportunities for developing PCK. As a result of this study, we have realized that the time spent in class on reflection of the video cases is crucial to maximizing PSTs' opportunities to learn.

A second change we intend to make moving forward is to further refine our approach for examining PCK; specifically by using the construct of mathematical knowledge for teaching (MKT) (Hill, Sleep, et al., 2007; Ball, Thames, & Phelps, 2008). For example, the one significant result indicated use of video cases had a significant effect on PST's ability to recognize student indicators of reform-based teaching practices: students using a variety of tools, student-student mathematical interactions, and students use of procedural and conceptual explanations. We argue that these are aspects of, or are related to, PSTs' Knowledge of Content and Students (knowledge of the ways students think and learn) and Common Content Knowledge (the mathematical content teachers are responsible for teaching to students). We can also use MKT to frame our other results. For instance, we saw a change, although not significant, within aspects of Specialized Content Knowledge (mathematical knowledge used for teaching but not taught directly to students) as well as Knowledge of Content and Teaching (knowledge of task selection, appropriate representations, questioning strategies, etc.). In further iterations of this study, it is our intent not only to use MKT in our analysis, but to gear the discussions of the video cases within the methods course around MKT as well.

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TEACHER CHANGE IN MATHEMATICS PROFESSIONAL DEVELOPMENT: UNDERSTANDING EXPERIMENTATION IN CLASSROOMS

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As a field, we know little about how teachers take up practices and sustain them (or not) due to professional development. This project provided professional development based on CGI and studied change in classroom practices over two years with K-3 teachers in an urban district with a high population of ELLs. Findings illustrate the complex nature of changing mathematics classrooms.

As a field, we know little about teacher experimentation in classrooms based on participation in practice-based professional education (PBPE) (Kazemi, 2008). Questions remain about which practices are taken up easily or which ones are more complex to bring across the boundary of professional development (PD), how practices are adapted or not taken up at all, which practices are sustained over time or which are left behind. At the same time, teachers do not have sole control of their classrooms.

In this paper, we present 2 years of analysis on a PD effort in an urban district in Arizona with a high population of English Language Learners (ELLs). The professional development in this study focused on embedding Cognitively Guided Instruction (CGI) within classrooms in this district. The purpose of the PD was to enable teachers to gain insight into how their students think, understand a variety of mathematical concepts, use that knowledge to inform their practices, and learn practices for supporting ELL students in mathematics.

The analysis entails two years of observations focused on the mathematics practices that teachers were experimenting with in their classrooms. We looked at when new practices emerged and if they were sustained. This study adds a detailed look at mathematics instructional change and the impact of PD on classroom practice. We begin by reviewing work that looks at the effects of change efforts on teachers' practice. The paper then shifts to consider what research says about how educators can best serve ELLs in mathematics.

Reform Efforts and Teacher Change

The research on effects of change efforts often focuses on the top down nature of PD efforts and teacher fidelity to the reform. This section looks at teachers' responses to change efforts as well as how teachers experiment within PD.

Teacher interpretation often gets framed negatively because reform efforts could be attending to rigid notions of fidelity, but not all adaptations are resistance or contrary to change efforts. Certainly, PD goes through a process of translation as it moves across levels of the school system (Eisenhart, Cuthbert, Shrum, & Harding, 2001). For instance, teachers can adapt policies according to their own students' needs (McLaughlin & Talbert, 1993, Tyack & Cuban, 1995). The adaption of PD to students is consistent with CGI, which is not meant to have rigid fidelity. In CGI professional development it is understood that teachers are best situated to adapt what they are learning to their classrooms. However, translation of change can also occur because teachers have little understanding of the implementation (Cuthbert, 1984). In that case, we would agree that teacher translation is problematic. When change efforts align with teacher beliefs and allow teachers to implement practices, PD can spur teachers to implement practices more aligned with children's home lives (Heath, 1983). Across this work, teachers adapt policy for student needs, connecting to families, but also for a lack of understanding.

Within this space of understanding teachers' responses to change, we understand very little about classroom experimentation during implementation and how this changes over time (Kazemi, 2008). We do not know which practices change first, which change last, how teachers implement and adapt new practices, and how or whether various practices are sustained over time. Bringing practices across the boundary between professional development and classrooms is extremely complex (Authors, 2008). It is not a simple issue of fidelity to a predetermined model since we expect teachers to design mathematics instruction in a way that is tailored to the needs of students. However, in bringing practices across this boundary, we know little about what makes practices easy or complex to translate into classroom practice and how sustainable particular instructional practices are over time. Understanding these micro-changes in practice could help professional developers design PD better by understanding how teachers respond to the demands that learning new practices places on them.

Supporting Latinos and ELLs in the Mathematics PD

With the increasing concern about Latino students and ELL students in particular, there is a growing body of work focused on instructional practices that engage Latinos in learning mathematics with understanding (DeAvila, 1988; Flores, 1997; Gutiérrez, 2002; Khisty, 1995; Moschkovich, 1999). While the current study engaged teachers of Latino students, this work is at the intersection of providing mathematics professional development in schools with a high population of Latinos who are designated as ELLs as well as those who are not. A major gap in the mathematics education literature is professional development that integrates learning about teaching mathematics for understanding and ways to support the learning of ELLs. In this project, we integrated some of the research literature about instruction that serves ELL students to support teachers in their mathematics teaching. We briefly review some of this work below.

In focusing on instruction for ELLs, we often see an overemphasis placed on mathematical terms without mathematical meaning (Khisty, 2006). While acquiring vocabulary is an important part of learning to use the register of mathematics, students' ability to "construct multiple meanings, negotiate meanings through interactions with peers and teachers, and participate in mathematical communication" is even more important (Moschkovich, 2007, p. 89). This is particularly important in mathematics, since many words hold different meanings in the mathematical register as compared to natural language (Khisty, 1995). Moreover, by placing an emphasis on vocabulary without properly contextualizing the mathematical terms, we run the risk of perpetuating deficiency models of students since focus is placed on what students cannot do rather than on what they can. As Moschkovich (2007, p. 90) argues,

> If all we see are students who don't speak English, mispronounce English words, or don't know vocabulary, instruction will focus on these deficiencies. If, instead, we learn to recognize the mathematical ideas these students express in spite of their accents, code switching, or missing vocabulary, then instruction can build on students' competencies and resources.

A related finding in this research is that using and valuing home language is critical. This practice is linked to higher achievement, especially in mathematics (Khisty, 2006). Beyond using students' home language, instruction must also draw on cultural understandings. Secada (1991) found that when the forms of language and culture in word problems do not match students' experiences, they have a difficult time developing solution strategies. In a study of think-alouds used in mathematics classrooms, Celedon-Pattichis (1999) determined that ELLs needed to translate the problem to Spanish, read it at least twice, attribute meaning, translate the meaning into mathematical symbols, and ignore irrelevant words. This research demonstrates the complexity of the cognitive activity that ELLs engage in when participating mathematically in classrooms.

Across this work on providing effective mathematics instruction to ELLs, we find that it needs to provide numerous opportunities to engage in mathematical discussion, speaking, listening, reading and writing the content as well as encourage meaning making within relatable contexts (Moschkovich, 1999). Additionally, to support this form of instruction, teachers need to be able to change curriculum to meet the needs of their students and reject deficit models framing students as intellectually inferior. This professional development attempted to integrate these ideas with more established research on CGI professional development.

Methods

This research study takes place in Monroe elementary school district, an urban school district in Arizona composed of four schools. One of the schools chose not to participate in the project and did not receive any treatment. The remaining three schools participated in mathematics professional development based on Cognitively Guided Instruction (CGI).

Participants and Background

These schools have been identified as having "not met" the adequate yearly progress for No Child Left Behind (NCLB). In the second year of the project two schools were in year two of not meeting, while one was in year one. A third year would make them candidates for being taken over by the state. In 2009, 52% of students in the district tested proficient in mathematics on the Arizona Instrument to Measure Standards (AIMS) as opposed to 62% the year before (2009 state average was 72%). On the Terra Nova, 35% of third graders tested proficient in mathematics compared to 40% in 2008 (the national average was 50%).

According to the Arizona District Report Card 2006-07, 45.7% of the teachers have been in the district three years or less. On the other hand, 33.1% of the district teachers had been working there for ten or more years. Also, 19% of the teachers held a provisional certification. The percentage of female teachers participating was 82%.

Teachers participated in the project during different periods: kindergarten and first grade teachers participated during the first two years. Second and third grade teachers participated during the second year. The project is ongoing, but only the years are included in this paper.

Twenty-one kindergarten and first grade teachers participated in the first year (cohort 1). Due to teacher turnover, we lost 7 teachers from cohort 1 in year 2. Two teachers also moved grade levels. In year 2, 23 new teachers began participating (cohort 2). Eight of those teachers replaced cohort 1 teachers in kindergarten and 1st grade. A total of 44 teachers participated in the study.

Professional Development Context

The professional development consisted of on-site monthly meetings and on-site weekly visits. The professional development focused on the principles of CGI, combining the earlier work on student strategies and problem types (Carpenter, Fennema, Franke, & Levi, 1999) with more recent work on algebraic thinking (Carpenter, Franke, & Levi, 2003) and counting (Schwerdtfeger & Chan, 2007). This broader take on the work allowed the professional development to be designed around the content in which teachers were currently working.

Workgroup meetings

The meetings lasted approximately 1.5 hours and were held at one of the three participating schools. The professional development focused on gaining knowledge of student thinking as a way to develop instruction towards teaching mathematics for understanding.

On-Site Visits

In addition to the workgroup meetings, the teachers and professional developers worked together at the school sites. We spent approximately one day each week on-site. During the weekly visits, the professional developers observed classes, worked with teachers and students, and were available for informal conversation about whatever the teachers wanted to talk about. Our goals were to support teachers, engage with them informally, and to explicitly draw on teachers' practices in designing future PD. Our time at the schools was important to remind teachers to experiment with new practices and to build relationships with students and teachers.

Measures and Analysis

Data for this paper were collected during a two-year period. We collected data using three measures: a teacher knowledge assessment, field notes, and interviews. This paper only details the classroom observations collected through field notes to examine teacher experimentation. We detail this measure below.

Classrooms' Observations

For the duration of the project, teachers were visited once a week during their math time by professional developers. The visits lasted from 30 to 50 minutes. During the observations, field notes were taken focusing on the new practices the teachers implemented when teaching mathematics, specifically those that were based on student thinking and tied to the content of the professional development. The professional developers dialogued about the practices and what they noticed with the teachers to clarify ideas and discuss future directions for the lessons. These notes included the kinds of participation facilitated in the class (for example, retrieving memorized answers vs. eliciting strategies), types of activities (word problems vs. drill worksheets, activities designed by the teacher or adapted from the textbook, or just the using the problems in the book), and types of interactions between teacher and students.

We looked across the field notes, noting themes and practices that appeared repeatedly across classrooms and teachers. Those practices became categories that were contrasted again with the notes, in order to confirm their validity. The authors then took notes on the frequency of appearance of those practices, as well as when they became a regular part of these teachers' classroom. The practices were coded according to semester they emerged and were only included in the analysis if the teachers maintained them in the classroom. These codes were analyzed and examined across years 1 and 2 for cohort 1 and for year 1 for cohort 2.

After a detailed reading of the field notes collected through two years 10 categories of interest emerged: (i) use of word problems, (ii) journals, (iii) student sharing, (iv) counting, (v) breaking apart numbers, (vi) use of tools (e.g. manipulatives), (vii) attending to the details of student thinking (e.g. organizing activities around student strategies), (viii) adapting the textbook, (ix) designing their own problems and (x) questioning student thinking. These categories arose from the content of the professional development sessions as well as the classroom practices observed. In addition, since the PD was informed by research, these categories represent mathematics teaching practices aimed at teaching for understanding. We cannot completely expand on these categories, but we briefly explain them below.

We drew on the original CGI work in focusing on use of word problems, student sharing, tool use, teachers designing problems tailored to student thinking, and attending to the details of student thinking (Carpenter, Fennema, Franke, 1996; Carpenter et al., 2003). In this same vein, using textbooks to meet the needs of particular students in the classrooms means monitoring "curriculum use and selection" (Franke & Grouws, 1997, pg. 334). In the words, there is no perfect curriculum and teachers must adapt curricula to meet the specific needs of their students. We also drew on more recent work on counting collections (Schwerdtfeger & Chan, 2007) and the algebraic thinking work that focused on breaking apart numbers while using the equal sign more flexibly (Carpenter et al., 2003). Another important code was teacher questioning as a follow-up to student thinking (Franke, Webb, Chan, Ing, Freund, & Battey, 2009). In the case of this paper, this merely means that teachers posed questions to students after a shared strategy. In this sense, it is a less detailed form of attending to the details of student thinking mentioned earlier. Lastly, we focused on journals as a way of recording student solution strategies and student thinking. This is a way of making student thinking explicit for teachers (Kazemi & Franke, 2004). These ten codes served as a way to look at the micro-changes that occur as teachers attempt to teach mathematics for understanding.

Results

New Mathematical Practices of Cohort One and Cohort Two in First-year Participation

In the coming paragraphs we review how the two cohorts of teachers behaved after being part of the professional development project for one year. The fact that cohort one consisted of teachers of kinder and 1st grade, and cohort had primarily 2nd and 3rd grade teachers should be considered when comparing these teacher groups.

In year 1, cohort one added a number of instructional practices into their mathematics classrooms. When appropriate we also present the percent of teachers that used the practices before the professional development. The practices cohort 1 engaged in the most were counting collections (44% of teachers, 6% already used this), tool use (38%, 13%), breaking apart numbers (25%), and using word problems (25%, 6%) (see table 1). To be clear, the teachers who used the practice are not included in the teachers who began the practice. For instance, 50% (44%+6%) of cohort one teachers counted collections by the end of year one (51% for tool use). Conversely, the least used practices were journaling (13%), editing the textbook (13%), designing their own problems (13%) and questioning student thinking (13%). While in some ways these might not seem like overly impressive percentages, 44% of cohort one began and maintained counting practices throughout the first year of the professional development in addition to the 6% who already used this practice. It is important to note again, that if teachers tried the practice once but did not continue it, it was not included in this analysis.

Teacher Group	Word Problem Use	Journals	Student Sharing	Counting Collections	Breaking Apart Numbers	Tool Use	Attention to Student Thinking	Editing the Textbook	Questioning Student Thinking	Designing Problems
	Practices									
Cohort 1 year 1 [a]	25%	13	19	44	25	38	19	13	13	13
Cohort 2 year 1 [b]	4	4	22	7	7	4	22	4	26	7

[a] n= 27, [b] n = 16

Table 1: Percentage of Teachers Engaging in New Instructional Practices

In year 1 for cohort 2, the instructional practices embedded in their classrooms more were student sharing (22%, 19% already used this practice), questioning student thinking (26%, 7%), and attending to the details of student thinking (22%, 4%). The practices that cohort two incorporated the least were word problems (4%, 4%), tool use (4%, 19%), journaling (4%, 15%), editing the textbook (4%, 4%), breaking apart numbers (7%), counting (7%, 4%), and designing word problems (7%, 4%). For seven of the ten practices, no more than 7% of the teachers engaged in them during instruction. However, it should be noted that the practices that cohort two engaged in more centered on student thinking and that 19% of teachers already practiced some form of student sharing.

The two cohorts were similar in taking on practices around attending to student thinking, student sharing, journaling, editing the textbook, and designing problems. The only difference that favored cohort two was questioning student thinking, but this difference was fairly minor. On four practices however, cohort 1 significantly incorporated them more into their classroom practice. Across word problems, counting, breaking apart numbers, and use of tools, between 18 and 37% more cohort 1 teachers incorporated these practices into their classrooms.

New Mathematical Practices across Years One and Two for Cohort One

The question guiding this section is how did their participation change in terms of the 10 practices in their second year of the PD? In looking across years one and two for cohort one, teachers increased their level of participation very little. While cohort one teachers did not decrease their use of any practices, they minimally increased their use in two categories. We discuss possible reasons for this in the discussion.

In table 2, you can see that the two practices that more teachers took on were counting (from 44% to 50%) and questioning student thinking (from 13% to 19%). None of the use of other practices changed. To say this is disappointing would be an understatement. The cohort that took more practices on in their first year of participation seemed to stop changing. Remember this was occurring at the same time as cohort two year one. So during the same year of professional development, while cohort one stopped taking on new practices, cohort two also did not take on many new classroom practices. The pattern seemed to point towards it being more difficult for teachers to take new practices on in the second year of the project across cohorts.

Teacher Group	Practices									
	Word Problem Use	Journals	Student Sharing	Counting Collections	Breaking Apart Numbers	Tool Use	Attention to Student Thinking	Editing the Textbook	Questioning Student Thinking	Designing Problems
Cohort 1, year 1 [b]	25	13	19	44	25	38	19	13	13	13
Cohort 1, Year 2 [b]	25	13	19	50	25	38	19	13	19	13

[a] n = 16, [b] n= 27

Table 2: Percentage of Teachers Using New Instructional Practices

Discussion

The classroom practices data shows that the first year teachers were trying and sustaining practices. For sustaining practices, we think these numbers are quite realistic. About 40% of the first year teachers maintained two practices (counting and tool use). Another 25% of teachers maintained two practices (breaking apart numbers, and using word problems) and 19% of teachers maintained two other practices (student sharing and attention to student thinking). There are a number of factors that could be limiting teacher change in year 2.

It is possible the project did not engage the teachers as well in year two. While this is a possibility, the same professional developer provided the training. Additionally, our relationships with teachers grew and we came to better know their classrooms. Another possibility is that we did not integrate the ELL practices well with CGI principles. Since it was the first time we attempted this, it is a possibility, but teachers changed I year 1 with this integration. The most likely explanation is that new educational policy in the state affected teacher practice of ELL students. This impacted all of out teachers in year 2 of the project and could have limited new practices they were willing to try in classrooms. We discuss this elsewhere (Authors, 2010).

Here however, we want to focus on the nature of experimenting in classrooms. One idea to be pulled out from this data is that practices that can be used as entire activities seemed to be taken up easier. Counting collections, breaking apart numbers, and word problem use are all practices that are supplemental activities to add on to lessons. This makes sense, especially if we see these activities as holding their own norms and ways of engaging students.

Practices that cut across lessons such as editing the textbook, designing problems, and questioning student thinking were slower to be taken up by teachers. These practices are ways of interacting in the math classroom across differing activities, from whole class to group work to individual practice. This may require more fundamental change from teachers since we asked them to change these practices to begin teaching for understanding.

If we take this notion of the types of practices that are easier for teachers to take up, we pose two questions for the field. Should we begin PD with practices that are add on activities for teachers to be successful with the principles of professional development early on? This is one way to interpret these findings and might be a way to begin to build trust with new instructional methods. Can we incorporate these more difficult practices in routinized ways so teachers can practice the norms of questioning students and designing problems? The notion of routinizing practices does not mean that we endorse simplifying them, but rather that we think teachers need to build standard ways around these practices before they can take on the complexity of more in depth student questioning. These ways of approaching PD may give us more insight into how to support teachers as they transition from learning with colleagues into learning with students in the classrooms.

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UNPACKING START-UP DIFFICULTIES OF MATHEMATICS COACHING

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This study, representing a small part of a larger six-year mathematics-coaching project, focuses on the difficulties that can arise as schools take on a coaching program. The program studied provided guidance to district personnel regarding how to begin a coaching program. Sixteen schools were chosen at random and separated into two groups that were designated as having higher or lower student growth. Results indicate that schools that were more closely aligned with the program protocol had fewer difficulties and greater student growth.

Mathematics coaching used as a model for professional development has gained momentum (McGatha, 1999). Results are mixed as to how coaching influences teachers' practice and student mathematics achievement. Little is known about what factors determine whether coaching works. The roles and responsibilities of a mathematics coach are many and varied. For this paper, we define mathematics coaches as those who team with teachers in classrooms with students daily for six weeks before moving on to the next set of teachers (Brosnan & Erchick, 2010). The purpose of this report is to reveal some of the many issues that arise before coaching even begins.

The context for this study is a mathematics-coaching program that has been in existence for six years and has shown many compelling results. Having served close to 200 schools, many, but not all, schools have shown increases in both teacher and student mathematics knowledge (Brosnan & Erchick, 2010). Mathematics coaching can be both challenging and rewarding, but starting without careful planning can be futile.

Purpose of Study

The purpose of this study is to unpack what has proven to be a myriad of stumbling blocks in the establishment of mathematics coaching in schools. The mixed results that published papers have reported, may be related to how many of the to-be-noted difficulties schools, coaches, and teachers have at the beginning of the process. These difficulties that can and have arisen in establishing mathematics coaching in the program discussed here, will reveal how starting off in the wrong direction may lead to little or no measurable improvement in mathematics achievement.

The work discussed in this paper is part of a larger research study and professional development program aimed at preparing a cadre of mathematics coaches to work in classrooms with teachers to enact research-based instructional strategies. The goals of the larger project are 1) To increase coach and teacher mathematics content knowledge; 2) To increase coach and teacher mathematics pedagogical knowledge; and 3) To increase student mathematics achievement on standardized tests. Because the results of this larger project are mixed, and in an attempt to help schools that did not increase achievement, a study to determine factors that prevent coaching success from the outset was warranted. To prepare schools for entering into mathematics coaching, the project provides building principals and coaches with three important documents a *Coach Selection & Information Packet*, *Program Assurances* and a *Start-Up Packet*.

These three documents were designed to assist school administrators and coaches in getting the coaching program off to a good start. The *Coach Selection & Information Packet* includes information about the program, program results, coach qualifications, suggested coach selection procedures, and a coach interview protocol. Project members offer to assist in the selection of a coach and/or to administer and score a test in mathematics and pedagogy. Coach qualifications listed in the packet for administrators to consider include such things as degrees and/or coursework in mathematics and mathematics education, successful teaching of mathematics, successful teaching of students who struggle, experience working with adults, good personal relation and communication skills, and commitment towards student learning of mathematics.

Our *Program Assurances* document has proven to be a very important guide to program fidelity. This document is a signed contract between school administrators, the state department of education, and program directors. Program expectations ensure that each participant knows what is expected from them and for them while engaged in the work of the coach. For example, project personnel are expected to provide professional development for coaches, facilitators, site visitors, and administrators; administrators are expected to assist the coach in arranging teachers' schedules to align with the coach's schedule, not use coaches as substitute teachers, and build a school-wide positive disposition for mathematics; coaches are expected to team teach with four teachers daily for six weeks, have pre- and post-conferences with teachers, and administer program assessments; and teachers are expected to team with the coach, conference with the coach at least once per week, and be willing to try new approaches in practice.

The *Start-up Packet* provides facilitators and coaches the tasks that must be completed before coaching begins. These tasks are broken into immediate and on-going categories. The immediate tasks include things such as building relationships with teachers; aligning curriculum to the current standards; taking inventory of school supplies, equipment, and manipulatives; administering and scoring pretests for teachers and students; collecting, synthesizing, and analyzing student data records; and other important start-up tasks. The on-going tasks include working with data; collecting student work samples; building trust with teachers and principals; making reports to program facilitators and directors; and other important management tasks.

Theoretical Framework

The framework's approach depicts a learner-responsive pedagogy comprised of elements from three areas, 1) mathematics content; 2) pedagogy; and 3) socio-cultural contexts. The mathematical elements include a rigorous, integrated procedural/conceptual perspective on the content (Baroody, Feil, and Johnson, 2007), processes for understanding, and a content that is richly connected (NCTM, 2000). The MCP pedagogical elements include student-centered interactions, where assessment has an informative focus, and grounds instruction incorporating problem-based tasks. These elements are drawn from scholarship of those who focus on the critical features of instruction (Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Wearne, D., Murray, H., Oliver, A., & Human, P., 1997), teacher content knowledge and content knowledge needed for teaching (Ball, Hill & Bass, 2005; Leinhardt & Smith, 1985; Lampert, 1990; Ma, 1999), and teacher pedagogical content knowledge (Leinhardt & Smith, 1985; Ball, 2000; Shulman, 1986, 1987; Wilson, Shulman & Richert, 1987). These scholars recognized the complexity of teacher knowledge and the role of both content knowledge and pedagogical content knowledge (PCK) in effective instruction. PCK embodies an understanding of how to draw from one's own knowledge of the subject matter to make the content accessible to children, make sense of students' thinking and draw on their ideas to orchestrate classroom instruction so to advance mathematical learning through continuous formative assessment. Finally, the contextual elements for coaches and teachers include an understanding of the learner in terms of both cognition and culture, and a learning environment where authority is shared between the learner and the teacher. A commitment to action distinguishes the equity, diversity and social justice element, where knowledge of the learner and understanding content and pedagogy are not enough. Using that knowledge to further the growth of the learner is an intentional commitment.

The MCP instructional approach is grounded in the situated cognition (Lave & Wenger, 1991) perspective. As such, it incorporates Liping Ma's (1999) notion of positioning the content development of teachers' in the context of student learning and classroom practice. In mathematics education this perspective has been supported by a number of scholars of teacher education and preparation. For instance, Ball & Cohen, 1999- cited in Smith (2001), acknowledged professional development should be situated in the context of practice in order for learning to be meaningful and transferrable. Others have argued that such learning should provide teachers with enactment strategies that would allow them to enhance learning for all students (Manouchehri & Goodman, 2000).

What We Know About Coaching in the Literature

Coaching is grounded in the notion of building capacity for positive change in teaching practice and having highly trained practitioners *working with their colleagues* to promote learning and improve practice (Becker, 2001). Working with teams of teachers, these coaches share their expertise and provide feedback and assistance as teachers learn new teaching skills and solve classroom-related problems (Galbraith & Anstrom, 1995; Borko, Davinroy, Bliem, & Cumbo, 2000; Breyfogle, 2005; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998). While researchers indicate that teacher coaching can be practiced using a variety of structural approaches, it has not yet been studied extensively (Biancarosa, Bryk, & Dexter, 2010; McGatha, 2009). Other studies have been designed to examine what coaches do but have not yet shown much connection to actual improved student achievement (Campbell & Malkus, 2009; Wei, Darling-Hammond, Andree, Richardson, & Orphanos, 2009). In studies in which university-based researchers have taken on the role of coaches and reported on their interactions with teachers in schools (Olson & Barrett, 2004), there has been only limited information about how and what type of professional skills have been developed at the coach level, and then effectively communicated with teachers in ways that knowledge is enacted in classroom practice.

Methods

Participants

In the larger study, the participants include approximately 185 schools, principals, coaches, and teachers. Most of the schools were in School Improvement Status and Title I served. Grade levels in buildings varied from K-8. Some buildings housed K-5 students, some K-3, 5-6, 6-8, and other groupings. In this smaller study, a random sample of 16 schools, principals, coaches, and teachers were selected. These 16 schools were then placed in rank order by student achievement growth. Half were placed in the higher growth school group and half in the lower growth school group.

Data Collection

For this report we use data from *Facilitator* and *Site Visitor Reports*. In addition, teachers and principals were interviewed once during the school year using a semi-structured interview protocol. These interviews focused on how teachers and principals understood the reform work and how they chose to follow program initiation protocols.

Facilitators are mathematics specialists who were chosen by the geographic area of the state in which they worked as university faculty or as state agency professional development providers. They are hired by the program to work four days per month. Two days per month, they attend the training as their coaches do to learn the program as presented, and the other two days per month they meet with small groups of coaches who work in the same geographic area. These small group sessions are designed to detail the work in enacting their duties as a coach. *Site visitors* are also mathematics specialists who work in state agencies around the state and are hired to work for our program to visit all of the schools participating in the mathematics-coaching program. Both the *Site Visitors* and *Facilitators* used the same measuring instrument, the *Site Visit Inventory*, to document what coaches actually do or report doing in the schools. This instrument serves as a tool to document evidence of the progress, challenges and issues that arise in coaching. The instrument was designed with the program framework, assurances, and the envisioned roles and responsibilities in mind to determine fidelity of implementation.

Data Analysis

The process of analysis was the same for each set of data, with the analytic procedures falling under constant comparison methods. First, data were entered into a qualitative research software program and reviewed by project researchers to discover emergent themes. Emergent themes in each data set determined the creation of additional codes added to pre-determined research-based codes. Data was then coded using NVivo qualitative analysis software, and sorted into the coded categories.

Findings

Mathematics coaching is a challenging endeavor. Careful planning at the outset makes a difference as to whether coaching may be successful in terms of improved student mathematics achievement. After analyzing the reports from the 16 randomly selected schools, the categories that emerged from the data include 1) Coach selection criteria, 2) Compliance with program *Assurances*, and 3) Completion of Start-Up Tasks.

Coach Selection

Although there were few data sets that actually revealed how coaches were selected, what we did learn from interviews was that in schools that began in a less-than-optimal way, the coaches hired by school districts had no greater mathematics knowledge than the teachers they were coaching; some either disliked or never taught mathematics; several union contracts required hiring by seniority only, resulting in several coaches who sought alternatives to classroom responsibilities; and others did not have the necessary leadership nor human relation skills optimal for the job of coach. In one situation we learned that the principal selected her “worst” teacher (principal’s term) in the building to be the coach. It would appear obvious that none of these situations would proffer much hope for a successful start to a coaching program.

In one of the principal interviews when asked about her coach’s leadership skills, she said, “He leads by example. At no time does he expect more from his teachers than he is willing to give.” Another principal said that he hired his coach from a small group of applicants, in a time crunch. He thought he made the right choice, given the applicants.

Compliance with Assurances

Engaging in large-scale school reform requires participants to be ever so vigilant in following guidelines gleaned from the literature and experience. Following our set of Assurances was our proactive approach to waylaying some of the usual pitfalls.

Assurance Constituent	Schools with Higher/Growth Mathematics Achievement	Schools with Lower/Growth Mathematics Achievement
Administrators are expected to		
• Assist the coach in arranging teachers' schedules to align with the coach's schedule	75%	50%
• Not use coaches as substitute teachers	87.5%	50%
• Build a school-wide positive disposition for mathematics in the building	62.5%	50%
Coaches are expected to		
• Team teach with four teachers daily for six weeks	62.5%	37.5%
• Have pre- and post-conferences with teachers	75%	25%
• Administer program assessments	100%	87.5%
Teachers are expected to		
• Team with the coach	75%	62.5%
• Conference with the coach at least once per week	75%	37.5%
• Be willing to try new approaches in practice	87.5%	50%

Table 1. Compliance with program assurances comparison

Data from Table 1 indicated that those schools that had greater levels of compliance with program assurances had a better likelihood of achieving greater growth. The reports also showed that those who started out in alignment with the assurances also demonstrated a greater sense of enthusiasm in the school at large resulting in building strong relationships from the start.

Site visitors reported that one coach works with four teachers but for only 30 minutes each. Another coach moves to a new building every two to three weeks and may work with 2-3 teachers at a time, depending on the situation. Another site visitor reported that some coaches are given far too many non-coaching responsibilities. For example one has breakfast, lunch and dismissal duties and is also part of the district Improvement Process team and is in charge of AIMS web for both mathematics and reading. She is the contact for two math/computer programs, Alex & Math Tracks. She also has the responsibility of reordering math materials and the like. Another coach is in charge of AIMS web testing, grade 2 and 5 oral reading fluency testing, and Orleans H Algebra Prognosis Tests.

Regarding one coach who did not like relinquishing control of the teaching to the classroom teachers, the site visitor wrote, "If she is the only one planning and teaching, she should not be so puzzled by the fact that when she does leave, they are discontinuing the use of the strategies." This same site visitor reported asking a coach why she was not coaching in more than one classroom, the coach replied, "… Their schedules had already been developed, with many of the teachers teaching math at the same time. And, they didn't want to change their schedules for the year." As a consequence, the principal told the site visitor, "You are not going to see what you want to see today, but next year we are going to fix it. There are 6 math teachers and next year she will interact with them like we need her to."

All of these statements indicate how important the assurances can be to whether a coach is able to their job well under certain conditions.

Start-Up Tasks

These tasks were designed to prepare coaches to coach. We felt that coaches need to know and align the curriculum with standards, know and understand student records and culture, and take time to complete test administration and build teacher relationships.

Task Description
Immediate • Building relationships with teachers; • Aligning curriculum to the current standards; • Taking inventory of school supplies, equipment, and manipulatives; • Administering and scoring pretests for teachers and students; • Collecting, synthesizing, and analyzing student data records; and • Other important start-up tasks.
Ongoing • Working with data; • Collecting student work samples; • Building trust with teachers and principals; • Making reports to program facilitators and directors; • Other important management tasks

Table 2. Sample Start-Up Tasks

Program Start-Up Tasks were important for our mathematics coaches to complete before they began coaching. First, these tasks are necessary for the coach to learn about curriculum, school data, and classroom culture. Secondly, teachers are so grateful to have someone do work for them that is very time consuming, and finally because of the many tasks done ahead of time contribute greatly to building rapport with teachers. The following excerpts from the facilitator reports confirm these observations.

Principal: “Teachers tend to be surprised when working through this analysis with [the coach]…it helps them understand the curricular needs of their students.”

“He has worked very hard to build a positive rapport with the teachers by doing tasks that are very helpful to their teaching but they just don’t have time for…such as the gap analysis sheets, vocabulary, matching lessons to the SST goals, etc. He works with grades 3-4-5 and he did a gap analysis study for them to go with the new series. He also found activities and lessons to fill-in the gaps not covered.”

“[She] gives students a pretest and goes over the results, making a spreadsheet identifying the strengths and weaknesses of the students.”

“I shared data from the [state] tests with the teachers. This was the first eye-opener for the teachers. The teachers don’t have time to analyze the test questions.”

Conclusions

The evidence in this study shows that following the prescribed program protocol is important to establishing a mathematics-coaching program. The selection process used in hiring a coach cannot be understated. Teachers having greater mathematics content knowledge, successful teaching records, good relationship building and leadership skills prove to be necessary characteristics in beginning coaches. In cases where there was little choice in how a coach was hired, i.e. union contract seniority regulations, there was more opportunity for a non-optimal selection.

One of the unique elements of the mathematics-coaching program studied was the design and adherence to program assurances. It is often the case that when university and school partnerships go awry, lack of implementation fidelity can certainly lead to a lack of program impact. The assurances in this study that seemed to make a difference in whether a school started off in better ways was whether the coach was able to coach as prescribed; whether the coach was stretched between buildings; or if the coach was given too many extra-program responsibilities. Also, in the assurances category, if teachers were reluctant to team with the coach, the possibility for the coach to build capacity was clearly diminished.

The program start-up tasks proved to be useful in getting coaches ready to do their job. For example, aligning the school curriculum to the current standards and synthesizing classroom student data reports were the two most highly touted ‘helps’ for teachers who say, they never have ‘time’ to do such things and the information is so important and useful.

School districts interested in establishing a mathematics-coaching program should consider using the program protocols described in this paper. Coaching is difficult in any capacity let alone falling prey to some of the pitfalls described herein.

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PROPORTIONAL REASONING INSTRUCTION: A COGNITIVELY GUIDED APPROACH TO PROFESSIONAL DEVELOPMENT

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Content-specific cognitively guided instruction (CGI) professional development programs have been shown to lead to positive changes in instruction in the elementary grades. This paper presents the results of a study that investigates how teachers use new knowledge gained from a CGI professional development workshop, on proportional reasoning in the middle grades, to inform their instructional decisions. Four teachers' instruction and their rationales for their instructional decisions were examined before and after the workshop intervention. All four teachers' instruction changed to become more cognitively guided.

Introduction

The National Council of Teachers of Mathematics [NCTM] (2000) recommends that proportional reasoning become a major focus of the middle grades curriculum by interweaving it with percents, similarity, scaling, linear equations, graphs, probability, histograms, and problem solving. Furthermore, proportionality is involved in at least one of the focal points for each of the middle grades. Students who fail to develop an understanding of proportional reasoning in middle school are likely to struggle in algebra and other higher level mathematics (Langrall & Swafford, 2000).

Although proportional reasoning and facility with rational numbers are so important, research shows that many students struggle with them (Misailidou & Williams, 2003; Singh, 2000). In fact, preservice and inservice teachers often lack a deep understanding of proportionality (Bezuk, 1988; Conner, Harel, & Behr, 1988). Furthermore, teachers often have weak pedagogical content knowledge (Hines & McMahon, 2005) and knowledge of their students (Ruchti, 2005) related to proportional reasoning.

In order to understand the true breadth of the area of proportional reasoning, teachers must understand both the subtle (e.g. number structure) and obvious (e.g. context) nuances of proportion problems (Franke, Fennema, & Carpenter, 1997). Additionally, teachers need to be able to analyze a problem to determine the range of strategies that students are likely use when solving it. This ability better enables them to determine the difficulty level of a problem. According to Carpenter and Fennema (1991), "analyzing problems, strategies, and development allows for the ability to select critical problems both to differentiate between levels of performance but also to help children progress to the next level" (p. 9).

In 1989, Carpenter, Fennema, Peterson, Chiang, and Loef developed an approach to professional development, cognitively guided instruction (CGI), which focused on increasing teachers' pedagogical content knowledge and knowledge of their students in a specific content area. Since, participation in a content-specific CGI program has been shown to have a positive effect on teachers' pedagogical content knowledge and lead to changes in classroom practice towards instruction guided by students' thinking.

This study investigates how teachers use new knowledge gained from a CGI professional development program on proportional reasoning to inform their instructional decisions. Particularly, the ways that the teachers generated student thinking prior to and after gaining access to research-based models of students' thinking about proportional reasoning were examined. This study is intended to extend the CGI studies, which focused primarily on operations in the elementary grades, to proportional reasoning in the middle grades.

Theoretical Framework

A few assumptions about the goals of instruction and what instruction should look like underlie the CGI philosophy (Carpenter & Fennema, 1991). First, the goal of instruction is to increase students' understanding by connecting to their prior knowledge (Donovan & Bransford, 2005). To effectively structure instruction to build off students' prior knowledge and experiences, teachers must have strong pedagogical content knowledge, as well as a deep understanding of their students.

Second, effective instruction requires teachers to make informed instructional decisions, by continually reflecting on their instruction and students' learning. Thus, teachers must continually assess their students' understandings through using appropriate tasks and questions that enable students to showcase their thinking.

Third, for learning to occur students need to develop mathematical explanations and justifications and connect them to their existing body of knowledge. The teacher's role is to choose meaningful exploration activities and provide a learning environment where mathematical conjectures, multiple techniques, and mathematical dialog are valued.

Methodology

The professional development program which provides the context for this study is briefly described here. Data collection and analysis methods will also be discussed.

Professional Development Program

The context of this study was a two-day professional development workshop, followed by ongoing implementation support over the six subsequent months. The workshop design was inspired by the CGI studies. During the workshop, we discussed the research findings in the area of proportional reasoning that are related to (a) students' strategies; (b) problem types; (c) factors influencing students' success and strategy choices; (d) prerequisites to the development of proportional reasoning; and (e) developmental theories, specifically those of Piaget (Piaget & Inhelder, 1969), Noelting (1980a, 1980b), Milsailidou and Williams (2003), Lesh, Behr, and Post (1987) and Karplus et al. (1983a).

Written and video cases illustrating real classroom teaching episodes were also used to illustrate CGI. The teachers were asked to analyze the pedagogy, questioning, student thinking, and teacher's role within the cases.

Researchers have found that teachers are more likely to change their instruction as a result of professional development when supported (Steele, 2001). Here, when initiated by the teachers, the researcher served as a mentor to aid in planning and analyzing students' strategies.

Participants

The participants in this study, who volunteered, were four middle school teachers from the same school in a rural part of Virginia. The teachers had varying levels of experience from 1 to 22 years. There was one teacher from each grade, 6-8, and one who taught in eighth grade for the first half of this study and seventh grade for the second. The sixth grade teacher (Bob), seventh grade teacher (Abby), and the seventh/eighth grade teacher (Julie) were all in their first year teaching mathematics. However, Julie had four years of science teaching experience. The eighth grade, and veteran, teacher (Sue) had taught mathematics for 22 years. All of the mathematics teachers in the school participated in the workshop.

Data Sources

The data analyzed here were collected primarily through classroom observations, interviews, and document compilation. Each participant was observed and interviewed prior to the workshop to examine their instructional style and bases for making instructional decisions. In addition, documents were collected that illustrated how each teacher had taught or planned to teach proportion concepts.

After the workshop, each teacher was observed teaching ratio and proportion concepts on several occasions, ranging from 4 to 16 instances. The large variation in the number of observations per teacher was due to differences in the depth of curriculum coverage of proportions between sixth and eighth grade. Detailed field notes were compiled during each observation which focused on (a) what the teachers *did* and *said* during instruction, (b) how the students reacted to the instruction, and (c) how the teacher interacted with the students.

Following each observation, the teachers were interviewed by the researcher to determine their rationales for their instructional decisions. The interviews were recorded and later transcribed to capture the teachers' responses accurately. These interviews were semi-structured; they began with a predetermined list of questions, and appropriate follow up questions, and ended with questions related specifically to the observations.

Data Analysis

A case study and cross-case analysis was conducted to describe each teacher's instruction and rationale before and after the workshop. Systematic data analysis was used to derive causal descriptions and lawful relationships among the data. The three frameworks that guided the initial coding were derived from: (a) the content of the CGI workshop on proportional reasoning, (b) Franke, Carpenter, Levi, and Fennema's (2001) schema for analyzing teachers' actions on a CGI scale, and (c) Stein, Smith, Henningsen, and Silver's (2000) schema defining levels of cognitive demand. To ensure validity, the data was triangulated across methods, member checking was used, and colleagues of the author were asked to evaluate the author's interpretations of the data.

Findings

The purpose of this study was to answer the following research questions: (1) How did the four teachers generate student thinking through their instructional decisions *prior* to participating in the workshop? (2) What were the four teachers' instructional decisions based on prior to the workshop? (3) How did the four teachers change in order to generate student thinking through their instructional decisions after participating in the workshop?

How Was Student Thinking Generated through the Teachers' Instructional Decisions Prior to the Workshop?

All four teachers involved their students in their lessons to varying degrees before the workshop, but they all focused their tasks and questions on applying and remembering procedures rather than developing connections and understandings. Thus, the teachers rarely generated student thinking through their instructional decisions.

All four teachers used questioning to involve their students, but all their questions were categorized as placing low cognitive demands on students. For example, the teachers commonly asked students to: state the next step when applying a procedure, give an answer, or perform a computation. Two teachers were observed presenting a task with a high-level of cognitive demand, but their treatment of the task had the effects of lowering the cognitive demand. For example, one teacher had “tricky” problems, which were different than those the students had seen before, at the end of a worksheet. Those tasks were initially categorized as high-level tasks because the students need to form connections or “do mathematics” (Stein et al., 2000). However, the teacher immediately pointed out the tasks as “tricky” and prescribed another procedure for solving them, ultimately lowering the demands of the tasks to “procedures without connections”.

What Were the Four Teachers’ Instructional Decisions Based on Prior to the Workshop?

Sources. Prior to the workshop, the novice teachers relied on others for the majority of their lesson ideas: two teachers almost exclusively relied on their textbook while the other relied on worksheets and materials provided by colleagues. Unlike the novice teachers, the veteran teacher was not entirely dependent on other sources for lesson ideas.

The novice teachers lacked confidence in their ability to plan appropriate lessons to meet the Virginia Standard of Learning (SOL) objectives, which likely stemmed from their lack of experience teaching mathematics and their unfamiliarity with the content and pedagogical techniques for teaching. One teacher exclusively used the materials that came with the textbook. The role of the textbook was also significant for the veteran teacher. However, she created her own example problems to relate the new concepts to her students’ background knowledge and to be relevant to their lives.

Rationales. All four teachers followed the general sequencing determined by their pacing guides or the textbook, but the three novice teachers did so blindly. The three novice teachers did not have an overview of the general sequencing of their courses, indicating that no consideration was given to the order in which they introduced different mathematics topics. One admitted that she planned her lessons one day or one week at a time and followed the pacing guide. When I asked her how she typically planned her lessons, she replied, “Day-by-day. We have our planning guides, they are really helpful. It tells you day-by-day exactly what they need to know.”

All four teachers’ planning decisions regarding which types of tasks to include in their lessons, as well as how to sequence them, were influenced largely by what appeared in their lesson planning sources and by their goal to have students memorize and master procedures to prepare for state-mandated testing. However, the veteran teacher referenced what her students knew or where they were in their development of understandings when describing her reasons for her instructional decisions, while the other teachers did not.

How did the Four Teachers Change in order to Generate Student Thinking Through Their Instructional Decisions after Participating in the Workshop?

Student involvement and questioning. All four teachers changed, to varying degrees, to encourage more individual high-level thinking from their students. They did so by asking students to solve problems rather than exercises exclusively and by asking high-level questions in addition to low-level questions. They all asked more questions and used more tasks that I categorized as requiring high-levels of cognitive demand; unlike prior to the workshop, when all four teachers posed only low-level questions and tasks. All four teachers allowed their students to solve problems without any prior instruction on how to do so, but Abby and Bob always presented procedures after students were given the opportunity to solve the problems on their own. Prior to the workshop, none of the teachers allowed their students to solve problems without prior instruction.

Sources. The sources that Sue, the veteran teacher, and Bob used for lesson ideas only changed minimally, while the sources the other two teachers used changed more substantially. The novice teachers all still relied on the pacing guide to determine their overall instructional sequence of topics, while the veteran teacher was observed referring to the textbook for the general sequence of topics.

Immediately after the workshop, Julie became less reliant on the textbook materials. She began creating her own warm-up exercises to meet her instructional goals and to highlight her students' relevant background knowledge. Although she initially depended on the textbook to provide her with appropriately sequenced problems, she chose which problems to include in her lessons. In her second time teaching proportion concepts, after moving to the seventh grade, she relied even less on the textbook. Instead, she used what she learned in the workshop, together with problems from a SOL Test Bank, to plan her lessons.

In Abby's case, there was a substantial change in her planning sources. She crafted her own problems for worksheets on proportions, percents, and cross multiplication where before she only used worksheets others had created.

Similar to before the workshop, all the novice teachers said that the overall sequencing of their lessons was determined by the outlined sequence and timing given in their pacing guides. They did not seem to have an overview of the topics and sequence of their courses. They planned their lessons on a weekly basis, often without looking ahead in their pacing guides. There were occasions when they could not tell me what was coming up next or generally when they would be teaching a specific topic.

Rationales. After the workshop, three of the teachers expressed that they had sequenced tasks in order to further their students' development. Of these three teachers, the increase in Julie's attention to the sequencing of the tasks used was the most profound. The fourth teacher, Bob, generally articulated no particular reason for the sequences of tasks he was observed presenting, besides following the sequence in the textbook. However, all four gave some consideration to the numerical structure of the tasks.

Changes in the teachers' levels on the CGI scale. All the teachers increased at least one level according to Franke and colleague's (2001) CGI schema indicating that their instruction was more cognitively guided, after the workshop.

Julie had the most significant changes in her instruction and beliefs, from Level 1 to Level 3 or Level 4A. In Franke and colleague's (2001) schema, Level 1 corresponds to instruction that is not cognitively guided and Level 4B corresponds to instruction that is completely cognitively guided. Julie was at Level 1, prior to the workshop, for several reasons. First, she believed that her students were not capable of solving problems without the help of delineated procedures or their textbooks. During my observation, she did not allow her students to figure out how to solve problems on their own and from her description of her standard lesson plan, it was clear that Julie did not allow her students to invent their own strategies for solving problems. In fact, she never posed a "problem," instead she used tasks where procedures were given or exercises that were similar to previously solved tasks. Like the tasks Julie used prior to the workshop, the questions she asked her students required low-level cognitive demands of students; she did not ask them about their thinking.

After participating in the workshop, Julie satisfied all of the characteristics of a teacher at Level 3. First, while there was no change in Julie's expressed beliefs about how students learn mathematics best, her teaching style changed to corroborate these beliefs. In both instances she indicated that discovery was the best method for learning and that teachers should always allow their students to solve problems, but Julie commented that she learned how to teach in this way through the workshop. Post workshop, Julie *never* taught a procedure before allowing her students to discover their own ways first; in fact, in most cases she avoided teaching procedures all together.

Second, after the workshop, Julie encouraged her students to solve problems in different ways. She asked, "Did anyone do it differently?" so frequently that her students began to volunteer to share alternative approaches. Third, Julie used a variety of different types of problems after the workshop. She attempted to teach in a developmentally appropriate way by creating progressions of problems according to their difficulty levels or the strategies they were likely to elicit from her students. For instance, one of her lessons on proportions progressed from problems involving an integer factor of change to problems without integer factors of change and with large numbers, to encourage her students to first develop a factor of change strategy and then advance to cross multiplication (CM). She said, "I deliberately gave them a problem where factoring was going to blow their minds so that they would be forced into the other [developing CM]." She explained that she "wanted to lead them [her students] into discovering CM or needing the CM algorithm."

Fourth, Julie expected her students to share their thinking with the class, after the workshop. *Before* the workshop, the only questions Julie was observed asking had low-level demands, but *after* the workshop Julie also asked questions with high-level demands. In fact, as more time passed after the conclusion of the workshop, Julie asked more and more questions placing high-level demands on students. Immediately after the workshop, Julie asked her students to share their solution strategies with the class, but she focused more on the processes than on her students' reasons for their actions or why their strategies worked. Over time she asked more questions about her students' thinking, such as, "Why does this work?" or "Why did you put the numbers there [within a proportion]?" Julie's eighth-grade students really responded well to their new responsibilities involving solving problems and sharing their thinking. One of her students went to the board and announced, "All eyes on me. Ok…" and then she proceeded to explain her solution process. This was a drastic change; before the workshop, Julie's students were far more disruptive than participatory. Julie noticed this too and she attributed the change in her students to requiring them to *think*:

> Yeah ever since I switched it [switched her teaching to problem based rather than read in the book and fill in the blanks], he has been into it. Because he is using his mind. And you know, S [another student] did too [got into the lesson].

For these reasons, Julie was at least at Level 3 on the CGI scale after the workshop. But Julie also possessed some of the characteristics of a teacher at Level 4A. Julie may have been in transition to this level. She focused her lessons around problem solving. The majority of the tasks Julie posed were word problems after the workshop. She always asked her students to solve problems in their own ways and to share their approaches with her class. As I mentioned earlier, she based her instructional decisions on her students thinking, attempting to push them further along in their development of understandings. There was also at least one instance when she based an instructional decision during class on her students' thinking. This instance was when one of her students explained that he was writing percent proportions in the order in which the values appeared in the problem. In response, Julie posed a problem where this students' misconception would be highlighted. Later she explained to me, "They wrote it [a student said, 'It has to go 78 over 58 because it is 78% of 58'] and I thought I have to use the other [another] problem to show that it may not go in order."

Sue changed her instruction from Level 2 to Level 3. Prior to the workshop, Sue was at Level 2 for several reasons. Unlike the other teachers, Sue occasionally asked her students to solve problems on their own and repeatedly stressed the importance of relating new knowledge to students' background knowledge (e.g. "If they can tie it to something that they have learned previously, it tends to stick a lot better."). She also described being open to a variety of approaches to solving problems, which was characteristic of a teacher at Level 2. However, that behavior was not observed through her teaching. Because she showed her students how to solve problems with her anticipatory set, note-taking guides, and approach to dealing with students' difficulties, Sue was not at Level 3 prior to the workshop.

After the workshop, Sue's instruction changed in several ways to become more cognitively guided. The following changes led to a Level 3 characterization: (a) Unlike before, Sue allowed her students to solve problems in their own ways without first asking them to read about procedures for solving similar problems. (b) When her students had trouble figuring out a way to solve a novel problem, Sue posed questions to help them discover a solution strategy. She did not resort to dictating a strategy, like she did prior to the workshop. (c) She demonstrated the openness to a variety of strategies that she expressed previously. Her students also perceived her receptivity to different approaches; During one of my observations, Sue asked her students, "Do you have to do it that way?" and one student replied in a mimicking tone, "No, whichever way works for you." (d) Sue asked questions placing higher-level cognitive demands on her students, such as, "Why can we do that?" or "What is another way you could do this?" These types of questions encouraged her students to discuss their thinking, as well as develop connections and understandings.

Both Abby and Bob progressed from Level 1 to Level 2. Prior to the workshop, neither teacher allowed their students to solve problems on their own. Instead, they demonstrated how to solve problems and then asked their students to solve similar exercises. For this reason, it appears that they did not believe that their students were capable of figuring out their own ways to solve problems.

Both teachers were not observed posing questions or tasks that placed high-level demands on their students prior to the workshop. Actually, Abby and Bob used tasks and questions that could have been deemed high-level, but their treatment of them had the effect of lowering the tasks' cognitive demands. Neither Abby nor Bob asked their students about their thinking.

After the workshop, both teachers progressed to Level 2 on the CGI scale. They occasionally allowed their students to solve problems on their own, which they did not do previously. However, they both always taught procedures for solving problems after their students had been given a chance to solve them on their own. Bob recognized that there were a variety of proportion problem types and that those types can have an effect on students' thinking. He varied the types of problems he included on his proportion assessment and spoke about his decision being influenced by the fact that different numerical structures would encourage different strategies. Abby also considered different problem types and varied the numerical structures and contexts she chose to include in her lessons. However, she only allowed her students to use a factor of change or CM strategy. Additionally, both teachers elicited more thinking from their students after the workshop and asked some questions with high-levels of cognitive demand. Abby and Bob were not at Level 3 due to the fact that neither teacher freely allowed their students to solve problems in their own ways.

Conclusions

A content specific professional development program, regardless of content or grade-level, can lead to positive changes in teachers' instruction to become more cognitively guided. Although this workshop intervention was brief, all four teachers changed their instruction to become more cognitively guided afterwards. It is reasonable to assume that a longer professional development opportunity may have lead to greater positive changes in instruction.

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THE ROLE OF TECHNLOGY FOR IN-SERVICES TEACHERS: IMPROVING GRADUATE EDUCATION THROUGH INFORMED DECISIONS

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GRADUATE EDUCATION PROGRAMS FOR IN-SERVICE TEACHERS HAVE BEGUN FOCUSING ON THE USE OF TECHNOLOGY IN MIDDLE AND SECONDARY MATHEMATICS SETTINGS TO OFFER OPPORTUNITIES FOR STUDENT LEARNING. DISCONNECTS BETWEEN IN-SERVICE TEACHERS AND TEACHER EDUCATORS REGARDING AVAILABLE TECHNOLOGY CAN HINDER EFFORTS TO SUPPORT STUDENT LEARNING THROUGH TECHNOLOGICAL MEANS. IN THIS STUDY, WE IDENTIFIED THE ACCESSIBLE TECHNOLOGY IN-SERVICE TEACHERS IN A MASTER'S PROGRAM HAVE IN THEIR CLASSROOMS, AND PROVIDED THE INFORMATION TO THE TEACHER EDUCATORS OF THE GRADUATE PROGRAM. AFTER GATHERING TEACHER EDUCATOR PERCEPTIONS OF THE TEACHER PARTICIPANT TECHNOLOGICAL DATA, WE SUMMARIZED HOW THIS PROCESS MIGHT REDUCE SUCH DISCONNECTS AND PROMOTE MEANINGFUL TECHNOLOGY INCORPORATION IN THE PROGRAM.

INTRODUCTION

The challenge of providing teacher education that encompasses appropriate use of technology for student mathematical learning spans traditional discipline borders in graduate programs (NCTM, 2000; Signer, 2008). Cuban (1986) and Lloyd and McRobbie (2005) make a case for professional development that provides opportunities to learn about pedagogically appropriate technology, focusing on activities similar to teachers' current classroom practice. However, graduate education and professional development programs are struggling to follow such advice, in part due to varying advice on how technology education can best influence classroom practice (Dede, Ketelhut, Whitehouse, Breit, & McCloskey, 2009; Wilson & Wright, 2010). This paper describes a study aimed to improve technology education within a graduate program designed for in-service mathematics teachers.

Graduate and professional development opportunities for teachers are central to national improvement efforts, and effective classroom implementation of new teacher learning is seen as a determining factor of success (Lemke & Fadel, 2006; O'Dwyer, Russell, & Bebell, 2004; Penuel, 2006). Putnam and Borko (2000) explained, "professional knowledge is developed in context, stored together with characteristic features of the classrooms and activities, organized around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations" (p.13). Also, professional development with technologies related to local curricula, standards, and current teaching practices are more likely to have a positive impact on implementation of that technology (Martin et al., 2010; Penuel, 2006).

Since Cuban (1986) first emphasized the importance of making explicit connections between technology and teachers' specific types of instruction, the subject has received substantial attention. In particular, professional development involving technology can impact teacher practice in significant ways, and has been shown both to increase and fundamentally change student learning (Pea, 1985; Penuel, 2006; Ravitz, 2009; Ringstaff & Kelley, 2002; Stevens, To, Harris, & Dwyer, 2008). The work of Stevens et al. (2008) and Martin et al. (2010) addressed the connection between a teacher's professional development about technology and student learning outcomes. Stevens et al. (2008) found teachers' self-efficacy and self-determination improved through participation in a professional development program focused on technology and was correlated with "positive educational outcomes for students" (p. 215). In Martin et al.'s (2010) study, after participating in a technologically-focused professional development program, teachers' use of mathematics technology had a strong and positive relationship to student achievement across elementary and middle school levels. While these results are positive, researchers emphasize the importance of keeping technology tied to classroom practices (Glazer & Hannafin, 2006; Mouza, 2009).

The purpose of this study was to examine perceptions of teacher educators regarding availability and use of technology for mathematics teachers enrolled in their graduate classes. Specifically, we sought to answer the following research questions: (1) What original assumptions and beliefs do teacher educators have about the availability and use of technology in the K-12 classroom? (2) How are teacher educators' plans for instruction with technology influenced by knowledge of the availability and use of technology in the K-12 classroom? and (3) What are the felt needs about technology expressed by teachers and teacher educators in a technology-rich professional development setting?

Methodology

Participants and Setting

This study focused on a two-year master's program in mathematics for secondary teachers[1].

Teacher Educator	*Subject Matter Expertise*	*Number of Courses Taught in the Program*
Dr. Heilmann	Mathematics Education	More than 3
Dr. Torres	Applied Mathematics	None
Dr. Selvidge	Mathematics	More than 3
Dr. Atteberry	Mathematics	1

Table 1: Teacher Educator alias and description.

Designed for middle and high school mathematics teachers, courses are offered year-round through online and blended educational settings. The program incorporates a variety of online technologies during the school year and combines face-to-face and distance instruction in the summer. At the time of the study, participating teachers were primarily from two states in the Rocky Mountain region, and will hence be called teacher participants.

While this study summarized data from the 35 teacher participants taking courses during the summer, our main focus was on the four teacher educators who taught the courses. The four courses offered were Mathematical Problem Solving, Modern Geometry, Teaching Geometry, and History of Mathematics, with each teacher educator instructing one course (see Table 1).

Data Collection and Analysis

We investigated the research questions using a combination of survey and interview methods with both teacher participants and teacher educators. First, we administered an open-ended survey to the teacher educators, the Technology Input Survey (TIS). This seven-item survey asked about beliefs regarding the technology available to teacher participants. Two questions gathered information on how teacher educators planned to incorporate technological instruction, and how the instruction would assist teacher participants in teaching with technology in their own classrooms. All four teacher educators completed the survey.

We then developed and administered a second survey, the Classroom Technology Survey (CTS). Completed anonymously, online, by all teacher participants in the month before summer classes began, this survey consisted of nine open-ended questions about what technology was available and used in teacher participants' classrooms and how participants characterized the use of technology in their classrooms. The response rate for the CTS was 22 out of 35 (63%).

In addition to the surveys, we conducted semi-structured interviews with each teacher educator of the summer courses. A summarized version of the CTS data was shared with the teacher educators shortly before the beginning of the summer term. Interview questions were asked to elicit responses on the implications of the CTS data, such as "What is your initial reaction to the information presented? Does the information we have shared with you influence your plan to teach the course this summer? If so, how? If not, why?" The individual interviews took place two weeks prior to the summer courses started and were audio recorded and transcribed.

The TIS results were first summarized for each teacher educator, and responses from the question, "What information about teachers' use of technology would be helpful to you as an instructor" were incorporated into the CTS. Other responses were summarized and grouped according to the research questions they addressed. Following collection of the CTS, descriptive statistics were calculated regarding the type and frequency of technology teachers reported was available and used by them. Completely opened-ended questions (such as how teachers characterized the use of technology in their classroom) were open-coded (Creswell, 2007) then arranged into categories before being summarized into the examples provided in the summary given to teacher educators (see Figure 1). In a similar process, instructor interview transcripts were open coded to distinguish themes among instructor perspectives about the CTS data. Researcher and data triangulation occurred through the TIS and interview analysis process.

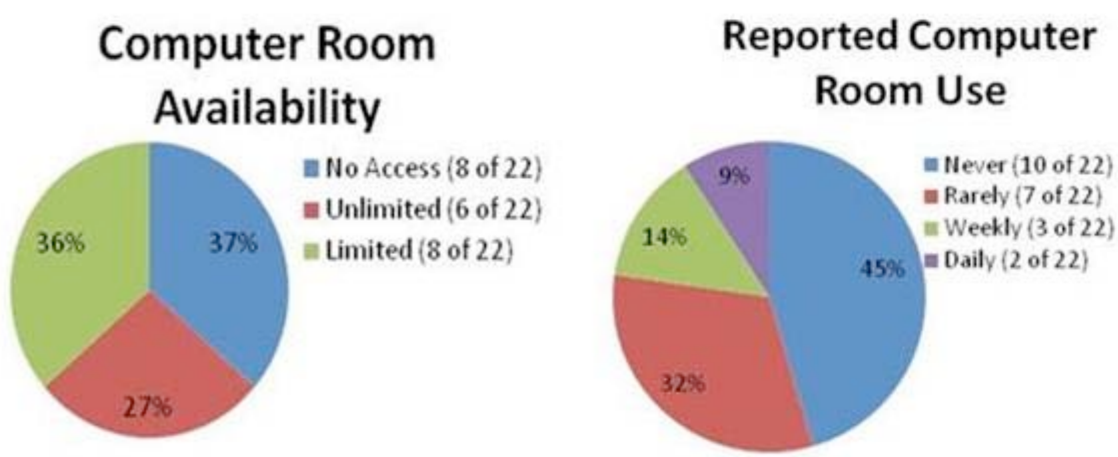

Figure 1: An example item from the CTS summary of teacher participants' reported availability and use of computer rooms.

Results

Teacher educators' beliefs and assumptions about the availability and use of technology in teacher participant classrooms were altered after presentation of the CTS data. Prior to seeing this data, each teacher educator believed all teachers would have access to graphing calculators and that most would have access to a computer lab with dynamic geometry software. Three of the teacher educators reported they believed SMART Boards would be available to some of the teacher participants. Drs. Atteberry, Selvidge, and Heilmann, all having prior experience with in-service teachers as learners, reported making the assumptions based on conversations with teachers and previous teaching experience. Dr. Torres based her assumptions on her experiences working with other in-service teachers, attending sessions at conferences dealing with classroom technology, and reading reports of classroom technology availability. After receiving the CTS data, all four teacher educators indicated being surprised by the high number of interactive whiteboards available to the teacher participants; specific reactions for each teacher educator are described below.

Dr. Atteberry believed his role as an instructor was minimal in connecting technology to the teacher participants' classrooms. Dr. Atteberry also expressed his belief that calculators and other technology were overused in classrooms. He stated, "I worry about people creating graphs so much with technology they can't do it anymore...we have a lot of evidence of that...when I taught calc I [to undergraduate students last year]." Dr. Atteberry said he planned to use technology in his course, "only minimally - I would concentrate on showing different kinds of problems that can be facilitated with the computer or graphing calculator."

After reading the results of the CTS, Dr. Atteberry expressed surprise at the prevalence of whiteboard and digital projectors in the teacher participants' classrooms. He asserted that the cost of the technology must be less of a hindrance for having the technology than he originally believed, but that the information seemed unlikely to change his instruction during the summer course. Dr. Atteberry said he still had numerous questions, and offered suggestions for the next steps in continuing to provide information to teacher educators: "The glaring thing that's missing on these kinds of things are, 'did you feel that using the technology was helpful for your students in high school?...Did they learn better, are they more motivated? What is your evidence? What was good about it?" Dr. Atteberry felt that asking the teacher participants to gather data from their classes about students' perception of technology would be quite valuable, since students may be 'engaged' with a type of technology such as a computer because they thought of it more as a video game than mathematics. Dr. Atteberry also recommended asking teacher participants the question, "what technology would you wish someone would invent?" because, he said, "they'll probably have an idea of something they need and it may be invented and just not available or nobody bought it."

Dr. Torres thought SMART Boards and programs such as Geometer's Sketchpad were used in geometry instruction, and would like technology to be used in teacher participants' classrooms in ways that support mathematical conversations with students and provide powerful demonstrations for students. She also believed in-service teachers do not integrate technology into teaching because they fear their students will be prohibited from using the technology on required exams or in college classes (e.g., where calculators are not allowed). Dr. Torres planned to incorporate video cameras for individual recordings during her class, with possible use of Geometers Sketchpad and Excel, but she claimed this technology use was far removed from the context of the teacher participants' classrooms. Dr. Torres planned to use assessments that incorporated electronic submissions, which could be connected to technology in teacher participants' classrooms.

When given the information from the CTS, Dr. Torres said the data were useful in helping her make decisions about what technology to use in instruction. For example, Dr. Torres mentioned she would like to know if teachers can get past the firewall at their respective schools – if they cannot, then showing YouTube videos might not be a tool she should use. When asked how, specifically, the information presented might affect her instruction, she said that, due to the nature of the course, it probably would not affect her plan for the way the course would be taught. However, Dr. Torres said that if it were a different course, more information about software would have been helpful. She deduced from the information we gathered that "new sites, new software" are things the teacher participants want to learn with the condition that they will have access to them in their classroom. Similar to Dr. Atteberry, Dr. Torres also wanted to know how computer programs such as Excel were being used in the classroom for the preparation of documents or for doing mathematical problems with equations.

Like Dr. Torres, Dr. Selvidge wanted teacher participants to use technology in their classrooms to support student exploration of mathematical ideas. Dr. Selvidge stated she had no specific plans for how to make the connection between technology in her course and the teacher participants' classrooms, but she planned to emphasize GeoGebra because she believed the teacher participants had access to computer rooms and the program was free for all users.

Upon receiving the CTS data, Dr. Selvidge indicated she was encouraged to think about technology use in more "innovative" ways, given the prevalence of technology such as interactive whiteboards. Like Dr. Torres, Dr. Selvidge wondered about the comfort level the teacher participants might have regarding technology and openness to learning about new technologies. Dr. Selvidge was also curious to know how various technologies can be used in mathematics classrooms to enhance learning. She mentioned specific interest in learning how technologies such as SMART Boards, writing tablets, iPads, document scanners, and programs such as GeoGebra, Dropbox, and Google Docs might be applied by teacher participants in their own classrooms to enhance learning.

Dr. Heilmann's assumptions about teacher participants' access to technology were similar to Dr. Selvidge's. He believed if teacher participants had access to SMART Boards then they would use them daily. Dr. Heilmann stated he would like technology to be "fully integrated into [teacher participants'] classrooms" and listed several technologies he would like teacher participants to use (e.g., networked graphing calculators, web 2.0 computer applications). Dr. Heilmann reported his plan to teach the summer course using GeoGebra because teacher participants could take this technology back into their classrooms. Dr. Heilmann also had plans to poll his students to determine how many had access to computers in order to best incorporate the GeoGebra lessons.

After examining the results of the CTS, Dr. Heilmann stated "there is more technology in the classrooms than I thought...it's very helpful to know what's there and the potential for use." Dr. Heilmann added his surprise about the prevalence of GeoGebra: "eight of the 22 [teachers] already use a technology that was planned to...introduce. So...when it comes to that lesson I may have to do some differentiated instruction. And so now I'm thinking to myself, do I really know enough to really push people that use GeoGebra in their instruction? So that's what I learned from this survey." He made clear that the data we provided him would be used to make decisions, and concluded, "It makes me pause and ask myself, 'do I know more technology than the teachers in the classroom that I'm going to be having?'" Dr. Heilmann commented on a specific change he would also be making in his course: he no longer needed to poll the teachers about the availability of computers in their classrooms. Dr. Heilmann's only other comment was to mention how the survey data could be inflated, and he suggested observations to verify the data.

Educational Implications

Providing graduate education to positively influence mathematics classroom practice is a major hurdle for program instructors and administrators. In this study, we followed Cuban's (1986) guidelines that technology education should be grounded in teachers' current practice in order to help link education to practice (Putnam & Borko, 2000). Teacher educators reported the data on K-12 technology use and availability was useful in (a) answering their original questions stated in the Technology Input Survey, (b) aligning their beliefs about teachers' availability to technology, and (c) self-reflecting on the appropriateness of graduate program technology instruction intended for transfer to teachers' classrooms.

Teacher educators all reported being surprised by at least one shared from the CTS. For instance, all four teacher educators were surprised at the large number[2] of teacher participants with unlimited access to an interactive whiteboard. Stemming from this revelation, all four noted that the information made them (re)consider the role of technology planned in the courses they would teach. For instance, Dr. Selvidge said that she had "not been thinking a lot about how to use it to enhance instruction...I sort of feel like after having reading this that I need to step it up a little bit." Even though Dr. Atteberry and Dr. Torres stated this information would not change how they planned to use technology in their summer courses, it did make them consider aspects of technology and ask additional questions about the topic. New questions brought up by teacher educators included the role the technology had in the K-12 classroom, how technology use supported student learning, and how technologically adept the teacher participants considered themselves. The teacher educators commented that these questions might only be answered through observations or interviews with teacher participants.

Data on teacher participants' reports on use and availability of classroom technology was offered to teacher educators to support their reflection on their own college instructional practice. The offer was timely in that it happened in the month before they taught a summer course that had a significant technology component – teacher participants were using technology as learners in the summer courses. The goal was to provide teacher educators an occasion to become more aligned with teacher classroom practice and to be more reflective about their instruction. We recommend future studies continue this effort, and expand on the reported survey and interview data by including classroom observations of both teacher educators and teacher participants. The efforts in this study were worthwhile in providing teacher educators information that can help structure graduate education in beneficial ways for in-service teachers. In making the connection from learning to practice clearer for educators, we hope to improve teacher education and subsequently student learning.

End Notes

1. This material is based upon work supported by the National Science Foundation under Grant No. DUE0832026. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

2. Eighteen of 22 teacher participants reported having unlimited access to an interactive whiteboard, such as a SMART Board, in their classroom.

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MATHEMATICS COACHING PROGRAM FIDELITY AND ITS IMPACT ON STUDENT ACHIEVEMENT

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This paper reports results of an analysis of data from a mathematics coaching professional development program. The goal of the study was to investigate whether a relationship exists between program implementation fidelity and student achievement. Data were obtained from Mathematics Coaching Program (MCP) documents collected during the 2009-2010 school year. Documents included site visitor inventories, end-of-year inventories, and facilitator weekly logs. From these data emerged four categories for future analysis: 1) coach leadership; 2) alignment with MCP assurances; 3) coach's role; and 4) coach and teacher emphasis on student thinking. Initial results indicate program implementation fidelity impacts student achievement.

Mathematics education in the United States has been unfavorably compared to that of other countries for decades (Miller, Sen, & Malley, 2007; Nelson, 2003). Numerous reasons are suggested as to why a majority of the students in U.S. schools seem to lack both an understanding of and an appreciation for mathematical literacy. By the time many of them reach high school, mathematics is one of their least favorite subjects and they fail to see its relevance to any aspect of their lives. Therefore, it becomes critical to ensure that mathematical instruction at the elementary school level encourages and inspires students while providing them with the necessary knowledge to not only be successful, but also be prepared for the rigors of middle school and high school mathematics.

Even though elementary mathematics education sets the foundation for each student's future success, most students at this level are taught by teachers with limited mathematical preparation as few were probably required to take more than two or three mathematics content courses and possibly one mathematics content pedagogy course (Fennell, 2007). Literature confirms that teacher preparation programs and professional development need to emphasize both mathematical content knowledge and mathematical pedagogical content knowledge (Cavanagh, 2008; Hill et al., 2005; Li, 2008). In light of these suggestions, calls for designing professional development programs that target teachers' subject matter understanding while increasing their sensitivity to children's thinking have been wide spread (Becker & Pence, 2003; Bruce & Ross, 2008; Cavanagh, 2008; Hill et al., 2005; Li, 2008; Murray, Ma, & Mazur, 2009).

This report is part of a larger longitudinal research project that closely studies the link between a mathematics coaching professional development model and student learning (Brosnan & Erchick, 2010). While there are several studies examining how mathematics coaches impact teachers (Bruce & Ross, 2008; Gerretson, Bosnick, & Schofield, 2008; Lord, Cress, & B. Miller, 2008; McGatha, 2008; Murray et al., 2009), there remains a concern that is yet to be addressed as to whether or not mathematics coaches impact student achievement (Murray et al., 2009).

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Results indicate that the (MCP) model, which has been in schools throughout Ohio in partnership with The Ohio State University (OSU), is having a positive impact on student achievement (Brosnan & Erchick, 2010). The purpose of this paper is to report results of a more focused yet preliminary analysis of program data to determine whether there was a relationship between program implementation fidelity and student achievement.

Theoretical Framework

The theoretical framework for this study was based on the roles and responsibilities of a mathematics coach as discussed in the literature and on the conceptual framework of the Mathematics Coaching Program model (Brosnan & Erchick, 2010). The roles and responsibilities of a mathematics coach take on many forms and change from day-to-day. Literature (Fennell, 2007; McGatha, 2008; Reddell, 2004) summarizes most of the roles and responsibilities of mathematics coaches into the following categories: 1) implementing standards and research; 2) planning and facilitating professional development; 3) working collaboratively with classroom teachers and administrators; 4) analyzing and interpreting data; and 5) organizing and coordinating mathematics resources. To ensure that students are receiving necessary mathematical instruction, mathematics coaches align their curriculum with local, state, and national standards and encourage classroom teachers to carry this into instructional practice. Coaches also need to be familiar with the most up-to-date research about teaching and learning so that they can influence teachers to use more effective methods of instruction. Since content knowledge and pedagogy are weaknesses for many elementary mathematics teachers, coaches are responsible for planning and facilitating professional development opportunities that will provide teachers with the necessary knowledge and instructional practices to be effective in the classroom. Collaboration with classroom teachers is critical and includes such activities as co-planning, co-teaching, reflecting on, and occasionally modeling a lesson. Mathematics coaches work with teachers and administrators to examine data and use it to guide future instructional and assessment practices. They also serve as the resource person responsible for gathering, organizing and coordinating the use of mathematical resources such as manipulatives, calculators, and rulers. These roles and responsibilities, while not exhaustive, provided a general framework from which implementation fidelity of the MCP model was investigated.

Additionally, the conceptual framework of the MCP model guided our study of the relation between implementation fidelity and student achievement. This framework focuses on learner responsive mathematics education with respect to both coach-teacher interaction and teacher-student interaction, consisting of elements from mathematical, pedagogical, and socio-cultural context (Brosnan & Erchick, 2010).

Background

The MCP coaching project that is the focus of this paper was developed over six years ago at OSU as a means by which to provide embedded professional development to mathematics teachers in the state of Ohio. The original recruitment for the program targeted low-performing schools around the state as determined by results of the Ohio Achievement Test and Ohio Proficiency Test. Three years of professional development and administrative support for the program are funded through grants from the National Science Foundation and the Ohio Department of Education. While school districts are responsible for funding the mathematics coaches, the monthly professional development is provided at no cost to schools. Coaches attend two consecutive days of training each month and two non-consecutive days of collaboration with other coaches clustered geographically and led by an MCP facilitator. Quantitative and qualitative data are collected within various aspects of the program.

The MCP model consists of the following four areas: 1) MCP conceptual framework; 2) coaches' professional development; 3) design and structure for implementation; and 4) accountability and quality control (Brosnan & Erchick, 2010). For the purpose of this paper, data analysis will focus on the two latter areas of design and structure for implementation and accountability and quality control.

Methodology

Participants in this study were ten coaches in schools participating in the Mathematics Coaching Program that showed the greatest and the least growth between the school years of 2008-2009 and 2009-2010. To determine the top five and bottom five schools in terms of student achievement growth, a spreadsheet analysis of Ohio Achievement Test (OAT) results was used to compare scores from the year prior to the first year of participation (2008-2009) in MCP with those at the end of the first year of participation (2009-2010) in the program. Difference scores (09/10 – 08/09) between each grade level in the building were calculated and the sum total of the differences was identified. The top five schools were those with the largest positive sums and the bottom five were those with the largest negative sums. Some schools for which OAT results were not available for both years were eliminated from consideration as well as those schools whose scores were extremely inconsistent across grade levels.

Data Sources

This paper presents results from an analysis of documents obtained through the MCP during the 2009-2010 school year. Data collected and considered for this research included: 1) site visitor inventories; 2) end-of-year inventories; and 3) facilitator weekly logs.

The site visitor inventory is an extensive document completed by MCP personnel during a full-day visit and observation at each school staffed with an MCP mathematics coach. As part of the site visitor inventory, classroom observations are documented, interactions between coaches and teachers are noted, and a series of pre-written interview questions regarding their work as coaches are administered. Site visitors received training on how to conduct site visits.

End-of-year inventories are completed by facilitators - MCP personnel in charge of overseeing small groups of coaches - towards the end of each school year. The contents of the end-of-year inventory are virtually identical to the site visit inventory, but do not include components that require the facilitator to actually visit schools. Facilitators work with coaches on a more personal level than do the site visitors, making the end-of-year inventory a contrasting perspective to the site visitor inventory.

Facilitator weekly logs are completed by facilitators, and are intended to document encounters with coaches at small group meetings as well as other discussions and conversations between facilitator and coach.

Analysis

Employing an interpretive case-study approach, analysis of the data took place through documenting emerging themes and recursive evaluation of findings (Merriam, 1988). Categories for qualitative analysis emerged from two sources: review of MCP data and existing literature on implementing coaching models. Issues of validity, trustworthiness, and credibility, as recommended by Lincoln and Guba (1985), were dealt with by utilizing multiple data sources, as well as independent coding by three researchers.

Results

The role of mathematics coaches in their first year of work with MCP creates many new challenges. Coaches are not only responsible for learning the program, but are also tasked with developing rapport with teachers and administrators, promoting their role as a coach within their building, balancing their time between planning, teaching, data analysis, collaboration with teachers, providing professional development and attending their own professional development sessions each month. Becoming a coach is a complex process, but one that can be mutually beneficial and can create a positive learning community among coaches, teachers, administrators, and, most importantly, students.

Prior research categories related to coaching and iterative analysis of the multiple data sources suggested several categories of differences between those schools showing the greatest and the least growth in student achievement. The four most prominent, recurring themes were 1) coach leadership, 2) alignment with MCP assurances, 3) role of the coach, and 4) coach and teacher emphasis on student thinking. Coach leadership refers to the demonstration of leadership by a coach within the school. Alignment with MCP assurances refers to how well a coach's school followed the MCP guiding principles for coaches and administrators. Role of the coach refers to the types of activities for which the coach felt responsible. Finally, coach and teacher emphasis on student thinking refers to the evidence describing the types of opportunities in which coaches and teachers allowed children to serve as problem solvers and use their own mathematical ideas.. Brief case studies of the five schools showing the least growth, as well as the five schools showing the greatest growth, will be used to demonstrate instances of these four themes.

Cases from Schools Showing the Least Growth

A commonality among the schools demonstrating the least growth was that the coaches in those schools often demonstrated a lack of leadership; not leading professional development at their schools, passively participating with fellow coaches, and an unwillingness or passiveness to engage with uncooperative principals. What follows are samples from the various data-source documents that give examples of coach leadership in the schools showing the least growth:

- Facilitator End-of-year Inventory on Coach X: Coach X is a pleaser. She does not tend to take the lead during Columbus PD. She has a quiet spirit, but gains approval from her peers by listening and understanding their needs. Coach X needs to stand up to her principal and qualify what she is doing and what she needs to take off her plate. She tends to follow her table-mates rather than make suggestions to solve problems.
- Site visitor form for Coach Z: I believe Coach Z has a lot that could be shared with the teachers, if only she took the opportunity.
- Facilitator End-of-year Inventory on Coach Y: Coach Y needs to model professional behavior by not being on her tech devices during meetings or conducting side-bar conversations when inappropriate, this gives the perception that she is not engaged with the other coaches and is not on task.

For the schools demonstrating the least growth, the most prevalent trend was the tendency to stray from the MCP assurances document. Principals are expected to support coaches in their work with teachers, and are asked to not interfere with the proposed coaching schedule by requiring coaches to do office work and non-coaching related tasks. Coaches are also asked to work with 3 or 4 teachers over a 6-week period of time. What follows are samples from the various data-source documents that give examples of the types of challenges coaches encountered with respect to following the MCP assurances:

- Site visitor form on Coach X: This coach is constantly being asked to do "chores outside her MCP assignment". She is asked to serve on building and district committees, collate OAA questions for teachers, deliver packets of "to-dos" to teachers that are inconsistent with MCP. However, when the coach is coaching, she is following the MCP model. Just that her time is limited. Quote from Coach X: I prefer to be with kids and not doing office work!
- Site visitor form for Coach Y: Things Coach Y does that are NOT COACHING:
    - AM duty 7:25-7:45 which is Hall duty; PM duty 2:00-2:15 Dismissal duty
    - Design Team committee which meets 2 times a month for 2 hours after school and once a month all day
    - Out of the building [beyond the 4 days required by MCP].
- Site visitor form for Coach Z: The principal tells her certain groups/targeted grades. She visits those classrooms first…So….she has 11 total teachers to work with and she is coaching in 5 different classrooms.
- Facilitator End-of-year Inventory on Coach X: The principal did not support all of the assurances of the MCP initiative. Coach X was caught in the middle between MCP directives and her assignments from the school principal. Coach X stayed true to MCP when allowed to do so. When the principal over-rode her assignment, it was difficult for Coach X to coach as needed.

Coaches in schools showing the least growth also demonstrated two additional themes; confusion about the role of the coach, as well as a struggle to emphasize student thinking in the classes in which they worked. This included modeling how to teach without coaching teachers on how to learn from their modeling, serving only as an intervention teacher, and feeling uncertainty about how to approach coaching. It also included a heavy emphasis on students working in groups, but failed to focus on developing more substantive mathematical tasks for students. What follows are samples from the various data-source documents that give examples of coach struggles in schools showing the least growth. These examples define the role of the coach and focus on student thinking:

- Site visitor form on Coach X: I got the idea that in the K-4 classrooms when students are in "groups" Coach X is taking a "group" and working with them, but not really changing much else.
- Site visitor form on Coach Z: The visit in the 3rd grade classroom did not show me an intellectually engaging lesson….it was all question/answer test prep.

She is not confident in her abilities and second guesses her coaching
I did not see evidence of her 1) Coaching the teacher in learning how students think, 2) Coaching the teacher in how to use the mathematical knowledge in the room, 3) Coaching the teacher in questioning techniques, and 4) Coaching the teacher in using the process standards regularly.

Cases from Schools Showing the Greatest Growth

The following cases describe the most common aspects seen within schools showing the highest margin of growth after one year of mathematics coaching. The coaches in these schools possessed similar traits, approached challenges in similar ways, provided leadership consistently, and followed the MCP model most closely.

Coaches within high growth schools showed leadership qualities both within their schools and during monthly professional development sessions. Within their schools, these coaches understood and promoted their roles as coaches. They ensured that teachers understood they were co-teachers and not evaluators. As well, these coaches took measures to build trust with teachers while also communicating to their administrators that they would discuss general information about student achievement, but would not betray the confidence of their colleagues.

- Facilitator End-of-Year Inventory on Coach A: Coach A has approached her job by establishing trust with her staff and taking things slowly. This method has worked for her in that more and more teachers are seeking to work with her.
- Facilitator End-of-Year Inventory on Coach B: What does the coach find most challenging in the coaching process? At first, getting everyone (administrators, staff) on the same page. The district's expectations of the math coach were different from MCP at first.

At monthly MCP professional development sessions, these coaches were highly engaged, modeled leadership qualities and set professional examples for their colleagues.

- Facilitator End-of-Year Inventory on Coach C: Coach C participates at every meeting. I commend her for not following others at her table and getting onto her computer…her professional approach to the MCP PD is to be commended

Schools with the highest growth consistently showed evidence of adherence to the MCP assurances document. Coaches worked with three to four teachers every six weeks co-teaching rather than modeling, principals offered strong program support, and coaches engaged in few additional duties beyond their coaching duties.

- Site Visitor Form on Coach C: Coach C is in four rooms each day. She also has time built in where she takes an enrichment group, she has one planning time and lunch. According to her schedule and discussions, she spends the majority of her day coaching.
- Site Visitor Form on Coach A: She plans, co-teaches, models and debriefs. She has built capacity with the teachers.
- Site Visitor Form on Coach C: Coach C suggests when coaching, that she and the teacher go back and forth in facilitating the lesson.
- Site Visitor Form on Coach D: Coach D said she is coaching 4-5 hours, or most of the day.

Another commonality among schools with the highest growth was the structure of the coaching process. The MCP coaching model is designed to include a pre-lesson and post-lesson conference in order to encourage collaboration and reflective practice between coaches and teachers. Coaches in high growth schools spent more time in pre-lesson and post-lesson conferencing. Further, these conferences were *planned* and intentionally included in the coaches' daily schedule.

- Facilitator End-Of-Year Inventory on Coach A: Coach A puts time in her weekly schedule for pre and post conferences with the teachers she is working with, and for planning activities. She spends time debriefing, especially with her one afternoon teacher.
- Facilitator End-of-Year Inventory on Coach C: She allows the teacher to realize where the lesson could be improved and guides the conversation to encourage self-discovery of transforming their classroom to a more student-centered culture.

Finally, coaches and teachers in high growth schools consistently emphasized the MCP principle of focusing on student thinking.

- Site Visit Form on Coach A: Coach A helps remind me the kids need to do the talking and thinking. I say so and so has a question….who can answer for them?...the kids help each other.
- Site Visit Form on Coach D: One teacher told Coach D….you taught me it is ok for students to be loud and communicate." "A lot of the teachers are letting kids loose."

General Observations

Utilizing the scoring rubric completed by facilitators and site visitors on their respective inventories, trends emerged demonstrating differences between the coaches in schools showing the most versus the least growth. Coaches in schools showing the most growth scored as well or better than coaches in schools showing the least growth with respect to: 1) co-planning with participating teachers; 2) co-teaching with participating teachers; 3) coaching related to student thinking; 4) coaching related to using mathematical knowledge in the classroom; 5) coaching on questioning strategies; 6) coaching on process standards; 7) coaching on identifying rich problems and activities; and 8) debriefing lessons with teachers. In addition to differences on the preceding coaching categories, the scoring rubrics also indicated that site visitors and facilitators did not view the two groups of coaches to be different in their content knowledge and knowledge of coaching. However, coaches in the higher growth schools received far more favorable scores with respect to pedagogical knowledge and comfort level with the program.

Discussion

Results from this study, although preliminary, show potentially useful areas that may be worthwhile to explore in order to better understand the relationship between the implementation of effective coaching and student achievement gains. All participating coaches were part of the same coaching program, yet each had unique experiences with the participating schools, teachers, and project goals that impacted their ability to effectively encourage instructional change and student achievement gains.

The MCP assurances document is designed as a mechanism to help ensure that coaches are able to effectively do the job of a mathematics coach. The situations showing the greatest alignment with the MCP assurances were also those showing the largest student achievement gains. Specific components of the MCP assurances document, such as the role of the coach, occurred frequently in the analysis. Coaches showing evidence of the viewpoint that it is the role and responsibility of the coach to plan, co-teach, and debrief lessons with participating teachers were far more likely to be coaching in a school showing the largest student achievement gains. Although the actual substance of the planning, co-teaching, and debriefing procedures enacted by coaches is not a component of this analysis, the discrepancy in viewpoints about the role of the coach found in this analysis indicates that a commitment to a coaching approach that honors all components of the teaching process (planning, teaching, and reflection) could be related to instructional change and subsequent student achievement gains.

Conclusion and Implications

As a result of this investigation, it is apparent that further research is necessary to form a more concrete understanding of the factors that most deeply impact student achievement in schools with mathematics coaches. Further, from this data emerged four a priori categories for future analysis within this data set: 1) coach leadership, 2) alignment with MCP assurances, 3) role of the coach, and 4) coach and teacher emphasis on student thinking. As well, it is proposed that at least one of the data sources be improved – the Site Visitor form – to create a more provocative instrument that can produce detailed accounts of the impacts of these proposed categories upon the coaches' ability to do their jobs. It is also proposed that a greater number of MCP participating schools be investigated. Additionally, an increased number of observations of coaches in those schools could begin to more clearly elaborate the components of the assurances that appear to have the greatest impact on student achievement.

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SECONDARY MATHEMATICS TEACHERS INCORPORATING DIGITAL TECHNOLOGIES INTO THEIR CLASSROOM PRACTICE

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The work we present here contributes to research on teacher learning on the issue of incorporating mathematics technology into classroom practice, approaching relatively new in-service teacher training design and using Zbieck & Hollebrands (2008) revisited theoretical framework of analysis related to introducing innovation in school. The main data were obtained from videos of participant teachers taken during their classroom practice. These teaching videos revealed five different ways that participant teachers achieved to incorporate mathematics technology into their classrooms.

Introduction

According to Cobo (2009), there are many new digital artefacts that have the potential to be used in classrooms. From his point of view, the educational field has benefited from the irruption of digital technologies, especially with the emergence of the Web 2.0.

However, as the same author set out, the advent of new technologies generates the possibility for new student abilities and skills. Then, it makes teachers face real challenges: they have to be acquainted with new digital resources and should learn how to integrate these technological tools into their classrooms.

In this context it is plausible reconsider the following research questions.

i) In reality, what has been the influence of digital technologies in school, especially regarding mathematics learning?

ii) What do teachers think about using digital technologies in mathematics classrooms?

iii) How do mathematics teachers achieve using digital technologies into their classrooms?

The work we are presenting here contributes to answering the third question, building upon previous work accomplished mainly by Zbieck and Hollebrands (2008), Ruthven (2007; 2002) and Ruthven and Hennessy (2002).

Specifically, we have already set up an exploratory study on the practice achievements of in-service secondary teachers, as they try to incorporate mathematics technology into their classroom.

Finally, in relation with the two first questions, we will only revise here some relevant research results mainly noticed by Olive et al. (2010), Ruthven (2007; 2002) and Zbieck & Hollebrands (2008).

Background

Technological Tools in School

In according with Olive et al. (2010), at the end of 80's and during the 90's there were great expectation in relation with the potential of new technologies to transform the way mathematics could be taught and learned (Howson & Kahane, 1986, cited in Olive et al, 2010).

However, as Ruthven (2007) points out, in opinion of revolutionary software designers (as S. Papert, the LOGO creator), the uses of technology in education have often simply replaced paper with computer screens without changing tasks. More over, computers have been used to "simply transfer the traditional curriculum from print to computer screen" (Kaput 1992, p. 516), in ways that resemble traditional worksheets and structured learning environments, rather than working to transform learning (Tyack and Cuban 1995. Cited by Ruthven, 2007).

Even it happens that *"the most insightful… teachers working in conventional schools understand what they are doing today… [as] not being the ideal they wish for"* (Papert, 1997. Cited by Ruthven, 2007). Papert then suggests (Ruthven, 2007) *that "as ideas multiply and as the ubiquitous computer presence solidifies, the prospects of deep change become more real*" (Papert, 1997. Cited by Ruthven, 2007).

In his analysis of the relationship between teachers, technologies and structures of schooling, Ruthven (2007) mention that, indeed, contemporary theories of educational change, just like those of technological innovation, acknowledge how these processes, like change for integration of technology into classrooms are shaped by the sense-making of the agents involved (Spillane, Reiser & Reimer, 2002. Cited by Ruthven, 2007).

Teachers´ Conceptions about Using Digital Technologies in Mathematics Classrooms

Specifically, in this section we will only refer to the work published in 2002 by K. Ruthven & S. Hennessy on teachers' ideas about their own experience of successful classroom use of computer-based tools and resources.

These authors obtained teacher accounts that were elicited through focus group interviews involving mathematics departments in secondary schools. They then analyzed these interviews qualitatively and quantitatively so as to identify central themes and primary relationships.

Then, in relation with the use of technology they found that teachers thought that it can serve as a means of: (i) *Enhancing ambience through changing the general* form and feel of classroom activity; (ii) *Assisting tinkering by helping to correct errors* and experiment with possibilities; (iii) *Facilitating routine by enabling subordinate tasks* to be carried out easily, rapidly and reliably; and (iv) *Accentuating features by providing* vivid images and striking effects which highlight properties and relations.

Ruthven & Henessy also obtained teacher information in relation with those themes more directly related to major teaching goals: (v) *Intensifying engagement related to securing the participation of students in classroom* activity; (vi) *Effecting activity related to maintaining the pace and productivity of students* during lessons; (vii) *Establishing ideas related to supporting the development of student* understanding and capability.

Finally, these authors also obtained information related with key learning topics connected with teaching uses and goals: (vii) *Improving motivation through generating student enjoyment and interest, and (viii)* building student confidence; (ix) *Alleviating restraints through mitigating factors which* inhibit student participation such as the laboriousness of tasks, the requirement for -and the demands imposed by- pencil-and-paper presentation, and vulnerability to mistakes being exposed; and (x) *Raising attention through creating the conditions for* students to focus on overarching issues.

It is worth to mention that Ruthven (2007) points out to the fact that each of 10 mentioned constructs represented a desirable state of affairs which teachers seek to bring about in the classroom, to which they see the use of technology as capable of contributing. But, as Ruthven says, that is only a model based on teachers' de-contextualized accounts of what they saw as successful practice, not supported by examination of actual classroom events (Ibidem, p.56).

In the work we are presenting here, we report the teaching practices that 15 secondary math teachers achieved into their classrooms, after they participated in an online comprehensive course on the use of technology and to do mathematics with technology (based on a problem-solving approach).

Methodology

Design of an Online Training Course

From the work realized by Zbieck and Hollebrands (2008), which synthesize tenth years of research on the topic of incorporating mathematics technology into classroom teachers´ practice, we extracted a structure to set up an online training course (OTC) of six months on mathematics and information technology topics for secondary mathematics in-service teachers. Specifically useful in this aspect was Zbieck and Hollebrands' (2008) re-conceptualization of the PURIA model of Beaudin & Bowers (1997), which also allowed us to analyze the data obtained as those derived from the introduction of technological innovation.

The general context we set the OTC up in was within the official educational policies current in most of the world's countries. According with them, in-service teachers should learn how to incorporate mathematics technology into their classrooms (see for example, Assude et al. 2006). In Mexico too, official educational policies are supporting the integration of new technologies into teaching, with particular attention to the teaching of mathematics.

To train teachers so that they could meet the task of incorporating mathematics technology into their classrooms practice, we implemented an exploratory online training course (see: http://upn.sems.gob.mx) of six months with 15 in-service secondary teachers that would allow the participants to learn to use technology and learn to do mathematics with technology.

Both of those mentioned aspects (learn to use technology and learn to do mathematics with technology) constitute important modes in the PURIA model of development along a learning continuum (Zbiek & Hollebrands, 2008). In fact, that model implies that teachers experiment different modes or development states to advance toward successfully incorporating technology into classrooms: The Play, Use, Reccomend, Incorporate and Asses modes.

As Zbiek and Hollebrands said, the growth during the P and U modes includes the transition of the technology as the developer's tool into the teacher´s instrument for doing mathematics.

Moreover, in the Incorporate and the Assess modes, the teacher's attention turns, implicitly or consciously, toward the use of technology as a pedagogical tool, including the development of instructional orchestrations (Trouche, 2000. Cited by Zbieck & Hollebrands, 2008), or elaborated plans regarding use of technology in the social dimensions of classrooms. Finally, the Recommends mode seems marked by a transition between mathematical and pedagogical aspects of the technology (Zbieck & Hollebrands, Ibidem, p.295).

Data Collection

On the Different Technological Resources Displayed throughout the Online Training Course (OTC)

The OTC's topics we addressed for teachers learning to use technology were related with computer programming, particularly an introduction to HTML and JavaScript programming, design of algorithms and their representations, algorithm development, flow charts, and codification. These topics were taken up with the specific purpose that the teachers could experiment a change in the way they approached to study mathematical algorithms.

The training course also included reviewing sequences of activities on the use of interactive software (eg. Logo, Geogebra, Aplusix, Excel, RecCon, FunDer); and exploration of a wide range of digital possibilities put in the Internet, for example, the library of virtual materials (eg. http://nlvm.usu.edu/en/nav/vlibrary.html) of Utah University (USA).

On the Mathematical and Pedagogical Tasks along the OTC

Some of the maths topics studied (with the purpose of teachers did mathematics with technology) were introductions to the fundamental arithmetic theorem, Goldbach's conjecture, graph theory, and calculating roots of polynomials (with the bisection method and the *Regula Falsi* method, the secant and Newton's method) purposed to review some of the algorithms important to secondary math contexts, as well as the distinct possibilities of representing them in mathematics and computing.

Although we can observe that those topics are notably more advanced than the included in the standard secondary mathematics curriculum, they were planned so as to present the teachers with a challenge to their existing knowledge and to solve problems on topics they do not necessarily already master.

In relation to the pedagogical activities displayed throughout the OTC, it was included four weeks of activities (two at the end of the first 10, and two at the end of the next 10 weeks) where teachers had to:
(a) Choose one high school mathematical topic, along with the software, tools, or digital materials they thought it would be useful to use on teaching the chosen topic; (b) orchestrate a classroom work session with their students in a convenient way according to their chosen digital material; (c) video-record that work session; (d) upload a seven minute version of that recording to YouTube; and (e) finally, upload to the training platform a descriptive report of the video's content together with its URL.

In fact, it was from this last segment of activities that were extracted the main data that generate the results we are presenting here.

Results

By the online training course (OTC) that we could instrument, we obtained evidence that showed teachers displaying the PURIA modes of Play, Use, and Recommend, and we also could observe how they started to display the Incorporate mode to integrate mathematics technology into the classroom.

Description of Teacher's Different Ways of Incorporating Mathematics Technology in Classrooms

There were only 9 teachers (from the total of 15 that participated in the group of observation) that accomplished all of the tasks required during the online training course. We summarize the participant executions in the following table. They are classified focusing on the topic and the digital tool teachers chose and the way they realized how to incorporate mathematics technology into their classroom practice.

	General data (a) Teacher initials (b) City of residence	Topic and digital tool chosen & Video URL	Method of incorporation into classroom practice.
1	(a) HA (b) Veracruz	- Resolution of equations - PPT software http://www.youtube.com/watch?v=PlLYsIO-Vh0&feature=related	Classic pattern of teaching, in the sense that the teacher uses a LCD, a laptop, and some software to explain or introduce a math topic.
2	(a) AG (b) Baja California	- Relation between a function and its derivative - Geogebra http://www.youtube.com/watch?v=Lk2yVHDjexA	Classic pattern of teaching (see the specification above)
3	(a) AM (b) Guanajuato	- Simplification of rational algebraic expressions - Java and HTML http://www.youtube.com/watch?gl=MX&hl=es-MX&v=N1FwbEo5KGI	The teacher adds to a classic pattern of teaching, questioning students on related math topics.
4	(a) HM (b) Baja California	- Calculation of the area of geometrical figures (2D) - Geogebra http://www.clipshack.com/Clip.aspx?key=CDF72468862861A8	The teacher adds to a classic pattern of teaching choosing appropriated digital tools to justify or confirm complex calculations.
5	(a) FM (b) Veracruz	- Graphics and ecuations of functions - Geogebra http://www.youtube.com/watch?v=BXAE2b5U3M4	The teacher adds to a classic pattern of teaching choosing appropriated digital tools to justify or confirm complex calculations.
6	(a) AL (b) Sinaloa	- Design of geometrical figures and calculation of areas - Geogebra http://www.youtube.com/watch?gl=ES&hl=es&v=yhXs8BLMFlM	The teacher is able to orchestrate student computational work, driving student activity by means of a work template.
7	(a) SM (b) Colima	- Equation of a straight line - Geogebra http://www.youtube.com/watch?gl=MX&hl=es-MX&v=X4c8IHEzQsM	The teacher is able to orchestrate student computational work, driving student activity by means of a work template.
8	(a) OV (b) Baja California	- Resolution of inequations - Aplusix http://mx.youtube.com/watch?v=gwGcPtyXYbs	The teacher orchestrates the use of both digital tools and paper and pencil in order for students to compare affordances and results issued from both contexts
9	(a) FG (b) Hermosillo	- Mathematical proprieties of instruments in Physics - PPT software http://fcogurrola.blogspot.com/	The teacher is able to orchestrate student computational autonomous work, based

			on the organization of student projects and small group cooperation.

Summarizing, the analysis of the teaching videos revealed five different ways of teaching that showed how teachers started to use mathematics technology in classrooms:

(a) A pattern of incorporation of the technology probably derived from the classic approach to teaching (see cases 1 and 2 in the table). This is to say that the teacher uses a LCD, laptop, and software to explain or introduce a math topic. One could name that way of using technology as *a classic pattern* of introducing mathematics technology in the classroom.

(b) A modified version of the classic pattern that added teacher interaction with the students, basically by teacher questioning to the whole class (see case 3). Or the teacher adds to a *classic pattern*, choosing appropriate digital tools to justify or confirm complex calculations (cases 4 and 5).

(c) An instrumental approach of the activity (Verillon and Rabardel, 1995; Assude et al., 2006) mainly directed by the use of a script. In these cases (see #6 and #7 in the table) the teacher is able to orchestrate some student computational work.

(d) An orchestration (Trouche, 2004) of the activity using different instruments or artifacts, plus group negotiation of meaning. In this orchestration the teacher uses both digital tools and paper and pencil in order for the students to compare both executions and results (case 8).

(e) An organization of cooperative work centered on student appropriation of technology. Here, the teacher is able to orchestrate student computational autonomous work, based on student project work and small group cooperation.

In order to see some raw data of each type found, readers may access the URLs in the table. Next we will show some pictures of them, and also the conclusions.

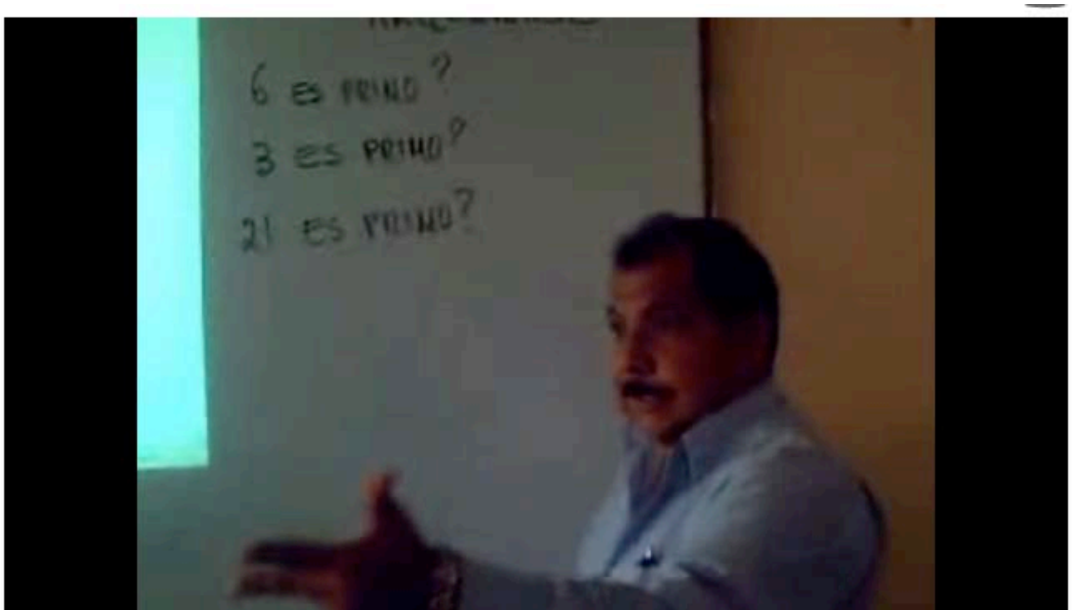

Figure 1. Classic pattern of teaching, adding teacher questioning to the whole class (see case 3 in the table).

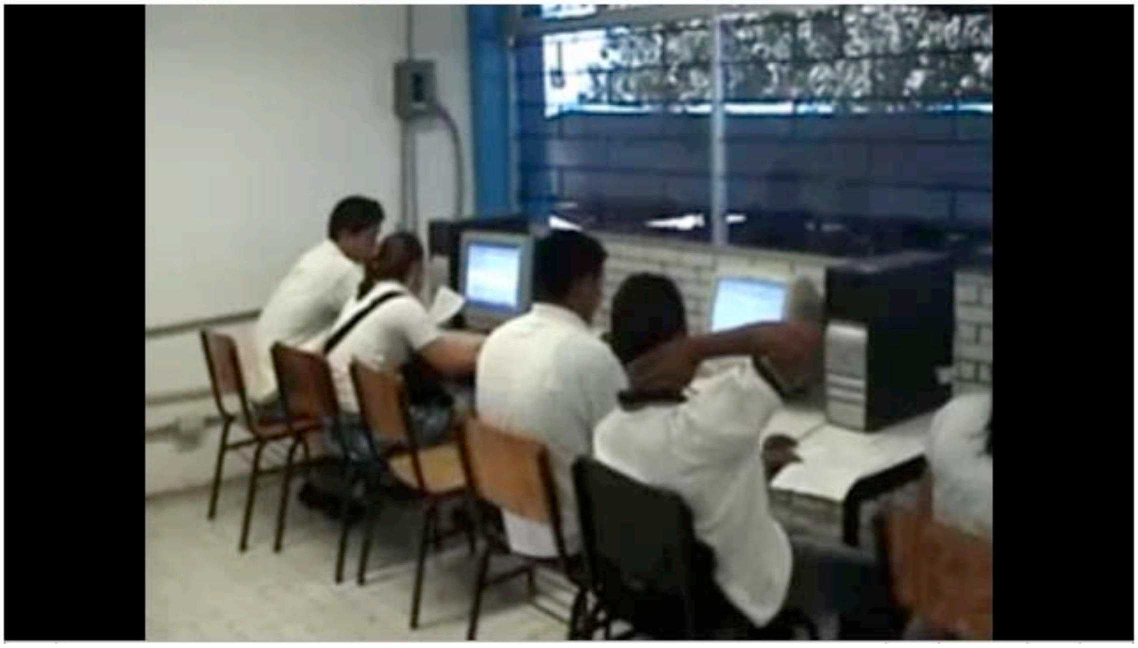

Figure 2. The teacher drives an instrumentalization of the activity led by the use of a work template (see cases 6 and 7).

Conclusion and Work Prospective

The different ways teachers displayed their integration of mathematics technology into classroom instruction allows us to make a qualitative appraisal of their craft knowledge (Ruthven, 2002) related with the use of technology in their teaching practices, which came after they participated in the online training course already mentioned before.

Thus it becomes feasible to predict teacher progress in the process of learning to use technology for teaching mathematics. According to the extended PURIA Model (Zbieck and Hollebrands, 2008), participant teachers would pass to the last mode of using technology (the Assess one) through involving themselves in assessing or noticing their students' mathematical thinking (see, for example Jacobs, Lamb and Philipp, 2010; or Herbst, 2010).

Finally, a base for attaining the next stage in teacher development, mainly in cases 1 to 5 in the table (which are less advanced than the others teaching types, basically because in these cases teachers do not turn to see what their students are learning by using technology) could be conformed by three principal components that would be used as methodological or pedagogical tools to develop professional teacher knowledge.

i) Teacher working on the construction and/or adaptation of digital resources to incorporate them into their mathematics classrooms (Gueudet &Trouche, 2010);

ii) Teacher planning and video-recording their own practice, plus uploading their videos to the digital platform already in use (Hoyos, 2010);

iii) Teacher involvement in noticing student mathematical thinking (Jacobs, Lamb and Philipp, 2010), and reflecting from a teaching representation approach (Herbst, 2010).

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THE TEN DIMENSIONS AND PEER COACHING: A COLLABORATING COMBINATION FOR PROFESSIONAL DEVELOPMENT AMONG ELEMENTARY MATHEMATICS TEACHERS

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The combination of peer coaching, and the resources and professional focus provided by the Ten Dimensions framework can prove to be effective model of professional development. Through teacher interviews and pre- and post-observation meetings, this study determined how teachers engaged in peer coaching, and what components of the framework support the process. The pairing of a Grade 1 and 3 teacher found that initial hesitancy in having a colleague observe them teaching was eased by supplementary resources. The teachers felt that peer coaching made them more aware of their teaching practices and they enjoyed working with a colleague.

Mathematics education reform documents (NCTM, 1989, 1991, 2000) emphasize the need for change in the mathematics classroom environment from one in which the teacher transmits knowledge to the students to one in which teachers and students interact as a community of learners in mathematical investigation and exploration. There are many challenges involved in creating such a learning community and also in determining the extent to which teachers progress towards this ideal of math reform. Teachers do adapt to change, but they require instruction on the nature of the required changes. Mathematics reforms are complex, however, and most reform efforts underestimate the complexities. Teachers can help each other change, and peer coaching has been used to help teachers improve their teaching practices (Loucks-Horsley, Love, Stiles, Mundry & Hewson, 2003).

This study aimed to discover how elementary school teachers engaged in a peer coaching model of professional development. The research questions were:

1. How can the Ten Dimensions of Mathematics Education act as a framework to focus efforts of professional development?
2. What components of the peer coaching process nurture professional growth?
3. What are the perceived benefits and challenges of the peer coaching process?

Conceptual Framework

For profession development to be effective, it needs to be intentional, ongoing and systemic (Guskey, 2000). A peer coaching model of professional development along with the support of a framework such as the Ten Dimensions of Mathematics Education (McDougall, 2004) has all three of these characteristics.

Peer coaching involves the pairing of two colleagues in a session with classroom observation, giving feedback and discussion to allow both members a chance to learn from one another (Loucks-Horsley et al., 2003). Both participants have a chance to reflect on what they observe along with their own teaching practices in order to grow. This reciprocal gain is one of the major benefits of peer coaching. Having an extra set of eyes in the classroom gives teachers another perspective that they may not have come up with on their own reflections (Guskey, 2000). The colleague can help identify both strengths and weaknesses for the teacher to focus on along with seeing practices that they may not have had exposure to beforehand.

At first, there may be hesitation to engage in the process as the teacher may be uneasy at the prospect of being observed and evaluated, but being paired up with a colleague that they trust will calm some fears (Arnau, Kahrs, & Kruskamp, 2004). An environment of "trust, collegiality, and continuous growth" (Loucks-Horsley et al., 2003, p. 208) will ensure that participants feel more comfortable. Each member of the pair must acknowledge the body of knowledge that they bring to the partnership as well as the expertise that their colleague can contribute to the dialogue (ross, Rolheiser, & Hogaboam-Gray, 1999). As long as both participants focus on the goal of professional development and maintain a level of respect or their peers, the advantages will certainly outweigh the disadvantages (Slater, & Simmons, 2001).

This study uses the Ten Dimensions of Mathematics Education (McDougall, 2004) as a conceptual framework to improve the quality of mathematics education in elementary classrooms. The Ten Dimensions of Mathematics Education framework was developed through a multi-year research project (McDougall et al., 2006; McDougall, 2004; McDougall & Fantilli, 2008; Ross et al., 2002). The framework was created to give teachers areas in which they can focus their attention in order to improve their teaching, knowing that many of the Dimensions are interlinked and focusing on one invariably leads to improvement in other areas.

The Ten Dimensions are: (i) Program Scope and Planning (focusing on the inclusion and integration of all strands of mathematics in the classroom); (ii) Meeting Individual Needs (teachers should use different teaching strategies and techniques to cater to the needs of the diverse student body); (iii) Learning Environment (consideration of classroom organization, cooperative groups, teacher feedback and student input/choice improves student achievement); (iv) Student Tasks (teachers should use rich tasks within the mathematics classroom); (v) Constructing Knowledge (a variety of instructional strategies and effective questioning techniques should be used); (vi) Communicating with Parents (parents should be involved with their child's mathematics learning); (vii) Manipulatives and Technology (students should be allowed to use and experiment with these tools); (viii) Students' Mathematical Communication (teachers should provide opportunities for students to use both oral and written forms of communication); (ix) Assessment (teachers should use diagnostic, formative and summative assessment to report on student achievement); and (x) Teacher's Attitude and Comfort with Mathematics (teachers should develop a comfort level with mathematics and promote a positive attitude towards the subject).

The School Leadership Handbook for Elementary Mathematics (McDougall, 2004) includes resources for educators who are using the Ten Dimensions. An interview protocol includes questions that can be asked before an after the lesson. There are general questions about the lesson format and content, and specific questions related to key areas of each of the ten dimensions. These questions help the observer to understand the thought processes of the teacher as well as to prompt the teacher to reflect on their own practices. Through this questioning, participants will gain a better understanding of the dimensions and the qualities that need to be focused on for improvement in these domains. An observation template lists 'look-fors' and guiding questions specific to each dimension. This template can be written on during the lesson and as a prompt for the observer's own field notes.

The Ten Dimensions framework encourages teachers to focus on key areas that will generate higher levels of student achievement, giving teachers an intentional approach to their growth. With set goals and areas for improvement that are selected by the teacher, the personal interest and investment towards professional change is undoubtedly present. Teachers focus on one or two dimensions at a time. In order for a substantial consideration to be made in each of the ten dimensions, ongoing commitment for professional development is required. Thus, the intentional and ongoing commitment to professional growth fostered by the Ten Dimensions coupled with a systemic change through peer coaching facilitates an effective vehicle for professional development.

Method

This study spanned two years and concluded in November of 2008. Four schools from a large urban city in Western Canada participated in the study. The teachers and administrators in the participating schools attended a professional development session about the Ten Dimensions of Mathematics Education. In this session, teachers learned about the Ten Dimensions framework, were introduced to this project about peer coaching and were invited to join.

The teachers and administrators then completed the Attitude and Beliefs Survey (Ross, Hogaboam-Gray, McDougall, & Le Sage, 2003) to determine how their current practices fared with respect to current mathematics education trends. Each of the 20 questions on the survey is related specifically to one of the Ten Dimensions and the higher the score on each dimension, the more the educator's attitude and teaching practices are aligned to reform trends. A low score may be impetus for the educator to focus on that particular dimension for personal growth. Individual teachers could decide which areas they wanted to concentrate on for personal improvement.

Participants

Within each school, colleagues were paired up. There were some teacher-teacher pairings as well as teacher-administrator pairings. There were a total of 10 pairings within the four schools.

In this paper, I discuss the findings from the pairing of Grade 1 teacher, Meredith and Grade 3 teacher, Christina. Meredith has been teaching for 25 years in the Primary division and also has experience teaching Kindergarten. Christina has taught her entire career at Grace School. In her 19 years of teaching, she has taught at both the Primary and Junior level.

Teacher Interviews

Participants were individually interviewed at the beginning of the study. These semi-structured interviews were approximately 45-minutes long and teachers were asked questions about their professional background, views and beliefs as a teacher, and goals as a mathematics educator. As well, interviews about the peer coaching process itself were conducted immediately after each peer coaching session.

Peer Coaching Sessions

The peer-coaching sessions consisted of three parts: a pre-observation conference, classroom observation, and a post-observation debrief.

In the pre-observation conference, the observer would ask the teacher a series of guiding questions from the Ten Dimensions of Mathematics Education resources (McDougall, 2004). These questions uncovered the lesson topic, structure, activities, and goals. Guiding questions specific to the teacher's chosen dimension were also asked and the teacher shared the area in which (s)he wanted the observer to focus.

During the classroom observation, the observer could use the observation template provided in the Ten Dimensions resources or could make informal notes of their choosing. The observer could interact with the students if they wished.

The post-observation debrief followed a similar format to the pre-observation conference. Again, a series of guiding questions from the Ten Dimensions resources were asked and teachers reflected on their lesson through the lens of the dimension of focus. Observers shared their observations of the lesson, specifically mentioning points that relate to areas that were highlighted for focus by the teacher. Judgments about the teacher were withheld unless the teacher asked for suggestions for improvement. A final question asking the teacher about their next steps concluded the post-observation debrief.

Data Collection and Analysis

There were four peer coaching sessions in total, such that each teacher observed her colleague twice and was observed twice. All interviews, pre-observation conferences, and post-observation debriefs, were audio-taped and transcribed by the researcher. Qualitative analysis was completed using nVivo8 software.

This study used a grounded theory approach to determine the effects of peer coaching on the participating teachers, thus, I allowed the data to dictate the outcome as the themes that emerged from the data analysis became the foundation for the major findings of the study (Glaser & Strauss, 1967). Coding was completed by first highlighting key ideas and these ideas were then grouped into themes.

Findings

In examining the pairing of Meredith and Christina, insight was gained from their experiences with the peer coaching model of professional development. They shared their feelings about the process itself, the benefits and challenges of the peer coaching model and how the Ten Dimensions framework and its resources anchored their efforts of professional development.

The Ten Dimensions Framework

Both Meredith and Christina selected dimensions that were of personal interest to them. Meredith wanted to select dimensions which she though had the most impact on students and of personal interest. She shares:

> To me, [Dimension 5] seemed very important. They are all important, but to me, [Dimension 5] gets back to how we present this, how we allow them to explore, how we allow them to tell use what they know already and go from what they know already to where they are going. I thought that was a really important one. We use manipulatives so much because it helps them in their concept development so I just thought it was really important. I just picked the ones that were really important to me. I think that [Dimension 5] is one of the key things and manipulatives fit into everything that I do in math. (Teacher interview, November 28, 2008)

By selecting two dimensions to concentrate on, Meredith was able to improve in a focused way in an area in which she was passionate.

Christina also selected Dimensions 5 and 7 for professional growth. As she wanted to improve her teaching practices, Christina explained that the Ten Dimensions of Mathematics Education framework gave her specific areas in which she can improve her teaching.

Christina shared that the resources, including the Attitudes and Beliefs Survey and the School Mathematics Improvement: Leadership Handbook were able to provide valuable information about the areas of her teaching in which she wanted to improve. She says: “[The Ten Dimensions] have made me more aware of how I can give [the students] an activity whether it be a problem of the day or a mega problem of the week or whatever it is and then give them the time to come up with their own strategies or connections” (Teacher interview, November 28, 2008). She adds, “It has actually made me more aware of how I am teaching” (Teacher interview, November 28, 2008).

With a clear focus for areas of professional growth, both teachers were ready to engage in the peer coaching sessions. Neither teacher had experience with the peer coaching process and both were apprehensive about how to organize and use the process to improve their teaching practice. The Ten Dimensions provided areas in which the teachers could focus for professional growth as well as resources that could guide them during peer coaching sessions.

The Peer Coaching Process

The teachers enjoyed the structure of the peer coaching process. The pre-conference allowed the teachers to focus on their goal of professional growth before heading into the classroom. Each member of the pairing could ease into the process through asking, answering and reflecting on the questions that were specific to the dimension chosen by the teacher.

The teachers used the guiding questions provided by the protocol instead of having to create their own questions. With their busy schedules, the teacher shared that they were grateful for the resources provided by the Leadership Handbook.

The interview protocol helped the participants to explore teaching behaviours and practice. These questions were specific to their chosen dimension and allowed observers to ask relevant questions even if they were unfamiliar with key components of the dimension. The guiding questions allowed both observer and teacher to instantly frame their thinking within the guidelines of the dimension. As Christina puts it, the guiding questions allowed her to “pinpoint” key areas pertinent to the chosen dimension (Teacher interview, November 28, 2008).

The observation template helped to anchor the observer’s thoughts. Meredith thought that she should focus her notes on the areas in which Christina wanted feedback. As such, she used the area of focus as the main heading for her notes and added secondary comments that related to the prompts on the observation template only if they were relevant to Christina’s improvement plan. She explained that the guiding questions and observations template helped her to focus on what to look for specific to the dimensions, however, she said it was “most helpful to know what the teacher wants me to look for” (Teacher interview, November 28, 2008).

Christina used the observation template as more than just a prompting device. She thought that the template was easy to use and simplified the process. Christina explained that “it was really well laid out and we could just quickly jot down what we were looking for” (Teacher interview, November 28, 2008).

Both teachers found the resources provided to be helpful in the peer coaching process. As they had never engaged in such a process before, Meredith and Christina valued having these tools to guide them and focus their thinking to parallel with their chosen dimensions.

Challenges of the process. A challenge that the teachers faced was the amount of time that the process required. Even before engaging in the peer-coaching process, Christina expressed that "just having the time to meet, to share, to go into each other's classes to observe, and talk about what was really great and what we need to do further" would be of benefit to her development (Teacher interview, October 5, 2007). After engaging in the process, she reiterated that it was a challenge to find time for the peer coaching process. In order to save time, the actual observation would often take place during the observer's prep time, however this was complicated to schedule. Christina shared one example of how they were able to schedule a session:

> This week, we did it during our prep time. While my students were at gym, I went into [Meredith's] room. We coordinated so that she would have her math class at that time, and, she did the same during her music time. I taught math in the afternoon, where I would not normally teach math because I like to [teach math] in the morning. (Peer coaching session, May 15, 2008)

The teachers' initial hesitation towards the peer coaching process quickly disappeared after going through the process for the first time. Although both teachers expressed concern for having a colleague and a researcher observe them teach, they quickly realized that they were so engaged in their teaching that observers would not affect nor effect their work. Meredith shared: "After awhile, I did not even know you guys were in the room. I am too busy and I am trying to get to as many kids as possible" (Peer coaching session, January 23, 2008).

The teachers are hesitant about being judged by observers, however, their overwhelming desire to improve reinforced to them that peer coaching can illuminate their practices and allow them to improve in the areas in which they were focusing on. The teachers' positive outlook towards this process allowed them to focus on the benefits rather than focus on the challenges.

Benefits of the process. Both teachers admit that the benefits of the peer coaching process clearly outweigh the challenges. Meredith and Christina shared that one of the biggest benefits was the chance to dialogue with a colleague. The teachers appreciated getting feedback after being observed. Meredith shared about this component on several occasions. First, she said she likes to have "something that someone else could look for" (Peer coaching session, January 23, 2008) and later she said, "It is good to get the feedback. [Christina] notices things that I may do naturally, but I do not notice, so that is good" (Teacher interview, November 28, 2008).

It was the act of collaboration that had profound effects on the teachers. The teachers saw that they were more than just sharing their observations with their colleague, but were able to work together to improve each other's practices. There was a reciprocal investment in their growth in that each teacher contributed ideas and suggestions for improvement.

Going into a colleague's classroom as an observer served as both a chance to provide feedback to the teacher, but also a chance for the observer to watch another professional in action. Meredith enthusiastically talked about this benefit: "Christina is such a good role model. I really enjoyed [observing her], because I see her way of going about it" (Teacher interview, November 28, 2008). In watching a colleague the teachers could further reflect on their own practices. Meredith said that watching Christina made her think about what she was doing in her own classroom.

In her final interview, Christina summarized the benefits of the peer coaching process:

> We enjoyed doing [peer coaching] together and it was nice to be a part of the learning process on both ends. Not only being in the spotlight, but also having somebody else come in and say, 'I really like the way you did that, and this is what I observed' and reinforcing what

you may have already known or maybe we have a tendency to be a little bit too harsh on our own self, being the perfectionist that we think that we are. (Teacher interview, November 28, 2008)

Conclusions

In this research project, the goal was to discover how a peer coaching model of professional development effects elementary school teachers and how the Ten Dimensions of Mathematics Education acted as a framework for professional growth. Through teacher interviews and in-class observations of peer coaching sessions, it was found that teachers benefited from engaging in a peer coaching model. Although the participants admit that there are challenges in engaging in professional development opportunities, their underlying goal of being better teachers to improve the quality of learning of their students and the support that they receive from their principal and each other is worth the hardship.

For professional development to be effective, it needs to be situated in school and classroom based contexts (Walker, 2007; Zwart, Webbels, Bergen, & Bolhuis, 2009). The peer coaching model allows for feedback to be shared between colleagues as a result of classroom observations and invites teachers into each others classrooms to observe teaching strategies in practice. The participants in this study described that the quality of their reflections was deepened based on the feedback that they received from their colleague and that they learned new strategies from observing their colleague in the classroom.

In the peer coaching process, the participants shared that it was stressful to be observed by a colleague, however, found it reassuring to work with a colleague whom they trusted and knew would be respectful of their shortcomings. These findings are consistent of the work of previous research emphasizing the need for teachers to take a respectful responsibility to collaborate with their peers (Arnau et al., 2005; Klingner, 2004). A guideline of the peer coaching model used in this study asked the observer to state observations rather than judgments of what they saw in the classroom. This guidance allowed the teachers to focus on improving the areas in which they wanted to improve rather than promote a feeling of evaluation (Slater & Simmons, 2001). Both participants shared that they welcomed additional feedback from their partner, thus giving evidence that there was mutual respect between the teachers.

In this project, peer coaching was examined as model for professional development. The use of a framework such as the Ten Dimensions of Mathematics Education (McDougall, 2004) allowed the model to be more effective by focusing the efforts of the participants. Resources from the Leadership Handbook facilitate the peer coaching process, alleviating some of the challenges that arise from never having gone through the process before and general pressures of the field. Teachers reported that peer coaching had many benefits. Among them include a chance to watch a colleague in action, a chance to see other students in other classrooms and/or grade levels, a chance to get feedback, and reflect on their teaching. These benefits, however, were outshone by the joy of being able to work with a colleague, to share, discuss and collaborate with a fellow educator in an often solitary profession.

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TEACHER COLLABORATION FOR THE IMPROVEMENT OF GRADE 9 APPLIED MATHEMATICS PROGRAMS

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In secondary schools, multiple teachers often teach the same course. The Collaborative Teacher Inquiry Project invited teachers, department heads and heads of schools to work together to improve their Grade 9 Applied Mathematics programs. The focus of this study was to find out the benefits and challenges of working collaboratively. Teacher interviews resulted in findings that highlight that collaboration can happen in many ways. In order for the collaborative team to be successful, members need to have and work towards the same goals. A conceptual framework such as the Ten Dimensions of Mathematics Education can provide this focus.

Traditionally, teachers work alone in their classrooms, planning in solitude and delivering material as they see fit (Hargreaves, 1994; Lortie, 1975). Although pre-service teachers work collaboratively with peers and mentor teachers to create activities, lessons, and unit plans, once out in the workplace, they are often left to their own devices.

The Learning Consortium's Collaborative Teacher Inquiry Project had three purposes:

1. To improve the teaching the learning of Grade 9 Applied Mathematics;
2. To investigate the use of the Ten Dimensions of Mathematics Education framework to improve student achievement in secondary school mathematics; and
3. To investigate collaborative inquiry as a professional development strategy.

For the purposes of this paper, the authors wish to focus on the third purpose. The Learning Consortium is a partnership between the University of Toronto and four local school districts in the Greater Toronto Area. From these four school districts, 11 schools participated in the project, which spanned three semesters from January 2009 to May 2010. Participants included teachers, department heads/curriculum leaders, and administrators.

Conceptual Framework

The relatively formal structure of subject departments is a long-term and widespread phenomenon at the secondary school level. Such a structure serves administrative, communicative and pedagogical purposes. Not surprisingly, professional development at the secondary level, collaboratively-based or otherwise, has been traditionally department-based. It has been assumed that the natural cohesiveness of subject departments provides a logical starting place for collaboration among teachers.

A number of studies have confirmed that, in secondary schools, the most efficacious communities tend to reside at the level of subject-matter departments (Talbert & McLaughlin, 2002; Siskin & Little, 1995). There is evidence that these loosely knit teacher communities can influence approaches to classroom practice (Gutiérrez, 1996; Johnson, 1990), as well as shape responses, positive or otherwise, to reform (Ball & Bowe, 1992; Coburn, 2001; Little, 1995).

Within a department, teachers are socialized into a tradition with its own norms of teaching, learning, grouping, and assessment (Siskin, 1997), which in turn shape their work and their students' classroom experiences (Gutiérrez, 1996). Departments typically have an infrastructure in place to allow for collegial work on professional projects like course modifications, implementation of new curriculum or learning about technology. Subgroups of teachers often arise to meet short-term demands, such as modifying or even creating new courses of study.

Most research into teacher collaboration has focused on the logistical parameters of delivery and evaluation. Recent efforts have looked at the potential for improving "on-the-job" collaborative opportunities for teachers in a seamless, situated fashion.

Guskey (2000) defines professional development as a process that needs to be intentional. In order for professional development to be effective, participants need to have clearly stated goals and expectations. These explicit purposes and goals focus participants to ensure that they will work together with the same purpose as opposed to working against each other towards divergent goals. These goals must be meaningful and applicable to those involved in the professional development. As such, all parties will feel involved and the personal commitment towards their goals will ensure a deeper investment towards the process. If it is to be worthwhile, professional development must be developed with participants who are motivated, based on clearly defined goals that can be measured.

The Ten Dimensions of Mathematics Education (McDougall, 2004) is a framework consisting of ten research-based domains of mathematics education that help teacher to focus their professional development efforts. This framework encourages teachers to focus on key areas that will generate higher levels of student achievement, giving teachers an intentional approach to their growth. The ten dimensions are: (i) Program Scope and Planning (encouraging teachers to consider all strands, expectations/outcomes and key ideas of the mathematics curriculum); (ii) Meeting Individual Needs (teachers should vary lessons and instruction to cater to the needs of all students); (iii) Learning Environment (varied student groupings and student input should be used); (iv) Student Tasks (teachers should vary the types of tasks that are being used in lessons and all tasks should be meaningful); (v) Constructing Knowledge (multiple instructional strategies and thoughtful questioning techniques help students construct knowledge); (vi) Communicating with Parents (parents are influential in student achievement and as such, should be kept informed); (vii) Manipulatives and Technology (these teaching tools enhance student learning); (viii) Students' Mathematical Communication (students should experience oral, written and physical forms of communication); (ix) Assessment (teachers should use a variety of assessment strategies to gain diagnostic, formative and summative data about their students); and (x) Teacher's Attitude and Comfort Mathematics (teachers affect student perception and should project positive attitudes towards mathematics). Although educators may select just one or two dimensions for school-wide focus, the dimensions are not discrete and a focused effort to improve in one area will inevitably improve their teaching practices more globally.

Method

The Collaborative Teacher Inquiry Project (McDougall, Jao, Maguire, Stoilescu, & Gunawardena, 2010) spanned three semesters from January 2009 to May 2010. There were 11 schools from four school boards across the Greater Toronto Area involved with the project. Participants included: teachers who taught Grade 9 Applied Mathematics for at least one semester during the project; mathematics department heads from each of the participating schools and a member of the school's administrative team.

The timeline of the project is as follows:

Semester 1: Introduction to the Collaborative Teacher Inquiry Project, the Ten Dimensions of Mathematics Education, completion of the Attitudes and Beliefs Survey (Ross, Hogaboam-Gray, McDougall, & Le Sage, 2003), selection of dimensions for department and personal foci, Learning Consortium organized professional development workshops focused by dimension, and individual reflection of participation in the project.

Semester 2: Learning Consortium organized professional development in-service sessions, co-planning and collaboration within math implementation teams at each school, and individual interviews with participating teachers.

Semester 3: Learning Consortium organized workshops, individual and team collaboration, and school team and individual reflections on the Collaborative Teacher Inquiry Project. A final celebration with a panel and school presentation was held at the end of the project.

Back at their home schools, participants team-taught classes, observed their colleagues teaching, and had group meetings for program planning. Participants were asked to complete reflections at the end in June 2009 and May 2010 to reflect on their journey throughout the project. Data was collected in two different ways: teacher interviews and the reflection surveys.

Data Collection

Teachers and department head/curriculum leaders were individually interviewed to determine participants' professional background, beliefs, goals and visions for education. These semi-structured interviews were 45-minutes in length, audiotaped and transcribed.

Teachers were asked to complete reflection forms that asked questions to specifically focus on the benefits and challenges of the collaborative process, the affect of the Ten Dimensions framework throughout their journey, future plans, and general reflections about the Collaborative Teacher Inquiry Project.

Data Analysis

A qualitative approach was used to gain a deeper, authentic and descriptive perspective of the issues being explored (Denzin & Lincoln, 2000). Qualitative analysis software, Nvivo9, was used to efficiently organize and perform a multi-faceted data analysis to uncover trends and patterns in the data. The data analysis followed a series of coding cycles. For this study, an open coding format was used where data were examined, compared and categorized (Strauss & Corbin, 1990).

This open coding process involved the reading of transcripts during which time key ideas were highlighted and labeled. The key ideas were grouped by similarity to form sub-categories and sub-categories were further re-grouped by similarity (Strauss & Corbin, 1990). These components were called concepts and categories respectively (Glaser & Strauss, 1967). The categories were considered the over-arching themes extracted from the data.

This study followed a grounded theory approach (Glaser & Strauss, 1967) and as such, the data collected were the source of the major findings of the study, however, categories and subcategories were created with components of existing literature in mind to be able to generate a possible hypothesis that could be relevant to current research and practices (Glaser & Strauss, 1967).

Findings

The data re-iterated that teachers face many challenges. Participants shared that the Collaborative Teacher Inquiry Project was a mode of professional development that allowed teachers to work collaboratively to improve their teaching practice in often a solitary profession.

Participants reported that the project acted as a vehicle for collaboration in several modes: with other teachers within their mathematics department, across departments within their school, with other participants within their school board and with those from other boards involved in the project.

Collaboration within the Department

The Collaborative Teacher Inquiry Project showed that teacher inquiry within the department could take many forms. Some school teams met as a group, scheduling times to meet, plan together, create consistent activities and assessments to use in each of their classes, and discuss challenges that they faced in their classes. Other school teams collaborated by meeting before class to plan their lessons and then engaging in team teaching. Other schools had teachers observe one another teach to learn new strategies and reflect upon their own teaching practice.

One challenge reported by school teams was the change of teaching staff within the school team. Teachers said that school teams would have been more successful had the same teachers been teaching the Grade 9 Applied Mathematics course for each of the three semesters of the project. Teams found that it took time to get the new members of the team involved and would have been able to more efficiently use their time if the members were consistent throughout the project. Although teams wanted to work collaboratively, they were unable to create a cohesive unit with when their group was continuously changing.

Teachers also stressed that, in order for collaboration to be successful, participants need to respect and be respectful of one another. It also helped if all teachers have the same vision for mathematics. This, of course is not always possible, so to facilitate this, the educators appreciated the Ten Dimensions of Mathematics Education as a framework to guide their improvement plan. This ensured that although teachers may have had different professional goals, they could still all work towards the same school-wide goals.

In general, teachers appreciated the change to work together. The following quote from a participant describes how her school collaborated and the positive impact it had on their team: "We established a core team to work together on everything from lesson planning to writing new culminating activities to coordinated EQAO (Provincial testing) preparations. Our new Grade 9 Applied program is more cohesive and purposeful to address student engagement and achievement gaps" (Maria Kovalevskaya School, Team reflection, May 27, 2010).

Collaboration within the School

School teams also collaborated with other educators within their school. This type of collaboration came in two forms: support from the administration and collaboration with other departments.

Teachers shared that a supportive administration encouraged their team to put a positive attitude towards the project. Teachers shared that they appreciated when they felt that their administration valued the work that they were doing and were invested and showed a strong desire to improve their school's mathematics program.

Other departments that got involved with the school teams' efforts include special education and English. Participants reported that these collaborations were especially beneficial as support during the provincial standardized exam organized by the Education Quality and Accountability Office (EQAO). This collaboration also strengthened the professional community at the school. Many of the participating schools have a high number of students with documented learning exceptionalities and school teams attributed part of their increase in EQAO results to this improved relationship.

Collaboration within the School Board

Participants reported that a very useful collaboration, and one that they believed that they should further develop, is with feeder schools. In the Ontario school context, secondary school starts at Grade 9 and therefore educators do not have the natural lines of communication between Grade 8 and 9 that exist with teachers within the same school. One school reported on a board-wide initiative between secondary schools and their feeder schools. It was explained that this program opens the lines of communication between secondary and feeder school teachers to ensure that students have a smooth transition between the two divisions. While this initiative has a student focus, teachers reported that they had a better understanding of the skill level of their students entering Grade 9 and could therefore better frame their Grade 9 Applied Mathematics program to meet their students' needs.

The Learning Consortium organized professional development sessions were the only structured opportunities for teachers within the same school board to meet and share ideas. Teachers appreciated collaborating with teachers from other schools and having the change to learn about the teaching strategies they have tried and to have worked for them. They also appreciated the chance to clarify and find consistency between schools for different board-wide initiatives.

Teachers expressed that it was a challenge to find time to collaborate teachers at other schools, given teachers' busy schedules and the distance that teachers would have to travel to meet one another. An online platform was created for the Collaborative Teacher Inquiry project for this very reason. The wiki was, unfortunately, not used to its full potential. Teachers reported that they liked the idea of a wiki as a platform to share ideas with other participants, however, not many participants posted resources, nor did any indicate why this resource was not used more often. This should be a platform that is further expanded on in future studies as it could overcome many logistically barriers to teacher collaboration.

Collaboration between School Boards

The Collaborative Teacher Inquiry Project allowed teachers to work with teachers from other school boards. The reported benefits of this type of collaboration are similar to those shared about collaboration between schools.

One participant described how this project opened her eyes to the possibility of teacher collaboration. She shared how she wanted more opportunities to collaborate with other educators, discuss ideas, trade resources, share success stories and hear about what is happening in other schools and school boards.

> I wish when we went to [more] meetings with the four school boards, I wish we could do more and see more student work. To invite me to a meeting and get me to do mathematics is not something I find useful. I really wish that we could sit down together and mark students' work together to see if we are all marking fairly and marking - if I am marking fairly. I would like to see the work that other students do in other schools. Because I only have my students to base my evaluations on. I do not ever see the work of other kids, good or bad. Just meeting the teachers from the other boards and hearing that they are dealing with the same types of students that we have that are coming to us without skills and for us to be teaching them curriculum that is often abstract and might be obvious to us as adults but is totally not obvious to the children. (MK2, Interview, November 26, 2009)

Another teacher shared the impact that the Collaborative Teacher Project and the chance to dialogue with teachers across the four school boards has made on their professional practice:

> The Learning Consortium has been a good eye-opener for a lot of us. I did not really know what was going on in other schools, and other schools did not know what was going on with our school. The fact that we had that time to collaborate, and share our resources, and discuss how we are running programs and how we are achieving success in our different courses. I think that is fantastic. For sure I think that was one of the good things about how schools can be more effective in the way they help students be more successful in math. Especially with our EQAO results, we jumped a lot from the year before to last year and again it was most because of collaboration with other schools, other boards, other teachers. Finding out what they are doing to help their kids to be successful, and same thing back, we are providing all the information that we have, sharing assessments, all that stuff, sharing our workbook even. I think the collaboration is what is helping us understand what can make math or how to make math more successful for the kids. (FC3, Interview, November 26, 2009)

Significance of the Study

This paper reports on the challenges and benefits of teacher collaboration on teachers' professional development. As such, the participants were able to improve their teaching practices through this dedicated time to focus on their Grade 9 Applied Mathematics program. Participants shared that their self-efficacy increased and were able to develop their instructional and assessment strategies. Extending out of this study, the findings can help other educators, department heads/curriculum leaders and school administrators who also want to improve their Grade 9 Applied Mathematics program. Although this study specifically focused on teachers of the Grade 9 Applied Mathematics course, similar challenges and benefits will most likely occur in other secondary grade levels and even other subject areas. This study showed that teacher collaboration is a worthy professional development strategy of mathematics educators across four school boards across the Greater Toronto Area.

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USING PREDICTION FOR MATHEMATICS TEACHER DEVELOPMEMENT

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This study investigated using "prediction" as a means of mathematics teacher professional development. Throughout the study, one middle school classroom teacher developed prediction questions around the main mathematical ideas of each lesson, posed those questions to students at the launch of that lesson, and orchestrated a classroom discussion on students' predictions and reasoning. Developing such prediction questions was a challenging task that required a careful examination of the mathematics being taught as well as students' mathematical thinking. It also facilitated well-planned instruction and teacher reflection on teaching. The results show the teacher's improved understanding of the content and changes in her teaching practice, which supports the use of prediction for mathematics teacher development.

The teaching and learning of mathematics greatly depends on the teacher. Everyday on-site instructional decisions that teachers make influence the quality of mathematics instruction that the students experience. For example, the task a teacher chooses to discuss in the whole group determines the kinds of mathematical strategies and reasoning that are shared. In order to make appropriate instructional decisions, teachers need to know the big ideas of the main instructional task, connections between the task and other tasks before and after, potential difficulties students might have, and useful mathematical representations of the big ideas. It has been widely acknowledged that teacher preparation at the college level is not sufficient to help develop the kinds of teacher knowledge required for effective mathematics teaching, and thus continued teacher professional development is very important (Loucks-Horsley, 1997). Because of this need, various teacher professional development activities have been developed, implemented, and examined. This case study developed, implemented, and examined such activities with an initiative to use prediction as a teacher professional development context for middle school mathematics. What prediction is and why this particular procedure is suggested is described below.

What Is Prediction?

In this paper, *prediction* refers to a type of reasoning, more specifically, reasoning about the mathematical ideas of the lesson at the launch by using prior knowledge, patterns found, or connections made from related concepts (Kim & Kasmer, 2009). Peirce's (1998) notion of abduction supports the importance of prediction in developing human knowledge. According to Peirce, abduction is forming a prediction without any positive assurance, is the only way in which people are introduced to a new idea, and makes a logical connection between deduction and induction.

An example from Kim and Kasmer (2009) further explains what prediction is like in the mathematics classroom. Before exploring a problem on linear relationships, students are asked to predict what effect increasing walking rates will have on the table, the graph, and the equation as they examine the relationship between distance and time. When making such predictions, students have to look back on what they already know (i.e., what walking rates mean, and how those rates are represented in a table, a graph, or an equation) and use that to reason about what will happen when a rate is increased. Even though students' predictions will vary and often will not be accurate, such an opportunity to make a prediction encourages students to check their prior knowledge and concepts in relation to the mathematical ideas embedded in the prediction question and its context. This facilitates thinking about the mathematical ideas in the problem they solve and making connections between what they already know and a new exploration.

Why Is Prediction Used for Professional Development?

Because of the usefulness of prediction, in various disciplines, prediction has been investigated as a means of promoting students' learning. In reading education, for example, students were asked to make a prediction in reading activities using questions such as, "What do you know about this character that helps you predict what he or she will do next?" and "Given the situation in the story, what will possibly happen next?" The results showed that asking students to make such a prediction helped increase their reading comprehension (e.g., Block, Rodgers, & Johnson, 2004). Gunstone and White (1981) incorporated prediction in science teaching and suggested a prediction-observation-explanation model, which was utilized in several other studies (e.g., Lavoie, 1999; Palmer, 1995). Interestingly, de Bruin, Rikers, and Schmidt (2007) asked college students to make a prediction when they learned how to play chess in a computer game setting. They found that students who were asked to predict and explain their predictions learned chess principles better than those who only predicted chess moves and those who observed games without making predictions.

In mathematics education, while exploring students' learning in geometry, Battista (1999) found benefits in having students make predictions. According to him, a discrepancy between predictions and actual answers made students reflect on their strategies and helped build useful mental models for problem solving. Buendía and Cordero (2005) viewed prediction as a social practice that supports the construction of mathematical knowledge, and provided such evidence in the context of periodic functions. In fact, the analysis of state mathematics standards revealed that prediction was the most prevalent reasoning expectation across grades as well as content strands (Kim & Kasmer, 2006). Kim and Kasmer (2007, 2009) and Kasmer and Kim (2011) also investigated the potential of using prediction for the development of students' mathematical understanding and reasoning in the middle school algebra context over years. Using prediction provided an opportunity for students to make connections among mathematical ideas that they learned and to develop the meaning of algebraic concepts. As a result, students who participated in prediction activities developed better mathematical understanding and reasoning around algebraic concepts than those who did not. Moreover, using prediction offered additional classroom benefits, such as promoting classroom discussion on mathematical ideas, helping teachers assess students' thinking informally and formatively during instruction, and encouraging teachers to reflect more often on what they plan and teach.

Even though the advantages of using prediction in the teaching and learning of various subjects including mathematics have been reported, the focus has mainly been on student learning outcomes. Little has been investigated in terms of teaching practice and teacher development. Given the effectiveness of using prediction on student learning and the potential benefits for teachers (e.g., prediction serving as a means to promote classroom discussion rather than direct teaching, and quick informal assessment on student thinking during instruction), further investigation is needed on how utilizing prediction can prompt teacher professional development.

Research Questions and Analytic Framework

Based on the above rationale, this study addresses an overarching question, *How does using prediction promote teacher professional development*? In this study, I define professional development as Therefore, to analyze the impact of using prediction on professional development, two aspects are closely examined: teacher knowledge and teaching practice—whether positive changes occur in each aspect and, if so, what the changes look like. Two research questions are formulated to address each of the two aspects.

1. Does using prediction help increase teacher knowledge? If so, what aspects of knowledge?
2. Does using prediction help improve teaching practice? If so, what aspects of teaching practice?

First, teacher knowledge is accounted for with respect to (1) understanding of big ideas in tasks and lessons, (2) understanding questioning strategies, and (3) content knowledge measured on a standardized exam relative to a norm-referenced group. Second, teaching practice is examined with respect to (1) the characteristics of classroom discussion, and interactions with and among students, (2) kinds of questions asked by the teacher, and (3) ways in which the teacher reflects on her own teaching practices.

Professional Development Procedure

The main procedure of this professional development centers on a middle school teacher's involvement in activities, such as examining the content of the lessons to teach, developing one or multiple prediction questions for each lesson, asking them at the launch of each lesson and organizing discussions on students' predictions and reasoning, and reflecting on teaching. The detailed process is given below.

Teacher Selection

One middle school mathematics teacher was selected based on her interest in the study as well as her teaching and professional development experiences. She taught middle school mathematics for 12 years. Throughout her teaching career, she used Connected Mathematics Program (CMP). At the time of the study, she had been using the second edition of the curriculum (CMP2) for one year. She was an experienced teacher and confident about her teaching. She, however, wanted to identify areas to improve and welcomed an opportunity for professional development. In particular, her decision to participate in this study was based on the fact that she felt that she was losing her initial enthusiasm and settling into routines, as she had taught for many years. It was also based on her desire to discuss the mathematics that she taught and pedagogical strategies that she wanted to be equipped with.

Content Area

Algebra was chosen for the target content area of the mathematics lessons because it is one of the most important and difficult topics in middle school mathematics. Also, algebra units are taught throughout the year. The unit taught during the study was *The Shapes of Algebra* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2009) from CMP2, which was about linear systems and inequalities. Because of the abstract nature of the content and extensive use of symbols, teaching this unit was very challenging, especially in helping students understand what they were doing and why when they manipulated symbols, and what that meant in the problem context.

Teacher Professional Development Activities Using Prediction

There were three stages of teacher professional development activities using prediction during the study. In the first stage, I provided the teacher with prediction questions of each algebra lesson so she could use them in her teaching. During this stage, the teacher became accustomed to using them in class and examining how prediction questions provoked students' thinking and guided her teaching. In the second stage, the teacher and I developed prediction questions together, and the teacher used them in her teaching. In the final stage, the teacher developed prediction questions on her own to use in her lessons. Before and after each stage, the teacher and I discussed the big ideas across the lessons, how prediction questions would stimulate or stimulated students' thinking, advantages and disadvantages of particular prediction questions, and possible alternatives of the prediction questions proposed or used. Each stage had three lessons, and nine total lessons were taught during a one-month period. (Each lesson often needed more than one day. The teacher also used practice problems for a few days in between lessons to help improve the students' ability of symbol manipulation.)

At the individual lesson level, the following process repeated throughout the professional development:

- Examine mathematical ideas of a lesson and create prediction questions for instruction
- Pose prediction questions at the launch of the lesson and elicit students' responses and their reasoning
- Use information to adjust lesson plan
- Reflect on teaching, plan for next day, and revise prediction questions for future use

Data Collection and Analysis

During the project, data were collected to measure the growth of teacher knowledge and the improvement of teaching practice. Data collection and analysis of each data source are described below.

Paper-and-Pencil Measures

Teacher knowledge was examined by using a paper-and-pencil measure, which was developed and tested by a research group at the University of Michigan (Content Knowledge for Teaching Mathematics [CKTM] measures). The CKTM measures have two forms that measure middle school teachers' knowledge of algebra. One form was used at the beginning of the project and the other at the end of the project. Each form contained about 20 items. Questions about the items were asked during the pre- and post-interviews looking for teacher reasoning and further explanations regarding particular choices made. The two test responses were compared using the reference group provided by the test developers.

Interviews and Discussions

During the study, through interviews (pre- and post-interviews) and discussions (before and after each stage) with the teacher, I had opportunities to clarify the teacher's thinking in terms of teaching and her own understanding of the mathematics being taught. During the interviews and discussions, the teacher was asked about big ideas of tasks and lessons, the purpose of particular prediction questions and advantages and disadvantages of those questions, and her instructional decisions made on site. All the interviews and discussions were audiotaped and transcribed. These transcripts were analyzed for evidence of change in her knowledge and teaching practice.

Classroom Observations

During the project, the teacher's instruction of the unit was videotaped three times, one per stage. Those lessons videotaped were used for the analysis of characteristics of classroom discussion, and interaction with and among students by looking at related teaching actions, such as kinds of questions asked after prediction questions and other main questions, kinds of responses given to students, and ways in which student responses were used in teaching. In particular, the *Coding Rubric for Measuring the Quality of Mathematics in Instruction* (QMI instrument, Learning Mathematics for Teaching, 2006) was used to analyze whole-group discussion periods, for the purpose of looking at kinds of classroom discussion and how the teacher organized discussions. This instrument was designed to measure teacher knowledge revealed in instruction, and therefore it was helpful to account for what kind of teacher knowledge was used in teaching and in what ways. Each coding element required an indication of whether evidence was "present" or "not present" and whether it was "appropriate" or "inappropriate" for every 5-minute segment. Important segments identified during this coding process were transcribed for further analysis. Once every lesson was analyzed, characteristics of each stage were summarized and then the three stages were compared.

Another coding tool used was the *Reformed Teaching Observation Protocol* (RTOP) (Piburn, Sawada, Falconer, Turley, Benford, & Bloom, 2000), which was developed as an observation instrument to provide a standardized means for detecting the degree to which K-20 classroom instruction in mathematics or science is reformed. For the purpose of the study, the coding elements on classroom culture (communicative interactions and student/teacher relationships) were used. Each element was coded by a 5-point Likert-scale (0 being *never occurred* and 4 being *very descriptive*). Once the coding was done, a score for each lesson was assigned by finding the mean of the codes. The scores of the three videotaped lessons were compared to determine any changes in classroom culture.

Lesson Plans

The teacher provided one lesson plan using prediction during each stage. Each lesson plan was provided before the actual lesson took place. The teacher chose the format of the lesson plans. Once the lesson was taught, the teacher provided a reflection on the lesson. These lesson plans were analyzed to account for whether each lesson was formulated to highlight the big ideas of the lesson, whether prediction questions were used to help students make connections between the main ideas of the lesson and previous concepts, whether the overall lesson sequence as well as specific questions and guidance encouraged students to develop the big ideas of the lesson, and what strengths and weaknesses the teacher identified in her lessons and lesson plans.

Teacher Journal Entries

The teacher kept a weekly journal during the study. Although the teacher had the freedom to choose any lesson of the week to write about, each journal entry had specific components: big mathematical ideas of the lessons of the week, prediction questions asked, how prediction questions helped instruction (from the perspectives of teaching and learning, respectively), what went well and what did not and possible reasons, and what she would do differently next time. When analyzing teacher journal entries, patterns of reflections in each of the three stages were searched in terms of ways in which the teacher reflected on her teaching, and the breadth and width of her reflection.

Results

The results are described to answer the two research questions, and highlight the areas in which using prediction positively impacted the teacher's professional development.

Teacher Knowledge

The results indicate that using prediction helped increase teacher knowledge, especially knowledge of the content that she taught and knowledge of questioning in teaching.

Content Knowledge

The teacher and I discussed the content and its representations in-depth throughout the study in order to create good prediction questions to use in instruction. Teacher interviews and discussions show that creating and using prediction questions challenged the teacher's knowledge base. The teacher revealed her strong and weak areas, developed a better understanding in the weak areas, and realized things that she did not think about before. For example, during the first discussion, the teacher talked about the content of three lessons and what she would focus on. She understood the importance of using graphical representations for the solutions of linear inequalities, and yet did not see the connection between "a number-line graph" and "graphs of two linear functions" on the coordinate plane. (The number-line graph is, in fact, shown on the x-axis of the coordinate plane.) Therefore, she focused on the number-line representation of linear inequality solutions and did not plan to spend time on the graphical representation of the coordinate plane. When further examining the latter during the discussion, she saw that the number-line graph was embedded in the graphical representation on the coordinate plane and how the two representations were related, and she gained confidence in using both of them in instruction. As the study progressed, the teacher began to focus more on big ideas and meaning along with symbolic solutions and to make more solid connections. Previously, she was concerned about her students' skills on symbol manipulation ("The biggest priority is, can you solve a linear inequality?") rather than whether they understood what they were doing. Standardized measures, CKTM-middle school algebra, also indicated that her knowledge on the content increased. IRT scores on pre- and post-CKTM were 1.1295 and 1.8781, respectively. Both were high scores according to the norm group, but the latter was the highest, whereas the former was the fourth highest.

Knowledge of Questioning

The teacher explicitly mentioned several times that she tried to improve her questioning skills. In fact, a comparison of the first day and the last day observed as well as a review of the lesson plans shows that, at the end of the study, the teacher demonstrated greater knowledge of what to ask and how to probe students' thinking. By her choice, one of the components in her lesson plans was "questions I will try to ask to extend our discussion." The last lesson plan included a longer list of questions, even with rationale, including: "How does this process (what students just did) relate to the type of process we learned yesterday? Can you do this for all problems of this type? Is it possible to solve this system using x= instead of y=? What happens if you do solve it for x=?" She asked these questions to encourage students to make connections between what they did and previous learning, to make conjectures, and to think about alternatives.

In fact, it is difficult to formulate good prediction questions when the content is very abstract, as in this unit. Nevertheless, she was able to create good prediction questions on her own later in the study. For example, the problem context was as follows: "The eighth graders are selling T-shirts and caps to raise money for their end-of-the-year party. They earn a profit of \$5 per shirt and \$10 per cap." One of the questions she posed was, "What do you predict will be the equation that shows how the profit is related to the number of shirts and caps sold? Explain your reasoning." The students had experienced linear "only in the $y = ax + b$ form" and $ax + by = c$ was new to them. She realized the benefit in using equations for the situation that students came up with to guide her next steps during the lesson. It also would provoke students to think about what they knew already and relate that to this situation—whether the standard form ($y = ax + b$) would work or if they would need a different form to represent the problem context.

Teaching Practice

The results reveal that using prediction was effective in improving classroom practice. The areas positively influenced include classroom culture, questioning skills, and teacher reflection.

Characteristics of Classroom Discussion

First of all, more discussion time was spent during instruction as the study progressed. In the first observation, class discussion lasted for 5 minutes except for discussions on prediction. The lesson in the second stage included 10 minutes of discussion, and in the last lesson observed, more than half of the 50-minute class period was for discussion. Even when checking homework answers, the teacher initiated discussion on important mathematical ideas behind the solutions. Second, she devoted much time to discussing the meaning and reasoning behind the procedures. During teacher interviews and discussions, she repeatedly expressed that she had difficulty juggling between increasing students' symbol manipulation skills and focusing on the big ideas she identified. For example, it took time to help students understand $ax + by = c$ as linear, as opposed to focusing on symbolical problem solving alone. Often students graphed it by putting points and connecting them, not realizing that this new form also represented a linear relationship between x and y. Likewise, students focused on solving for x or y, not understanding what that x- or y-value meant. This year, the teacher spent a great deal of time helping students understand the rationale behind the procedure or symbolic manipulation that they were engaged in. She also helped students to make sense of what they did by connecting back to the problem situations when appropriate. For such purposes, she asked many questions, such as "Why do you solve for x? What does that mean? What's next? Which equation should be used to substitute x? What would happen if the other one is used? What does that mean? Why do they give the same y?"

RTOP results also indicated that her classroom culture slightly changed (first, second, and third stages scored 2.8, 3.0, and 3.1, respectively). In particular, student-teacher relationships remained the same (3.2), and yet communicative interactions changed a bit (2.5, 2.8, 3.0) as the study progressed. Areas that improved include:

- The teacher's questions triggered divergent modes of thinking.
- There was a high proportion of student talk and a significant amount of it occurred between and among students.
- Student questions and comments often determined the focus and direction of classroom discourse.

Questioning Strategies

Lesson plans and classroom videotapes showed improved teacher questioning skills. As shown in earlier examples, the teacher asked more "why" questions along with "how" and more probing questions, and asked more about the meaning behind procedures to help students develop algebraic reasoning with symbols.

Reflection on teaching. The teacher reflected on teaching more often, ranging in topics from fundamental teaching philosophy to specific teaching strategies. Such regular ongoing reflection helped her improve her teaching practice. Some teacher reflections at various points are as follows:

- *It* [using prediction] *helped me revisit and refine my philosophy of mathematics teaching – something that had become a bit dusty from underuse… It strengthened teaching skills.*
- *It forced me to think about their responses, the flow of the lesson, the types of questions I needed to ask if they got or didn't get the question.*
- *It also caused me to reflect more on the lesson before teaching it and, of course, after teaching it, something that I've lost along the way for lack of time to devote to this process.*

Conclusion and Significance

This is a case study involving one teacher. During one month of the study, the teacher had extensive and in-depth discussions with me about the content she taught and prediction questions to use in teaching the content, as well as pedagogical challenges and strategies, such as questioning skills. The results show not only that using prediction for professional development is feasible, but also that such professional development activities are effective. Using prediction provided a significant context for professional development. To create good prediction questions, the teacher had to closely examine lesson content, which led her to focus on big ideas of the lessons and connections among those ideas for her students and to build on her own learning of the content that she was teaching. To use those prediction questions in teaching, she had to think about appropriate probing questions and to organize classroom discussions around meaning of and reasoning about mathematical ideas and procedures. The next step is to design professional development activities for multiple teachers working together over a longer period of time. A group of teachers engaging in this type of professional development would benefit from discussing mathematical ideas, formulating prediction questions, and reflecting on their teaching as a group.

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MEASURING THE EFFECTIVENESS OF A HYBRID PROFESSIONAL DEVELOPMENT PROGRAM FOR MATHEMATICS TEACHERS

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This study explores how a hybrid professional development model may support quality and success of mathematics teachers. The primary focus of this study is on the experiences and the opportunities, which on-line discussion and face-to-face workshops provide for mathematics teachers' professional learning. The findings present the impact of the program on teachers' practice, as well as the nature of the virtual interactions including frequency and content of virtual communication. The conceptual discussion includes potentials and limitations to be considered in further dialogue, development, and research in designing and evaluating a hybrid program for teacher success in deep learning and practice.

Purposes of the Study

This study is based on a year-long professional development program for middle school mathematics teachers. Our project program was designed considering the following issues: How should the professional development program be designed to maximize teacher learning and improve instructional practice?; What types of professional learning experiences should teachers experience in the program?; and How should the professional development program foster professional interactions? This paper investigates the nature of participants' interactions in a virtual learning community, as well as the impact of the professional development program on teachers' view about mathematics and their practice.

Conceptual Framework

Studies comparing online and traditional courses report no significant differences between the two programs. The professional development program of this paper was designed under this assumption that there is little difference between these two models. Consideration for a teacher's busy schedule and the need for on-going teacher interaction led the professional development program in this study to adopt a Blended Learning System, which forms a harmonious combination rather than random cost efficiency (Graham, 2005; Heinz & Procter, 2005). "Blended Learning is learning that is facilitated by the effective combination of different modes of delivery, models of teaching and styles of learning, and founded on transparent communication amongst all parties involved with a course" (Heinze & Procter, 2004, p. 12).

Since Dewey, the social nature of all human learning and the role of communication ability/skill in the human development process are the most often discussed concepts in higher education (Feldman, 2000; Heinze & Procter, 2005). Learning in Communities of Practice, networks of practitioners, relies on communication between individuals and occurs as an interaction among practitioners takes place (Heinze & Procter, 2005, Kahan, 2004). Communities of Practice "are creating a Zone of Proximal Development with capable peers. There is no one on the stage who is the knowledge source, but all individuals have an equal right to share their experience, and their stories are valuable contributions to the community" (Heinze & Procter, 2005, ¶ 3).

When we discuss students' learning, our question should not be whether students can or cannot achieve mathematical skills but about whether students will elicit maximum success in mathematics. "Such recommendations call for an approach to mathematics teaching that allows students to communicate, problem solve, and engage in conceptual mathematics. … From a professional development standpoint, this perspective suggests that programs should provide opportunities for teachers to learn mathematics around specific content and teaching situations that may arise in practice." (Brown & Benkenp, 2009, p. 56). The key concerns, therefore, for online professional development are to ensure peer exchange of ideas and information (Hammond, 1998); the creation and assimilation of knowledge (Thomas, 1992); and providing a climate of accelerated change in which participants need to access up-to-date knowledge and apply new skills flexibly in changing circumstances (Hargreaves, 1994). The learning community should be more than a forum for the exchange of information (Chapman & Ramondt, 2003). It must evolve in response to the diverse needs of learners and the communities in which they work.

The project being investigated is developed to provide a blended learning option. The project incorporated a combination of face-to-face and online communication that has the potential of increasing the participants' sense of community (Rovai & Jordan, 2004) and continuation of professional learning. It was recognized that a sense of community was critical for sustaining teacher interest. Participants in the program came from various backgrounds and diverse experiences, which made bonding a challenge. The assumption was that if teachers could form a community in supporting and learning from each other, then they would be more likely to succeed with their learning.

Methods

The professional development program of this study was designed to improve student mathematics knowledge and attitude by (1) increasing teacher content knowledge and (2) improving instructional skills to teach mathematics concepts in an environment rich in context, community, and connections. The major activities of the project consisted of three components: face-to-face workshop courses, implementation, and sustaining a professional community. The online conversations between face-to-face meetings facilitated the building of a professional community among project participants. Desire2Learn was used to sustain the professional network during and after the project period.

Twenty-nine teachers in grades 5 - 9 participated, 22 first year and 7 continuing participants. A team of 2-4 mathematics teachers from each district was recruited. Each participant received 9 graduate credits, and all participants engaged in 73 hours of face-to-face workshop meetings, at least 18 hours of an online workshop course, and co-teaching with project instructors. Teachers explored how algebra is taught at different grade levels, is related to other math content, and applies to our daily life. The workshops were conducted through discussions; collaborative group work; hands-on activities; problem-solving opportunities; and presentations by participants. Assignments included readings in the textbooks; reflecting on activities introduced during the workshop; developing lesson plans; providing critical comments on lesson plans developed by peers; solving mathematical problems using invented ways; employing various technology to solve problems and perform tasks and mentoring; observing peer participants classroom practice and provide productive comment on their practice; on-going communication with colleagues through Carmen (Web-based course tool); and evaluating web-based resources.

In order to answer research questions, pre- and post- surveys, interviews, field observations, and online discussion messages were analyzed. Surveys, interviews, and filed observation notes were analyzed, mean scores compared, and synthesized in the Results section. All 1149 online messages were analyzed and coded. Three coders were trained and checked for inter-rater reliability at the end of training by using a sample set of coding. The Assignment 1) postings (237 messages) were coded by all three coders and checked for inter-rater reliability. For the assignments 2, 3, 4, and 5, each coder was assigned a series of threads to code. Coders and the researcher had regular meetings to discuss issues and questions.

Results

Changes in Teachers' Opinions about Mathematics and Classroom Practices

Teachers had a good understanding of effective questioning techniques and their use in the classroom; the average rating on this item increased significantly (from 1.96 to 1.54); a significant increase in those agreeing that they had the methods necessary to teach math concepts effectively (up from 74% to 96% and the mean went from 2.26 to 1.73); significant improvements on their understanding of authentic assessment methods used to measure student performance (mean went from 2.37 to 2.00) *(Lower mean score indicates a higher level of agreement, as "Strongly Agree" was represented by the value of 1 and "Strongly Disagree" had a value of 5).* Although all started the program saying that they had a good understanding of fundamental core content in their discipline, this dropped slightly by year-end (89%), which may suggest that the course made them aware of some gaps in their knowledge.

The teachers expressed the highest level of agreement with, "Classroom interaction involves a dialogue among teacher and students" (74%), and by year-end, most (92%) confirmed that this was true of their classes; there was a significant movement on the average response (up from 3.96 to 4.46). There was also a significant increase in those saying that the "Student role is to apply inquiry and problem solving skills to discover solutions to problems" (increased from 56% to 81% and average was up from 3.70 to 4.23). Participants were significantly less likely to report that students worked independently (down to 8% from 19% and the average decreased to 2.23 from 2.89) and that instruction emphasized broad coverage of information (down to 4% from 12% and the average decreased to 2.27 from 2.65). Finally, increased use of alternative assessment methods was confirmed in the significant gain on the average use reported (up from 2.37 to 2.77).

Participating teachers showed initial variation in teachers' comfort with some of the listed practices. Significant differences in participants' use of specific instructional practices in math and professional activities are highlighted in Table 1. Teachers' responses pointed to increases in their use of inquiry-based instructional practices and alternative assessment. They had also expanded their professional development activities at their schools and online. Teachers emphasized the changes that they made to their instructional approach. "*I try very hard not to be the primary speaker in the classroom. As often as possible, I have the students working together, explaining to each other and answering each other's questions. I try to stress the importance of being able to explain how an answer was found and the fact that there are many different paths to a correct answer."*

Individual teachers made a lot of change in what they were doing and more teachers choosing more student-centered approaches. The conversation was much deeper and their questioning skills got advanced. Teachers were looking for ways to question and find out now to get the ideas into the classroom. The biggest change is that the project encouraged teachers to do were random responses and encourage all students to be listened to and heard. Instead of responding to students who had their hands raised, they were randomly selecting students and giving opportunities for those students to respond. Every student was given the opportunity to summarize, and if confused, to ultimately resolve that confusion. The teacher was constantly asking the question "why" and not simply accepting an answer but requesting support for that answer. A much higher level of learning was occurring.

Table 1: Changes in Teacher Practices and Professional Activities (Sorted by Mean Difference)

	Percentage Difference			**Mean Difference**[1]		
	Pre-survey	**Post-survey**	**More Frequent**	**Pre-survey**	**Post-survey**	**More Frequent**
I utilize the project website during and after the project.	14.8	32.0	N.S.	2.58	1.88***	-0.70
I encourage students to engage with materials in a more in-depth way for longer periods of time.	7.4	53.8***	46.4	2.11	1.46***	-0.65
I include opportunities for students to engage in active and inquiry-based learning.	29.6	80.8***	51.2	1.78	1.19***	-0.59
I actively participate in classrooms and virtual communications.	14.8	38.5	N.S.	2.40	1.85**	-0.55
I match instructional strategies and materials to identified students' learning experiences.	37.0	61.5**	24.5	1.78	1.42***	-0.36
I develop a resource binder, collection of ORC lessons, instructional materials, and readings.	37.0	61.5**	24.5	1.89	1.54***	-0.35
I use problems for which there is no immediately obvious method or solution.	18.5	46.2**	27.7	1.96	1.65***	-0.31
I monitor student performance in mathematics by applying rubrics and keeping records of alternative assessment tools.	7.4	30.8**	23.4	2.22	1.92*	-0.30
I encourage students to debate ideas or otherwise explain reasoning.	55.6	80.8*	25.2	1.48	1.19*	-0.29
I conduct a team-designed professional development in own school district.	3.7	12.0	N.S.	2.81	2.60*	-0.21
I develop inquiry and problem solving activities to help students associate the subject matter with their own experience and redefine the concepts.	22.2	50.0**	27.8	1.81	1.50	N.S.

[1]Lower mean score indicates a higher level of frequency, as "Frequently" was represented by the value of 1 and "Never" had a value of 3. N.S. = Not Significant

Table 2: Changes in Teacher Opinions about Mathematics (Sorted by Mean Difference)

	Percentage Difference			Mean Difference[1]		
	Pre-survey	Post-survey	Change in Agreement	Pre-survey	Post-survey	Change in Agreement
It is a good use of time to work on mathematics problems that have no precise answers.	66.7	92.3	N.S.	4.26	5.27***	1.01
Strategies help a student learn mathematics.	96.3	100.0	N.S.	5.00	5.31*	0.31
It is a waste of time to work on mathematics problems that have no exact answers.	18.5	0.0*	-18.5	2.00	1.42*	-0.58
Mathematics is primarily a solitary activity, done by individuals in isolation.	14.8	3.8	N.S.	1.93	1.27*	-0.66

[1]Positive difference indicates a higher level of agreement. Negative difference indicates less agreement.
N.S. = Not Significant

Table 2 reveals teachers' opinions about mathematics. Responses on teachers' confidence about their preparation to answer students' questions indicated that teachers were more comfortable about their skills in mathematics at year-end. Teachers' view of math has shifted, particularly with respect to the value of working on problems that do not have a precise answer. They also increasingly view strategies as a way to help students learn math. Finally, they were less likely to view math as a solitary activity. This is consistent with the results with another survey in which participants were significantly more likely to have students working in groups.

The greatest change in the classrooms was the expectation that all students actively participate. One shared, "*I have a better idea of how to get the students involved and know better now how to help the students do more and on their own."* Others stressed how they have made their classes more student-centered. Teachers noted, however, that students were often initially resistant. A participant explained, "*They sometimes have a difficult time accepting the fact that I am not just giving them the answers or telling them exactly how to do the problems*." Encouragingly, by year-end, several revealed that they had succeeded in getting their students to buy-in to the changes.

Virtual Communication Frequency

Three structured online assignment topics were assigned during the academic months from March to May, and two assignments were given from September to November. Online discussion occurred between monthly face-to-face meetings. During the summer, there were intensive face-to-face meetings for a week.

Figures 1 and 2 depict the average number of messages per individual teacher. Approximately 10 of 22 teachers posted more than 8 messages while all 2nd year teachers posted more than 8 messages. Overall, more contributions in online communication were made by 2nd year teachers than 1st year teachers: an average of 8.9 messages by the first year teachers and 12.4 messages by the 2nd year teachers.

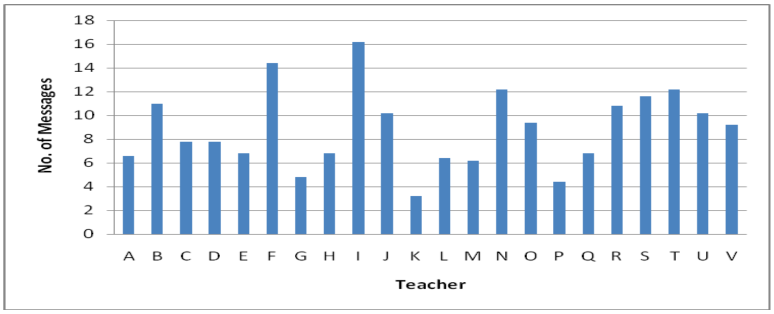

Figure 1. Average No. of Messages Posted by Year 1 Teachers

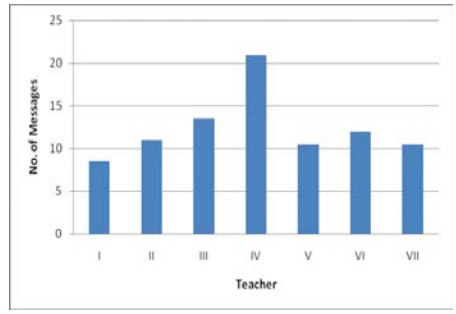

Figure 2. Average No. of Messages Posted by Year 2 Teachers

Figure 3 compares average number of messages posted per assignment. The first year teachers posted the most number of messages for the Assignment 1) Book Reading Discussion and Assignment 2) Video Critique. Compare to the first two assignments, relatively lesser number of messages were posted for three Implementing discovery-based lessons, Assignments 3), 4), & 5), which required creation of lessons and implementation, and commenting peer teachers' work. (*no messages by the 2nd year teachers for the first three assignments, because they joined in the program for the last quarter*). The second year teachers' contribution was much more active and interactive than the first year teachers'.

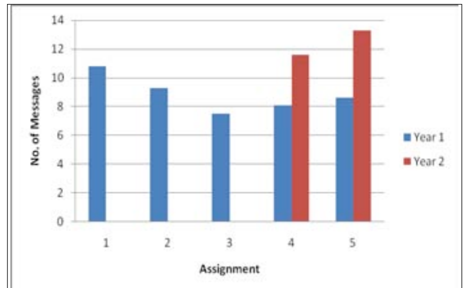

Figure 3. Average No. of Messages per Assignment

Virtual Communication: Purpose and Content

In addition to communication frequency, our study analyzed the data to find the purpose of messages and their content. Our codes were founded in Gareis and Nussbaum-Beach (2007), Interstate New Teacher Assessment and Support Consortium (1992), and Joint Committee on Standards for Educational Evaluation (1988) and finalized after two initial analyses. Table 3 provides final codes and sample messages.

Table 3. Discussion Messages Analysis Codes and Sample Messages

Codes	**Sample messages**
Reflection	I should be more forth coming making sure they hold the spaghetti towards the end of the spaghetti and maybe they would have achieved better results.
Sharing experience	I did this activity with my Algebra class. I found this lesson through the Math Portal site. The lesson took two class periods, which are about 42 minutes in length
Issue/problem	but I always have a few kids who just write one or two words
Question/Suggestion	With all of the variables that prevented a linear graph, was it worth it or would you use a different route?
Instruction (not based on personal exp. but used information from other sources)	Geometry Made Simple (High School Edition), Frank Schaffer Publications – This is where I found the Logic 101 questions. www.printable-puzzles.com – This is where I found the logic puzzle. This is a great website with a lot of different types of puzzles (crosswords, sudoku, cryptograms, word searches, etc.).
Students (extended message about students)	They always want to help so and so add to their answer- no one ever tells another student "No, you're wrong".
Guided Advice	I covered my box with metallic Contact paper, which by the way is dry erase. I

	then used a piece of sparkle craft foam for the screen and white craft foam for the dial. Lastly, I used a pair of old school (large) headphones for the top. Easy and durable!
Simple Agreement	I agree that the number line is a great way to get the year started
Acknowledgement	I am interested in seeing this in action. It does sound like a good activity

Even though the first assignment was reading reflection, the main focus of discussions was not the information presented in the reading chapter. Teachers' interacted discussions were mostly about how to work with certain math concepts, sharing their experience as well as asking questions for instructional delivery including how to relate math concepts to the real world; how well a specific strategy works; games and websites that could help their students with learning concepts; creative ways to demonstrate concepts to their students; the most effective way to use manipulatives in the classroom; how to allow for more student centered time in the classroom; being able to make mistakes in math, not being afraid to do this, and allowing students to help teachers correct them by collaborating with them or turning the mistake into its own mini lesson; the connections made between the different concepts covered in math and all the standards that need to be addressed; the use of vocabulary as a way to help with math communication; and how to overcome time constraints to allow them to teach the "why" as much as they would like to.

Teachers also discussed the growth that they saw in their students, such as asking questions and working more efficiently. They expressed excitement about each student being able to be the "teacher" and also having every student present themselves as a star in the classroom at some point. They discussed the importance of mathematical wording in the classroom, feedback from students on what they think makes a good teacher and also classroom control and how difficult it is for teachers to give that up to allow for a more student directed classroom at times.

Teachers shared struggles that they have faced with math and how they can and have turned that around to help their students. They discussed ways they have made personal connections from their own learning in school and also things that they still want to learn so that they can help the students to hopefully more easily get past the things that they had a lot of difficulty with when they were in school. They also talked about how even teachers are scared about not knowing how to explain the "why" to students, which is the most important part.

Some discussions were about inclusion teachers and general education teachers. They were taught so differently when they were in school that they don't feel comfortable with teaching certain concepts to their own students. They were taught that there is only one way to do problems, but they want to be able to teach their students that there are multiple ways of solving and learning. That was one of the main things they were hoping to gain insight into from this class. They also discussed with interest the similarities teachers share with students in that they all have different ways of learning, such as hands-on, visual, auditory, etc.

There was also a great deal of discussion about how important it is for a teacher to be able to continue to learn and make changes in the classroom, but to try to do this in small amounts (they suggested 10% at a time) to make it easier and more feasible. They also discussed how important it is to find time to talk to and work with your colleagues. (*Quantified results of the content analysis will be shared during the presentation*.)

Discussion

This study investigated the effectiveness of a hybrid professional development program and online communications. Approximately half of the group at the first workshop session revealed they were most comfortable with traditional instructional practices. Nevertheless, teachers overwhelmingly expressed an openness and desire to improve their math instruction in ways that would address the needs of their students. At the end of the summer course, teachers emphasized their appreciation for exposure to additional ideas about how to teach algebra to their students. Participants stressed that they increased their understanding of math content and learned about practical strategies that are applicable to their classrooms. They also valued the sheer number of new hands-on and inquiry-based activities that were covered. By year-end, it was evident that participants had made an effort to make their classrooms more student-centered and inquiry-based. Students were doing more group work and the teachers had learned strategies that resulted in their involving all students in their classrooms. They had begun requiring students to explain their answers and to respond to “why” inquiries if their explanations were not sufficiently clear. They had also begun incorporating more alternative assessments.

Overall, teacher participants increased their math content knowledge and improved their instructional skills. Furthermore, teachers enthusiastically described changes that they have made in their classrooms as part of enriching their students’ math learning. They have made their classes more student-centered, including giving students more opportunities to do inquiry-based, exploratory activities in which they have to present and explain their answers to the class. Teachers also learned additional questioning strategies, which they have applied in their classes. Finally, teachers were more frequently having their students working in groups and using a wider variety of manipulatives than in the past.

With regards to the effectiveness of the online part of the program, there were some trivial findings and unexpected outcomes that can provide an insight into mathematics teacher educators. Most face-to-face discussions centered on the activity or mathematics problems that were presented in class; whereas online discussions reflected on and shared experiences in their own classrooms. The content of online discussions made no distinction between different assignments. The main focus of discussions was sharing their experience about teaching mathematics instead of focusing on what they were required to discuss. It would be important to investigate how different online discussion topics and assignments contribute to teachers’ active learning and participation.

Another interesting finding was the differences in the total number of messages for each assignment. First year teachers had more to share when they were asked to reflect on book readings or video-taped best practices than to exchange their own ideas and experiences about planning and implementing a discovery-based lesson. What teacher educators can learn from this finding is the sequential order of the online professional development program. Practitioners also need plenty time for learning the website and technology that the online PD uses.

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SUPPORT FOR PRACTICING TEACHERS IN THEIR USE OF RESEARCH-BASED FRAMEWORKS OF STUDENTS' MEASUREMENT KNOWLEDGE

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This study investigated the ways in which a teacher developed conceptions of measurement teaching and learning as she collaborated with a researcher to learn and implement a measurement learning trajectory (MLT) (Clements & Sarama, 2009) with two of her students over the course of 18 weeks. The results indicated that several of the components of the professional development program were critical to the teacher's growth from a Level 3 to a Level 4-A on the CGI framework for teacher knowledge developed by Fennema and her colleagues (1996).

Objectives

This study sought to determine which aspects of a professional development experience were most important to an elementary teacher's professional growth by developing accounts of practice for a teacher as she extended her pedagogical content knowledge of measure. More specifically, this study sought to develop a model for a professional development opportunity which incorporates research on student understanding of measurement into a teacher's instructional practices. The researcher sought to offer a unique opportunity involving collaboration, relevant research, teaching sessions with individual students, weekly professional development tasks and reflection to encourage and support a practicing teacher to unpack her own students' conceptual knowledge of measure. A Measurement Learning Trajectory (MLT) for length and area (Clements & Sarama, 2009; Barrett & Clements, 2003) was used as the research-based framework of student knowledge that the teacher used as a lens to view her students' understanding and misconceptions of measure. With a focus on ways of guiding a practicing teacher to understand and students' ways of thinking about mathematics, the Cognitively Guided Instruction theory for professional development provided a model for designing this study within which the measurement learning trajectories served as the researched-based framework for identifying and addressing students' misconceptions (Fennema et al., 1996; Carpenter & Fennema, 1992).

Theoretical Framework

This study combined aspects of both a Teacher Development Experiment (TDE) and a Three-Tiered Teaching Experiment to form a collaborative teaching experiment model for professional development. A TDE (Simon, 1999) is one tool for conducting a professional development program that allows for in-depth analysis of the procedures involved in the professional development experience and the ensuing teacher growth. Because this methodology allows researchers to closely study and foster the development of teachers through cycles of analysis and intervention, the TDE offered a framework that could be used to gain insight into a teacher's developing expertise. In a TDE the focus is not only on the teacher's mathematical development, but also on the generation of models that describe how teachers develop pedagogical knowledge (Simon, 1999). The researcher used the TDE to help determine which aspects of our interaction seemed to be most helpful to the teacher and allowed for the creation of a model for how the teacher grew. This study attended to the teacher's development of pedagogical content knowledge in the area of measurement as she collaborated with a mentor/researcher and as she interacted closely with two students in teaching experiments to develop their measurement knowledge. The teacher's pedagogical content knowledge was developed through the process of learning a measurement trajectory and using it as a tool for guiding tutoring sessions with two of the teacher's students.

In order to attend to the teacher's growth, as well as my own growth as a professional developer I employed a modified Three-Tiered Teaching Experiment model, placing emphasis on the researcher and teacher tiers (Lesh & Kelly, 2000). Teaching experiments (TE) (Steffe, 1988) are the basis of a variety of research designs and "focus on development that occurs within conceptually rich environments that are explicitly designed to optimize the chances that relevant developments will occur in forms that can be observed" (Lesh & Kelly, 2000, p. 192). The Three-Tiered Teaching Experiment allows attention to focus on students' developing knowledge, teachers' developing pedagogical content knowledge, and the researcher's developing understanding of the students' and teachers' developing conceptions. See Table 1.

Tier 3: The Researcher Level	Researchers develop models to make sense of teachers' and students' modeling activities. They reveal their interpretations as they create learning situations for teachers and students and as they describe, explain, and predict teachers' and students' behaviors.
Tier 2: The Teacher Level	As teachers develop shared tools (such as observation forms or guidelines for assessing students' responses) and as they describe, explain, and predict students' behaviors, they construct and refine models to make sense of students' modeling activities.
Tier 1: The Student Level	Three-person teams of students may work on a series of model-eliciting activities, in which the goals include constructing and refining models (descriptions, explanations, justifications) that reveal partly how they are interpreting the situation.

Table 1 Three-Tiered Teaching Experiment (Lesh & Kelly, 2000)

Three-Tiered Teaching Experiment allowed the researcher to focus on hypothesizing about the processes and mechanisms that promoted growth through the collaborative teaching experiment. In a teaching experiment a researcher generates and hypotheses about a student's thinking strategies and then tests those hypotheses in individual interviews (Steffe, 1988). In this study the teacher was encouraged to assume the lead role in this process as she used the measurement learning trajectories for length and area to guide her development of teaching experiments with two students.

The researcher focused the teacher's attention on weekly decision-making activities that involved explanations or justification of predictions that exhibited how the teacher was interpreting situations and problems (Lesh & Kelly, 2000). Through these weekly professional development tasks (PDTs), the researcher hypothesized, tested, and refined my own developing ideas about the mechanisms that were promoting the teacher's growth and adapted a model of her thinking accordingly. An account of the teacher's practice was thus formed in the first three weeks of the collaborative teaching experiment and was used as the benchmark for determining if this initial account of her practice changed throughout the 18 week collaborative teaching experiment (Heinz, Kinzel, Simon, & Tzur, 2000).

The Cognitively Guided Instruction Professional Development framework (Fennema et al., 1996) was used as a tool to guide the decisions made during the process of creating the model of the teacher's measurement instruction. After identifying the major components of instruction and beliefs fundamental to the goals of the CGI program, Fennema et al. (1996) defined four instructional levels. These levels were used in the current study as a lens for viewing teacher change. Although every math lesson was not observed, the four pre- and four post-lessons, as well as the three measurement lessons the teacher invited me to observe were viewed through this lens. This framework identifies four major levels of instruction that are fundamental to the philosophy of Cognitively Guided Instruction (Fennema et al., 1996). See Table 2.

Level	Description
1	Provides few, if any, opportunities for children to engage in measurement problem solving or to share their thinking in measurement.
2	Provides limited opportunities for children to engage in measurement problem solving or to share their thinking in measurement. Elicits or attends to children's measurement thinking or uses what they share in a very limited way.
3	Provides opportunities for children to solve measurement problems and share their measurement thinking. Beginning to elicit and attend to what children share but doesn't use what is shared to make instructional decisions.
4-A	Provides opportunities for children to solve a variety of measurement problems, elicits their thinking, and provides time for sharing their thinking. Instructional decisions are usually driven by general knowledge about his or her students' measurement thinking, but not by individual children's measurement thinking.
4-B	Provides opportunities for children to be involved in a variety of measurement problem-solving activities. Elicits children's measurement thinking, attends to children sharing their thinking, and adapts instruction according to what is shared. Instruction is driven by teacher's knowledge about individual children's measurement thinking in the classroom.

Table 2 Levels of Cognitively Guided Instruction (Adapted from Fennema et al., 1996)

In this study, the researcher blends aspects of the Teacher Development Experiment and the Three-Tiered Teaching Experiment and proposes a different approach to professional development. This trajectory-focused approach is called a Trajectory-Based Collaborative Teaching Experiment (TBCTE). Different from Simon's Teacher Development Experiment, the TCTE is a close collaboration working one-on-one with a teacher as they learn to use learning trajectories specific to the measurement content domain. The CGI levels of professional development were used as a lens for viewing teacher change as the researcher supported the teacher as she developed tasks and assessed two of her students.

Methodology

In order to identify essential aspects of the TBCTE I worked with one, fifth grade teacher from the rural Midwest over the course of 18 weeks. Because the Measurement Learning Trajectory (MLT) was the focus of the professional development experience, the teacher was given opportunities to explore her own concepts of measurement and students' measurement misconceptions prior to learning the MLT. Weekly Professional Development Tasks (PDTs) were used to provide the opportunity for the researcher to hypothesize about the teacher's responses and refine and reorganize conceptions about the mechanisms that were promoting the teacher's growth, ultimately helped the researcher adjust and refine the model of the teacher's thinking. Such mechanisms included: critiquing example problems for strengths and weaknesses, role-playing, presenting her own problems to the entire class, determining which trajectory levels different tasks addressed, and writing performance assessment tasks for her students.

Before the teacher was introduced to the MLT her students were given a measurement assessment. She was also asked to read related literature and reflect on those readings prior to learning about the MLT. The researcher observed two whole class lessons and used these observations to document an initial interpretation of the teacher's CGI Level in an attempt to build an initial model of her thinking. Weekly PDTs were then created and posed to the teacher. The researcher made predictions about how the teacher would respond to the tasks so these predictions could be prepared to the teacher's actual responses. This cyclical process of model building through predictions and examinations using PDTs continued throughout the study as the researcher gradually refined, expanded, or rejected the current interpretations of the teacher's growth (Lesh & Kelly, 2000). Readings, reflections, meetings with the researcher, and PDTs continued throughout the entire 18 weeks of the professional development experience.

In week four the MLT was introduced as a tool for assessing student knowledge of length and area. Two students were chosen based on parent permission and their initial assessments. After the teacher had time to practice using the MLT to analyze other students' thinking strategies and discuss her thoughts with the researcher, they individually reviewed the two focus students' initial assessments in an effort to determine the level at which the two students were operating for the majority of the length and area tasks. Once an initial level of student thinking was identified, the teacher predicted how she thought the student would respond to new tasks that were presented in individual interview sessions. Following each interview the teacher was encouraged to make a claim about the student's length or area knowledge based on the learning trajectory. This cycle of planning, predicting, and checking started in week seven and continued for six sessions per student, requiring 12 weeks to complete this cycle with the pair of students as the teacher gradually reorganized, expanded, or rejected her current interpretations of their thinking level.

Results

Weeks 1-3 CGI Level 3

Based on my observations and the teacher's reflections from the first three weeks, the initial model of the teacher's measurement instruction was at a Level 3 on the CGI professional development framework because she "elicits and attends to what children share but doesn't use what is shared to make instructional decisions." The researcher looked for changes in the teacher's ability to elicit student thinking in the individual interviews and her use of that information to inform whole-class instruction and assessment. Such changes informed decisions to make new models of the teacher's pedagogical content knowledge throughout the remaining fifteen weeks of the professional development program.

Weeks 4-7 Level 4-A

Following these first three weeks, the teacher was given weekly PDTs to assess her use of measurement content knowledge and her knowledge of the measurement learning trajectories to guide instruction and assessment. These tasks were used to predict her responses based on the knowledge the researcher gained in week three. Her actual responses were then used to guide the creation of the next PDT. Along with these tasks the teacher was introduced to the trajectories and began planning and conducting the individual interview sessions with the two focus students. In the following weeks the teacher also shared ways that she had changed whole-class instruction and invited the researcher into her classroom on two occasions to observe measurement lessons she had created. These events form the context for the adaptation and refinement of the model of her thinking at various CGI instructional levels.

During weeks four through seven, the teacher was given many opportunities to use the MLT as a tool for analyzing assessment problems as well as her students' answers to measurement tasks. Also during this time frame, without a prompt from the researcher, the teacher significantly altered a measurement lesson to incorporate the information she had learned through the trajectory use and the measurement articles she had read. At the conclusion of week seven the teacher's actions demonstrated instruction at the 4-A Level because, on more than one occasion, she had provided "opportunities for children to solve a variety of measurement problems and elicited their thinking."

Another indication of Level 4-A was when the teacher explained how she had discovered that her students did not know that a meter stick was comprised of 100 centimeters. She explained that she addressed the meter stick misconceptions the following day by engaging students in a discussion about the units on the meter stick, which indicated that she had based instructional decisions on general knowledge about her students' measurement thinking. Over the course of the next 11 weeks, the teacher continued to read measurement articles, complete weekly PDTs, and began the interviews with the two focus students.

Weeks 8-11 Level 4-A

The teacher's responses to the PDTs for weeks eight through ten indicated that she indeed understood the importance of the problem statement. She changed two problems to better assess her students' knowledge of length. These changes were a result of her knowledge of measurement misconceptions she had learned about from the readings. The model of her thinking was not changed as a result of these three weeks and remained at Level 4-A because she had changed measurement problems to allow "opportunities for children to solve a variety of measurement problems and elicited their thinking."

During week eight the teacher described a change in a task she had done in previous years. She changed this task due to what she had learned about one focus student's difficulties with understanding the difference between area and perimeter when given a rectangular grid instead of labels on the sides of a rectangle. The teacher asked her class to make a rectangle that had a perimeter of 12 units using square tiles. No student was able to correctly do this task. Her findings are consistent with prior findings (Chappell & Thompson, 1999) that illustrate the difficulty students have when asked to produce a drawing to show a specific perimeter. Although the teacher suspected there would be some students who struggled with this task, she was surprised that no student could do this task correctly.

Weeks 12-15 Level 4-A

Over the course of these four weeks, the length and perimeter TEs were completed and work with area began. By this time in the program, the teacher had gained confidence in her measurement knowledge and was taking on more of a lead role in the TE task creation by asking more probing questions during the TEs. For a PDT task, she was asked to find an area problem she typically assigns as homework and change the task to make it a better task for informing her about students' knowledge of area. The problem the teacher chose to change showed a three by four grid made of toothpicks and asked students to determine how many toothpicks formed the perimeter and to find the area of the figure. She changed this task to include the length of the tooth picks as 1 cm long and asked her students to draw a line that would be equal to the length of the perimeter of the rectangle. In addition to having the students find the area, she asked her students to draw a square that was the same size as one of the area units. The teacher explained the rationale for her changes. "I would like to make this into a problem where the student was using construction to help them 'see' or 'understand' the difference between perimeter and area units." Her attention to perimeter and area beyond formula use indicated that she had a better understanding of how to address her students' misconceptions about these formulas.

The changes the teacher made to the homework question indicated that her knowledge of student misconceptions motivated her to focus students' attention on the notion that perimeter was composed of length units and area was composed of square units. There was not substantial evidence to indicate there was a change in the teacher's model for teaching measurement during these four weeks. She had changed measurement problems to "elicit students' thinking" and had used information about Eliza's struggles with an area task to make a decision about how to rewrite the task for the whole-class final assessment. These actions fit with Level 4-A, which is indicated when "Instructional decisions are usually driven by general knowledge about her students' measurement thinking, but not by individual children's measurement thinking."

Weeks 12-15 Level 4-A

During the last TEs the teacher exhibited a great deal of confidence with her knowledge of students' misconceptions and carried out the interviews with little assistance. She made decisions based on her understanding of the role each task played in her quest to assess students' understanding of measurement and move them into higher levels of thinking. The post-questionnaire exhibited the teacher's changing conceptions about measurement teaching and learning. Her responses to the post-questionnaire focused on the role the MLT should play in her teaching practices. She used more specific measurement ideas to explain her understanding of the big ideas of measurement in the post-questionnaire. She wrote, "The importance of equality of length, conserving space and iterating units need to be mastered before I can expect them to understand more advanced ideas. Looking at my questioning techniques from a different angle, not only as a probe, but also as a way to encourage them to cross over into that next level conceptually is something I think about more often." There was no evidence to indicate that the teacher had moved beyond Level 4-A and into Level 4-B which would require her to base instructional decisions on her "knowledge about individual children's measurement thinking in the classroom."

Discussion

The findings of this study indicate that although the teacher's measurement content knowledge changed little, her understanding of students' misconceptions about measurement conceptions improved. The critical aspects of the professional development experience were the measurement readings, the PDTs, the teacher's reflection practices, the MLTs, the cyclical nature of the TEs, and the weekly collaborative meetings. The order in which these components were incorporated into the experience was also important to the manner in which the components impacted each other. For example, the findings indicate that the readings were critical in sparking the teacher's curiosity about her own students' measurement thinking and learning and validated the need for the trajectories. This helped in the teacher's acceptance of the MLTs and prompted her to develop measurement lessons on her own. Later the intense cycles of reading and applying the information in the form of TEs helped the teacher make sense of the MLTs. After the MLTs were introduced, the researcher found that the PDTs served as a means to give the teacher experiences that supported her developing understanding of and use of the MLTs to assess student responses.

Through the teacher's participation in the professional development experience using the MLTs, she shifted on the CGI scale for teacher knowledge from a Level 3 to a Level 4-A. By utilizing both a CGI model for professional development and a learning trajectory for length and area, this study offers unique insight into how this combination might impact a teacher's professional growth. The researcher anticipated that this model of a professional development program would give readers a clear picture of the developmental processes involved when a teacher and a researcher collaborated in an attempt to help the teacher and researcher grow as professionals.

As researchers and professional developers work to determine ways in which practicing teachers might incorporate research into their own classrooms this teacher's incorporation of the ideas she read about into her own classroom, indicate one possible avenue. Also, the research-based learning trajectories were accepted and used by the teacher as a lens to view and make sense of assessment and student learning. The findings indicate that the cyclical nature of the readings and the application of that material into the TEs helped the teacher grow as a professional. This study offers a promising method for incorporating research into teachers' everyday practices.

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A PERSPECTIVE OF CHANGE: A TEACHER'S EVOLUTION THROUGH PROFESSIONAL DEVELOPMENT

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This study discusses the evolution of one teacher as a result of participating in professional development focused on children's mathematical thinking. Viewing her interactions with students in video-taped problem solving interviews from the beginning and third year of her participation in the professional development, prompted her to reflect on the changes that she had made. She reported that she had begun to capitalize on mathematics as it came up throughout the day, as well as valued student strategies rather than their success at solving problems. These perceptions were consistent with previous research findings even though different student populations were examined.

In an attempt to improve teacher quality in mathematics, the National Council of Teachers of Mathematics (NCTM) developed and set forth their 1991 *Professional Standards for Teaching Mathematics,* which provide goals to guide teachers in continuous learning and improvement. As part of these goals, teachers are encouraged to participate in "appropriate and ongoing professional development" (NCTM, 1991). The rational for encouraging teachers to participate in continuous learning is to enhance their reflective and metacognitive processes. Ideally, they will create opportunities for their students' metacognitive mathematical thinking, as well as share best practices for teaching mathematics with other colleagues (NCTM, 2000).

To have the greatest impact on teacher improvement and student achievement, prior work suggests that professional development programs be rigorous, ongoing, and provide teachers with the opportunity to engage in collegial exchanges (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009). Stigler and Hiebert (cited in NCTM, 2000, p.18) have found that "collaborating with colleagues regularly to observe, analyze, and discuss teaching and students' thinking or to do 'lesson study' is a powerful, yet neglected, form of professional development in American schools". Fortunately, there has been a push to increase the number of reform based professional development programs and investigations of the impact of those programs on teacher practice. This study examined the effects of one such reform based professional development program (PDP) and provides insight into the changes of one teacher's mathematical instruction in an urban school district in northern California.

Theoretical Framework

Our study investigated the effects of a PDP using the principles of Cognitively Guided Instruction (CGI), (Carpenter, Fennema, Peterson, Chiang, & Loef, 1990). In professional development programs using CGI, leaders share "an organized set of frameworks" (Franke & Kazemi, 2001, p. 103) of the different types of mathematical problems children solve and the

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invented solution strategies they employ. Not only is there a focus on the different invented solution strategies, but also on the explanations students provide to justify those strategies. If teachers are aware of different solution strategies and how students explain their thinking, they can help students verbalize their thinking, make more sense out of what their students are thinking, and more effectively plan instruction and guide classroom discussions (Franke, Carpenter, Levi, & Fennema, 2001).

Research investigating teachers, who have participated in CGI PDPs, and their pedagogical beliefs have shown that by the end of the professional development, teachers had stronger beliefs in students' abilities to solve problems without explicit instruction. These teachers also tended to created environments in which students were encouraged to develop their own solution strategies and share them with others. They also saw their role as a guide who questioned children and supported their thinking when the need arose (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). This research is over 15 years old now and conditions for teachers have changed tremendously during that time. This study examines the degree to which CGI continues to foster this kind of change for teachers.

The research questions that guided our investigation were:

- How does one teacher in a CGI PDP perceive that her participation has impacted her teaching?
- What is one teachers' perception of the evolution of her interaction with her students?

In order to investigate these questions, we interviewed a participant from our CGI PDP to gain insights into her evolution as a mathematics teacher. Given the variability between teachers, such as differing personal and professional experiences, levels of education, or previous professional development, we focused on one particular teacher who took advantage of what was offered, a sort of best case scenario. Mrs. K, a first grade teacher who has taught for 12 years and is bilingual, was chosen to be interviewed because she had participated in all three years of our CGI PDP. She often provided thoughtful comments during after school meetings and she implemented many of the strategies discussed in the professional development which was evident from classroom observation and student interviews.

Methods

In the spring of the 2007-2008 school year, professional development began in a small urban school district in northern California with a high percentage of English Language Learners. This three year grant-supported program invited kindergarten through second grade teachers from around the district to join each year. Unlike previous CGI PDPs, our program situated interviews and other professional development material within the district and teachers' classrooms instead of relying on original CGI materials, which depicted homogeneous classrooms quite different than those found within the teachers' district. This is done to change teachers' preconceive notions and promote the idea that children from diverse ability, cultural, economic, and language backgrounds are capable of problem solving without direct instruction.

As part of the professional development, teachers attended monthly after school meetings to discuss students' mathematical thinking by viewing video-clips of their students solving problems; listening to audio-clips of students in classrooms sharing solution strategies; and

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learning ways to scaffold critical thinking during problem solving. During the after school meetings, teachers were given story problems to present to their students and a member of the professional development team scheduled a time to visit each teacher's classroom to observe and audio-tape student solution strategies. The teachers also selected three students to interview during the fall and spring of each year, which provided them with an opportunity to implement strategies learned in the professional development in a one on one session with their students. These interviews provided teachers with an opportunity to develop elicitation and questioning strategies. Because these interviews were videotaped, they, as well as classroom observations and audio recordings, formed the basis of our instructional material to present during meetings to elicit authentic, meaningful comments and begin dialog amongst the teachers.

In the summer of 2010, we invited Mrs. K to discuss her impressions of the PDP as well as perceived changes in her teaching as a result of her participation in the program. In the summer interview, the two authors of this paper used a semi-structured interview protocol to discuss with Mrs. K her experiences in the professional development (Merriam, 2009). Videos of past interviews with students were also used as artifacts to spark discussions and also to provide a diachronic report, which described phases of development or change that captured her evolution as a teacher (Weiss, 1994). The interview occurred at the university office of one of the authors, which contained the audio/video equipment used to play the student interviews. The interview lasted about two hours, and was audio recorded in order to retain the original data as well as to ensure that our interpretation and analysis was accurately constructed out of the participant's words (Seidman, 2006).

Our goal was to provide an account of one teacher's evolution as a result of participating in a child centered professional development. The interview was transcribed to enable the authors to make judgments about the data without imposing their own reference frame too early in the analytic process (Seidman, 2006). After transcribing, the transcripts were read in more detail but particular attention was devoted to segments when Mrs. K specifically discussed changes in her practice and/or perceived differences in her interactions with children.

Results

Reflecting on Practice

We began our interview with Mrs. K by asking her to reflect on her three years in the PDP, and in particular what changes she had noticed in her instruction. Throughout the interview, Mrs. K described her realization that mathematical concepts and ideas were not only occurring during her math lessons, but that there were opportunities to explore these concepts throughout the school day. She described her process of math instruction before the professional development as, "I got to teach math, let's get our lesson, let's get through it, and then we're moving on to a new subject…The lesson is just one day and we turn the page and move on." Her instructional process was one that focused on efficiency and getting through the material to move on to something different and that by doing this, she believed that her students understood the material.

After participating in the professional development and reflecting on how her math instruction had changed, she stated,

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> I really have noticed that there are math concept ideas throughout the day, it's not just math time. You look for opportunities to do story problems. I really recognize moments throughout the day where it's just so easy to pose a question…

Mrs. K recognized that although it may seem more efficient to follow a pacing guide or the text book and quickly move through topics, it is insufficient and that to accurately gauge how well her students were making sense of the mathematics, she had to take the time to incorporate the math throughout the day, listen, and observe her students. She also realized that the way she taught previously was underestimating what her students were able to accomplish.

She discussed a previous perception of her role as a classroom teacher indicating, "I pretty much knew everything and I'm going to tell you everything and then you'll walk away knowing things." But after engaging in the PDP, she stated that, "I teach completely differently as a result and I give kids more opportunities. We are learning together. It's kind of exciting. It takes a lot of pressure off." Here we see Mrs. K's shift in perceptions of her role as transmitter of information to facilitator. Where other teachers may find it more difficult to guide students through the problem solving process, Mrs. K found this process to be exhilarating, less stressful, and lead students to anticipate math throughout the day. She explained that her students "are always looking for a way to solve the problems themselves. Before I even know that that's what I want to do, they're already doing the numbers." This process slowly became a classroom norm and children began to anticipate mathematics on a daily basis; without prompting, children began mathematizing for themselves.

Not only did Mrs. K describe the evolution in her teaching but she also mentioned how her students responded in a positive way to her teaching methods. She described her students as becoming more confident in their mathematical abilities and enjoying the process.

> Everything is a way to do math and they love it. It's surprising how much they actually enjoy it. It really surprised me, they look forward to it…I've noticed I can let go and students are learning from each other. And they feel good about it. They look forward to it. They are looking forward for opportunities to show what they know.

With this change in teaching practices and in her role as a teacher, Mrs. K created an environment where student thinking was valued and students were able to take charge of their math learning.

Viewing of Student Interviews

After discussing the perceptions in her practice, we then asked Mrs. K to view video-clips from two sets of her student interviews. One set was recorded after her first year participating in the PDP and the second set was recorded after her third year. We decided to view student interviews that were recorded after her third year of participation first for her to identify potential differences. Each set of interviews were about 15 minutes in length and we watched short segments and asked her to discuss her interpretation of the child's thinking and/or her scaffolding techniques. The following results will focus on a particular type of story problem, a Compare Difference Unknown (CDU) problem, selected because these problems have been found to be especially difficult for children. (Fuson, Carroll, & Landis, 1996)

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This first segment, recorded after her third year in the PDP, was with a child Luis, a designated English Language Learner (ELL). Luis struggled at the beginning of the school year but by the end of the year, he had improved academically even though his English language was developing. The first text box describes what occurred in the video-clip as he solved the following story problem:

- Julissa has ten flowers and Brianna has 16 flowers. How many more flowers does Brianna have than Julissa?

[Luis began counting out cubes to represent each set. He made a set of eight which he miscounted as ten and connected them together to make a long stick of cubes and then he made another long stick of 16. He then stopped, reread the problem, and lined up the two sticks next to each other.]

L: They both have one, they both have two, they both have three, they both have four, they both have five, they both have six, they both have seven, they both have eight.

[He realized that he was missing two more cubes on Julissa's row and added two more cubes.]

L: They both have nine, they both have ten. *[He stopped and looked at the remaining cubes.]* Julissa has ten and Brianna has 16.

Mrs. K: Right, how many more does Brianna have than Julissa?

[He reread the problem out loud.]

Mrs. K: She has 16, Julissa has ten, so that means Brianna has more right? How many more does she have? How many more flowers?

L: Two flowers.

Mrs. K: Well, what are you holding in your hand? We are pretending those are what?

L: Flowers

Mrs. K: Ok, show me Julissa's flowers. *[He held up the stick with ten cubes.]* Show me Brianna's flowers. *[He held up the stick with 16 cubes.]* So when you put them next to each other, who has more flowers?

L: Brianna

Mrs. K: How many more flowers does she have?

[He looked at the cubes which were next to each other and counted the ones that were extra.]

L: Six

Teacher-Student Interview: Spring 2010

Mrs. K actively interacted with Luis. She revisited the specifics of the problem with him, asking him to consider the relative size of the quantities. When he gave her the wrong answer, "2" she asked him to consider how his cubes related back to the problem, "what are you holding in your hand?" She then evoked the work that he had previously done ("they both have one, they both have two…" by asking him, "when you put them next to each other, who has more flowers?" Mrs. K's interactions with Luis were not extensive but it is clear from her questions to him that she was actively listening to his comments and closely observing his work. She was

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actively supporting him without taking over his thinking. The next text box is Mrs. K's reaction to the previous video-clip.

> **Mrs. K**: He knows how to compare but he just doesn't know. He knows how to set it up but doesn't know where to go from there. He's on his way.
> **Researcher**: He kind of lost his flow. So I think that shows that comparison problem isn't that solid for him yet. You didn't tell him what to do, he figured it out finally.
>
> Researcher-Teacher Interview: Summer 2010

In this reflection of the video, Mrs. K is not being critical of the child and she is noticing that he could build the sets but did not know what to do with them. She also realizes that Luis needs more experience to be facile with this problem type.

In the next segment we examined Mrs. K's reaction to an interview during her first year in the PDP. The student, Daniel, a native English speaker, was solving the following problem:

- Erica has 12 stuffed animals. Melissa has nine stuffed animals. How many more stuffed animals does Erica have than Melissa?

> *[Mrs. K read the problem twice for the student. After hearing the problem the second time Daniel wrote on his paper 12+. Mrs. K then read the problem a third time.]*
> **D**: Ok.
>
> Teacher-Student Interview: Spring 2008

> **Mrs. K**: Me rereading the problem doesn't do anything. Poor kid, oh my gosh!
>
> Researcher-Teacher Interview: Summer 2010

While watching her interview, Mrs. K responded with the above quote which acknowledged that her interaction was not productive. The next text box continues with the teacher and student interview.

> *[He crossed out the plus sign next to the 12 and wrote the number nine under the 12 and then wrote a six under the nine.]*
> **D**: Twelve minus nine is six.
> **Mrs. K**: Is that right? You want to check with the cubes? You can.
> *[He started counting out cubes.]*
> **Mrs. K**: Count out loud so I can hear you.
> **D**: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, *12 [He counted out loud as he connected the cubes together to make a stick. He then started making another stick of six. He lined them up and counted backward starting at 12 and ending at six. He then looked up at Mrs. K.]*
> **Mrs. K**: Would you like me to read it again? *[They read the problem again together and then she reread it two more times.]*
>
> Teacher-Student Interview: Spring 2008

In this episode, Mrs. K only repeated the problem to Daniel as opposed to the first episode with Luis where she questioned and supported him in figuring out the answer. At this point in her

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professional development she did not know how to offer support and so took a passive role in her interaction with the student.

> **Mrs. K**: Oh my gosh! I'm torturing him. This is horrid! I'm not even questioning why he's holding on to that six…I'm not giving him any direction you know, there's nothing. He's just out there trying to do something.
>
> Researcher-Teacher Interview: Summer 2010

This observation suggests that she sees other ways she could have supported his thinking rather than just repeating the problem. She recognizes that she could have asked him about his strategy and thinking, especially in the six. At this point in her professional development, she realized that questioning the student's strategy/thinking can inform her interactions with the child. As she noted, "For me I was just looking for the correct answer not the process in getting there."

By observing these two episodes of her student interviews, Mrs. K could see the difference in her interactions. She reported that she was unclear as to what her role was in the interaction, so she chose to restrict her activity to restating the problem. After viewing this second episode, Mrs. K discussed how she was thankful to have seen these two episodes, especially at that point as opposed to after her first year because "if I would have seen it, I wouldn't have known what to compare it to. So I think this is actually effective. For me looking at it, I really see a difference." This suggests that her interpretations of interactions with children had changed significantly from the beginning to the end of the PDP, with her having a much better sense of how to support children in figuring things out for themselves at the end of her three years. She now values students' mathematical thinking rather than looking exclusively at their success in solving problems.

Discussion

In her reflection of the changes she had noticed, Mrs. K mentioned that she had perceived changes in her role as a teacher, especially in the way she approached her mathematical instruction and in the way her students participated in mathematics in the classroom. Two of the most important roles for teachers in reform based classrooms are creating environments where students explore, justify their solutions, and make sense of the mathematics; and guiding activities and discussions while not taking over students' thinking (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, et al., 1997). Instead of teaching in a didactic way using algorithms and direct instruction, teachers should encourage students to develop their own solution strategies and help make connections between manipulatives, mental arithmetic, and written methods (Verschaffel, Greer, & DeCorte, 2007; Fuson & Burghardt, 2003). Mrs. K, with the help of the PDP, viewed herself as a facilitator as opposed to a distributor of knowledge and she created an environment in which she and her students were learning together. Her comment "I have taken the time to listen to them…and time to observe the strategies that kids use," highlights this change in her role and the new focus of her instruction.

By examining student interviews from Mrs. K's first and third year of participation in the PDP, we were able to compare and visualize the evolution which she had undergone while in the

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program as opposed to relying on self assessment. It is unclear if this evolution would have been evident had we only used the original CGI professional development materials. The CDU problems were of particular interests because research has shown that these problems are one of the most difficult problem types for children to solve, as there is no clear action stated within the problem (Fuson, Carroll, & Landis, 1996). Also, children have limited experience with this type of problem as they are more accustomed to joining, separating, sharing, counting groups of items, and identifying greater/lesser quantities, but rarely do they ever calculate the difference. Because of this lack of action in the problem, teachers too find it difficult to scaffold students without explicitly teaching them how to solve them. In the case of Luis, Mrs. K was able to skillfully question him so that he was able to figure out the solution for himself without her telling him how to do so directly.

Through the interview with Mrs. K, it appears that the PDP was a valuable experience that had a positive effect on her mathematical instruction as well as having beneficial effects on her scaffolding skills and interactions with their students. Although only one teacher's experience was presented here, several other teachers have expressed similar perceptions and experiences in their classrooms and interactions with their students. We have continued our professional development in this school district by incorporating third and fourth grade teachers into the program. We anticipate similar outcomes from this program for developing teachers.

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MAKING SENSE OF STUDENT WORK WITH LEARNING TRAJECTORIES

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This paper investigates a group of elementary teachers' uses of a learning trajectory to making sense of children's mathematics during professional development. Teacher learning activities focused on understanding students through observation and analysis of written work, as well as considering next instructional steps in response. Results suggest that mathematics learning trajectories can support teachers in identifying, interpreting, and responding to evidence of student cognition.

As stated by *Principles and Standards for School Mathematics* (NCTM, 2000), assessment "should be an integral part of instruction that informs and guides teachers as they make instructional decisions" (p. 22). Though work to improve large-scale assessment systems and accountability models has received much attention recently, Black and colleagues (2004) remind us of the importance of considering what happens "inside the black box" of classrooms. Informal classroom assessments, such as observing students engaging in mathematical tasks or reviewing their written work, have the potential to improve student learning provided teachers use these assessments to guide instruction.

Some researchers have reported the benefits of having a framework for understanding the mathematics of children to interpret students' mathematical work (e.g. Carpenter, Fennema, Franke, Levi, & Empson, 1999). Such structures for making sense of students' approaches to mathematical concepts assist teachers in identifying likely understandings and misconceptions that may direct teachers' instructional choices and ultimately support greater student achievement (Carpenter et al. 1999). One such structure currently receiving attention is the construct of learning trajectories (LTs). Some have conjectured such positive benefits of LTs (Confrey & Maloney, in press; Corcoran, Mosher, & Rogat, 2009), and early research on the ways in which teachers may use their understandings of LTs in classrooms suggests that they may serve as a framework for understanding students' mathematics (Wilson, Mojica, & Confrey, 2010).

The purpose of this paper is to explore teachers' uses of an LT in identifying, interpreting, and responding to evidence of students' mathematical thinking when observing students and when analyzing written work on instructional tasks. The research reported describes the ways that a group of practicing elementary teachers used their knowledge of an LT to make sense of students' mathematical activity by addressing the following research questions: In what ways do elementary mathematics teachers in a professional development setting use their knowledge of a learning trajectory to understand students through the classroom assessment practices of observation and analysis of written work? In what ways do they respond instructionally?

Perspectives

The current notion of learning trajectories is related to Simon's (1995) seminal work on the Hypothetical Learning Trajectory, the idea that a teacher coordinates his or her knowledge, learning goals, and plans for instructional activities to form a hypothesis about a student's learning process. In recent years, researchers have built upon this idea by synthesizing research and empirically supporting descriptions of the ways student understanding evolves over longer periods of time (Confrey & Maloney, in press). Though multiple definitions exist among researchers, most agree that LTs are descriptions of the varying levels of reasoning sophistication held by students as they refine their informal understandings into more adequate, complex conceptions. This study uses Confrey et al.'s (2009) definition which states that a LT is "a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction, and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (p. 347). From this perspective, learning is closely related to students' instructional experiences, highlighting the teacher's role in supporting student learning (Confrey & Maloney, in press).

In *Knowing What Students Know* (NRC, 2001), the authors of the report cite a common principle for all assessment to be *reasoning based on evidence*. Assessments serve various purposes in various contexts, from the large-scale evaluation of programs to the examination of individual student achievement. Of particular relevance to this paper is the report's point concerning classroom assessment. It describes classroom assessments as "informal methods for determining how students are progressing in their learning" that are used by teachers "to inform day-to-day and month-to-month decisions about next steps for instruction"; such assessments include "classroom observations, written work, homework, and conversations with and among students" (pp. 37-38).

The report presents a three-part framework that underlies the assessment process called the Assessment Triangle (figure 1). The framework illustrates how three facets of assessment, one at each vertex of the triangle, are integrally related. The Observation vertex represents the "a set of beliefs about the kinds of observations that will provide evidence of the students' competencies" (p. 44). The Interpretation vertex represents the process by which one makes sense of evidence. The Cognition vertex represents a model of cognition and learning for a particular domain of knowledge. Each of these facets works together for an assessment to draw conclusions about what students know based on their performance.

This paper explores the possibilities of LTs serving as a model for student cognition at the Cognition vertex. It seeks to characterize the observations and interpretations teachers make during classroom assessments when an LT is taken as a model of cognition. As classroom assessments may be made for the purpose of informing and guiding instruction, it also investigates possible steps that teachers might take in response to their assessment of student learning.

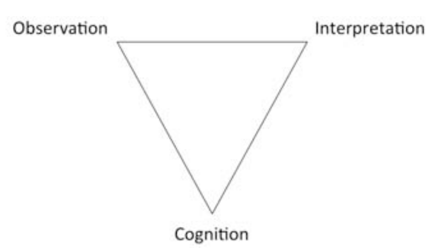

Figure 18. Assessment Triangle (NRC, 2001).

Methods

Design studies methodology was used to investigate the ways in which teachers used their understandings of an LT to observe students and to analyze written work. Design studies seek to engineer learning environments and study the learning that results from them (Confrey, 2006). They are based on the iterative design, implementation, and refinement of conjectures of how a particular form of learning unfolds (Cobb, Confrey, diSessa, Lehrer, and Schauble, 2003; Confrey & Lachance, 2000). For this study, a professional development experience was designed for elementary teachers to support their transition to newly adopted state standards that represented increased content knowledge on rational number by focusing on a single learning trajectory for rational number.

A learning trajectory for equipartitioning (EPLT) (Confrey et al., 2009) was selected as a context for studying the ways teachers might use knowledge of an LT. Confrey and her research group define equipartitioning as the set of cognitive behaviors that have the goal of producing equal-sized groups or parts from collections, single wholes, or multiple wholes such as is typically encountered by children in the context of 'fair sharing' (Confrey et al., 2009). The EPLT describes how children's experiences with fair sharing develop into an understanding of partitive division across the elementary years. Its levels describe the strategies, mathematical practices, emerging relationships, generalizations, and misconceptions that children may encounter as they gain proficiency in an understanding of $a \div b$ as b equal-sized parts of a (Confrey, Maloney, Wilson, & Nguyen, 2010). A central part of the EPLT is a description of the ways that children work to coordinate three equipartitioning criteria. When equipartitioning, students must (1) create the correct number of groups or parts, (2) create equal-sized groups or parts, and (3) exhaust the whole or collection. The EPLT outlines how different parameters, such as the number of parts created (e.g. repeated halving is easier than creating odd numbers of parts) or the type of whole (collections, a single whole, or multiple wholes) affect the level of difficulty of instructional tasks.

Twenty hours of professional development on the EPLT was designed for elementary teachers. In five 2-hour after-school sessions, teachers studied the EPLT as well as engaged in instructional practices focused on elucidating, understanding, and using student thinking in instruction. These practices included observing and conducting clinical interviews with students, selecting and adapting instructional tasks, and analyzing students' written work. The remaining ten hours involved classroom-based activities with students pertaining to these practices (Wilson, 2009).

Thirty-three K-2 teachers from two rural schools in the southeast participated in the study: 11 Kindergarten, 11 First Grade, and 11 Second Grade. Of these 33 participating teachers, 88% received undergraduate training in elementary education and 24% held a master's degree in education. Their average number of undergraduate credit hours was 6.4 hours in mathematics (SD = 4.1 hours) and 4.4 in mathematics methods courses (*SD* = 2.1 hours). Six of the participating teachers were currently enrolled in a graduate program in elementary education. Two of the teachers were certified by the National Board of Professional Teaching Standards in Elementary Education. On average, the teachers had 12.0 years of experience (*SD* = 7.2 year), 9.2 of which were with students in grades K-2 (*SD* = 5.8 years). Upon completion of the professional development, participants earned two continuing education units toward license renewal.

Primary data sources for this investigation are three activities from the professional development. In the Session III Student Work Analysis activity, grade-level groups of teachers analyzed various hypothetical written work samples from students engaged in fairly sharing circles and rectangles among various numbers of people. They were guided by a series of questions about what the students may know or not know about equipartitioning, what kinds of feedback they would provide to the student, and what next instructional moves they might take with the student. Later in the professional development, teachers met in cross-grade groups during their final meeting (Session V) to analyze two samples of written work from a Kindergartener, Emma, who had fairly shared a circle among three and four people. Their discussions were guided by questions similar to those in the previous written work analysis activity. The third source was an Interview Analysis activity that immediately followed the Session V Student Work Analysis activity. Teachers viewed a clinical interview of Emma fairly sharing rectangles and circles among two, four, three, and six people. Before each task, the recording was paused and teachers anticipated how they believed Emma would complete the next task. After viewing the episode, the recording was again paused while teachers summarized what they believed Emma knew about fair sharing and predicted her approach for the next task.

Participants' written responses to the three activities served as data for the analysis. In addition, video and audio recordings of each of the three activities were viewed to identify critical moments (Powell, Francisco, & Maher, 2002) for transcription. Written responses and the transcripts were coded for teachers' observations, interpretations, and conjectured responses. Analysis was conducted using the constant comparison method (Glaser, 1992), allowing for the identification and distillation of patterns within each of these categories.

Results

Observation

An analysis of the instances of observation indicates that experience with the EPLT during the professional development *sensitized* teachers to students' behaviors and verbalizations that were indicative of particular approaches to instructional tasks. In the Session V Student Work Analysis activity for example, the work samples had been inscribed by the student with "1 = 3" on each unequal-sized part when using vertical lines to share among three people and "1 = 4" on each fourth when sharing among four. Some of the teacher groups noticed this inscription and rather than dismissing it as mathematically incorrect, the teachers used this as evidence to further refine their understanding of Emma's thinking about equipartitioning. As one teacher in a group stated, "I think she's trying to say one of the four instead..." The teacher's comment may indicate that she believed that Emma was attempting to meet the three equipartitioning criteria by exhausting the whole and making the correct number of equal-sized parts as described by the EPLT.

Also, the teachers were sensitive to students' verbalizations. Recall that the interview viewed by teachers depicted Emma creating written work for sharing among three that the teachers had just examined. During the Interview Analysis activity, many of the teachers commented on Emma's utterance, "it's bigger than you think," believing that this indicated Emma's understanding of the need for equal-sized parts when sharing among three but her inability to create them on a circle. Drawing on their understanding of the three equipartitioning criteria and the task parameters, one teacher stated, "Odds, such as three, seem to be more difficult and circles seem more difficult than squares. She knew it was thirds, so she understood parts, but she didn't know how to get them equal. Saying 'it is bigger than you think' implies she knew it [one of the parts] wasn't big enough." With the EPLT as a model of student cognition, the teachers were sensitive to evidence in students' work and words when observing students and analyzing written work.

Interpretation

The data also suggest that teachers were able to *locate* a particular student's understanding within a range of understanding described by the EPLT. They used their knowledge of the trajectory to describe the ideas a student was likely to already understand and which ideas were likely to develop next. For example in the Interview Analysis activity, when asked to anticipate how Emma would share a rectangular cake among three people, one teacher recalled the increasing difficulty of splits from the EPLT and stated, "The student is not sure how to divide the cake to make equal parts when there are an odd number of pieces. She understands how to divide the cake if it cut into an even number of pieces." The teacher used the EPLT as a model of how students progressed through increasingly difficult equipartitioning tasks together with her observations to locate Emma's understanding within this range.

In the Session III Student Work Analysis activity, a group of Kindergarten teachers considered a sample of student work on equipartitioning a circle into three parts. They discussed:

T1: The student was asked to share a key lime pie among three people. What do you think the student understands?

T2: They understand a half - they started with the half. They knew they had to have three pieces… so they halved the other one, so…

T3: The evidence is they had a line down the middle. And they have another line to create three pieces.

T1: [writing] "…has a diameter cut and another cut."

T3: What do you think the student does not understand?

T2: Equal parts.

T1: He doesn't understand how to cut a circle into thirds.

The teachers establish that the student can halve and that he or she may understand that there needs to be a one-to-one correspondence between the number of pieces created and the number of people sharing the cakes but is still in the process of coordinating the three equipartitioning criteria when creating thirds. In these and other examples, teachers used the EPLT when interpreting evidence to situate a students' conception in terms of what the student was likely to already know and what the student might learn next.

Response

Suggesting next moves. In many instances, teachers were able to make decisions about their next instructional steps based on their location of student's conceptions along the EPLT. One type of instructional decision involved altering the parameters of a task, such as changing the shape from a circle to a rectangle in order to focus the student on creating a particular number of equal-sized parts. For instance in the Interview Analysis activity, teachers were asked what they would do next with Emma after she did not create equal-sized pieces when sharing a circular cake among three. One teacher wrote, "I would work on fair sharing from the stand point of equal. I would show Emma how to divide for a square, rectangle, then circles." Another teacher suggested, "…have her practice sharing with three and six people – she would catch on to this quickly. She needs other ways to share than just halving."

Similarly in the Session III Student Work Analysis activity, teachers were asked what they would do next with the student based on the work sample. Some of the groups of teachers suggested changing from a circle to a rectangle as their next move or changing the number of people sharing from six to four to decrease the difficulty of the task. For example, when suggesting next moves for the student whose work indicated that he or she made four equidistant cuts on a rectangle (creating five parts) while attempting to share among four people, one group discussed:

> T1: Well, you could ask, "do you have four pieces?" He's going to say, 'No, you have five,' and then…
>
> T1: What are you going to do about that last piece? How are you going to make it equal? How will you divide it fairly?
>
> T2: That kind of answers the next question, too.
>
> T3: What will you do next with the student? Let them eat cake.
>
> T3: Would you? Give him a new cake? Or slab of play dough or whatever
>
> T2: And say try again and see if you can divide this, all of the cake, between four people without a left over piece.

Other teachers suggested revisiting previous proficiency levels of the EPLT when planning follow-up activities. During the Interview Analysis activity, one teacher recommended, "Go back to sharing cookies, gold, etc. [collections] with 3 people." Similarly, another teacher wrote, "I would give her objects like counters and ask her to fair share among different numbers of people." Here, the teachers were recommending that the next instructional task for the student should be aligned with a lower proficiency level of the EPLT. As a model for thinking and learning, the EPLT not only assisted teachers in observing and interpreting but also suggested instructional steps to move a student from their current understanding to the next as described by the trajectory.

Predictability. Also based on how they located a student's understanding along the trajectory, teachers were able to anticipate what evidence they might see within the student's actions, work, or words on subsequent tasks. That is, the EPLT allowed teachers to make predictions as to how a student might approach a new task. For instance, after each segment of the video recording of the Interview Analysis activity, teachers were asked to predict how Emma would solve more difficult equipartitioning tasks, and many of the teachers suggested strategies described by the trajectory; in particular, creating six equal-sized parts of a circle tends to be less difficult than creating three (Confrey et al., 2010). After viewing a clip of Emma unable to share a rectangular cake fairly among three, one teacher wrote, "I believe Emma will be able to share among six people based on the attempts for three. She begins with cuts in half which makes her chances better since six is an even number." Later, after viewing segments of Emma's attempts at sharing a rectangular cake among three, six, and at sharing a circular cake among three, another teacher predicted this about Emma's sharing of a circular cake for six:

> She will cut the cake in half diagonally and then again to create 4 parts. She will then draw a line through the middle to make unequal-sized parts. I think she'll do this because she sees things in halves and only makes cuts that result in halving the section. Emma will make a diagonal cut through the cake and will probably cut though the mid-line again to make six sections. If she can't make six sections (on a rectangle), she probably won't be able to do six (on a circle).

In both examples, the teachers used their observations and interpretations in relation to the ELPT to anticipate the way that Emma would approach the next task. Thus, the sensitivity and ability to locate a student's understanding within the EPLT afforded by the model of cognition also provided teachers with the ability to predict of how students would engage in a task. In turn, these anticipations would likely sensitize teachers to evidence of understanding on subsequent tasks.

Discussion

One promise of learning trajectories for teachers is the "ability to respond appropriately to evidence of their students' differing stages of progress by adapting their instruction to what each student needs in order to stay on track and make progress toward the ultimate learning goals" (Corcoran et al., 2009, p. 19). In this paper, an LT was taken as a model for student cognition and learning using the Assessment Triangle framework to investigate teachers' observations and interpretations of evidence as well as the ways in which they might adapt instruction as a result. As depicted in Figure 2, LTs may sensitize teachers to evidence of particular ways of thinking within a domain of knowledge and assist them in locating a student's understanding within a range of conceptual development through informing their interpretations of evidence. Additionally, these "models" of student thinking (Wilson, Mojica, & Confrey, 2010) inform teachers' instructional adaptations in response to evidence. Two ways that this might happen that are indicated by this study include: (1) the suggestion of targets for subsequent instructional activities believed to help the student move along the trajectory; and (2) the ability to predict a student's approach to a task, supporting the anticipation of more evidence for assessing a student's understanding. Thus, this paper offers an elaboration of the Observation and Interpretation vertices when an LT is taken as the Cognition vertex. Observations are enhanced as the LT brings sensitivity to evidence. Interpretations are informed by the LT in a way that locates a conception along a progression of growing understanding. Moreover, the findings begin to specify what it means to respond to evidence with knowledge of an LT.

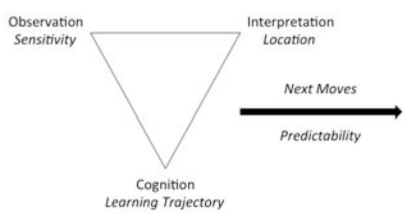

Figure 19. Results in relation to the Assessment Triangle (NRC, 2001).

Classroom assessment, the focus of the teachers' learning activities in this study, when used to guide instruction is one type of formative assessment, a practice widely accepted as a means of increasing student achievement (Black & Wiliam, 1998). In *Learning Progressions: Supporting Instruction and Formative Assessment*, Heritage (2008) describes eliciting evidence as one of several elements of formative assessment for which LTs are "foundational." She states, "When formative assessments arise spontaneously in the course of a lesson, interpretations of how learning is evolving can be made based on the trajectory of learning represented in the progression" (p. 6). This study offers a look at the classroom assessment practices of student observation and analysis of written work as two examples of "spontaneous" formative assessment and describes the role that an LT may play in reasoning about how student learning is progressing based on evidence.

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LESSON PLANNING INFLUENCES: TESTING AS A MEDIATING ASPECT

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This study investigates how teachers plan lessons and what mediates their lesson planning and ultimately how they teach. Four elementary teachers were observed planning and teaching elementary lessons. In addition, they were interviewed during this process. The findings revealed that standardized testing strongly influenced the lesson planning process and even the teaching style. Three teachers focused on teaching skills and algorithms for the test, as opposed to teaching for conceptual understanding. In this paper, we propose a theory that teachers followed a testing trajectory as opposed to learning or teaching trajectory.

Background

Lesson planning is a component of the cyclical process of teaching (Carlson & Bloom, 2005) that occurs when teachers make decisions about what they will teach and how they will go about delivering the lesson content (John, 2006). This process allows teachers to make decisions about future learning based on past student performance, with the intention of promoting student achievement (Bage et al., 1999). To plan effective lessons, teachers should take time and think about their learning goals and outcomes for students (Morris, Hiebert, & Spitzer, 2009). The process of planning lessons is a key component to delivering effective lessons; however, limited research exists on this topic. As a result, it is important to thoroughly analyze the lesson planning process to gain an understanding of teacher planning practices and to understand how teachers use lesson plans as they teach.

Theoretical Perspectives

The purpose of lesson planning is to support the learning of children (Bage, Grosvenor, & Williams, 1999). Teachers should design lessons that will help students achieve learning goals by utilizing their knowledge of content and pedagogy (Panasuk & Todd, 2005; Morris et al., 2009; Simon, 1995). In the process of designing lessons, teachers decide what will happen during their teaching and they make instructional decisions that influence the lesson that takes place (Bumen, 2007; Gilbert & Musu, 2008; John, 2006; Panasuk & Todd, 2005). All of the decisions teachers make during the lesson planning process have the potential to affect the student learning that takes place in the classroom (Borko, Shavelson, & Stern, 1981). When teachers consider elements of effective teaching strategies as part of the lesson planning process student learning is increased (John, 2006).

Teachers may consider a multitude of issues when planning lessons, but often influences beyond the teacher's control play a role in shaping the lessons that are planned (Regis, 2008; Shavelson & Stern, 1981). These influences can originate at the national or state level; however, many of the influences that affect day-to-day planning are due to decisions made at the district or school level. As a result, teachers' original ideas are influenced by constraints and affordances, often beyond their control. As a result, understanding how teachers plan, the influences they encounter, and why they make specific decisions, provides insight into how lessons are constructed.

This information is important for teacher preparation programs and for administrators seeking to assist teachers in improving student learning. Additionally, this information is important for future student achievement because lesson planning is related to the learning that takes place (Panasuk & Todd, 2005).

Educators should understand how teachers are planning elementary mathematics lessons to know the affordances and constraints in the planning process, and to understand how teachers' considerations are shaping elementary mathematics lesson plans. Additionally, because lesson plans serve as a guide for teaching (Shavelson & Stern, 1981) and teaching influences learning (Franke et al., 2007), it is important to understand how teachers are making decisions in this process. As a result, this study addresses the following question related to lesson planning:

5. How do elementary teachers plan mathematics lessons?

Method

A multiple-case design of case study (Yin, 2009) was used to explain how four, fourth-grade teachers from the same school made decisions in the lesson planning process, why they made these decisions, and to discover the mediating influences of these decisions. This approach allowed for an in-depth study of the participants, their institutional contexts, and their practices. Case study was utilized to optimize the understanding of the situation by providing data to answer the specific research questions (Stake, 2005). The teachers worked at a school that had made Adequate Yearly Progress (AYP), according to state standardized tests, in mathematics for the past three consecutive years. The school district had adopted Everyday Math as the curriculum of use seven years prior to the study. Of the four teachers in the study, Amie was a first year teacher and Bonnie, Carla, and Dana had each been teaching for about twelve years.

Data was collected through interviews, written lesson plans, direct observation, and physical artifacts. All audio recordings were transcribed and data was analyzed using a grounded theory approach with the purpose of describing the practices of teachers and adding to the theory related to the topic (Corbin & Strauss, 2008).

Results

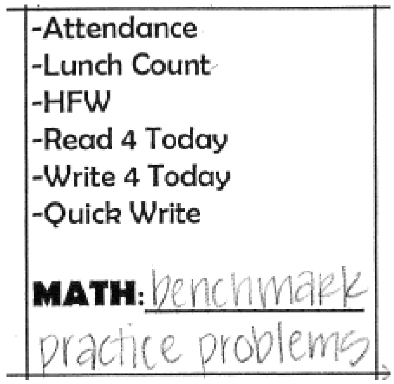
Even though the data for this analysis was collected during the end of the school year after the high stakes state CRT test, teachers continued to consider the high stakes assessments throughout their planning practices. In addition to the state mandated CRT examinations, students in the district where the data was collected annually took three District Benchmark Examinations, often referred to as "Benchmarks." The Benchmark tests were intended as formative assessments for teachers. However, the teachers viewed these exams as a summative assessment. The results from these examinations were entered into data analysis software and administrators and teachers were able to analyze results based on individual teacher, individual student, individual standard, or question type. The CRT and the Benchmarks tests influenced planning and teaching in the ways described below.

District Level Influences

The school district influenced the lesson planning process by publishing district-wide Benchmark assessments, adopting specific curricula, promoting the use of state standards in mathematics, and allocating resources.

Amie

Amie planned lessons to help students do well on the district Benchmark test, by planning to teach content that would be on the district Benchmark test. Her written lesson plan is evidence of this action (see Figure 1).


-Attendance
-Lunch Count
-HFW
-Read 4 Today
-Write 4 Today
-Quick Write

MATH: benchmark practice problems

Figure 1: Amie's written lesson plan for test preparation.

Amie devoted entire lessons to practicing problems similar to the questions that would be asked on the tests.

Carla

Carla was influenced by district standards as she worked to prepare students for the state test. When lesson planning, she focused on the district standards because the content in the standards was the content that would be tested. She was asked about lesson planning influences in mathematics.

> Researcher: District and state and national standards and policies?
>
> Carla: Uh, the, well, the standards, based on the fact that the kids are tested on the standards. So, I do, again, in long-range planning, make sure that those skills, I have covered in the classroom and that they will be tested on them.

Carla considered the standards as a set of skills that must be taught because the students would be tested on the standards. Therefore, the district standards mediated Carla's decision making while planning because she recognized that the plans she made should be based on the skills students would need for the tests.

The planning practices of all of the teachers were influenced by the district adopted standards in mathematics. These standards were used to write the district level Benchmark assessments and teachers were incorporating considerations for these tests into their planning. As a result, decisions made at the district level regarding standards and testing influenced teacher lesson planning.

Levels of Planning and Sequencing

The teachers used the Benchmark to guide their planning on a yearly level, so they were sequencing lessons over the course of a year based on the Benchmark topics. There were three Benchmark exams, so the teachers would spend the first third of the year teaching the content that was on the first Benchmark exam. They spent the second portion of the year teaching content that would be tested on the second Benchmark and the third portion of the year was spent teaching the content that would be on the third Benchmark exam. As a result, the test content determined how teachers planned on a yearly and daily level because the teachers were continually referencing the Benchmarks to determine the sequencing of mathematical topics within the context of the school year.

Carla

Carla looked at specific test questions and made lesson plans that aligned with the specific questions that would be asked, as opposed to following the Everyday Math curriculum.

> Researcher: Okay. How do you determine what in Everyday Math you are going to use or not use?
>
> Carla: Mmm, hmm. I (pause), umm, one, I look to see if it is even an idea that I like. For example, you know, finding the area of your skin. I, forget that. I am just like, "What are you freaking kidding me?" Because that's like, that activity is not how they test area on the test. They need to know the basic formula of a square or a rectangle to do that. So, activities like that, I am like, we, they take so long first of all. We don't have the time to be wasting on an activity that is not relevant to what we are going to be looked upon. Umm, so that's part of it. And, some of it, if it's, yeah, if it's just not what I would consider a worth-while activity—it is going to take forever, I can't see how the activity is going to relate to something that they are going to be tested on, then, it's uh, no. I don't do it. I know it probably develops that higher level thinking, but counting boxes on your hand is, I just, you know, or you measure the distance around your head. And, I know they are trying to get that manipulative, like get up and move, but it's, it's not, it's not worthwhile. It is not worthwhile. You know, there are four lessons in that fourth grade book on compasses. Well, you know, I throw that out as kind of like a center activity. Why am I going to waste four days on compasses when compasses does me no good anywhere else in the whole year?

Carla focused on the exact skills students would need for the test. As a result, when the reformed curriculum included a lesson to build conceptual understanding by measuring area of skin, Carla did not teach the lesson because fourth graders "only have to find the area of rectangles and squares on the state test." She made this decision even though she knew the lesson "develops that higher level thinking" because she considered whether or not the lessons seemed "worthwhile" as they related to the test. Therefore, the CRT examination and Benchmark examinations mediated Carla's curriculum decisions.

Dana

The Benchmark assessments influenced how Dana planned the order of mathematical topics she would teach. She considered the order in which the standards would be tested on the Benchmark Examinations and follow the Everyday Math order for those tested topics.

> Researcher: And how do you determine the exact content of the lesson you will teach?
>
> Dana: The standards. Yeah. I will pull….
>
> Researcher: Is there any specific order?
>
> Dana: I try to follow the standards as they are in the Everyday Math because they follow the Benchmark, the test that the kids take once every trimester, so I do try to stay, umm, up to whatever the standards are in the order they are presented in those tests.

Dana viewed the Benchmark as containing material that was in the standards, and she determined content based on the standards. Therefore, she referenced the Benchmark assessments to determine the order she would follow when sequencing lessons.

Curriculum Use

After the original adoption of Everyday Math, the teachers reported that they used the curriculum with fidelity; however, seven years after the adoption, the teachers were heavily supplementing, replicating lessons from others, and inventing lessons to attempt to have to their students do well on standardized tests. The CRT and Benchmark examinations mediated curriculum use because teachers only taught curriculum content that was tested because they wanted students to do well on the test. Therefore, material in Everyday Math was taught if it aligned with the testing standards. Furthermore, the teachers' desire for students to master test content mediated curriculum use because the teachers would repeat the content of a lesson until students had reached mastery, thus disregarding curricular pacing to continue with subsequent lessons.

Bonnie

Bonnie's fourth grade lesson planning was mediated by the fifth grade teachers' preference for teaching traditional algorithms, so that students would not be confused by multiple solution strategies. The fifth grade teachers at the school requested that Bonnie, a fourth grade teacher, teach traditional algorithms to her students, even though the district adopted Everyday Math taught multiple ways to solve problems, such as the lattice and partial quotients approach.

Researcher: Do you work with anyone when you plan your lessons?

Bonnie: Usually no. I mean sometimes we will discuss what we are doing with math. There was a big thing a couple of years ago where the fifth grade teachers were getting frustrated because the kids were going to them using lattice and like partial quotients and the varying algorithms that are taught in the Everyday Math program and that is not what our fifth grade teachers are teaching them, so the kids have no idea, as to, you know what they were talking about and weren't prepared when they were going there. So, there was kind of a consensus between fourth and fifth grade a couple years ago that we would teach them the traditional methods of multiplication and division as opposed to the other. Now, I know there are a couple teachers who still might teach the partial quotients, but I don't think anybody is still teaching the lattice. But, we try to, we try to listen to what fifth grade needs, as far as what they need to have, you know what their students need to be able to do, and be prepared for coming into fifth grade and we kind of do the same thing for coming into third grade. You know, as far as what those students, what we expect of them when they are coming in to fourth grade. Although, you know, that is not always you know, happening, so….(laughs).

The fifth grade teachers requested that Bonnie discontinue teaching multiple methods for multiplication and division, so that the students would be familiar with the traditional algorithms when they moved up to fifth grade. These teachers were focused on students getting correct answers, as opposed to helping student understand various mathematical processes. As a result, Bonnie reduced the conceptual focus of multiplication and division lessons by eliminating curriculum components at the request of the vertical grade level teachers. She disregarded the reformed curricula and chose to teach traditional algorithms for solving problems so that students would be procedurally sufficient at algorithm use for testing.

Dana

Dana planned for a teaching sequence that would take students from the use of a manipulative to demonstrate a procedure to more abstract representations of the procedure through the use of paper and pencil, while basing her yearly plan on the Benchmark examinations. Dana's knowledge regarding concrete and abstract learning influenced her planning because she considered the sequence of lessons for students along the concrete to abstract continuum.

Researcher: Okay, can you describe what you are teaching today?

Dana: Yep. So, today, moving from, I am taking the area from that irregular object, being their hand print, moving toward the formulas of the squares and the rectangles, and I will show them a model of a square foot, a square inch and a square centimeter and then we will

> pull in different, as a modeling, pull in different sized things to show them which would be the best thing to use for each item. Should we use the square foot for this? Should we use the square inch for this or the centimeter? And then show them, and then I will have lots of square inches cut up and take like a textbook and show them. Well, we can measure that with square inches and then lay them all out and show them how cumbersome that is and how long that takes and then try to guide them toward what would be a better way to do that, a faster way to do that, and then lead them into the measurement and then they will break up into partners again and then go out and measure the area of different squares and rectangles that they find in the classroom.

Dana considered how she would take students from the concrete representation of square manipulatives to determine area, and lead them to constructing the algorithm for area. Her purpose with using the squares was to make the point that finding the area with manipulatives is *cumbersome* and that the algorithm should be used instead. Consequently, the model she used was not conceptually based, but was focused on demonstrating the need for an algorithm to find the area. Dana's conceptualization of taking students from concrete to abstract was a focus on concrete representations that lead to memorized algorithm use, resulting in a skill focus for the context.

On a daily level, Carla and Dana acknowledged the learning process of taking students from manipulative use to paper and pencil practice, so they utilized this focus to sequence their lessons within a topic. However, their purpose of using this format was to lead students to the use of a specific algorithm to solve their problems, so that students would be proficient with skills when tested. The concrete representation was not viewed as a tool for building conceptual understanding, but was rather used to show students how "cumbersome" a model can be, which encouraged students to disregard the model and place their emphasis on the correct algorithm.

The teachers' use of the Everyday Mathematics curriculum decreased over the prior seven years and had been replaced with the incorporation of multiple supplemental materials that the teachers considered to align with the standards and would prepare students for tests. These replacement materials focused on skill mastery through repetitive practice of similar problem types. Bonnie disregarded the suggested use of Everyday Math altogether and incorporated a text that was not reformed based and focused on mastery. The changes the teachers made prioritized skill practice over conceptual understanding and all content was determined and sequenced based on testing content.

Discussion

The institutional context afforded a heightened emphasis on proficient test scores over the past several years. Specifically, in the state where the study took place, administrative pressure focused on increasing student achievement on the CRT examinations. The school district where the study took place published Benchmark examinations that teachers considered to be guidelines for how students would perform on the CRT examinations. As a result, the context of the study gave rise to teacher pressure for students to achieve high scores on both the Benchmarks and CRT examinations.

Due to the testing pressure, the teachers adjusted their teaching practices in an effort to improve student achievement. When reflecting on their individual teaching practices, and their students' achievement, the teachers believed that the adopted curriculum was not meeting the testing demands because a lack of focus on mastery. As a result, the teachers only taught pertinent curricula components, specifically content related to what would be tested. The teachers influenced student learning by focusing on specific curricula components over other curricula components (Remillard, 2005). In this study, teachers completely disregarded curricula components that they considered to be irrelevant for the standardized testing. Even though curricula guides what teachers teach (Remillard, 2005), the testing pressure on the teachers was so extensive that the teachers began to slowly eliminate all use of the adopted curricula.

While the changes in testing emphasis were taking place, teachers were faced with new demands they had not experienced in the past. Throughout this time, the teachers did not have district or school support focusing on using the Everyday Math curriculum to meet the testing demands. The only professional development on the curriculum had been conducted seven years prior, when the curriculum was purchased. As a result, the teachers were not confident in the ability of the curricula to meet the recent testing demands. Therefore, the teachers began using supplements for the majority of their lessons and they relied very little on the adopted Everyday Math curriculum. The elimination of the curriculum brought with it an increased emphasis on skill mastery. The teachers sought to incorporate curricular materials that would prepare students for testing by exposing them to repeated practice of skills that would be necessary for testing.

The supplements the teachers decided to use were focused on skill and practice through repetitive algorithm use. The teachers began using test preparatory practice in their lessons and taught lessons that were based solely on test content. Supplements became the main source of lesson content because teachers considered supplemental programs to better prepare students for the high stakes tests.

All four teachers in the study made decisions when planning that they believed would be the most beneficial for supporting student achievement. They were faced with challenges, such as preparing students for testing, and were forced to make decisions about what they would teach. These challenges were a result of the institutional context (Cobb et al., 2003) that fostered an environment with an emphasis on testing and student achievement on the tests. As a result, the teachers made the decisions they considered would be the most beneficial for the students and worked to prepare students for the assessments. The lesson planning that occurred was a result of the compensation and considerations the teachers made within the institutional context where they taught. These teachers were following a testing trajectory and making decisions based on the examinations, as opposed to following a learning trajectory (Simon, 1995).

Implications

This study highlights the need for professional development to support teaching for conceptual understanding. This means that teachers need to be familiarized with teaching concepts and need specific direction on how to teach for conceptual understanding. Likewise, it is important that teachers understand the benefits of making connections for students between and among concepts (Bransford et al., 2000) with the intention of forming links in student learning.

Additionally, it is important for teachers to understand how to use their curricula to meet the learning goals they have for students. Teachers with reformed curricula need support beyond the initial implementation of the curricula. Furthermore, when faced with changes in the institutional context, such as a heightened testing awareness, it is imperative that administrators are cognizant of the changes and provide teachers with support.

Finally, the study indicates a need for teacher support to prepare students for testing. The teachers made accommodations they considered would be helpful, but they did not have direction or support when making changes. As a result, preservice teachers should be familiarized with the pressures they will face when teaching and should be provided with support so when they begin teaching they are prepared to face the testing challenges. Practicing teachers also need support with how to prepare students for the examinations. This support should not be focused on skill and practice, but should instead highlight the importance of teaching for conceptual understanding.

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TOWARD THE IDENTIFICATION OF MATHEMATICAL HORIZON KNOWLEDGE OF PLACE VALUE CONCEPTS

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Horizon knowledge is still a relatively new and unexplored domain of mathematical knowledge for teaching. In preparation for an empirical study to take place in February and March of 2011, this paper will analyze research on student learning and teacher education to identify potential horizon knowledge as it relates to place value concepts.

Introduction

In the 1980s, Lee Schulman (1986) proposed a theoretical framework for conceptualizing the kind of knowledge needed for the practice of teaching. It has not been until the past decade, however, that researchers have empirically observed and designed measures to identify this kind of knowledge in teachers of mathematics (Ball, Thames, & Phelps, 2008). Ball et al. (2008) have identified six domains of subject matter and pedagogical content knowledge that accommodate a wide range of specialized *mathematical knowledge for teaching* (MKT). In their work to refine Schulman's framework for use in mathematics, Ball and colleagues provisionally included a category of *subject matter knowledge* (SMK) specific to the practice of teaching called *horizon knowledge*, that is neither common nor specialized mathematical knowledge (Ball et al., 2008; Ball & Bass, 2009).

At the present time, many important issues regarding horizon knowledge have yet to be addressed by the mathematics education research community (Ball & Bass, 2009). The purpose of this article is to contribute to both the emerging conceptualization of horizon knowledge and the development of a conceptual framework that will support an empirical study of horizon knowledge, specifically as it relates to place value concepts.

Theoretical Framework

Research supports the claim that possession of mathematical knowledge is a fundamental part of MKT (e.g., Ma, 1999). In her study of primary teachers' knowledge of elementary school level mathematics, Ma (1999) described and identified SMK called *profound understanding of fundamental mathematics* (PUFM). She concluded that the elementary teachers in her study possessing PUFM had developed it over the course of their teaching practice and that it was only present in teachers with substantial teaching experience. However, the development of this kind of knowledge does not occur easily nor does it necessarily happen at all, even in the presence of experience (Ball, 1990; Ball & Bass, 2000; Ma, 1999). How to measure and develop this knowledge in teachers, therefore, remains a significant challenge for researchers and educators.

In some cases, especially in studies using the number of university courses taken as a proxy for mathematical knowledge, advanced mathematical knowledge may not even have a significant effect on teaching knowledge (Fennema & Franke, 1992; Huang, Li, Kulm, & Willson, 2010). Only relatively recently have changes in methodology have allowed researchers to observe that teachers' mathematical knowledge does impact pedagogical decisions and by extension, must impact learning, although direct connections between teaching and learning remain unclear (Fennema & Franke, 1992, 149; Fuson et al., 1997).

The framework developed by Ball et al. (2008) has made a significant contribution to our conceptualization of MKT because it emphasizes knowledge of pedagogy, a component of teaching knowledge elevated by Schulman in the 1980s, in addition to SMK. Through empirical observations, Ball et al. (2008) have identified six domains of subject matter and pedagogical content knowledge. Five of these domains, horizon content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum, are specific to the practice of teaching while the sixth domain of MKT, common content knowledge, describes mathematical knowledge that is not unique to the practice of teaching but may be used in a variety of professional and/or nonprofessional settings (Ball et al., 2008).

Critics have argued that the MKT framework proposed by Ball and et al. (2008) ignores global mathematical knowledge in favor of isolated content knowledge (Clay, Silverman, & Fischer, 2010). At the time they proposed their framework, however, Ball and colleagues provisionally included a category of SMK specific to the practice of teaching, called horizon knowledge, which they described as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., p. 403). According to Ball and Bass (2009) their most recent conception of horizon knowledge has four constituent elements:

1) A sense of the mathematical environment surrounding the current "location" in instruction.
2) Major disciplinary ideas and structures
3) Key mathematical practices
4) Core mathematical values and sensibilities.

Although horizon knowledge is purely a mathematical domain of MKT, in practice, it is likely to have a significant influence on pedagogical decisions (Ball, 1993; Ball & Bass, 2009; Ball et al., 2008). As a result, no one is sure "how far out or in what direction the pedagogically relevant and useful horizon extends," nor does anyone "know how horizon knowledge can be helpfully acquired and developed" or "have ways to assess or measure it" (Ball & Bass, 2009). The goal of the study presented here, therefore, is to examine the extent to which available research on teaching and learning can guide researchers' conceptualizations of horizon knowledge and then utilize these conceptualizations to measure teacher knowledge.

In particular, a wealth of research on student learning, in addition to recent work on conceptions of prospective teachers, suggests that the area of place value is an important context in which to explore and define horizon knowledge and its pedagogical relevance (Baroody, 1990; Hiebert & Wearne, 1992; Mills & Thanheiser, 2010; Stacey et al., 2001; Thanheiser, 2009). Place value concepts arise in a variety of topics in the K-8 curriculum including reading and writing numbers, rounding and estimation, addition and subtraction of natural numbers, standard algorithms for the four operations, ordering numbers, representations of rational numbers, and scientific notation. Further, place value knowledge is also fairly difficult to construct and takes place over an extended period of time (Kamii, 1986).

Because our notational system is the product of social conventions and logico-mathematical knowledge (the construction of mental relationships), children cannot discover place value concepts (Kamii, 1986; Verschaffel, Greer, DeCorte, 2007). Place value refers to a system of notation in which the relative positions of the symbols is used to encode the value of a number. In the decimal place value system, the value of a numeral is given by the sum of the value of the individual digits; the value of each digit given by the product of its face value with its place value. The symbols 0-9 and the words zero, one, two, … used to represent numbers are social conventions. Similarly, by convention it is determined that the value of each column increases from right to left and adjacent positions are related by ten-for-one trades. Fractional values are represented to the right of the decimal. Making sense of place value notation requires constructing mental relationships such as connecting the column position of a numeral with its value (Kamii, 1986).

Research shows that constructing these mental relationships can be difficult for both children and preservice teachers (e.g. Thanheiser, 2009; Kamii, 1986) Many of these difficulties arise as a consequence of forming a strong unitary conception of number without developing a multi-unit one such as evidenced by a failure to differentiate between the complete value of a number and face value of the digits (Fuson, 1990; Varelas & Becker, 1997). For example, often both children and preservice teachers, while capable of identifying that 16 represents 16 ones, will also indicate that the 1 in the numeral 16 represents 1 one instead of 10 (or 1 ten) (Hiebert & Wearne, 1992; Kamii, 1986; Resnick et al., 1989; Thanheiser, 2009).

In contrast, a multiunit conception would allow for simultaneously viewing 16 in terms of reference units such as 16 ones or 1 ten and 6 ones. Thanheiser (2009), however, found that only 20% of the preservice teachers in her study could reliably conceive of a number in terms of reference units. Further, 67% of the preservice teachers in her study displayed concatenated conceptions of number where most or all digits were conceived of in terms of ones regardless of the position of the digit. These concatenated conceptions of number also appear in research on student learning of operations. For example, expressing the sum 38 + 24 as 512 and sometimes calling the answer "fifty twelve" is a frequent and widespread error (Fuson, 1990; Hiebert, 1992; Hiebert & Wearne, 1992; Kamii, 1986).

Many researchers have suggested that the English named-value spoken system of number words makes it more difficult to learn base 10 place value concepts, especially because a majority of the English number names between 1 and 100 do not correspond directly to the written form of the number (Fuson, 1990; Fuson & Briars, 1990; Miura & Okamoto, 1989; Verschaffel, Greer, DeCorte, 2007). In contrast to verbally expressing the number 40 as "forty," for example, Asian languages would read more like "four-ten," making it more likely for a child to "construct number representations that reflect the Base 10 numeration system" (Fuson et al., 1997; Miura & Okamoto, 1989, p. 109). The irregularity of the decade names may also contribute to the tendency for American children rely on unitary conceptions of two-digit numbers (Fuson, 1990). Unitary conceptions of two-digit numbers appear to be used frequently by adults and children who may use the nonstandard language "twenty ten" to refer to the year 2010 or read 2500 as "twenty-five hundred" (Fuson, 1990).

Studies also show that decimal concepts are difficult for children and preservice teachers (e.g. Hiebert, 1992; Stacey et al., 2001). Sakur-Grisvard and Leonard (1985) found that errors occurred as children utilized three intermediate rules to compare decimals beginning with treating decimals as whole numbers (longer number is larger) and then moving on to rules that take into account the decimal point but still ignore place value. Later, Resnick et al. (1989) showed that these intermediate rules appear to have resulted as students attempted to integrate new material with their prior knowledge of whole numbers, fractions, and zero. Although the use of these intermediate rules declines with age, evidence suggests that errors involving "shorter-is-larger" conceptions may persist in preservice teachers (Stacey et al., 2001). Further, preservice teachers may be unaware of the "shorter-is-larger" conception that children may develop as a result of prior experiences with fractions (Resnick et al., 1989; Sakur-Grisvard & Leonard, 1985; Stacey et al., 2001).

According to Ball and Bass (2000), students may expect symmetry in the place value names about the decimal. However, while there *is* a naming symmetry about the ones place value position, a similar symmetry does *not* exist for value. As stated above, in our base 10 notational system, the value of each column increases from right to left and adjacent positions are related by ten-for-one trades. This is true for both whole numbers and decimal numbers. Resnick et al. (1989) found, however, that children who use whole number rules for comparing decimals (decimal with more digits is larger) also demonstrate errors related to naming symmetry. For example, for "6 tenths and 2 hundredths," children may write 0.026 or 2.6, both of which could correspond to thinking of the positions of the whole values, 6 ten*s* and 2 hundred*s* (Resnick et al., 1989). Finally, preservice teachers in the study by Stacey et al. (2001) displayed errors when comparing decimals with zero suggesting that "many preservice teachers do not understand the relationships among decimals, whole numbers, fractions, zero, and negative numbers" (p. 222).

Place value concepts develop over a long period of time and across a wide range of mathematical content. Further, studies also suggest that teachers and programs/curriculum sequences impact the kinds of strategies children utilize in problem solving (Fuson et al., 1997; Resnick et al., 1989). Therefore, pedagogically useful horizon knowledge should include mathematical connections that can be used to bridge or direct a student's current understanding toward concepts to be acquired in the future. I argue here that it is possible to both gain insight into the kinds of horizon knowledge that is pedagogically useful and devise ways to measure and assess that knowledge in teachers by studying the literature on student learning and teacher knowledge, such as explored above.

Methods

The approach I took to determine potential horizon knowledge involved two phases. In the first phase I conducted a search of the existing literature on the teaching, learning and preservice teacher conceptions of place value. To learn about preservice teacher knowledge I used the recent work by Thanheiser (2009), Mills (Mills & Thanheiser, 2010), and Stacey et al. (2001) who have examined preservice teacher knowledge related to place value and decimals. To locate work on teaching and student learning of place value I used Google Scholar and educational databases such as ERIC to search for keywords, for example "place value" and "place value decimal." Additional works were obtained by repeatedly examining and obtaining literature found in the reference lists and citations within the papers I already possessed, beginning with the above works on preservice teacher knowledge and the results of the database searches.

The second phase involved a mathematical analysis of the literature obtained regarding the teaching and learning of place value concepts. I began by narrowing the literature to areas in which the conceptions of place value for preservice teachers and students were well documented, as outlined in the literature review above. I then analyzed these conceptions for mathematical connections in an effort to determine horizon knowledge that had a potential to impact pedagogical decisions regarding place value across the curriculum.

This mathematical analysis was then used as a basis to design a pilot study to be conducted in February and March of 2011. The participants of the pilot study are approximately 11 preservice teachers enrolled in a mathematics methods course for K-5 offered at a large public university in the southwestern United States. Each preservice teacher was asked to conduct one-on-one problem solving interviews with elementary school children in which the preservice teacher posed various addition/subtraction and multiplication/division problems, assessed the child's strategies, and designed and discussed ways in which they could extend and support the child's mathematical thinking. The resulting paper written by the preservice teachers will be analyzed to determine the extent to which preservice teachers display applicable horizon knowledge in their discussion. The preservice teachers will then be interviewed to further explore the extent of the mathematical connections they can identify as being pedagogically relevant to supporting and extending the mathematical thinking of the child in their interview.

Results/Discussion

In this section, I discuss some mathematical connections between whole numbers and decimal numbers that may comprise pedagogically useful horizon knowledge for place value. One potentially powerful example of horizon knowledge for place value is related to the conventions that allow for making connections between whole numbers and decimals. For example, in the verbal expression "twenty-five hundred," the 25 occupies the hundreds place and violates notational conventions because it indicates that there is more than one digit in the hundreds place value position, as in $\underline{25}00$. A similar problem presents itself if we want to write the decimal representing "one hundred twenty five thousandths" where all three digits can be thought of as occupying the thousands column, as in $0.00\underline{125}$. However, for all decimals we know that the values of each column increase from right to left. Further, each column in the standard decimal representation is related by ten-for-one trades, which is also true of the digits already comprising 25 and 125. Therefore, these conventions together tell us to trade each digit left until they occupy their own column, as in 2,500 and 0.125.

A similar example of horizon knowledge arises out of the meaning of the decimal; the values of the columns appearing to the right of the decimal are fractional. Additionally, we know that the contrapositive of the above must hold true for both whole numbers and decimals, column values decrease from left to right. If we consider 3.5%, we can view it as "three and five tenths of a hundredth." This can be represented as $0.0\underline{3.5}$ using the underline decimal notation introduced above. In this case, the hundredths represented in this decimal contains both whole and fractional parts. According to notational conventions then, we have three whole hundredths and the fractional part is traded to the right of the (original) decimal for five thousandths, giving 0.035. This concept extends naturally to numbers that are represented using decimals along with reference units, as in 12.5 million or similarly, using scientific notation as in 1.25×10^7. In the first case we have $\underline{12.5},000,000$ which has 12 whole millions in addition to a fractional 0.5 (or half) million. In the second case we have $\underline{1.25}0,000,000$ or 1 whole ten million and 25 hundredths of a ten million. In both cases, the notational conventions tell us that wholes trade with larger values to the left and fractional values trade with smaller values to the right, resulting in 12,500,000 in both cases.

The impact of a lack of horizon knowledge such as that described above is illustrated by the observations that children who displayed whole number rule errors while comparing decimals were likely to represent "two tenths and five hundredths" as 0.052 or "three hundredths" as 0.300 (Resnick et al., 1989). This may suggest that the rules utilized by the children in the study emphasized superficial features of whole numbers such as tens are positioned to the right of hundreds or the number three hundred is followed by two zeros (Resnick et al., 1989). The above examples, however, show a variety of connections that have a potential for being pedagogically powerful because they emphasize whole number concepts that are not unique to whole numbers but also carry over to decimals.

In the above examples, common fractions were not considered directly because it is often more efficient to determine a decimal representation using a division algorithm. Now I propose another potential example of horizon knowledge of place value involving connections between division algorithms, decimals, and the interpretation of whole number division as partitioning.

For example, consider 6625 ÷ 53 and 66.25 ÷ 5.3. An inefficient way to partition 6625 is to "deal out" 1 at a time into 53 groups until the 6625 is gone and then count the number of 1s in each group. A more sophisticated way to partition would be to "deal out" to each of the 53 groups 100 at a time, then 10, and so on. As a result we see that 6625 contains *1* set of 53 *hundreds*, *2* sets of 53 *tens*, and *5* sets of 53 *ones*, or 53 sets of 125. Similarly, 66.25 can be partitioned into sets of 5.3. We know that 66.25 can be partitioned into *1* set of 5.3 *tens*, *2* sets of 5.3 *ones*, and *5* sets of 5.3 *tenths*, or 5.3 sets of 12.5.

As with many computational algorithms, division can be described completely in terms of manipulations of symbols and no reference to place value. For example, computing 6625 ÷ 53 might start out like "53 goes into 6 zero times, 53 goes into 66 one time…". Further, when considered purely as symbolic manipulations, the whole number algorithm does not provide any clue as to how the algorithm must be adjusted to accommodate decimals. Instead, long division with decimals may begin by memorizing additional rules such as rewriting 66.25 ÷ 5.3 as 662.5 ÷ 53, placing the decimal in the quotient directly above the decimal in 662.5, then following the procedure for whole numbers. However, mathematical meaning lies not in the rules of symbol manipulation but in the construction of connections between the "quantities and the symbols and rules that represent them" (Hiebert, 1992, p. 297). In the example above, explicit references to place value in both whole number and decimal division illuminates the division algorithm as a sophisticated form of partitioning and eliminates additional rules used to modify whole number division for use with decimals. Therefore, it is conceivable that possession of this horizon knowledge would encourage a teaching approach that helps students construct a conceptual understanding of division of whole numbers as partitioning with respect to reference units that would support a conceptually similar concept of decimal division.

Finally, the last example of horizon knowledge that I propose here is related to the connections between place value and infinite series. Further, this knowledge can also extend to mathematical habits of mind as described by Cuoco, Goldenberg, and Mark (1996) including a disposition toward description and looking for patterns. As we considered the algorithm above, implicitly we realize that the quotient, 12.5, of 66.25 ÷ 5.3 is given by the sum 10 + 2 + 5/10. By using the division algorithm to find the decimal representation of a nonterminating decimal such as 1/3, we can easily recognize its equivalence to the infinite series 3/10 + 3/100 + 3/1000 + …. Further, this example illustrates the power decomposing numbers in various ways can create an awareness of interesting mathematical connections.

For example, solving the standard Towers of Hanoi puzzle gives another glimpse the power obtained by flexibility in the representation of numbers. The standard Towers of Hanoi puzzle uses a playing board with three pegs and n discs with increasing radii stacked on one of the pegs. The goal is to move all n discs from one peg to another, without placing a larger disc on top of a smaller one, in the least amount of moves. Table 2 shows the minimum number of moves, m, to solve the Towers of Hanoi puzzle for $n = 1,\ldots,5$ discs along with several representations of each number of moves.

n	m		
1	1	$2^1 - 1$	2^0
2	3	$2^2 - 1$	$2^0 + 2^1$
3	7	$2^3 - 1$	$2^0 + 2^1 + 2^2$
4	15	$2^4 - 1$	$2^0 + 2^1 + 2^2 + 2^3$
5	31	$2^5 - 1$	$2^0 + 2^1 + 2^2 + 2^3 + 2^4$

Table 2. Minimum number of moves, *m*, to solve Towers of Hanoi with *n* discs.

By representing the number of moves in different forms, we not only illuminate a functional relationship, $m = 2^n - 1$, between the number of moves and the number of discs but we also find that $\sum_{k=0}^{n-1} 2^k = 2^n - 1$, the closed form for the (n – 1)th partial sum of the power series $\sum_{k=0}^{\infty} 2^k$.

In conclusion, the above analysis of research on student learning and teacher knowledge suggest that place value concepts are important across the K-8 curriculum. Further, horizon knowledge of place value that appears to be pedagogically useful may extend quite far, both in terms of central mathematical ideas and practices. However, while the range of horizon knowledge is potentially vast, research on student learning can help narrow our conceptions of what may comprise the core of horizon knowledge we desire to find or develop in those involved in the teaching profession.

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ELEMENTARY TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE, EFFICACY, AND PROBLEM SOLVING ABILITIES IN ALTERNATIVE CERTIFICATION

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The purpose of this study was to understand teachers' mathematical content knowledge, efficacy, problem solving abilities, and teacher beliefs in an elementary education mathematics methods course for special education teachers in alternative certification programs. Findings revealed a significant increase in mathematical content knowledge and teacher self-efficacy. Additionally, teachers were found to have high self-efficacy at the end of the semester and strong problem solving abilities. Further, teachers generally found that helping students with disabilities learn mathematics was the biggest issue in their teaching, and that the use of technology and manipulatives were the most important topics addressed in their learning.

The purpose of this study is to provide an understanding of teachers' mathematical content knowledge, self-efficacy beliefs, level of problem solving abilities, and teacher beliefs in an elementary education inquiry-based mathematics methods course for special education teachers in the New York City Teaching Fellows (NYCTF) and Teach for America (TFA) programs. Understanding teacher knowledge is important because it is directly related to student achievement (Hill, Rowan, & Ball, 2005). Self-efficacy is a teacher's belief in his or her ability to teach effectively and positively affect student learning outcomes (Bandura, 1986; Enochs, Smith, & Huinker, 2000), and an important component for successful teaching. Further, it is recommended that mathematics should be taught through a problem solving perspective (NCTM, 2000; Schoenfeld, 1985).

Problem solving continues to be of high importance in mathematics education (NCTM, 2000; Posamentier & Krulik, 2008; Posamentier, Smith, & Stepelman, 2006). It is one of the five National Council of Teachers of Mathematics (NCTM) standards (NCTM, 2000), and is critically important in how students best learn mathematics (Posamentier et al., 2006). The National Council of Supervisors of Mathematics (NCSM) has considered problem solving to be the principal reason for studying mathematics (NCSM, 1978).

In order to understand what problem solving is, first it must be understood the definition of a mathematical "problem." Charles and Lester (1982) defined a mathematical problem as task in which (a) The person confronting it wants or needs to find a solution; (b) The person has no readily available procedure for finding the solution; and (c) The person must make an attempt to find a solution. According to Krulik and Rudnick (1989), problem solving is a process in which an individual using previously acquired knowledge, skills, and understanding to satisfy the demands of an unfamiliar situation. Polya (1945), in his seminal work *How to Solve It*, outlined a general problem solving strategy that consisted of (a) Understanding the problem; (b) Making a plan; (c) Carrying out the plan; and (d) Looking back.

Understanding the level of elementary school mathematics teachers' problem solving abilities is critical in supporting them to teach their students from a problem solving perspective. For example, teachers are critically important in developing abstract thinking in students in the problem solving process. However, Cai (2000) has shown that sixth grade U.S. students rarely used abstract strategies in their problem solving. Strong problem solving abilities among teachers are needed if teachers are to teach mathematics well because content knowledge by itself, while being necessary, is not sufficient for good teaching (Ball, Hill, & Bass, 2005; Ma, 1999). The NCTM (2000) said, “Problem solving is not only a goal of learning mathematics but also a major means of doing so” (p. 52). If there is interest in good student problem solving, teachers need to be more than proficient in their own problem solving abilities.

The theoretical framework of this study is based upon the emphasis of the importance of content knowledge for teachers (Ball et al., 2005). Additionally it is founded on Bandura’s (1986) idea that teacher self-efficacy can be subdivided into a teacher’s belief in his or her ability to teach well, and his or her belief in a student’s capacity to learn well from the teacher. Finally, the NCTM (2000) and Schoenfeld (1985) have emphasized problem solving as a way of teaching though an emphasis on problem solving as an important process standard.

Background on New York City Teaching Fellows

The NYCTF program is an alternative certification program developed in 2000 in conjunction with The New Teacher Project and the New York City Department of Education (Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007; NYCTF, 2008). The program goal was to recruit professionals from other fields to supply the large teacher shortages in New York City’s public schools with quality teachers. Prior to September 2003, New York State allowed for teachers to obtain temporary teaching licenses to help fill the teacher shortage. Teaching Fellows generally are recruited to teach in high needs schools throughout the city (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006).

Background on Teach for America

TFA is a non-profit organization formed in 1990 with the intention of sending college graduates to low-income schools to make a difference for the underserved students. Its founder, Wendy Kopp, was herself a new graduate of Princeton University looking to do something more with her life after graduation (Kopp, 2003). She considered that many recent college graduates at America’s top universities would consider teaching low-income students if given the opportunity. The idea was that there should be a teachers’ corps that would allow new graduates at top universities with an interest in teaching to quickly begin teaching students in underserved communities.

Research Questions

1. What differences existed between teachers’ mathematical content knowledge before and after an elementary mathematics methods course?
2. What differences existed between teachers’ concepts of self-efficacy before and after an elementary mathematics methods course? Further, what level of self-efficacy did teachers possess at the end of the semester?
3. What level of problem solving abilities did new teachers have in an elementary mathematics methods course?
4. What were teachers’ beliefs about teaching and learning mathematics?

Methodology

The methodology of this study involved both quantitative and qualitative methods. The sample in this study consisted of 24 new teachers in the NYCTF ($N = 9$) and TFA ($N = 15$) programs. One third was male and two thirds were female. The teachers in this study were selected due to availability and thus represented a convenience sample. Participants were enrolled in an inquiry-based elementary mathematics methods course for special education teachers that involved both pedagogical and content instruction.

Teachers were given mathematics content examinations and self-efficacy questionnaires at the beginning and the end of the semester, and were given a problem solving examination at the end of the semester. The mathematics content examination consisted of 20 multiple choice items that measured knowledge of number sense, fractions, decimals, and percents (6 items); probability and statistics (4 items); measurement and geometry (5 items); and algebra (5 items), and was based on the PRAXIS mathematics examination (ETS, 2005). Possible scores ranged from zero to 20 points.

The self-efficacy questionnaire was the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) developed by Enochs et al. (2000), and measured concepts of self-efficacy. The MTEBI was a 21-item five-point Likert scale instrument with choices of strongly agree, agree, uncertain, disagree, and strongly disagree, and was grounded in the theoretical framework of Bandura’s self-efficacy theory (1986). The MTEBI contained two subscales: Personal Mathematics Teaching Efficacy (PMTE) and Mathematics Teaching Outcome Expectancy (MTOE) with 13 and 8 items, respectively. Possible scores ranged from 13 to 65 on the PMTE, and 8 to 40 on the MTOE. The PMTE specifically measured a teacher’s self-concept of his or her ability to effectively teach mathematics. The MTOE specifically measured a teacher’s belief in his or her ability to directly affect student learning outcomes. Enochs et al. (2000) found the PMTE and MTOE had Cronbach alpha coefficients of 0.88 and 0.77, respectively.

The problem solving examination consisted of five problem solving situations as adapted from the literature (Krulik & Rudnick, 1989; NCTM, 2000; Posamentier & Krulik, 2008; Posamentier et al., 2006). It is important that students be unfamiliar with the problems because in order to be authentic problem solving students should encounter unfamiliar situations with no immediate solutions available. Each item was worth two points and possible scores ranged from zero to 10 points. Sample problems were as follows.

> A bicycle dealer just put together a shipment of two-wheel bicycles and three-wheel tricycles. He used 50 seats and 130 wheels. How many bicycles and how many tricycles did he make if he used all seats and wheels available?
>
> Jim bought a new camera that cost more than \$30 but less than \$40. He paid for the camera with \$5 bills and \$1 bills. He paid with the same number of each. What did the camera cost?

Teachers were required to keep reflective journals on their teaching and learning over the course of the semester, which provided qualitative data regarding their beliefs about teaching and learning mathematics.

Results

The first research question was answered using the mathematical content knowledge test, and data were analyzed using paired samples *t*-test (see Table 1). The results of the paired samples *t*-test revealed a statistically significant difference between pretest scores and posttest scores for the mathematics content knowledge test, and there was a large effect size.

The second research question was answered using the MTEBI with two subscales: PMTE and MTOE, and data were analyzed using paired samples *t*-tests (see Table 1). The results of the paired samples *t*-test revealed a statistically significant difference between pretest scores and posttest scores for the PMTE, and the effect size was moderate. Further, the results of another paired samples *t*-test revealed no statistically significant difference between pretest scores and posttest scores for the MTOE.

Table 1
Paired Samples t-Test Results for Content Test and Self-Efficacy Test (PMTE and MTOE)

Assessment	Mean	SD	*t*-value	*d*-value
Content Test				
Pretest	75.00	16.151	-3.778**	0.85
Posttest	87.08	11.971		
PMTE				
Pretest	2.70	0.504	-2.575*	0.45
Posttest	2.91	0.478		
MTOE				
Pretest	2.69	0.691	-0.213	
Posttest	2.71	0.666		

$N = 24$, $df = 23$, two-tailed
** $p < 0.01$
* $p < 0.05$

Further, the second part of the second research question was answered using independent samples *t*-tests to determine if the participants had significantly better concepts of self-efficacy at the end of the semester as compared to a neutral value coded as “2” on the survey sheet (see Table 2). For the PMTE and MTOE the results of an independent samples *t*-test revealed a statistically significant difference between efficacy scores and neutral scores, and the effect size was very large.

Table 2
Independent Samples t-Test Results for PMTE and MTOE Scores

Assessment	Mean	SD	*t*-value	*d*-value
PMTE Actual Scores	2.91	0.478	-9.310**	2.69
Neutral Scores	2.00	0.000		
MTOE Actual Scores	2.71	0.666	-5.249**	1.50
Neutral Scores	2.00	0.000		

$N = 24$, $df = 23$, two-tailed
Equal variances not assumed.
** $p < 0.01$

Descriptive statistics were used to answer research question three. At the end of the semester teachers had a mean score of 8.54 out of 10 possible points on the problem solving examination with a standard deviation of 1.615, which represents the teachers' level of problem solving abilities.

The fourth research question was answered using data collected from the teaching and learning journals. The teaching journal was used as a reflection upon the teachers' actual teaching and classroom experiences. As part of the mission of the school of education where this study took place, teachers are encouraged to be reflective practitioners. Similarly, the learning journal was used as a reflection on the learning in the mathematics methods course.

Analysis of the teaching journals revealed that the most commonly addressed topic was special education issues in the classroom. Teachers addressed their struggles and successes of working with students with disabilities. The next most frequently mentioned topic was mathematics anxiety. Teachers not only addressed the anxiety their students experienced, but also the anxiety they themselves held as teachers of mathematics. Teachers also expressed concerns about a lack of conceptual understanding in their students. Teachers lamented the difficulty in helping their students learn mathematics for understanding, as well as the lack of understanding students possessed prior to entering the classroom. Finally, teachers often reflected upon the emphasis placed upon standardized testing in their schools and the need to prepare students for the examinations, and they resented the emphasis on standardized testing. Surprisingly, very few teachers reported difficulty with classroom management.

Analysis of the learning journals revealed that the most commonly addressed topics were the use of technology and manipulatives in the classroom. Teachers appreciated exposure to these learning tools and implemented them in their own classroom. Manipulatives used in class included base ten blocks, Unifix cubes, Cuisenaire rods, and the Tower of Hanoi, among others. Next, teachers most often addressed their conceptual understanding of mathematics content. Teachers in the methods course appreciated the focus on understanding mathematics since many of them lacked conceptual understanding themselves. Recall that the teachers reflected upon their concern that their own students lacked conceptual understanding in their teaching journals. The topic of differentiation and numeracy were frequently mentioned in the journals. Additionally, microteaching was a topic that many teachers thought enhanced their learning in the course. Every teacher was required to present a 10 minute micro lesson that involved a "hook" to get students interested. Many teachers found observing other teachers teach to be a very valuable aspect of this course. Finally, problem solving was often mentioned in the reflective journals. Teachers were given a "problem of the day" problem solving situation that they were to solve in groups at the beginning of each class.

Discussion

It was found that elementary teachers increased their mathematical content knowledge and self-perception of their abilities to teach effectively over the course of a one semester inquiry-based mathematics methods course for special education teachers while teaching in the own classrooms. Further, it was found that at the end of the semester the teachers generally had high concepts of self-efficacy both in terms of their ability to teach well (as measured by the PMTE), as well as their ability to positively affect student outcomes (as measured by the MTOE). Additionally, teachers had relatively high problem solving abilities. Analysis of the reflective journals revealed that teachers revealed that the most commonly addressed topic was special education issues in the classroom. Teachers reflected on their struggles and successes of working with students with disabilities. Teachers found that the use of technology and manipulatives were the most important topics addressed in their learning.

Over the semester there was an increase in mathematics content knowledge, which is attributed to the combination of focus on content and pedagogy in the mathematics methods course, along with the teaching experience the teachers in this study were gaining. Future studies should focus on the contributions to an increase in content knowledge from the coursework and teaching components, separately.

The results of this study are consistent with the finding of Palmer (2006) and Swars, Hart, Smith, Smith, & Tolar (2007), who found that there was an increase in self-efficacy in terms of elementary pre-service teachers' ability to teach well and their ability to positively affect student outcomes. However, Palmer (2006) and Swars et al. (2007) examined preservice elementary school teachers, while in this study teachers were in-service and enrolled in alternative certification programs. Also, Palmer looked at efficacy using the STEBI-B for science, and Swars et al. used the MTEBI for mathematics, as was done in this present study. Soodak and Podell (1997) also found that preservice teachers exhibited higher levels of self-efficacy than when they began teaching in the field, but through experience the level of efficacy rose again. The results of this study were somewhat inconsistent with the results found by Hoy and Woolfolk (1990), who found a significant decline in beliefs to positively affect student learning outcomes during student teaching, which had been attributed to teachers' exposure to the realities of the classroom. However, in this present study teachers had an increase in this belief despite early encounters with the realities of the classroom. Perhaps there is something different about alternative certification preparation that could explain this difference. Change in self-efficacy should be further investigated, especially for alternative certification teachers.

It was found that teachers had relatively high problem solving abilities. This is in contrast to findings in the literature (Ball et al., 2005; Ma, 1999), and commonly held perceptions of teacher problem solving ability (Paulos, 1990). The translation of strong problem solving abilities into the classroom should be further investigated, and it is suggested that alternative certified elementary school teachers be evaluated for their implementation of their problem solving abilities into the classroom.

Since the teachers in this study were special education teachers, it was not surprising that teachers most often reflected upon the challenges and successes they had with their special education students in their teaching journals. It was surprising, however, that very few teachers reflected on classroom management issues. Classroom management is often considered a top concern among new teachers (Costigan, 2004; Cruickshank, Jenkins, & Metcalf, 2006). This should be examined in future studies.

Teachers expressed concern about lack of conceptual understanding in their students and an appreciation for the emphasis on understanding in the methods course. The method course in which these teachers were enrolled had a strong emphasis on teaching for understanding. It has been shown that many U.S. elementary teachers lacked the content understanding needed to teach well (Ma, 1999), and that this should not be a surprise given that these teachers graduated from the same system that researchers wish to improve (Ball et al., 2005). Hence, a cycle of non-understanding persists. Nurturing conceptual understanding in teachers is important given the relationship between teacher knowledge and student achievement (Hill et al., 2005).

Teachers frequently mentioned differentiation and numeracy in their learning journals. The school of education where this study takes place has a strong emphasis on differentiated instruction. It should be noted that teachers in this class had been exposed to differentiation prior to enrolling in the method course. Teachers were required to read Paulos' *Innumeracy: Mathematical Illiteracy and its Consequences* (1990). In this book Paulos addressed what it means to be numerate (i.e. mathematically literate) in a democratic society.

Teachers often discussed problem solving in their learning journals. Problem solving as a way of teaching was thoroughly addressed in the course with considerable time devoted to problem solving in the mathematics classroom as recommended in the literature (NCSM, 1978; NCTM, 2000; Schoenfeld, 1985).

Ball et al. (2005) found that teacher knowledge is an important variable in preventing the achievement gap from growing in high need urban schools. Additionally, self-efficacy impacts student learning (Soodak & Podell, 1997). As earlier referenced, it is possible that beliefs about self-efficacy may be a greater variable in quality teaching than content knowledge alone (Bandura, 1986; Ernest, 1989). Future studies should examine content knowledge, concepts of self-efficacy, and problem solving abilities in the special education context. Considering the high need for special education teachers, schools of education must be careful in determining the abilities and beliefs of special education teaching candidates. Further, considering the large number of alternative certification program candidates who teach in high need schools (Boyd, Grossman, Lankford, Loeb, & Wyckoff, 2006; Kopp, 2003), it is extremely important for the sake of these students that educational researchers understand teacher knowledge, concepts of self-efficacy, and problem solving abilities.

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A COMPARISON OF TWO ALTERNATIVE PATHWAY PROGRAMS IN SECONDARY MATHEMATICS TEACHER CERTIFICATION

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The purpose of this study was to compare the mathematics content knowledge, attitudes, and efficacy held by teachers in two alternative pathways to mathematics teacher certification: New York City Teaching Fellows and Teach for America (TFA). Differences were not found in content, attitudes, and efficacy, but learning and teaching journals revealed several differences between Teaching Fellows and TFA teachers. Particularly, social justice in the classroom was mentioned more often by TFA teachers, and Teaching Fellows found classroom management to not be as much an issue as had been expected.

The purpose of this study was to compare the mathematics content knowledge, attitudes toward mathematics, and concepts of self-efficacy held by teachers in two alternative pathways to mathematics teacher certification: New York City Teaching Fellows (NYCTF) and Teach for America (TFA). Secondly, the purpose was to determine differences in their attitudes toward their own learning and teaching as new mathematics teachers in New York City.

Strong teacher content knowledge is an important factor for teaching mathematics successfully (Ball, Hill, & Bass, 2005). Further, negative teacher attitudes toward mathematics often lead to avoidance of teaching strong mathematical content and affect students' attitudes and behaviors (Amato, 2004). Additionally, poor attitudes toward teaching are directly related to teacher retention issues (Costigan, 2004), and self-efficacy is an important component for successful teaching since self-efficacy is a teacher's belief in his or her ability to teach effectively and positively affect student learning outcomes (Bandura, 1986; Enochs, Smith, & Huinker, 2000).

Backgrounds on the NYCTF and TFA Programs

The NYCTF program is an alternative pathways program developed in 2000 in conjunction with The New Teacher Project and the New York City Department of Education (NYCTF, 2008; Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007). The program goal was to recruit professionals from other fields to supply the large teacher shortages in New York City's public schools with quality teachers. Prior to September 2003, New York State allowed teachers to obtain temporary teaching licenses to help fill the teacher shortage. The NYCTF program has grown very quickly since its inception in 2000: "Fellows grew from about 1 percent of newly hired teachers in 2000 to 33 percent of all new teachers in 2005" (Boyd, Lankford, Loeb, Rockoff, & Wyckoff, 2007, p. 10). Teaching Fellows represent 11 percent of all New York City public school teachers, and 26 percent of all mathematics teachers (NYCTF, 2010). NYCTF is the largest alternative pathways program in New York City (Kane, Rockoff, & Staiger, 2006).

TFA is a non-profit organization formed in 1990 with the intention of sending college graduates to low-income schools to make a positive difference for the underserved students. Its founder, Wendy Kopp, was herself a new graduate of Princeton University looking to do something more with her life after graduation (Kopp, 2003). She considered that many recent college graduates at top U.S. universities would consider teaching low-income students if given the opportunity. The idea was that there should be a teachers' corps that would allow new graduates at top universities with an interest in teaching to quickly begin teaching students in underserved communities. Kopp considered that her idea could be a Peace Corps for the 1990s, and that the teachers would either stay in education or go into other sectors and remain advocates for public education.

Literature Review

There have been several studies that compared different pathways to teacher certification with the primary focus on student achievement and teacher retention as measures of success (Boyd, Grossman, Lankford, Loeb, Michelli, & Wyckoff, 2006; Constantine et al., 2009; Rochkind, Ott, Immerwahr, Doble, & Johnson, 2007). Rochkind et al. (2007) documented teacher experiences from TFA, Troops to Teachers, and the New Teacher Project in Baltimore, who described their first year of teaching. However, this study was limited in scope. Recently the National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, through the U.S. Department of Education, conducted a comparison study of various pathways to teacher certification in elementary reading and mathematics (Constantine et al., 2009). Constantine et al. claimed that despite the rapid growth in alternative pathways programs, little evidence has been gathered to determine their effectiveness. Constantine et al. found that alternatively and traditionally prepared teachers did not have statistically significant content knowledge differences, or statistically significant differences in student achievement levels. The latter finding is in contrast to previous research in which differences were found (Boyd, Grossman, Lankford, Loeb, Michelli, & Wyckoff, 2006; Darling-Hammond, Holtzman, Gatlin, & Heilig, 2005; Laczko-Kerr & Berliner, 2002). Previous studies had indicated that teachers in alternative pathway programs had students who scored about as well as, if not better than, students of traditionally prepared teachers in achievement examinations after the initial certification process (Kane et al., 2006).

Plumer (2010) concluded that while alternative pathway teachers appeared to be slightly more successful in mathematics and science than in reading, the research had come to conflicting and mixed conclusions on the impact of alternative pathways teachers. In New York the NYCTF and TFA programs are two large alternative pathways programs, and this study compared the mathematics content knowledge, attitudes toward mathematics, and concepts of self-efficacy held by teachers in these two programs.

Theoretical Framework

Aiken (1970) was an early pioneer to examine the relationship between mathematical achievement and attitudes toward mathematics, and showed that attitudes and achievement in mathematics are reciprocal. Like Aiken, Ma and Kishor (1997) found a small but positive significant relationship between achievement and attitudes through meta-analysis. This relationship, along with Ball et al.'s (2005) emphasis on the importance of content knowledge for teachers, formed the framework of this study. Ball et al. said, "How well teachers know mathematics is central to their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about presentation, emphasis, and sequencing" (p. 14). Further, Ball et al. suggested that teachers with high content knowledge could help narrow the achievement gap in urban schools. Teaching Fellows and TFA teachers are placed in high need urban schools in New York City.

Additionally, Bandura's (1986) construct of self-efficacy theory framed this study's focus on self-efficacy in Teaching Fellows and TFA teachers. Bandura found that teacher self-efficacy can be subdivided into a teacher's belief in his or her ability to teach well, and his or her belief to affect student learning outcomes. Teachers who feel that they cannot effectively teach mathematics and affect student learning are more likely to avoid teaching from an inquiry and student-centered approach with real conceptual understanding (Swars, Daane, & Giesen, 2006).

Research Questions

5. What differences existed between Teaching Fellows and TFA mathematical content knowledge both at the beginning and end of a mathematics methods course?
6. What differences existed between Teaching Fellows and TFA attitudes toward mathematics both at the beginning and end of a mathematics methods course?
7. What differences existed between Teaching Fellows and TFA concepts of teaching self-efficacy?
8. What differences existed between Teaching Fellows and TFA attitudes as measured by learning and teaching journals?

Methodology

The study used both quantitative and qualitative methods, and participants consisted of 42 Teaching Fellows and 22 TFA teachers at a partnering university, a medium-sized urban university located in New York City. Teaching Fellows and TFA teachers were both at the beginning of their teaching careers teaching middle and high school mathematics while taking graduate education courses. For mathematical content knowledge, attitudes toward mathematics, and learning and teaching journals, the sample for TFA is the entire 22 teachers. However, for self-efficacy the sample is reduced to 19 participants because two teachers did not return their self-efficacy instruments, and one teacher left teaching, the partnering university, and the TFA program all together in the second year. All 42 Teaching Fellows were available for all measurements.

The teachers were enrolled in a mathematics methods course that was based upon reformed-based methods and addressed both pedagogy and content from a problem solving perspective. The course was one semester for Teaching Fellows, but it was a year long course for TFA. However, the same content and same amount of time was devoted to the mathematics content and methods in both programs. It was a year long course for TFA teachers because the course was combined with two other courses: assessment and literacy.

Both Teaching Fellows and TFA teachers took the New York State Content Specialty Test (CST) the summer before they began their program, which is required by the State of New York for teacher certification. This test assesses mathematical content knowledge for teacher certification and the scores range from 100 to 300 with a minimum passing score of 220. It consists of multiple choice items and a written assignment, and has six sub-areas: Mathematical Reasoning and Communication; Algebra; Trigonometry and Calculus; Measurement and Geometry; Data Analysis, Probability, Statistics and Discrete Mathematics; and Algebra Constructed Response. Further, both Teaching Fellows and TFA teachers took a mathematics content test at the beginning and end of their mathematics methods course. The mathematics content test consisted of 25 free response items ranging from algebra to calculus, and the test taken at the end of the course was similar in form and content to the one taken at the beginning.

Additionally, both Teaching Fellows and TFA teachers were given a survey instrument that measured attitudes toward mathematics at the beginning and end of the mathematics methods course. The attitudinal questionnaire was adapted from Tapia (1996) and had 39 items that measured attitudes toward mathematics including self-confidence, value, enjoyment, and motivation in mathematics. The instrument used a 5-point Likert scale with items strongly agree, agree, neutral, disagree, and strongly disagree.

Teaching Fellows were given self-efficacy surveys at the beginning and end of their mathematics methods course. However, TFA teachers were given the self-efficacy survey only once in their second year of teaching and enrollment in the graduate education program. The self-efficacy instrument was adapted from the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) developed by Enochs et al. (2000), and measured concepts of self-efficacy. The MTEBI is a 21-item five-point Likert scale instrument with choices of strongly agree, agree, uncertain, disagree, and strongly disagree, and is grounded in the theoretical framework of Bandura's (1986) self-efficacy theory. The MTEBI contains two subscales: Personal Mathematics Teaching Efficacy (PMTE) and Mathematics Teaching Outcome Expectancy (MTOE) with 13 and 8 items, respectively. Possible scores range from 13 to 65 on the PMTE, and 8 to 40 on the MTOE. The PMTE specifically measured a teacher's self-concept of his or her ability to effectively teach mathematics. The MTOE specifically measured a teacher's belief in his or her ability to directly affect student learning outcomes. Enochs et al. (2000) found the PMTE and MTOE had Cronbach alpha coefficients of 0.88 and 0.77, respectively.

Finally, Teaching Fellows and TFA teachers were required to keep reflective journals on their learning and teaching over the course of the methods course, which provided qualitative data of their attitudes toward learning and teaching mathematics. The learning journal had guiding questions such as: How has this course affected your teaching? What has been helpful? What are the most important concepts you've learned in this class? The teaching journal had guiding questions such as: How are your students learning? What challenges do you face? What successes have you had? Has your attitude toward teaching shifted over the course of the semester?

Results

Research questions one, two, and three were answered using independent samples *t*-tests with significance levels taken at the 0.05 level. For research question one, both pre- and post- mathematics content test scores and mathematics CST scores were used to determine if there were any significant differences between NYCTF and TFA teacher content knowledge. For research question two, both pre- and post- attitudes toward mathematics findings were used to determine if there were any significant differences between NYCTF and TFA teacher attitudes toward mathematics. Finally, for research question three, both pre- and post- MTEBI scores, separated for both PMTE and MTOE subscales, for Teaching Fellows were used with MTEBI scores for TFA teachers to determine if differences existed in teacher self-efficacy between the two programs.

Findings revealed no statistically significant differences between Teaching Fellows and TFA teachers on the mathematical content test, CST scores, attitudes toward mathematics instrument, and the MTEBI for both subscales: PMTE and MTOE. This means there are no differences between NYCTF and TFA mathematics content knowledge, attitudes toward mathematics, and concepts of teaching self-efficacy as measured by the instruments used in this study.

Analysis of the learning and teaching journals revealed similarities and differences between the two programs. For the learning journals it was found that both groups cited problem solving and numeracy in the methods course frequently. Techniques for motivating student learning were discussed in both NYCTF and TFA journals. Both Teaching Fellows and TFA teachers cited reflective teaching and literature critique reviews least often. While social justice was cited most frequently by TFA teachers, it was mentioned very infrequently by Teaching Fellows. While microteaching and learning about motivation techniques were two categories frequently mentioned by Teaching Fellows, TFA teachers rarely mentioned these. In both NYCTF and TFA methods classes teachers were expected to present a brief micro lesson that contained a motivator for the lesson, to their classmates.

For the teaching journals both Teaching Fellows and TFA teachers cited classroom management as the most frequently cited concern. However, it should be noted that while every TFA teacher reference to classroom management was citing a problem with classroom management, several Teaching Fellows mentioned classroom management as not being as problematic as they thought it would be. Also frequently referenced in both NYCTF and TFA journals were student motivation for learning, student attendance, and standardized state examination preparation as emphasized by their administrations. TFA cited unsupportive administration frequently as a concern, whereas NYCTF did not.

Discussion

No statistically significant differences were found between NYCTF and TFA teachers on mathematics content knowledge, attitudes toward mathematics, and concepts of self-efficacy. It is commonly claimed by TFA that their candidates come from the most highly ranked and selective universities in the United States (TFA, 2010; Xu, Hannaway, & Taylor, 2008), and the implication is that those among America's brightest become TFA teachers. However, the findings in this study indicated that NYCTF and TFA teachers are statistically similar in terms of content knowledge, attitudes, and beliefs. These results are quite surprising considering there is a common perception held by those working with the programs in New York that TFA teachers, while not staying in education quite as long at Teaching Fellows, do however have stronger mathematics content knowledge. Moreover, Constantine et al. (2009) claimed that of the various alternative pathways programs, NYCTF and TFA are the more selective of the other alternative pathways program in candidate selection. In 2008 only about 15 percent of NYCTF program applicants were accepted into the program, and over 8 percent of NYCTF applicants actually entered training in the summer before teaching with an additional 1 percent beginning early in the 2007/2008 academic year (NYCTF, 2008). In 2008 approximately 20 percent of TFA applicants were accepted into the program with about 15 percent of applicants actually starting the program (TFA, 2008). The mean grade point average in 2008 for new Teaching Fellows was 3.3 (NYCTF, 2008), while the mean grade point average for new TFA teachers was 3.6 (TFA, 2010). Statistically significant differences in grade point averages had not been examined.

Since the results of this study indicate there are no differences found in several variables that measured teacher quality between the two programs, the implication is that it should not matter in which program teachers are selected based upon content knowledge, attitudes toward mathematics, and concepts of self-efficacy. However, given results from prior studies that focused on teacher retention, perhaps NYCTF maintains an advantage over TFA using retention, a variable important in student success, as an important criterion for success. Sipe and D'Angelo (2006) found when surveying Teaching Fellows that about 70 percent of them intended to stay in education. NYCTF reports that 92 percent of Teaching Fellows completed their first year of teaching, 73 percent completed at least three years of teaching, and half have taught for at least five years (NYCTF, 2010). Boyd, Grossman, Lankford, Michelli, Loeb, and Wyckoff (2006) reported that about 46 percent of Teaching Fellows stayed in teaching after four years as compared to 55 to 63 percent of traditionally prepared teachers. Further, Kane et al. (2006) found that Teaching Fellows and traditionally prepared teachers had similar retention rates. However, Darling-Hammond et al. (2005) reported that upon becoming certified many TFA teachers leaving teaching. This is in contrast to TFA's own report of teacher retention. TFA claimed that about two-thirds of TFA teachers stayed in the field upon completing their time in the program, and half of those remained in teaching (TFA, 2008). This means that about one third stayed in the classroom upon fulfilling their commitment, which generally lasts several years. Another one third maintained non-teaching roles in education, such as in administration or advocacy (TFA, 2008). Lassonde (2010) claimed that only 11 percent of TFA teachers reported planning to teach 10 years or more. As of 2010 there were 17,000 TFA alumni (TFA, 2010), and according to TFA, over 5600 remained teaching in the classroom after their commitment ended (TFA, 2009). These findings indicate that approximately more than twice the percentage of Teaching Fellows stayed in the classroom compared to TFA teachers over a similar time period of several years. However, no statistical analysis had been conducted to determine significant differences.

In the learning journals teachers in both programs cited problem solving and numeracy frequently. However, more interestingly, social justice was cited most frequently by TFA teachers, but it was mentioned very infrequently by Teaching Fellows. This is an interesting finding given the emphasis on social justice in the school of education where this study took place. If social justice issues in the classroom are of high concern, then perhaps TFA teachers have an advantage over Teaching Fellows, in this respect. This should be further investigated.

In the teaching journals teachers in both programs cited, unsurprisingly, classroom management issues as the most frequently cited concern for new teachers, as documented in the literature (Cruickshank, Jenkins, & Metcalf, 2006). It was surprising, however, that NYCTF teachers found classroom management to not be as much of an issue, but TFA teachers found classroom management exclusively problematic.

An interesting finding was that Teaching Fellows cited learning about motivation techniques often in their learning journals, while TFA teachers did not, especially since both groups frequently mentioned student motivation in their teaching journals, a concept related to the microteaching required in the methods class.

Both Teaching Fellows and TFA teachers cited time needed for, and administrative emphasis on, standardized state testing. In the time of No Child Left Behind (NCLB, 2001) much emphasis on accountability means that schools and principals must ensure that teachers are raising standardized state test scores. It is no surprise that both NYCTF and TFA teachers rank this highly among their concerns.

The major implication from this study is that TFA is not stronger than NYCTF in content knowledge, attitudes, or in their confidence of their own teaching efficacy, as is often assumed given the reputation of TFA and caliber of college and university pursued by TFA in candidate recruitment, and it is hoped that this study changes this perception. From the literature it appeared that Teaching Fellows had an advantage in terms of teacher retention, but the results of this study indicated that TFA teachers had an advantage with concern for social justice in the classroom. However, more confirmatory research is needed. Further studies should compare differences between Teaching Fellows and TFA teachers in actual classroom practice in secondary mathematics teaching specifically, given the size of the programs. How does student achievement compare for the two groups among high need students? This study found no differences in content knowledge and beliefs for the two groups of teachers, but it would be beneficial to understand NYCTF and TFA direct impact in urban classrooms.

Educational researchers must continue to investigate the quality of alternative pathway programs. Since students in high need urban schools are often the ones who receive alternative pathways teachers in the classroom, it is imperative that educational researchers, professors of education, administrators, and policy makers ensure that these students are getting the highest level quality teachers they deserve. The question posed to administrators and policy makers is: Would you accept an alternative pathway teacher in your own child's classroom? We must continue to understand and improve alternative pathways until the answer is a resounding "yes."

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PROSPECTIVE MIDDLE GRADE MATHEMATICS TEACHERS' MISTAKES IN KNOWLEDGE OF ALGEBRA FOR TEACHING

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This study examined prospective middle grade mathematics teachers' mistakes in knowledge of algebra for teaching. 115 prospective teachers participated in a survey. It was found that the participants performed better in expressions and linear equations but revealed numerous mistakes in irrational functions and equations, and relevant advanced mathematics content area. They also showed limited knowledge in selecting appropriate function perspectives and representations.

Introduction

Equipping teachers with appropriate knowledge needed for teaching is the key to high-quality teaching that aims at achieving high student achievement (Conference Board of the Mathematical Sciences [CBMS], 2001; National Mathematics Advisory Panel [NMAP], 2008). Many studies revealed that U.S. teachers do not have adequate content knowledge and pedagogical content knowledge preparations (Kulm, 2008; Ma, 1999; Schmidt et al., 2007). Meanwhile, relevant recommendations for improving teacher preparations have been made (NMAP, 2008; RAND Mathematics Study Panel, 2003; Kilpatrick, Swafford, & Findell, 2001). In particular, CBMS (2001) recommended that prospective middle mathematics teachers should have at least twenty-one semester hours in mathematics, including at least twelve semester hours of fundamental ideas of mathematics appropriate for middles school teachers. Does taking much more mathematics or mathematics education courses matter for preparing qualified teachers? In this study, we examined prospective teachers from a program that meets the requirements (CBMS, 2001) how they responded to the survey which was designed to measure knowledge of algebra for teaching. By focusing on the particular content of algebra, we hope to provide a better understanding of teacher knowledge in the specific area and provide insight into mathematics teacher preparation in general. The purpose of this study is to examine the strengths and weakness of teachers' knowledge for teaching algebra with a particular attention to their mistakes.

Theoretical Framework

Teachers' Knowledge for Teaching Algebra

In the past decades, researchers have focused on conceptualizing and measuring particular mathematical knowledge needed for teaching (MKT) (e.g., Ball, Hill, & Bass, 2005; Hill, 2010; Kilpatrick et al., 2001). Recently, growing attention has been given to define and measure teachers' knowledge needed for teaching specific areas (e.g., Chinnappan & Lawson, 2005; Even, 1993, 1998; Floden & McCrory, 2007). Since algebra is a core component of school mathematics (National Council of Teachers of Mathematics [NCTM], 2000; NMAP, 2008), researchers have proposed different models defining teachers' knowledge for teaching algebra (e.g., Doerr, 2004; Even, 1993; Floden & McCrory, 2007). For example, Floden and McCrory

(2007) have developed a model to guide the development of a measure of teachers' knowledge of algebra for teaching (KAT). According to this model, the knowledge for teaching algebra includes three types of knowledge: school algebra knowledge, advanced algebra knowledge, and teaching algebra knowledge. Moreover, they also developed relevant instrument for measuring KAT. Thus, we developed our own instrument based on Floden and McCrory's (2007) framework and instrument.

Understanding of the Concept of Function

Process-object duality is a generally accepted model of mathematics concept development (Briedenbach, Dubinsky, Hawks, & Nichols, 1992; Schawarz &Yerushalmy, 1992; Sfard, 1991). With regard to the development of function concept, according to the process perspective, a function is perceived as linking x and y values: for each value of x, the function has only one corresponding y value. On the other hand, the object perspective regards functions or relations and any of its representation as entities (Moschkovich, Schoenfeld, & Arcavi, 1993). The term "representation refers both to process and product-to the act of capturing a mathematical concept or relationship in some form and to the form itself'' (NCTM, 2000, p. 67). Hence, representation is an essential part of the mathematical activity and a vehicle for capturing mathematical concepts (e.g., Goldin & Shteingold, 2001). Different representations play different roles in helping students understand the concept of function (e.g., Lesh, Post, & Beher, 1987; Schwarz &Yerushalmy, 1992). For example, while the algebraic representations can help understand a function as a *process*; the graphical representations help understand a function as an *object*. Thus, it is important to have sound understanding of these two perspectives (Clement, 1989; Even, 1998) and use appropriate representations with regard to different contexts (Cuoco 2001; NCTM, 2000, 2009).

Teachers' Knowledge for Teaching the Concept of Function

There are several studies on teachers' knowledge for teaching the concept of function (e.g., Even, 1993, 1998; Hitt, 1998). For example, based on 152 prospective secondary teachers' completion of an open-ended questionnaire concerning their knowledge about function, Even (1993) found that many prospective secondary teachers did not hold a modern conception of a function as univalent correspondence between two sets. These teachers tended to believe that functions are always represented by equations and that their graphs are well-behaved. In summary, teachers have difficulties in (1) understanding the concept of functions as univalent correspondence between two sets (sometimes, it is not presented by a formula, or it is not of discontinuous graph), (2) shifting different representations flexibly, and (3) relating formal function notions to contextual situations which produce the function.

Method

Data Sources

The data used for this study were taken from a comparative study (Huang, 2010) consisting of 115 U.S. prospective middle mathematics teachers' survey on knowledge of algebra for teaching and follow- up interviews with 5 participants (Larry, Jenny, Kerri, Alisa, and Stacy are pseudonym names of the interviewees). The participants were from an interdisciplinary program preparing math and science middle school teaches at a large research public University in South of the United States. Based on Floden and McCrory' (2007) questionnaire, we developed an instrument which included 17 multiple choice items and 8 open-ended items. The interviews were conducted individually during the week after completion of the survey. Each interview lasted about 20 minutes, and was audio recorded. The majority of the participants (87%) were junior and senior students and they had completed an average of 7 mathematics and mathematics courses.

Data Analysis

After quantifying the data, we first analyzed typical mistakes with regard to expressions, equations, and graphs (Items 1-17). Then, we analyzed the performance and typical mistakes in solving open-ended problems (items 18-25) from two perspectives: function perspectives and representations.

Results

Mistakes in Knowledge Related to Expressions, Equations, and Graphs

Some typical mistakes in knowledge of algebra for teaching were identified as follows: (1) neglecting the domain of independent variable when transforming and/solving irrational functions and equations; (2) inappropriate use of graphic representations; and (3) misconceptions in advanced knowledge.

Neglecting the domain of independent variables. In item 4, participants were required to find the equivalent expressions of the function $f(x)=\log_2 x^2$ from: i . $y=2\log_2 x$; ii . $y=2\log_2|x|$; iii. $y=2|\log_2 x|$. Their choices of A (i only), B (ii only), C (iii only), D (i and ii only), and E (i, ii, and iii), were 37%, 15%, 6%, 21%, and 0% respectively. This indicates that 37% of them did not consider the domain of independent variable x when transforming a function and also 21 % of them did not differentiate these four expressions. For another example, participants were asked to comment on the following process of solving the equation: $9^x-3^x-6=0$: "Peter denoted $y=3^x$ and gets the equation $y^2-y-6=0$, which has 2 different roots. He concluded that the given equation also has 2 different roots". 15% of them agreed with Peter's solution, and 54% of them believed the substitution method is wrong, only 18% of them realized the mistake in neglecting the range of y. These results suggest that participants were likely to neglect the domain of an independent variable□or range of a dependent variable□ when transformation irrational functions or expressions.

Inappropriate use of graphic representations. Many participants were not able to appropriately use geometrical representations to present algorithms, algebra relationships, relationship between time and velocity, and root of trigonometry functions. For example, only 30% of the participants realized the following relationships: " i. The equivalence of fractions and percents, e.g. $\frac{3}{5}=60\%$; ii. The distributive property of multiplication over addition: For all real numbers a, b, and c, we have $a(b+c)=ab+ac$; iii. The expansion of the square of a binomial: $(a+b)^2=a^2+2ab+b^2$" can be represented by rectangles. 26%, 22% and 16% of the participants did not realize that the distributive rules of multiplication over addition, the equivalence of fractions and percents, and the formula of square of a binomial can be presented by rectangles respectively. Thus, the participants did not have profound understanding of using geometrical representations to present algorithmic and algebraic relationships.

Misconception in advanced mathematics. We identified some essential mistakes in the operation rules across different number systems, mathematical induction, and slope of tangent line of a curve. For example, one question is to judge the statement that " *For all a and b in S, if ab = 0, then either a = 0 or b = 0"* is true in different number systems (S) such as real numbers (i), complex numbers (ii), and a set of 2 x 2 matrices with real number entries (iii). 44% of the participants selected only (i) (incorrect) and 22 % of selected only (i) and (ii) (correct answers), and 30% of them selected i, ii and iii (incorrect). It means that 44% of them are not able to generalize the rule to complex number system, while 30% of them inappropriately generalize the rule to matrix system. The invariance of operation rules in different number systems is a core concept; however, majority of them were not able to differentiate.

Inflexibilities or Mistakes in Quadratic Equations and Functions

The results are organized into two parts. One is related to the selection of perspectives of function concept. The other is about the flexible use of representations.

Weakness in selecting appropriate perspectives of function concept. Items 18, 24 and 25 are particularly used for measuring knowledge of understanding and applying function concept from different perspectives. Item 18 is in favor of using process perspective; item 24 is easily proved if adopting an object perspective. It is necessary to connect those two perspectives when solving item 25. The score distribution of these three items is shown in Table 1.

Table 1. Score Distribution of Items Related to the Adaption of Function Perspectives

Item	0 point (%)	1 point (%)	2 points (%)	3 points (%)	4 points (%)
Item 18	32.2	17.4	27.8	12.2	10.4
Item 24	98.3	1.7	0	0	0
Item 25	31.3	17.4	33	13	5.2

Take item 18 for example. The item is as follows: On a test a student marked both of the following as non-functions

(i) f: $\boldsymbol{R} \rightarrow \boldsymbol{R}$, $f(x) = 4$, where $\boldsymbol{R}$ is the set of all the real numbers.

(ii) $g(x) = x$ if x is a rational number, and $g(x) = 0$ if x is an irrational number.

(a) For each of (i) and (ii) above, decide whether the relation is a function;

(b) If you think the student was wrong to mark (i) or (ii) as a non-function, decide what he or she might have been thinking that could cause the mistake(s).Write your answer in the Answer Booklet.

About 23 % of the participants got a correct answer (10%) or almost correct answer with minor mistakes (12%). While 32% of them provided nothing or meaningless information about the solution and about 28% of them just gave correct answers without any interpretation or gave one correct answer and relevant explanations. The participants preferred to use diagrams to visualize the function relationship and then make a judgment of these two given relations

The perspectives used in participants' interpretations of the item 18 are listed in Table 2 (The percentages with wrong answers are omitted in the Table).

Table 2. Perspectives Adopted in the Responses to Item 18

Perspective	Description	Percent
Process	Corresponding relationship between domain and range (one-to-one; multiple-to-one)	6
Object	Expression feature (constant value; two expressions) Graphic features (one line, many holes/un-continuous)	9

With regard to this item, it is more appropriate to adopt the process perspective. However, the participants preferred using object perspective (9%), namely, basing on function expressions and graphic features to using essentially corresponding relationship features (6%). In responding to how they made their judgments, except for one Jenny, the others reported they used the vertical line test (Larry, Alisa, and Stacy) or diagrams presenting corresponding relationship between two sets (Kerri). Jenny made her wrong judgment based on visually graphical images. Since she had difficulty drawing the graph of the second relation, she believed it was not a function. However, when asked whether she heard of the vertical line test, she clearly stated that "one x value can only have one corresponding y value; one x value cannot be corresponded to two y-values." Kerri said she "is a visual learner, and likes using diagrams to represent the relationship between two sets (one-to-one or multiple-to-one, but not one-to-multiple)". Larry not only explained the vertical line test rule, but also showed an example ($x=y^2$) which cannot pass the vertical line test. Alisa and Stacy explained the rule by emphasizing "each input [value] should have only one [corresponding] value, but that does not mean that different [input] values cannot have same [corresponding] value". They also attributed students' mistakes to students' superficial understanding of the vertical line test rule (missing multiple x values correspond one y-value) or the confusion with "many holes", or the repeating output.

Inflexibility of Selecting Representations

The items 19, 21, 22, and 23 are deliberately designed for measuring knowledge for understanding and applying quadratic functions/equations/inequalities through flexibly using multiple representations. It is crucial for participants to flexibly use appropriate presentations and shift between different representations in order to solve them effectively. Regarding item 19, it is expected to have algebraic and graphic representations of equation and inequality. With regard to item 21, it is necessary to shift between algebraic and graphic representations in order to solve the problem. To solve the problem of item 22, it is necessary to have ability to translate graphic representations to algebraic representations. To solve the problem of item 23, it is required to use appropriate forms of algebraic expressions and transformations of different algebraic expressions, and translation between graphic and algebraic representations. The scores distribution of these four items is shown in Table 3.

Table 3. Score Distribution of Items Related to Flexible Use of Representations

Item	0 point (%)	1 point (%)	2 points (%)	3 points (%)	4 points (%)
Item 19	31.3	62.6	5.2	0.9	0
Item 21	84.3	14.8	0.9	0	0
Item 22	34.8	24.3	15.7	16.5	8.7
Item 23	80	16.5	0.9	0	2.6

Take item 19 for example. The question is as follows:

Solve the inequality $(x - 3)(x + 4) > 0$ in *two essentially* different ways.

Only one participant gave two essentially different solutions (i.e., algebraic and graphic methods). Around 60% gave an algebraic statement which did not result in a correct answer. In addition, more than one-third (31%) of the participants left it blank or gave some useless statements. Overall, the students performed worst in this problem. Moreover, many mistakes and misconceptions occurred as shown in Table 4.

Table 4. Misconceptions or Mistakes in Solving Inequality in Item 19

Mistake	Explanation	Example	Frequency (%)
1	Misconception: if ab>0, then a>0, b>0	(x-3)(x+4)>0→ x-3>0, x+4>0, then x>3, x>-4.	37
2	Only transforming into standard form	$x^2+x-12>0$ or $x^2+x>12$.	21
3	Transforming into standard form and getting stuck	$x^2+x-12>0$ or x(x+1)>12 or x(x+1)=12.	7.6
4	Working on the standard form with guess and check	$x^2+x-12>0 \rightarrow x(x+1)>12$; →x>12, x+1>12→x>12, x>11.	12
		$x^2+x-12>0$, or (x-3)(x+4)>0 →x_1=3, x_2=-4.	15
5	Drawing a number line	Find partial answer: x>3 or x<-4.	4
6	Using a table	x>3 (x=1, 2, 3, 4... or 0, -1, -2, -3,…).	4

The table shows that 37 % of the participants adopted the inference: if ab>0, then a>0, b>0. None of them realized that *a* and *b* are possibly negative. In addition, none of them care about the logical operations "or" or "and" between two logical propositions (such as a>0 and b>0 or a>0 or b>0). They also were satisfied with the solution "x>3, x>-4" without any intention to further intersect or combine.

In order to find another method to solve the inequality, an automatic alternative is to transform the factor form into standard form: $x^2+x-12>0$. 21% of them stopped with the standard form. 7.6 % of them were stuck with further algebraic operation: x(x+1)>12 or x(x+1)=12. Some of the participants want further with "guess and check strategies":

Mistake 1 (12%): $x^2+x-12>0 \rightarrow x(x+1)>12$; →x>12, x+1>12→x>12, x>11.

Mistake 2 (15%): $x^2+x-12>0$,or (x-3)(x+4)>0 →x_1=3, x_2=-4.

Participants' explanations. In the interview, Larry just simplified the factor form into standard form ($x^2+x-12>0$), and then moved forward by "guess and check" such as "$x^2>x-12$, and then $x>\sqrt{x-12}$" although it did not work. By analogizing the property of equation: (x-3) (x+4)=0 → x-3=0, or x+4=0, the remaining four interviewees made an inference as follows: $(x-3)(x+4)>0 \rightarrow (x-3)>0,\ (x+4)>0 \rightarrow x>3, x>-4$.None of them had intention to work on "x>3, x>-4" further, such as the logical operations "and" or "or" and the operations of intersection and combination of two sets. They seemed to be satisfied with the "solution". When asking if they can use a graphic method to solve equation or inequality, they recalled the graphs of quadratic equation. All the interviewees explained they did not know how to use quadratic function graph to solve inequality, although they knew the graphing method of solving a system of linear equation. They learned quadratic function first (probably late in middle schools or early in high schools) and then inequality late in high schools. These contents were taught separately. They were not taught how to use graphic representation to solve algebraic problems. They appreciated the method of integration of algebraic and graphic representations. The interview showed that participants may have relevant content knowledge but they did not have an interconnected knowledge network; did not have experience in flexibly using different kinds of knowledge and relevant representations

Take Item 21 for another example. The Item 21 is as follows:

If you substitute 1 for x in expression ax^2+bx+c (a, b and c are real numbers), you get a positive number, while substituting 6 gives a negative number. How many real solutions does the equation $ax^2+bx+c=0$ have? One student gives the following answer:

According to the given conditions, we can obtain the following inequalities:

$a+b+c>0$, and $36a+6b+c<0$.

Since it is impossible to find fixed values of a, b and c based on the previous inequalities, the original question is not solvable.

What do you think may be the reason for the students' answers? What are your suggestions to the student?

Only one participant gave a correct explanation and useful suggestions (there are two roots, and a graphic method is suggested). About 84% of the participants agreed with the student's explanation (actually, it is wrong), and they got stuck with the algebraic manipulation to find *a,b* and c, and had no idea about how to help the student get out of their difficulties. 15 % of the participants suggested plugging more value of *a, b*, and *c* (such as *a*=-10, *b*=-9, and *c*= 20) to see whether certain patterns can be found. They did not think to use graphic or geometrical representations to find the answers. Some of them (15%) had the following misconception: if you plug more numbers, you can find some patterns, and then you may find the solutions.

Participants' explanations. Three of the interviewees fully agreed with the student's (wrong) statement, namely "Since it is impossible to find out fixed values of *a, b* and *c* based on the previously given inequalities, the original question is not solvable". They tried to find out *a, b,* and c using algebraic transformation but it did not work. They had no ideas how to help the student find a solution. The other two interviewees felt that the problem may be solved, but they did not have any concrete ideas about how to solve it (Jenny and Stacy). What they could suggest is that the student is to "try different ways, such as plugging more numbers between 1 and 6."(Jenny) or "explore in different ways such as plugging more numbers to see whether they can find certain patterns, rather than being stuck"(Stacy).When asked whether they can try other methods such as graphical methods to solve, they sketched a graph and find possible roots. Four of them were successful in finding the number of roots by examining the intersection points of the quadratic function. All of them said that they had never realized such a way because they had not got such experiences in solving problems, but they realized the usefulness of the graphing method in algebra.

Discussion and Conclusion

This study showed that the participants from the interdisciplinary math and science middle teacher preparation program had relatively weak knowledge of algebra needed for teaching. Moreover, they also revealed their weakness in selecting appropriate perspectives of function and flexibly using of representations of quadratic functions. These findings are in line with other studies on teachers' knowledge for teaching algebra (Even, 1998; Hitt, 1998). The participants had limited advanced algebra knowledge although they had taken advanced courses such as calculus, linear algebra, and abstract algebra. The interviews revealed a lack of experience in using different representations, possibly because courses in mathematics and teaching methods were separately taught. This raises an issue about the curriculum for preparing mathematics teachers for middle and high schools. Equipping prospective teachers with appropriate knowledge for teaching is not a simple issue of adding more mathematics courses. It should include developing coherent curriculum that integrates content and pedagogy to help prospective teachers develop a well-structured algebra knowledge base with flexible use of different perspectives and representations. In order to implement mathematics curriculum that recommends providing middle and high school students with experience in developing connections and representational flexibility, prospective teachers should have the opportunity to develop relevant knowledge and skills needed for organizing classroom activities that promote this type of student learning (NCTM, 2000, 2009). The findings imply that beyond providing adequate number of math and math education courses, it is important to provide focal and coherent curricula from pre-university to university level. It is also important to have coherent teaching to emphasize knowledge development and form a well-structured and interconnected knowledge base.

This study has its own limitations. First, the instrument mainly focused on expression, equation, inequality, and functions which may not reflect a whole picture of algebra. Second, the participants were from a program in a single university, so caution is necessary in generalizing these findings.

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MATHEMATICS FOR TEACHING AND LEARNING: DEVELOPING TEACHERS' CONCEPTUAL UNDERSTANDING OF ELEMENTARY MATHEMATICS

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In this five-year study, 600 preservice teachers' understanding of mathematics as needed for classroom teaching was explored. Data sources included pre/post written open-response mathematics items, and pre/post semi-structured interviews. Results consistently showed that while participants were able to perform standard numeric calculations, conceptual understanding of elementary concepts was initially extremely weak. Moderate progress was made during the one-year program, with stronger growth for those teacher candidates who chose to enroll in the optional course in mathematics-for-teaching. Based on our results, an exam in "mathematics for teaching" is now a graduation requirement.

Objectives

Our previous anecdotal and observational experiences with upper elementary teacher candidates participating in mathematics methods courses were that what appeared to be weak understanding of mathematics concepts impacted preservice teachers' willingness and ability to embrace and develop problem-based lessons, even though participants were generally able to use a procedure to calculate a correct answer. Hence our research questions were: What mathematical understandings, both procedural and conceptual, are demonstrated by preservice teacher candidates at the beginning and end of their (in our case one-year) teacher preparation program?; What factors appear to influence such development?; What beliefs changes are observed?; Does extra course time in related mathematical concepts appear to better support such development?

Perspectives and Framework

This study used the bioecological model of Bronfenbrenner (1999) as a framework. Our research questions around teachers' beliefs and understandings formed the risk and protective factors thought to interact and influence growth.

Substantial literature (for example, Wilkins, 2008) links teacher beliefs with development of teacher knowledge, which in turn influences student achievement (Baumert et al., 2010). In particular, a specialised body of mathematical knowledge, variously dubbed *profound understanding of fundamental mathematics* (Ma, 1999), *mathematics for teaching* (Ball, Thames, & Phelps, 2008), and *pedagogical content knowledge* (Shulman, 1986; Baumert et al., 2010) has been argued as fundamental to the support of problem-based or inquiry-based classroom learning, the mainstay of suggested curriculum reforms over the past few decades (National Council of Teachers of Mathematics, 2000). It has been argued that "teachers' knowledge of mathematics alone is insufficient to support their attempts to teach for understanding" (Silverman & Thompson, 2008, p. 499), and further that such teaching is "predicated on coherent and generative understandings of the big mathematical ideas that make up the curriculum" (p. 501).

In particular, teachers need to understand the connection of appropriate classroom mathematical models to the concepts as well as have a knowledge of the underlying structure of procedures (Hill, 2010), in order to help students construct understanding and methods (including alternate or student-generated methods). Teachers also need to support such learning with appropriate questioning as well as the ability to diagnose student misconceptions and understand alternate methods (Ball et al., 2008). Such a depth of understanding provides significant challenges to new teacher-candidates.

Methods

The study used a mixed-methods design supported both quantitative and qualitative results. Preservice teachers for grades 4 to 10 (approximately 120 per year) were surveyed using a paper and pencil instrument (Kajander, 2007) at the beginning and end of their methods course. Mathematical understanding was measured as "procedural" as well as "conceptual"; questions relating to the former involved asking participants to perform a standard elementary procedure such as "3 x 1.6" or "5 – (-3)" using a method of their choice, while questions relating to the latter involved asking for an explanation, model, contextual example, or alternate method, all thought to be aspects of teachers' needed specialized knowledge (Hill, 2010). We use the term 'conceptual knowledge' to refer to those mathematical understandings that are mathematically specialized, yet may be developed without necessarily making a connection to students and teaching, as described by Hill (2010). As well a number of demographic variables were collected, such as undergraduate background, highest high school mathematics course taken, school curriculum experienced, and years out of university before entering the teacher education program. This same survey was used with several cohorts of in-service teachers during the first two years of the study. These data continue to be analysed statistically.

As well as the quantitative data, pre/post semi-structured interviews (50 pairs) were conducted during each of the most-recent 3 years of the study to support the survey results, and methods classroom artifacts such as assignment work and examination responses were also examined. Semi-structured interviews were conducted with 10 participating classroom teachers during the latter half of the study, and five were observed on at least one occasion while teaching. While all of this data forms the data-sources for the study, the current report focuses on the survey results related to preservice teachers' conceptual knowledge, and the preservice interview data related to beliefs about conceptual knowledge.

The written survey items were directly drawn from the questions used in Ma's (1999) landmark interviews and hence have arguably strong face validity. Reliability was also established during the first three years of the study (Kajander, 2010). Descriptive statistics from the survey are supported by examination of individual responses, as well as the participants' self-reports as drawn from the interview data. A final mandatory examination in "mathematics for teaching" was introduced to the program in year 3 based on the earlier data, and participant scores on this mandatory examination also support the written survey data.

Results

Our results consistently show that conceptual understanding of elementary mathematics concepts was weak to non-existent for teacher candidates at the pretest, which took place at the very beginning of the mathematics methods course. Also, nearly all of the participants interviewed mentioned some form of lack of comfort or insecurity with their mathematical understanding. The few who did feel slightly more comfortable tended to equate 'understanding' with procedural skill. Yet many participants who were almost completely unable to explain why standard numeracy methods worked, were still able to perform them accurately. While literature suggests that it is likely the case that teachers are better able to perform standard procedures than explain them (Hill, 2010), the extent of the problem, when we actually looked at multiple samples of prospective teachers' work, was staggering. According to our evidence, substantive numbers of teacher candidates were initially simply unable to provide *any* explanation, justification, model, or context for the standard algorithms they were generally able to perform. Explanations, if any, generally took the form of "because that is the rule". The mode score on conceptual knowledge on the pretest written survey was 0, with the mean each year hovering around 10%. Yet procedural knowledge on these same items had a much higher mean, over 70% (see Table 1).

Table 1

Initial Written Survey Mean Scores

Response Type	*M*	*SD*	*SE*
Procedural knowledge	7.1361	2.05917	.08602
Conceptual knowledge	1.2670	1.46291	.06111

Note: All knowledge scores are scaled out of 10 possible points. Pretest survey responses, *N*=573. *SE*=standard error of the mean.

The same written survey was re-administered at the end of each year. Substantial growth particularly in conceptual knowledge was shown over each of the five one-year cohorts (see Table 2). Results have been shown as significant for each of the first three years of the study (Kajander, 2010) and analysis for the final two years is underway.

Table 2
Final Written Survey Mean Scores

Response Type	*M*	*SD*	*SE*
Procedural knowledge	8.8586	1.58641	.06627
Conceptual knowledge	5.6021	2.49660	.10430

Note: All knowledge scores are scaled out of 10 possible points. Post-test survey responses, *N*=573. *SE*=standard error of the mean.

Yet the numbers alone fail to bring to light the issues as strongly as examining the actual participant survey responses. Using a semi-random selection process (every fourteenth response in the database was selected from year 5, as space permitted), a selection of specific responses to two of the survey questions is offered.

Two of the seven survey questions are chosen as illustrative. The instructions for each of these questions are the same. For Part a) the participants are asked to

***Answer** the question, showing your steps as needed to illustrate the method you used*

while in Part b) they are asked to

***Explain** what you can about why and how the method you used in a) works, using explanations, diagrams, models, and examples as appropriate. If possible, do the question another way.*

Part a) questions are scored as *procedural knowledge* items, while Part b) questions are scored as *conceptual knowledge*. Figures 1 and 2 show sample pretest responses selected from the year 5 data.

Selected Pretest Survey Responses to Integers Question

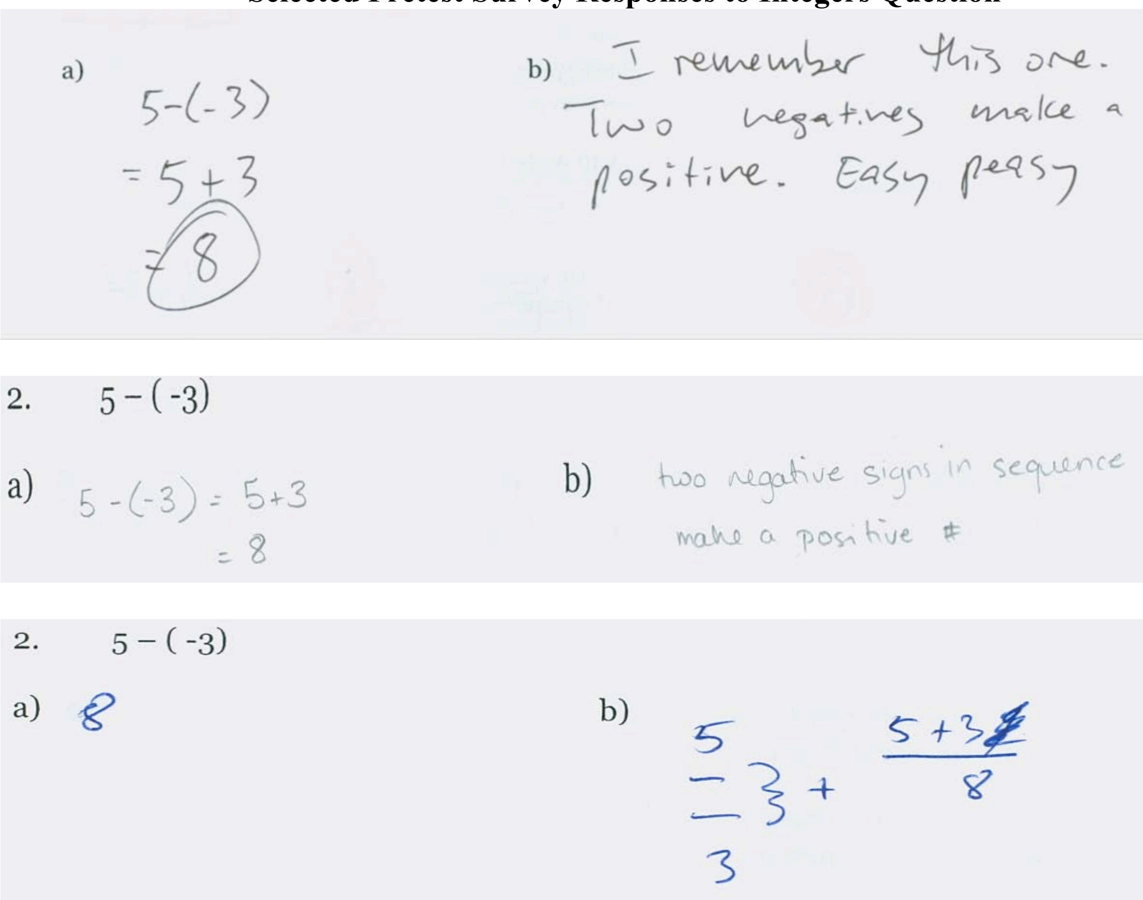


Figure 1. Pretest responses to 5-(-3). Part a) asks for answer and Part b) for explanation.

Selected Pretest Survey Responses to Fractions Question

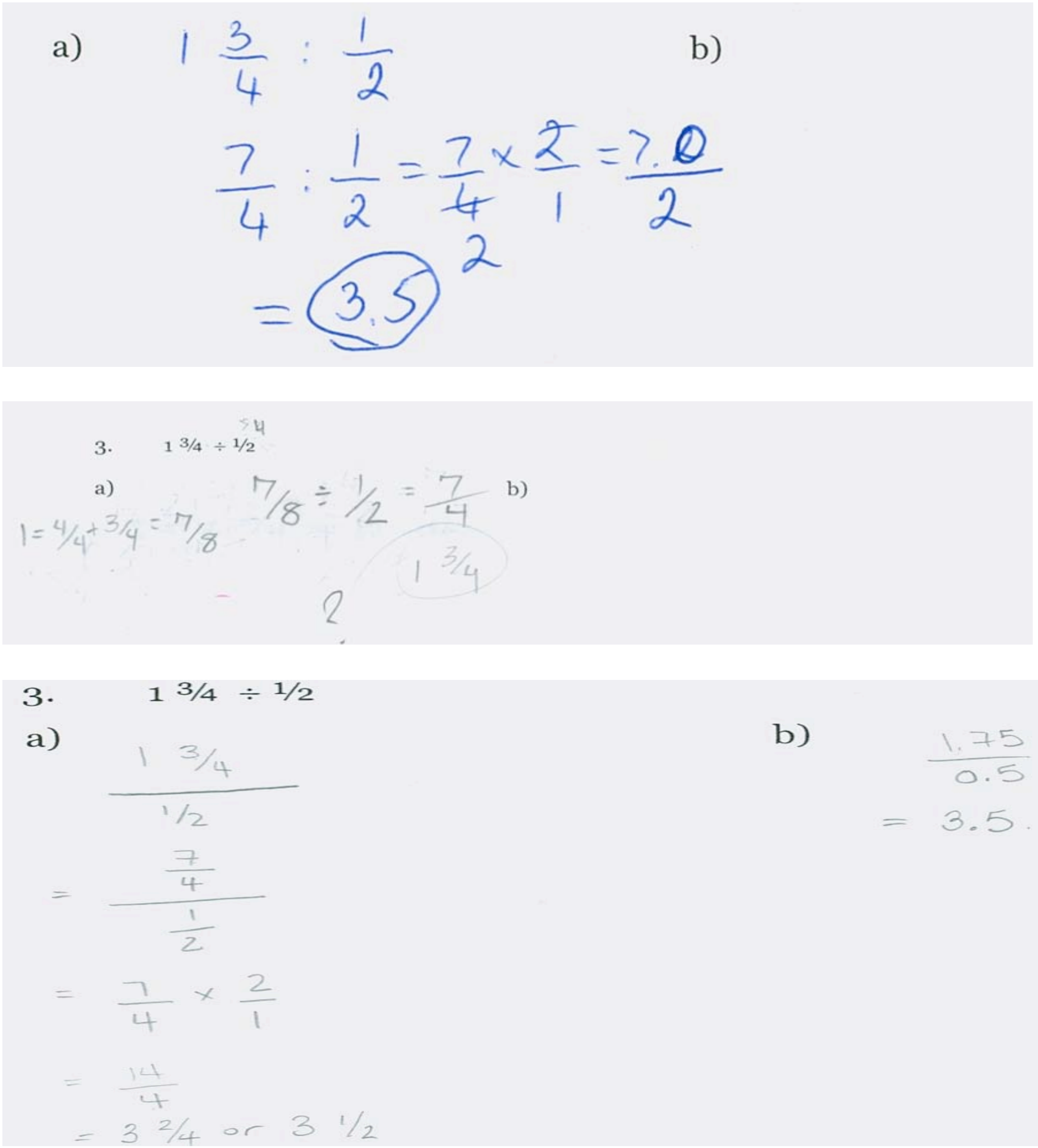


Figure 2. Pretest responses to 1 ¾ ÷ ½. Part a) asks for answer and Part b) for explanation.

The following samples (Figures 3 and 4) are post-test responses from the same respective participants as shown in Figures 1 and 2.

Selected Post-test Survey Responses to Integers Question

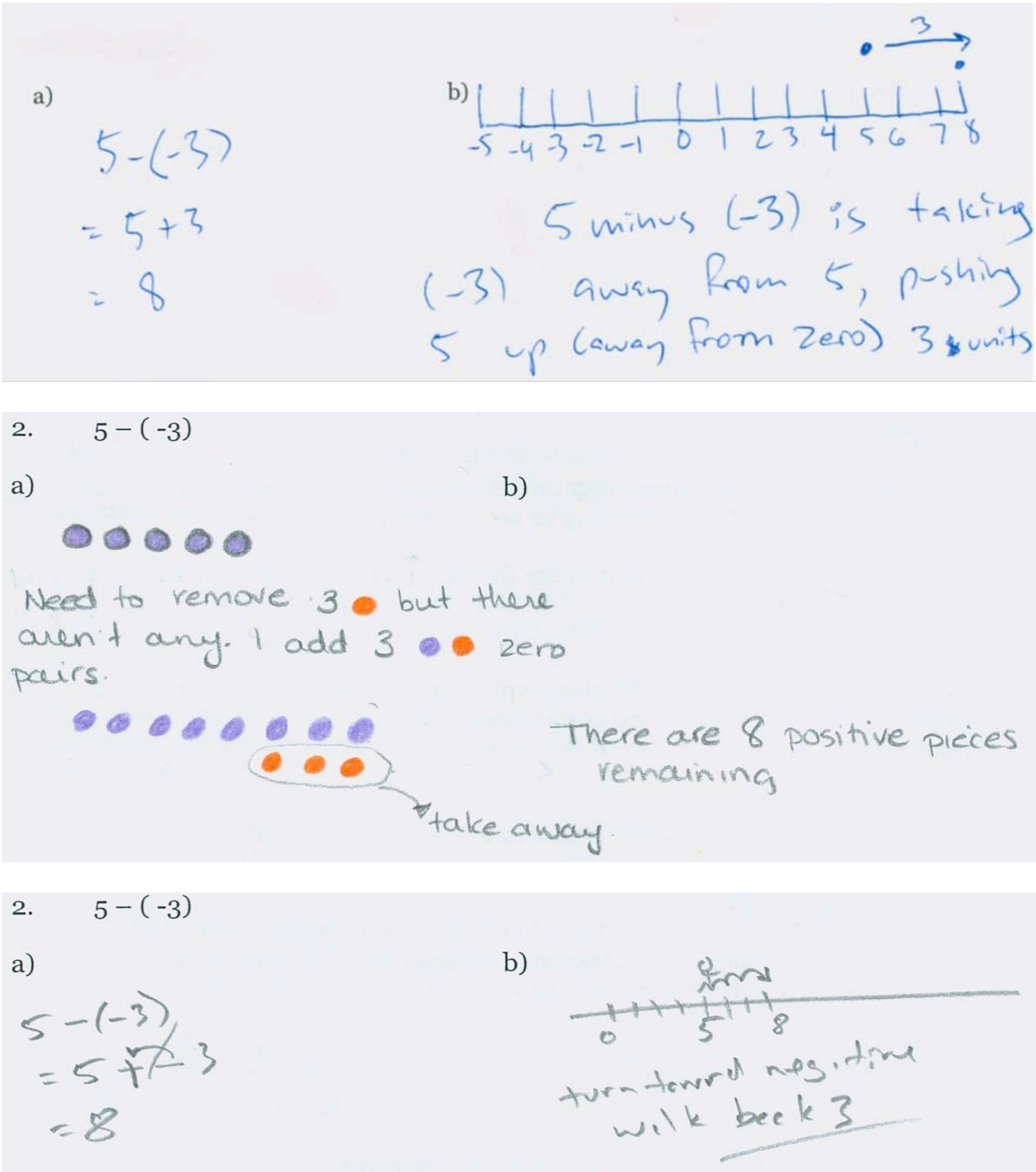
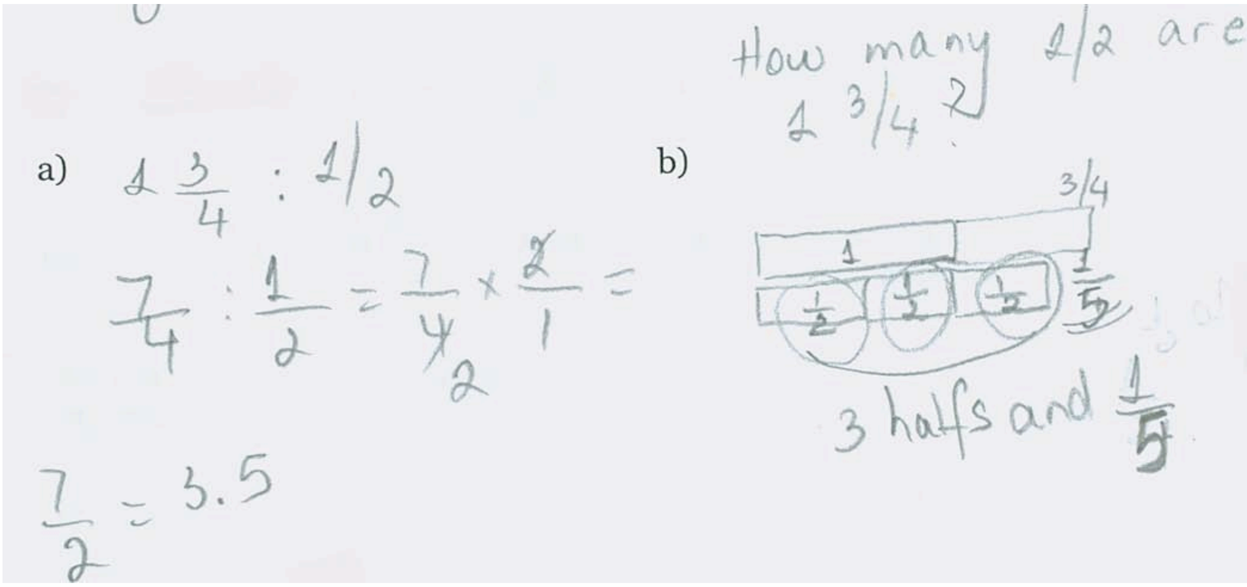
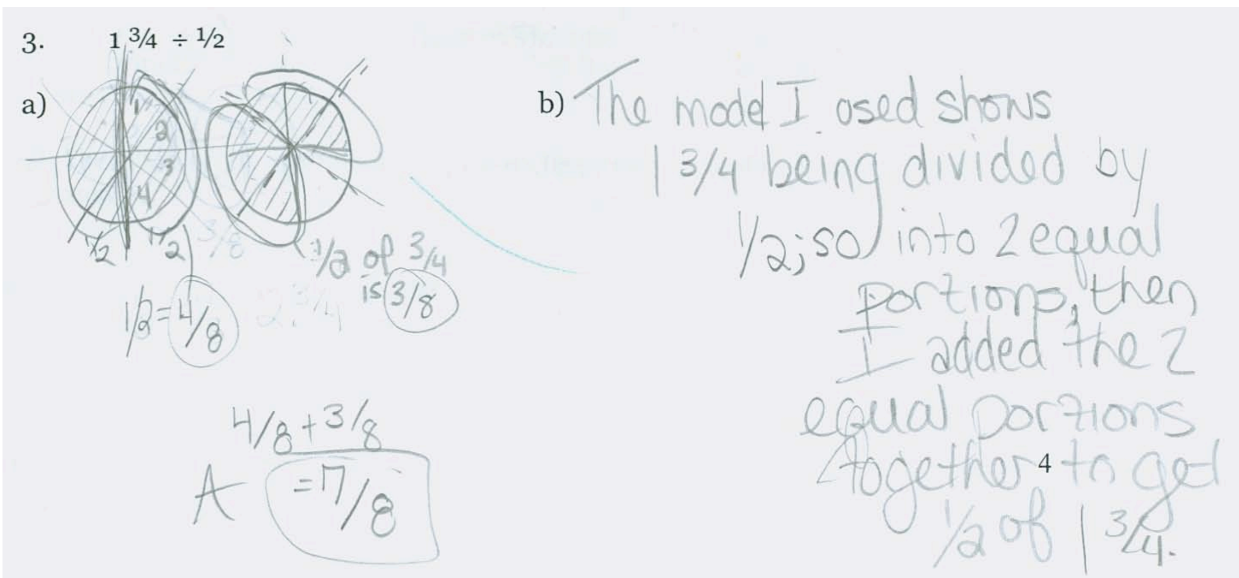


Figure 3. Post-test responses to 5-(-3). Part a) asks for answer and Part b) for explanation.

Selected Post-test Survey Responses to Fractions Question


a)
1 3/4 : 1/2
7/4 : 1/2 = 7/4 x 2/1 =
7/2 = 3.5
b)
How many 1/2 are 1 3/4 ?
3/4
1
1/2 1/2 1/2 1/5
3 halfs and 1/5


3. 1 3/4 ÷ 1/2
a)
1/2 of 3/4 is 3/8
1/2 = 4/8
4/8 + 3/8
A = 7/8
b) The model I osed shows 1 3/4 being divided by 1/2; so into 2 equal portions, then I added the 2 equal portions together to get 1/2 of 1 3/4.

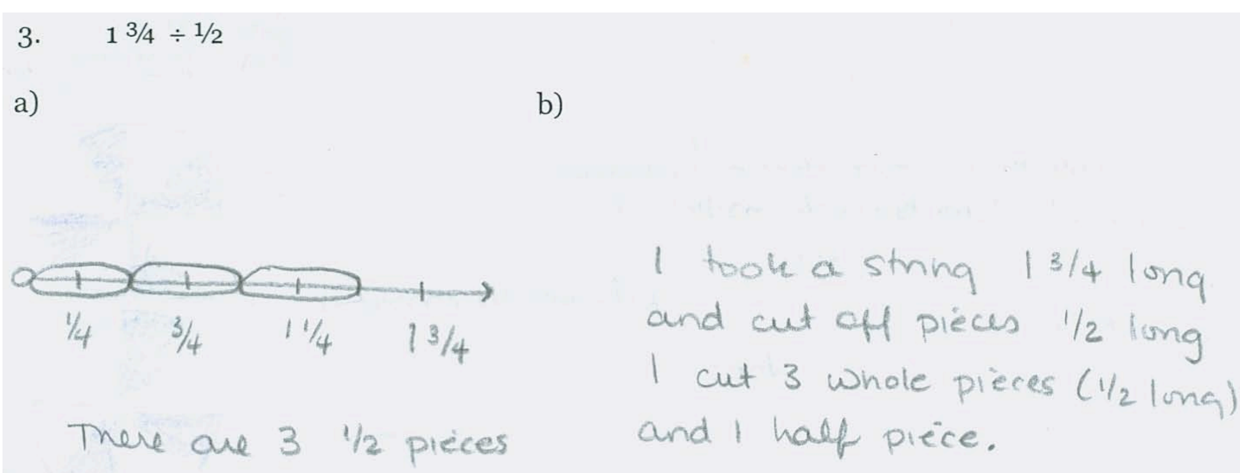

3. 1 3/4 ÷ 1/2
a)
1/4 3/4 1 1/4 1 3/4
There are 3 1/2 pieces
b)
I took a string 1 3/4 long and cut off pieces 1/2 long
I cut 3 whole pieces (1/2 long) and 1 half piece.

Figure 4. Post-test Responses to 1 ¾ ÷ ½. Part a) asks for answer and Part b) for explanation.

Interview data indicated that at the pretest nearly all participants viewed mathematics as "something to memorize". Interview responses at the post-test suggest that participants generally felt very good about their deepened understanding, yet there were still indications related to the fragility of the newly developed understanding. Many referred to their newly emerging conceptual understanding as a kind of revelation. For example, one participant stated "I feel like I am learning mathematics for the first time".

Discussion and Implications

An examination of the pretest data, both quantitative and qualitative, suggests to us that most participants were operating with an almost exclusively procedural view and capacity of standard elementary mathematical processes. The vast majority of survey respondents were virtually unable initially to offer any conceptual basis for the calculations they generally quite readily performed. Interview responses indicated that many participants were both aware of, and deeply troubled by, this situation. Significant improvement was noted in many cases by the post-test, even though the number of hours of mathematics coursework was less than ideal. While at times imperfect or incomplete, participants had generally made strong improvements. Even more encouraging was the sense that many participants now had a different view of mathematics, and their role in it.

Our results underscore the on-going and desperate need for specialised and supportive mathematical experiences for preservice teachers. We continue to be astonished that even with curriculum reform in Ontario, teacher candidates are entering education programs with virtually a purely procedural view of the subject and a sense that mathematics is a subject in which you have to follow the "steps" or memorize the "rule". While we had observed preservice teachers' struggles prior to this study, the extent of their initial lack of conceptual understanding was still a surprise. An awareness of the pretest results helped us be more sensitive to the struggle and effort required by participants as they developed so significantly during the one-year program.

Concurrently with the development of recommendations for preservice teachers in the U.S. (National Council on Teacher Quality, 2008), the Canadian *Working Group on Elementary Mathematics for Teaching* (Kajander & Jarvis, 2009) recommends that preservice teachers take a minimum of 100 hours of specialized mathematics courses. Since the mathematics methods course described is of a 36-hour duration, with only about half of the hours concentrated directly on mathematics concepts, we were far short of a desirable situation. Hence in year 3, an optional 20 hour course in "mathematics for teaching" was introduced, taken by about a quarter of the participants each year. As well, a mandatory examination in "mathematics for teaching" was introduced. Initial effects of these program changes seem promising, particularly for low-achieving students (Holm & Kajander, 2009). Further opportunities for participants to strengthen their conceptual mathematical understanding will likely be needed, however, before the full vision of mathematics teaching as is described in the *Standards* (NCTM, 2000) and elsewhere can be realized in their classrooms.

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EXPLORING SHIFTS IN A TEACHER'S KEY DEVELOPMENTAL UNDERSTANDINGS AND PEDAGOGICAL ACTIONS

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Research has identified mathematical knowledge for teaching (MKT) as specialized knowledge that contributes to a teacher's ability to teach for understanding. This report discusses one teacher's (Claudia) MKT as she implemented research-based precalculus materials. Claudia initially encountered difficulty implementing the materials in ways that supported her students' learning. As the semester progressed, Claudia's understanding of the central ideas of the course deepened and specific ideas emerged as key for understanding other ideas. Once she became aware of these connections she made more informed pedagogical choices that supported her students' thinking and learning.

Background

It is well established that although content knowledge is an important attribute of a successful teacher, content knowledge alone is not sufficient to support students' learning processes (Shulman, 1986; Silverman & Thompson, 2008). Explorations into relationships between a teacher's content knowledge and her/his practice have identified *mathematical knowledge for teaching* (MKT) as a critical link between content knowledge and supporting students' learning (Ball, 1990; Simon, 2006). Recent studies (Silverman & Thompson, 2008) have emphasized the importance of exploring teachers' ability to transform content knowledge from personal understandings to knowledge that has pedagogical power in the classroom.

Project Pathways is an ongoing initiative with a focus on improving secondary mathematics teachers' practice and secondary students' learning. The project's interventions (e.g., graduate courses, professional learning communities, and workshops) intend to improve teachers' content knowledge and their ability to achieve conceptually oriented classrooms that engage students in problem solving activity. During initial phases (years 1-4) of the project the research team observed gains in teachers' content knowledge, but shifts in the teachers' pedagogical actions lagged behind the content knowledge shifts. When investigating the obstacles preventing the teachers from shifting their practice, project members identified that the mathematical content in their curriculum was incompatible with the mathematical ideas developed during the project's interventions.

In an attempt to better support the teachers in improving the conceptual focus of their teaching, the project shifted the concentration of the summer work and professional learning communities (PLCs) to examining and modifying the teachers' curriculum, but the teachers found revising their curriculum to be a daunting and exhaustive task. The heavy burden of writing curriculum led the project leaders to offer a research-based, conceptually oriented precalculus curriculum (Carlson & Oehrtman, 2010) for use by one precalculus teacher (Claudia). We conjectured that the combination of the *Project Pathways* interventions and curriculum tasks would support Claudia in realizing the desired shifts in her practice and students' learning. Our report discusses the role of Claudia's MKT in her implementation of the curriculum. Claudia's difficulties and successes with the curriculum illustrate the relationship between Claudia's content knowledge, her transformation of this knowledge in ways that inform her practice, and her ability to support her students' learning processes.

Theoretical Perspective

We leverage the construct of *key developmental understanding* (KDU) (Silverman & Thompson, 2008; Simon, 2006) to examine MKT in the context of secondary precalculus mathematics. Like Silverman and Thompson (2008), we believe that a teacher's mathematical conceptions and reasoning abilities influence her/his pedagogical choices and actions. In order to describe understandings that support MKT, Simon (2006) and Silverman and Thompson (2008) characterized KDUs as one's understandings and ways of reasoning that connect various ideas, provide foundations for learning other ideas, and support solving novel but related problems as a consequence of these understandings.

Although KDUs are critical features of a person's mathematical knowledge, Silverman and Thompson (2008) highlighted that a teacher's development of KDUs does not imply these KDUs will inform her/his practice. Silverman and Thompson suggested that MKT is developed when a teacher transforms a KDU in a way that shapes and informs their interactions with students. Specifically, Silverman and Thompson described:

> A teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she (1) has developed a KDU within which that topic exists, (2) has constructed models of the variety of ways students may understand the content (decentering); (3) has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea; (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. (p. 508)

As the *Pathways* interventions transitioned to focus more on applying the teachers' content knowledge to their teaching practice through curriculum design, we conjectured that the interventions would better support the teachers in developing KDUs that informed their pedagogical practices. We expected that our providing Claudia with quality curriculum would better position her to construct the MKT necessary to achieve shifts in her practice. Specifically, we conjectured that the use of quality curriculum would lead to Claudia developing KDUs and transforming these understandings in ways that informed her pedagogical actions.

Research Question

How does a teacher's MKT impact her pedagogical choices and ability to support student learning? How does a teacher's MKT impact her ability to successfully implement research-based curriculum?

Methodology

During the semester in which Claudia used the Pathways curriculum, she attended a weekly 50-minute PLC composed of 4-6 trigonometry and precalculus teachers. She also sporadically attended a 3-hour trigonometry PLC after school with two other trigonometry teachers and a project member (the first author of the report) who served as the PLC facilitator. The PLC members primarily examined the knowledge involved in understanding and learning key ideas of their courses. The PLC members also spent time developing conceptually oriented tasks for classroom use.

Three *Project Pathways* members regularly observed Claudia's classroom and worked with Claudia to answer her questions about the curriculum. Claudia's classroom was videotaped two class periods a day and these videos were digitized for analysis. The project members took field notes during Claudia's PLC meetings, when observing Claudia's classroom, and during her meetings with project members. The project members directed the field notes towards Claudia's content knowledge as revealed by her interactions with her colleagues, her students, and the project members. As Claudia interacted with her students during class sessions, her responses and questioning offered insights into her mathematical understandings and thinking. Similarly, her interactions during the PLC meetings offered glimpses into her content knowledge, particularly when the PLC sessions focused on discussing how students learn various topics. Relative to Claudia's meetings with *Pathways* members, she often asked for clarification of the mathematical ideas driving the curriculum. Claudia also discussed questions she had about the solutions and reasoning that her students were providing. Collectively, the project members used these data sources to characterize Claudia's teaching practice by describing her mathematical content knowledge, pedagogical actions, and improvements in her students' learning.

As the semester progressed, the project members who observed Claudia's classroom noted shifts in her practice and her ability to scaffold her students' thinking. In order to characterize shifts in her teaching and her questioning, the research team analyzed her classroom videos to document relationships between her propensity to focus on student thinking and her questioning techniques (Teuscher, Moore, Marfai, Tallman, & Carlson, submitted). After analyzing her classroom sessions, the research team chose three particular sessions that were representative of her shifts over the course of the semester.

The present study extends the aforementioned study (Teuscher et al., submitted) by analyzing Claudia's thinking and MKT during the classroom sessions that the research team deemed to be representative of Claudia's practice (see Teuscher et al. for an explanation of the case selection). Compatible with Thompson's description of a conceptual analysis (Thompson, 2008), we analyzed Claudia's interactions with her students in an attempt to characterize the mental actions driving Claudia's pedagogical actions (e.g., the mathematical content knowledge that informed her questioning and utterances). We then compared the results of this analysis to the project members' field notes that were taken during PLC meetings in which the teachers discussed similar content topics. By comparing these data sources, we characterized information about Claudia's content knowledge and MKT and the influences of these knowledge bases on her pedagogical actions.

Results

In the following section we report the results of Claudia's interactions with her students when they were completing two in-class activities. We also discuss Claudia's meetings with project members and her PLC members in the context of her MKT and ability to support her students' learning.

During the first week of class Claudia's students completed The Diving Task (Figure 1) while working in groups of 3 or 4. The task asked the students to compare and contrast the meaning of average rate of change and average as an arithmetic mean. The previous lesson developed a meaning of average rate of change in the context of average speed (e.g., the constant rate of change needed to cover the same distance in the same amount of time). *Excerpt 1* presents Claudia's interactions with one group of students.

In a diving competition, a diver received the following scores from 4 judges after making a dive.

Judge 1	Judge 2	Judge 3	Judge 4
8.7	9.3	8.0	8.2

How does the meaning of the word *average* when computing a diver's average score compare with the meaning of the word *average* when computing a diver's average speed?

Figure 1 – The Diving Task

Excerpt 1.

S1:	Do you know the meaning of average in this problem?
C:	Tell me what you have so far.
S2:	If the average is 8.55…
C:	Okay, so…
S1:	I said that 8.55 would mean average would be like the constant or the, I don't want to say constant.
S2:	You said approximate.
S1:	The approximate score.
C:	Keep going.
S3:	Can we use mean to describe average?
C:	(*laughs*) No.
S1:	The approximate score to get your score.
S4:	No, cause…
C:	Your…
S4:	…the approximate score given by the judges for that dive…
C:	Keep going, the approximate score by the judges, what…
S4:	…for the dive.
C:	Okay, for what dive?
S2:	For the diver's dive.
C:	For just one dive?
S:	Yeah (*students in unison*).
S2:	They only did one dive.
S1:	If each, if one judge per dive or isn't it…
C:	Four judges for the one dive, okay, alright. Okay so tell me again what you said.
S4:	The approximate score given by the judges for the one dive.
C:	Okay, by which judges?
S1:	By the four judges.
C:	By what?
S2:	By the four judges. (*Claudia confirming the student's answer and moving on)*

During this interaction, Claudia's questioning did not result in her students making explicit connections between average as an arithmetic mean and average rate of change. For instance, she failed to leverage a student's reference to "constant" (lines 5-6) and she did not ask the students to expand on their meaning of "approximate" (lines 7-8, 12, & 15), nor did she ask them to compare and contrast their thinking with the outcomes of previous tasks. Additionally, Claudia's questions were mathematically shallow and she allowed the interaction to end with the students failing to develop the understandings as intended by the curriculum.

Claudia approached a project member after this class session and expressed an uncertainty of the ideas driving the diver task. She also expressed confusion about the tasks relation to previous tasks and claimed that her confusion about the intent of the questions prevented her from determining how to guide and support her students' thinking. Claudia's interactions with the project members, in combination with her interactions with her students in *Excerpt 1*, suggest that Claudia did not understand the content in ways compatible with the curriculum's intent. In lieu of her lacking the understandings necessary to make appropriate sense of the activity, she did not have the foundational knowledge necessary to identify ways to support her students in learning the key ideas of the lesson. Furthermore, Claudia's actions indicate that her understanding of arithmetic average consisted of a vague connection to an approximate value and that she had not cognitively distinguished notions of average rate of change and an arithmetic average in a way that she could support her students in making similar comparisons.

As the semester progressed, the project members observed an increase in the mathematical depth of Claudia's questioning in both her classroom and during her meetings with project members. During a lesson on linearity, during which the students created a graph, Claudia questioned her students in an attempt to draw their attention to determining changes of output for equal changes of input. The shifts the project members observed near the middle of the semester indicated that Claudia was developing understandings that enabled her to interpret the curriculum tasks in ways consistent with the curriculum's design. However, our analysis of Claudia's classroom also revealed that she was having difficulty engendering these understandings in her students (Teuscher et al., submitted). Such a finding suggests that Claudia had not transformed her understandings into mathematical knowledge that was useful in her teaching (e.g., MKT).

As the semester progressed the project members observed Claudia continuing to improve her content knowledge. The project members also observed a transition in Claudia's ability to support her students' learning. The concluding module of the precalculus curriculum introduces trigonometric functions, beginning with a lesson on angle measure. Following the lesson on angle measure, a lesson introduces the sine and cosine functions. Specifically, the curriculum introduces these two trigonometric functions using a circular motion context (e.g., a bug on the tip of a fan blade) and asks the students to determine how the bug's vertical distance above the center of the fan varies as the fan rotates. When beginning this problem, Claudia posed the following question: Does the angle measure change if the radius of the fan is changed, but the distance the bug travels (measured in radians) is not changed? *Excerpt 2* presents Claudia and two students discussing their response to this question.

Excerpt 2

S1:	How does the angle change if the radius changes? The angle stays the same.
S2:	Yeah, but if the distance traveled around stays the same, but the radius changes then it will be a different amount of radians.
S1:	Oh yeah.
S2:	It will be inversely proportional.
C:	It will be a different amount of radians?
S2:	The angle measure.
S1:	Well the distance the bug travels.
S2:	Because as the radius gets bigger, the arc length will be a lesser portion of the circle meaning a lesser angle.
C:	Ok, so draw, ok so let's take any two circles, if you were to cut out a 90 degree angle, actually, draw a circle inside of a circle for me.
S2:	The arc length is…
C:	Now draw another circle - yeah.
C:	Ok, now do you have a Wikki Stix, does anyone, you do?
C:	Ok, first for the inside circle I want you to mark off one radian of the

	circumference.
S2:	Can we just take a Wikki measure, do I have to use radians?
C:	She is trying to find the center of her circle though, is what she is doing. Here let me get you a smaller Wikki Stix. Here is a skinner one it might be easier to use.
S1:	It is two and half so it would be like in the middle
C:	Yeah so for the small circle mark off one radian of the circumference. Ok do the same thing for the bigger circle.
S2:	One radian?
C:	Yeah one radian
S2:	But the radian of an angle measure is not arc measure, the radian and arc measure are the same. No, the angle measure is the same…
S1:	One radian is the same…
C:	What is the same about one radian?
S1:	The same angle.
C:	Yes, what is the not the same about one radian?
S1:	The arc length.
C:	Yes, do you see that S2?
S1:	But then what is distance traveled? Is that the arc length?
S2:	Yes.

Claudia first asked the students to clarify their claim that the radian measure changes when the radius changes (line 6). Student 2 made an incorrect claim (lines 9-10) and in order to support the student in confronting his misconception, Claudia focused the students' attention on the idea of measuring arcs in radii (lines 11-29). Claudia's line of questioning led the students to realize that an arc of one radian, regardless of the size of the circle, corresponds to an angle with the same amount of openness (lines 28-34). As their interactions continued, Claudia probed her students to elaborate on the implications of measuring arcs relative to the radius, with Student 1 concluding, "The angle measure doesn't change if the radius is changed."

Later in the class session, the same group of students attempted to graph the bug's vertical distance above the center of the fan in relation to the bug's distance traveled around the fan. The students were experiencing difficulty relating the two quantities when Claudia joined their discussion (*Excerpt 3*).

Excerpt 3.

C:	Let's back up a step…Your bug's going to start moving. Now what's going to happen to the total distance?
S1:	It's going to increase.
C:	Ok, alright. And what's happening to the vertical distance, in general?
S1:	It's increasing at a decreasing rate.
C:	Ok, it's going to increase, and then, why did you say it's going to increase and then decrease (*referring to a student's statement given previous to this interaction*)?
S1:	'Cause once it gets to 12 o'clock, then it's going to, the vertical distance is going to decrease back to zero.
C:	Ok, what is the vertical distance at 12 o'clock.
S1:	About 2.6 feet.
C:	And how do you know that?
S1:	Because that's the radius.
C:	Ok, so, you know here (*pointing to the top of the fan*) that your vertical distance is

15		2.6 feet. And then what does it do?
16	S1:	It decreases.
17	C:	So, at a total of distance of zero, your vertical distance is zero, right?
18	S1:	(*nodding yes*)
19	C:	Now, what is your…
20	S1:	Then your total distance is 2.6 and your vertical distance is 2.6 as well.
21	C:	So when you get to the top of the circle, when you get to here (*pointing to the top*
22		*of the circle*), have you traveled one radian?
23	S1:	Ya.
24	C:	How many radians are there in a half circle?
25	S1:	Oh, um, pi.
26	C:	So how many radians have you traveled to here?
27	S1:	Pi over two.
28	C:	Ok, so what is that?
29	S1:	That is 1.57 (*identifying values on graph*).

In order to support the students in determining the graph, Claudia prompted the students to reason about variations in the total distance (lines 1-2). As the discussion progressed, Student 1 identified that the vertical distance increased and then decreased as the total distance increased (lines 5-16). Claudia then asked the student to describe specific pairs of values and Student 1 incorrectly identified that the bug travels an arc length of 2.6 feet from the starting position to the top of the circle (lines 17-20). In response, Claudia returned the student to reasoning about measuring the arc length in terms of a number of radii (lines 21-29), which supported the student in correcting her graph and creating the sine graph by the completion of this activity.

Claudia's continued focus (*Excerpts 2-3*) on measuring arcs in radii (which was covered in a previous lesson) suggests that she found this to be an important idea for understanding the sine and cosine functions. Claudia probed her students in ways that supported their measuring arcs in radii when determining the graph of a trigonometric function. These actions suggest that Claudia had developed an understanding of angle measure that she considered developmentally key for learning trigonometric functions (e.g., a KDU), and that she had transformed this understanding in a way that influenced her pedagogical actions and ability to support her students' learning (e.g., she developed MKT).Claudia's interactions with her trigonometry PLC further illustrated that her understanding of angle measure informed her pedagogical actions when teaching the sine and cosine functions. During the year prior to Claudia's use of the curriculum, it did not occur to her or her PLC members that reasoning about measuring arcs in radii is foundational for constructing understandings of the sine and cosine functions. After Claudia's use of the curriculum, her PLC returned to the topic of trigonometry and the PLC members chose to design a lesson exploring applications of trigonometric functions. Claudia became troubled about the topic choice and explained that she did not believe that they could teach applications of trigonometric functions conceptually without teaching angle measure in a way that supported students' understanding of trigonometric functions. Claudia's justification for her decision is consistent with our claim that Claudia's understanding of angle measure as measuring arc lengths in units of radii was key for the way she had come to conceptualize trigonometric functions, and that this understanding provided a basis for her pedagogical decisions.

Discussion and Conclusions

Claudia exhibited content knowledge shifts during the early phases of implementing the curriculum, but she had difficulty supporting her students' learning. At these early stages, her improved understandings did not include connections that supported her re-conceptualizing her teaching in terms of how her students may form similar understandings. However, Claudia's sustained focus on her students' thinking, along with the support she received from the project members, resulted in her building connections that she leveraged to support her students' learning. Specifically, as she developed and transformed KDUs to MKT, the research team observed Claudia scaffolding her students' learning by drawing connections between ideas. Future research should examine the supports (e.g., conceptual curriculum) necessary to help teachers transform their understandings to knowledge that has pedagogical power.

The findings from this study reveal that the nature of a teacher's understanding of an idea is key to her/his conceptualization of teaching that idea and her/his interactions with students when teaching that idea. Claudia's transitions over the course of the semester suggest that a teacher transforming her/his content knowledge in ways that inform her/his practice (e.g., developing MKT) is a complex process. As Stigler and Hiebert (1999) have noted, a majority of pedagogical shifts are superficial. However, Claudia's knowledge and pedagogical transitions were substantive. These shifts that we observed are consistent with Silverman and Thompson's (2008) claim that significant changes in teaching practices result from "pedagogical conceptualizations of the mathematics: both the sense they have made about the mathematics and their awareness of its conceptual development" (p. 507). Claudia was unable to support her students' learning in profound ways until she had acquired key developmental understandings (e.g., measuring arcs in radii to develop trigonometric functions) and used these understandings to scaffold her students' learning (e.g., she transformed these KDUs to MKT). The findings presented illustrate the necessity of future studies that explore the role of KDUs in teaching and the process by which teachers' transform KDUs to understandings (e.g., MKT) that improve their ability to scaffold and build upon their students' learning.

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A Case Study of Three Preservice Teachers on the Units and Unitizing of Fractions

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This study utilized a qualitative design of naturalistic inquiry to deeper and better understanding three preservice teachers' (PTs) knowledge on unit and unitizing of fractions. Findings demonstrated PTs had difficulties profoundly conceptualizing the unit whole and their unitizing abilities were limited. When responding to students' work, PTs identified their own misconceptions and were able to provide some basic instructional strategies for correcting students' misconceptions.

Introduction

For decades, a large percentage of studies have shown that teaching and learning fraction concepts is difficult (e.g., Ball, 1990, 1993; Lamon, 1996, 1999; Ma, 1999; Newton, 2008). Recent results of fourth graders' mathematics performance on the National Assessment of Educational Progress (NAEP, 2009) showed that only 25% of the participants could identify the fraction, 5/8 as the closest value to 1/2 when comparing it to other given values. In contrast, a majority (73%) chose incorrect multiple-choice answers of 1/6, 2/2, and 1/5 demonstrating participants' lack of fraction number sense. The results demonstrated many students did not acquire a strong understanding of fractional concepts during classrooms instruction. Additionally, this may reflect that teachers have failed to provide essential fractional knowledge for children.

Literature has documented that the problem of learning fractions was not limited to young children but adults as well. For instance, Ball (1990) and Ma (1999) acknowledged numerous inservice and preservice teachers (PTs) had insufficient knowledge of fractions necessary for classroom instruction. As a teacher plays a significant role in teaching and learning a particular concept, it is important for the teacher to have profound knowledge of basic concepts. Studying teachers' knowledge for teaching fractions is indeed important as the nature of the topic is complex to learn and teach. However, many past and current researchers were limited in their focus and overemphasized division of fractions (e.g., Ball, 1990, 1993; Li, 2008; Ma, 1999; Rizvi & Lawson, 2007; Tirosh, 2000). Much remains to be explored concerning other fractional concepts that are equally important for students' learning.

The concept of unit is important when a teacher first introduces fractions to students, however it has frequently been neglected in classroom discussions (Lamon, 1999). When teachers and textbook authors placed more emphasis on using the same unit such as one pizza, students assumed that a unit was always a single pizza (Lamon, 1999). Lamon further claimed that teachers' instruction failed to address the important concept of units during the teaching and learning of fractions possibly creating confusion for students. For example, Ball (1993) illustrated a 9-year-old student's misconceptions of the idea of unit when comparing 4/4 and 4/8. The student drew two rectangles (Figure 1) with unequal unit wholes and these models showed the values, 4/4 and 4/8 were equal.

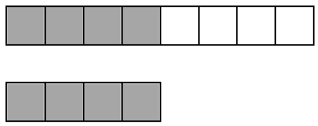

Figure 1. A representational model by a 9-year-old in Ball (1993).

Based on this representation, it appeared the student was not able to connect the concept of unit whole and role of a common unit when comparing two fractions. The student and classmates struggled to identify which one was a larger value from the drawing failing to understand that the rectangles must be the same size (same unit wholes) before making any comparisons. Ball's research findings supported Lamon's argument that many children tended to refer to a unit (one) as a single item (e.g., a single pizza). Most of them are not aware the unit may include more than one item or may consist of items packaged as one known as a composite unit. For example, a package (the unit) may contain two pizzas with the same or different kinds. In this case, it is called a one two-unit and not two one-unit pizzas. Children should be able to differentiate between these two types of units: simple and composite units.

Unitizing is another fundamental topic in teaching and learning fractions that is related to the concept of unit. Lamon (1996) defined unitizing as "The cognitive assignment of a unit of measurement to a given quantity; it refers to the size chunk one constructs in terms of which to think about a given commodity" (pg. 170). For instance, a case of cola can be referred to as a unit of 24 cans, 2 units of 12-cans, or 4 units of six-cans (Lamon, 1999). Lamon also found that the ability to think flexibly with different-sized units is largely dependent on understanding of simple and composite unit concepts. These two concepts were considered basic knowledge for teaching and learning fractions and have long been misinterpreted during classroom instruction. Teachers should first understand unit and unitizing concepts deeply and know how to teach the topics effectively to students.

Theoretical Framework

As fractions are difficult and complex concepts to teach, it is essential for teachers to have a profound mathematical knowledge to teach them effectively to ensure student understanding. The present study focused on PTs' knowledge for teaching and learning units and unitizing of fractions utilizing Ball, Thames, and Phelps' (2008) mathematical knowledge for teaching (MKT) framework emphasizing specialized content knowledge (SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT). According to Ball et al. (2008), SCK is the knowledge and skill specifically necessary for teaching mathematics and not necessary in other settings. It involves the understanding of different interpretations of mathematical operations, languages, representations, and models. KCS combines teachers' knowledge about mathematical content and students' knowledge concurrently. A teacher must always consider student thinking, conceptions, and misconceptions of a particular mathematical topic to make wise instructional decisions (Cooper, 2009). On the other hand, KCT is related to the interaction between teachers' understanding of mathematical content and pedagogical issues contributing to students' learning. For example, teachers need to evaluate which instructional representations are effective in teaching and learning fractions.

The purpose of this study was to examine prospective teachers' (PTs) understanding of unit and unitizing based on Ball et al.'s (2008) categories of teaching knowledge: SCK, KCS and KCT. The findings will provide some evidences of PTs' mathematical knowledge for teaching these concepts. Teacher educators may use this evidence to plan appropriate instructional strategies to improve PTs' learning so that they can acquire the knowledge of unit and unitizing in depth for use in their future classroom instruction.

Method

This study was guided by Lincoln and Guba's (1985) naturalistic-constructivist model of inquiry. Specifically, the qualitative data were gathered through a collective case study (Merriam, 2009) to gain a deeper and better understanding on the selected PTs' knowledge of units and unitizing of fractions. The research questions guiding this study were:
1) What knowledge did selected PTs possess about units and unitizing of fractions?
2) How did selected PTs respond to students' solutions of task-based problems on unit and unitizing of fractions?
3) What instructional strategies did selected PTs recommend for correcting students' misconceptions based on the solutions of task-based problems on unit and unitizing of fractions?

Initially, four female PTs at a southwestern public university were invited through email to participate in this qualitative research. Only, three PTs, Alisa, Talima, and Monica (pseudonyms) volunteered to be part of the study. These three cases were *particularistic* and *heuristic* in nature (Merriam, 2009). According to Merriam, particularistic means "case studies focus on a particular situation, event, program, or phenomenon" (p. 43) whereas heuristic provides readers with meaningful understanding of the context under study. Specifically, the present study focused on understanding PTs' conceptions and misconceptions of units and unitizing (SCK), ability to analyze students' responses (KCS), and PTs' teaching strategies for fraction instruction (KCT). All participants were undertaking a bachelor's degree in education and seeking Texas teacher certification. Alisa and Monica are in the elementary education (PreK-6) major with specialization in language arts and early childhood respectively. On the other hand, Talima is an international student with an emphasis in Grades 4-8 Math/Science. These three PTs completed most of their required coursework, will do their student teaching in the Fall, and graduate the following semester.

The selected fractional tasks were adapted from Lamon (1999) and were expected to disclose the context under investigation, the PTs' knowledge of unit and unitizing. A pilot test was administered to examine the tasks in terms of reliability and validity for assessing PTs' knowledge. All the tasks involved questions such as "how much?" versus "how many?" It was important to capture an individual's understanding of two kinds of questions that examined similar but different mathematical concepts (Sowder, Philipp, Armstrong, & Schappelle, 1998). Specifically, Task 1 focused on the concept of unit wholes in which the participants were expected to find the shaded part of a pizza that was eaten. Task 2, 3, and 4 investigated the knowledge of unitizing. Participants were required to verbalize their reasoning while solving the tasks. In addition, they were asked to analyze three to four children's responses to the same problems they had answered earlier (i.e., subsection 1b and 2b). Based on the children's responses, the participants were required to provide some instructional practices that may be used for teaching the concepts effectively.

The data were collected during Spring 2010 through a semi-structured, open-ended interview session based on task-based problems of fractions. Two participants agreed to be audio-taped, whereas notes were taken during a session with another PT who did not want to be audio-taped because of personal reason. Each session lasted about 30 to 40 minutes depending on the interviewee's responses when they were presented with fractional task-based problems. During the interview, the participants' behaviors and their working papers were noted for analysis as well. The audio-tapes and notes were transcribed and were sent to participants for member check. Also, PTs' working papers were used to triangulate the observation notes and the transcripts of the audio recording and note taking for the one who was not audio-taped. Then, the transcriptions were unitized and analyzed into smaller significant parts or ideas that developed during the examination of data using the constant comparison analysis (Leech & Onwuegbuzie, 2007). Common patterns in participants' responses were grouped into interpretable themes (i.e., SCK, KCS, KCT) in order to answer the research questions.

Results

As noted earlier, the findings were presented in terms of PTs' conceptions and misconceptions of unit and unitizing concepts (SCK), ability to analyze students' responses (KCS), and teaching strategies for fraction instruction (KCT). Specifically, the researchers used responses under category SCK to answer the first research questions. The remaining responses were grouped into categories KCS and KCT and were interpreted to support the final two research questions.

Knowledge of the Units and Unitizing (SCK)

The study showed participants had insufficient SCK about the units and unitizing. They could not distinguish between simple and composite units as defined by Lamon (1999). Specifically, the PTs were not able to recognize that the unit can be a group of objects. For example, in Task 1a the PTs were presented two pizzas using two similar size circles, which were divided into four parts and the shaded part denoted the amount of pizza eaten. All PTs agreed that the first shaded circle was one whole pizza and one of the four parts shaded denoted ¼ of another pizza. Monica indicated 1¼ units of pizza were eaten in contrast to Talima who wrote it in the form of an improper fraction, 5/4. Monica believed the two pizzas were two separate and could not see how the pizzas could be a whole unit or a composite unit. Interestingly, Alisa and Talima mentioned 5/8 (simple unit) as an alternative answer for this task demonstrating that their knowledge was not profoundly sufficient because they ultimately confused the value (5/8). Alisa believed that the pizza contained eight total parts and there were only five parts shaded. However, she canceled 5/8 and argued 1¼ was the best one after considering the case of two pizzas. A week later Alisa contacted the interviewer through an email and said, "I thought about the first math problem right after I left there. It was 5/8 of the pizza, because 'the pizza' refers to all the pizza". Based on Alisa's latter response, she seemed able to reflect on her own thinking about the concept of unit fraction - a group of two pizzas.

In addition, the PTs were presented Task 1c, which related two different sizes of pizza (i.e., small cheese pizza and medium pepperoni pizza) that were cut into eight equal-size pieces. Also, the participants were informed that two and three slices of the cheese and pepperoni pizzas were eaten respectively. When they were asked how much pizza was eaten, Alisa and Talima were able to identify the different sizes of the wholes (small versus medium) and to obtain the correct answer (i.e., 2/8 small pizza and 3/8 medium pizza). These same PTs said the sizes were not equal, therefore, one could not add both parts together to get 5/16. In this case, it seemed that they could understand the concept of common unit as discussed by Ball (1993). Alisa further argued that it was impossible to add two different sizes of pizza similar to adding apples and oranges. In contrast, Monica counted all the number of pieces eaten from two different sizes of pizza and provided 5/16 of the total pizza as the answer. Monica's latter response was contradictory with her earlier session for Task 1a in which she said that two items could not be put together to become a larger one unit. It showed that Monica had misconceptions about the concept of units because she could not associate the different sizes of pizza meaning unequal unit wholes. The select PTs' specialized content knowledge (SCK) of a unit-whole was inconsistent; they did not understand that a group of items of similar sizes could be one-unit whole.

On the other hand, in Task 3a the PTs' responses revealed their unitizing abilities were different. The task required PTs to decide which one was a better buy between two cereal boxes: 16 ounces ($3.36) or 12 ounces ($2.64). Alisa simply thought that the bigger one would be cheaper per ounce and upon this based her generalization even though she admitted that this may be applied 80% of the time. Talima unitized the number of ounces with the rounded price ($3) to obtain an estimated cost per dollar. She chose the larger box because she could get more ounces per dollar. Monica calculated the difference of prices ($0.70) and amount of cereal (4 ounces) between the two boxes. She estimated $0.60 and $0.80 per ounce for each box respectively deciding to choose the cheaper box if the calculation was correct. In contrast to Task 3a, PTs possessed a similar understanding of how they unitized a case of cola in Task 2a. The participants were provided a scenario, "Steve took a case of cola to Marcia's party, but it turned out that many of his friends drank water. He ended up taking three quarters of the cola home a. How much cola did he take home?"Alisa, Talima, and Monica agreed that the answer was ¾ of whatever the unit whole was. In addition, Alisa visualized her thinking of the case of cola by using a standard case of 12 cans, then multiplied ¾ with 12 and got 9 cans of cola as the answer. However, she still believed that ¾ of the case of cola was still the best answer as the task did not indicate the actual cans of cola in a case.

The "how many" question was quite easy for the PTs even though only Alisa got the answer correct during her first attempt. Alisa explained her working algorithmically,

> I just look at the total [writing 7⅓], how many ⅓ [writing ⅓ under "a second super piece free"] are in there? So, ok 21/3 plus 1/3, 21..7 times 3 [writing 7x3 on the left side of 21/3] equals 22/3 [writing 21/3 + 1/3 = 22/3]. Then 22 divided by 2 equals 11 [writing 22/2=11], 2 is what each customer can get....each customer will get [writing each customer will get] because there is buy one get one free [writing buy 1 get one free]. Ok, so 11 [circle 11] customers [writing customers as the label and underlining the word].

On the other hand, Monica and Talima used pictorial representation to answer this task. They drew eight circles, divided each into three pieces and shaded 7⅓ circles. Then, they paired every two parts and counted the number of pairs together which gave them 12 at first. Talima counted back and realized that she made a 'silly mistake' when adding the number. Although Monica did not check the final answer, she, Talima and Alisa had a clear understanding of 'how many' as compared to 'how much' in Task 1a, 1c, and 2a. When they were asked about the difference, Alisa mentioned the 'how many' question was easier to solve compared to the 'how much' question. Talima argued that 'how much' was related to measurement and 'how many' was for counting items. On the other hand, Monica thought that when a question asked 'how much pizza', it would mean number of slices and 'how many pizzas' referred to how many whole pizzas were involved. The select participants' responses were similar to results found by Sowder et al. (1998).

Knowledge of Content and Students' Responses (KCS)

The select PTs were able to analyze the children's responses by providing brief descriptions supporting their arguments albeit they had some difficulties on their own. The PTs could identify from the children's responses that they failed to label the final answer for Task 2b. For this reason, the PTs had difficulties interpreting what size chunks the children considered as a case of cola. At some points, the PTs were not sure what the children were thinking and how hard it was for them to predict the figures based on the children's unlabelled responses. In general, Monica explained,

> It seems like that they all assume something to be a whole, maybe I would ask them what they consider the whole to be..try to find out why and how they are getting these answers because it just give me numbers. I would ask them maybe to show their work, go back and show their work and then resubmit it or I will ask them to maybe verbally explain to me what they were thinking, what their thinking process was…to try to find out.

In contrast, Alisa and Talima were able to consider students' thinking, conceptions, and misconceptions when responding to this task. Specifically, they focused in much detail, on the number that children provided. Alisa and Talima tried to predict children's thinking with some drawings and working backward to solve the problem.

Through the observations, the interviewer noticed that Alisa and Monica slowly realized their own mistakes to their previous answer while they spoke aloud their reasoning about the children's responses in Task 1b. It showed they were able to make connections to what they already knew. Specifically, Alisa was able to detect her own misconceptions when reading Joe's response:

> Alisa: Ok, Kristin did 5/4 because 5 little ¼ pieces are shaded. She was thinking in the right direction that she forgot that it's ¼ of each pizza but there two pizzas so there 8 total pieces [I: Ok]. So 5 little one ¼ pieces, but what's the total? She..she put the total 4 but the total 8 of ¼ little pieces [I: Ok]. So, not correct, sorry Kristin [laughing and put a wrong sign beside Kristin's answer]. And then Jean…Jean got it correct 5/8 because there is eight pieces and 5 are colored. That's correct Jean. They got the total number and little pieces number. Tom, 1 and a bite. I guess that's correct but a bite is not as you know descriptive, that's a big bite I think. But guess you could call it correct because a bite could be ¼ of a pizza [writing ¼ denotes a bite]. Joe, it must be 5/8 because 5/4 is impossible. Ohhh..5/4 is [speak slowly]....[both were laughing]…now I think 5/4 is correct. Sorry 5/8 is not. [I: That's ok. That's ok] Oh my God I just realized that. I feel fall down [laughing]. Ok.
>
> I: Are you sure that Joe is correct? Yeah. Just..Just..yeah [Alisa: I know, it was difficult]. Just explain your….
>
> Alisa: Not I could think of it [drawing a big circle with 8 parts]. 5/8 is like..they could..I don't know... that will be like 8 and [shaded 5 parts] then 5 parts shaded, right. Now when I think of it 5/4, he's right because 4/4 equals 1 [writing it besides the big circle] and then the 1 left over will be the 1/4 [writing ¼ equals ¼ below 4/4=1]. So, may be 5/8 is not correct [laughing].
>
> I: 5/8 is not correct?
>
> Alisa: This is…this is confusing! [both were laughing]

These findings demonstrated that Alisa and Monica's thinking was easily influenced by the children's responses. They admitted that they were confused with their own answers after analyzing the responses supporting the notion of PTs' limited understanding of unit fractions. For example in Task 1b, Alisa and Monica believed the children's responses varied because the unit whole could be one object or more objects in the same context. They stated the children made choices about the unit they wanted to use. However, Lamon (1999) argued that deciding on the unit was not a personal choice and teachers should make sure students are aware that the unit may vary in every new context. Unfortunately, Talima was not aware of her own misconceptions while she was analyzing the responses. She strongly believed all the children did not understand the units and unitizing of fractions based on their responses.

Knowledge of Content and Teaching (KCT)

As mentioned earlier, KCT is related to the interaction between teachers' understanding of mathematical content and pedagogical issues contributing to students' learning. The findings found that the participants were able to suggest a number of basic instructional strategies that may be used in a classroom. As Monica, Alisa, and Talima acknowledged that the concept of unit and unitizing was important for teaching and learning fractions, they agreed with the use of concrete manipulatives, real-life application, and multiple representations such as pizzas and cake for learning fractions. Specifically, Talima strongly believed real-life examples made mathematical learning interesting for students when they shared ideas with their family. On the other hand, Alisa argued that teachers should use the same representations for basic fraction instruction so students can grasp the concept clearly.

Similarly, Monica thought pizza was a good representation because teachers could use it consistently in the classroom. However, she admitted that any representation would create problems and confuse the students if the concept was not taught clearly by the teacher. Monica made a connection to the students' responses explaining why the children considered using a different unit whole. She argued this might be due to a teacher's pedagogical style that children had gotten accustomed to. In terms of teaching, Monica further stated that a lesson "reteach" (Cooper, 2009) would be needed if students in her class could not understand the concept of unit wholes. She believed Task 1a was a great example for students to strengthen their understanding of whether to consider one or two pizzas as a unit.

Discussion

In conclusion the findings reveal the participants' knowledge (SCK, KCS, KCT) of teaching units and unitizing of fractions is limited. In regard to the first research question, the findings showed that PTs could not differentiate between 'how much' and 'how many' questions. This affirmed Lamon's (1999) argument that PTs' profound knowledge of the fractional unit was not clear. Even though the selected PTs seemed to slightly understand the concept, they had difficulty thinking during the interview session about the composite unit or unit that contained more than one same size objects. In contrast, the select PTs could unitize the items into different groups; however the strategies they used were mathematically inefficient and not flexible. They could not reconceptualize or reunitize the ounces into easier, faster, and more reasonably sized chunks (Lamon, 1996). Units and unitizing are essential concepts and important for teaching and learning of fractions; therefore, PTs must have adequate mathematical knowledge for teaching that should be developed during their teacher preparation program (Ball et al., 2008; Newton, 2008).

In relation to the final two research questions, the findings demonstrated that the PTs' knowledge of units and unitizing (SCK) might have an effect on their understanding KCS and KCT. Specifically, the select PTs' limited understanding of these concepts influenced their ability to identify students' misconceptions. Surprisingly, they were able to notice their own misconceptions while analyzing students' responses. Also, they offered some instructional strategies to help children limited to basic representational models such as pizza and cake. The select PTs appeared to have limited experiences with children, therefore, more exposure working directly and engaging in classroom teaching and learning is needed. This kind of experience will enhance and develop PTs' understanding of educational and mathematical theories into practice. Teacher educators are accountable for preparing prospective teachers with sufficient profound knowledge and experience necessary for teaching mathematics effectively. Future studies are needed to examine PTs' knowledge of fractions in more extensive and broad perspectives with these concepts, including quantitative data.

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MOVING BEYOND THE TRADITIONAL STRATEGY: PRESERVICE TEACHERS' RESPONSES TO STUDENTS' CORRECT AND INCORRECT STRATEGIES IN WHOLE NUBMER SUBTRACTION

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Recognizing meaning in students' mathematical ideas is challenging, especially when such ideas are different from standard mathematics. This study examined, through a teaching scenario task, the reasoning and responses of preservice elementary and secondary teachers to students' correct and incorrect strategies for whole number subtraction along with the standard algorithm. Findings revealed that although a large portion of the preservice teachers recognized the validity and generalizability of the student methods correctly, they justified it from the procedural aspect of each method. When they were asked to provide interventions and connect the student method to the standard method, a majority of preservice teachers tended to use what is called 'show-tell' responses.

Introduction

The idea of interpreting and responding to student thinking is highlighted as one of the central tasks of reform-minded mathematics teaching (NCTM, 2000). Yet, recognizing meanings in students' mathematical ideas is not easy task for teachers, especially when such ideas are different from standard mathematics. Van de Walle, Karp, and Bay-Williams (2010) assert that students who invent strategies or adopt them from classmates are involved intimately in the process of making sense of mathematics and gain confidence in their abilities. Along the same line, NCTM (2000) also stresses that teachers need to spend a significant amount of time and effort on student invented methods or their informal strategies that arise in a typical mathematics classroom and think about how to help students build on them before introducing the standard algorithms.

In this study I set out to investigate preservice elementary and secondary teachers' interpretation of and responses to students' correct and incorrect strategies involving whole number subtraction, and their teaching approaches in connecting student strategies and the standard algorithm. Although a growing body of researchers explored teachers' knowledge and their teaching strategies in whole number computation (e.g., Hill, Ball, & Schilling, 2008; Thanheiser, 2009), preservice teachers' responses and their strategies to student-invented methods has received limited attention in the research literature. In particular, we know little about how preservice teachers connect student-invented strategies to traditional algorithms. Whole number subtraction is known to be one of the most challenging operations in both calculation and justification of changes made to numbers in elementary school mathematics (Flowers, Kline, & Rubenstein, 2003). Not only children experience difficulty in understanding the reasons for different subtraction strategies, but also teachers (e.g., Huinker et al., 2003). It is important to examine how preservice teachers interpret different ideas that may emerge in the classroom and prepare them to deal with such ideas in connection with the standard algorithm.

In this study, I used a student-invented strategy reported by Carroll and Porter (1997) in which the difference is recorded as a deficit or a negative number (e.g., 40-70= -30) so that no "borrowing" or regrouping is necessary where the subtrahend is larger. According to the literature (Campbell, Rowan, & Suarez, 1998; Carroll & Porter, 1997), students often invent this method on their own with little difficulty. Nonetheless, this method may become cumbersome and difficult to use as the numbers get larger, which requires a need for a more efficient method (e.g., the traditional method). Thus, this study examined preservice elementary and secondary teachers' reasoning and responses to the correct and incorrect student-invented strategy and how they attempt to connect such a strategy to the traditional method. The research questions for the study were formulated as follows:

1. How do preservice teachers interpret and respond to student-invented strategies with whole number subtraction along with the traditional method?
2. How do preservice teachers connect the student-invented strategy to the traditional strategy?

Conceptual Framework

Research on Students' Invented Strategies in Whole Number Subtraction

Understanding flexible methods of computation with whole number subtraction requires sound understanding of (1) place value, which involves being able to group by tens and treating the groups as units, (2) properties pertaining to operations such as the associative of addition, commutative, and distributive property, and (3) the relationship between addition and subtraction. Table 1 shows the student-invented strategies for whole number subtraction reported from the literature with the example problem 41 – 25 (e.g., Huinker et al., 2003).

Strategies (subtracting by)	Characteristics	Examples (e.g., 41-25)
1. Adding up	Focus on the meaning of subtraction as the inverse of addition (think-addition strategies). Add up from the subtrahend to the minuend in groups of numbers by answering what I need to add to 25 to get 41.	25+<u>10</u> is 35; 35+*5* is 40; <u>1</u> more is 45. The difference between the two numbers is the sum of the numbers added: 5+10+1=16
2. Taking away	Break the subtrahend apart and taking each part away	41-25= 41- (20+5) 41-20→21-5=16
3. Compensating	Change either subtrahend or minuend as compatible number and then compensate	41-25→45-25=20→20- 4 =16 (add 4 to the minuend and then subtract 4)
4. *Partial differences*	Decompose the numbers based on the place value, subtract each part separately without regrouping, and add the partial differences.	41-25→ (40 +1) – (20+5)→ (40-20) + (1-5)=20-4=16

Table 1. Student-Invented Strategies for Whole Number Subtraction

The *subtracting by partial differences* strategy was chosen for the study since the method requires students to use negative numbers. This method allows students to subtract separately in each column without requiring regrouping, which lead students to use a negative number. Carroll and Porter (1997) point out that it is not important whether students actually think of these partial differences as negative and positive numbers and students may simply consider the negative as "being in the hole" or having a deficit of that quantity. However, this method would be become cumbersome and difficult to apply when applied to a problem involving larger numbers (e.g., 642-489). In addition, some students tend to express the negative number as '0' while using this method (see, video clip from the *Developing Mathematical Ideas*). For these reasons, I was curious about how preservice teachers interpret and respond to a student who used this method and help her/him connect this alternative strategy to the traditional algorithm.

Research on Teacher Knowledge and Approaches with Whole Number Subtraction

A growing body of researchers has explored teachers' knowledge and their teaching strategies in whole number computation. Flowers, Kline, and Rubenstein (2003), for example, investigated elementary school teachers' understanding of whole number subtraction problems. They reported that although most teachers could explain how they perform a procedure, the teachers had difficulty justifying why it works. Some teachers tried to use rules they did not fully understand or that did not apply to the situation to describe the reasons behind their procedures. In addition, Flowers et al. revealed that although teachers recognized the value in developing children's informal strategies, most of the teachers did not fully explain why it worked. Similarly, Thanheiser (2009) reported preservice teachers' lack of understanding about standard algorithms for multi-digit whole numbers and their inability to help the child develop conceptual understanding of regrouping in whole number subtraction. She suggested that teacher educators must be able to build on preservice teachers' initial conceptions and help them move beyond their misconceptions and be able to clearly explain the mathematics behind it. Although these studies have helped us gather important data on preservice teachers' understanding of whole numbers and computation, much research needs to be done concerning how preservice teachers interpret and respond to student-invented strategies and how they connect student-invented strategies to traditional algorithms.

Methods

This study investigated preservice elementary and secondary teachers' interpretation of and responses to student-invented strategies through a classroom scenario in which two hypothetical students come up with different solution methods to a two-digit whole number subtraction problem, as addressed in Figure 1. This task was modified based on an actual elementary student who appeared on a video clip from the *Developing Mathematical Ideas* professional development curriculum (Schifter, Bastable, & Russell, 1999). The first student, Tommy, used a student-invented (*subtracting by partial differences)* strategy and produced the correct answer. Sally, the second student, tried a method similar to Tommy's, but did not understand and failed to execute the procedure correctly.

<table>
<tr><td colspan="2">You are teaching whole number subtraction problem 62-25 to third graders. You asked students to solve it. After a few minutes, two students came to the board and explained their methods in a following way</td></tr>
<tr><td>Tommy's strategy

62
- 25
- 3
40
37

Tommy said:
You have 62 and 25. Take 5 away from the 2</td><td>Sally's strategy

62
-25
0
40
40
Sally said:
Well, I have a different answer from Tommy. The answer is 40. I put 0 in the ones place since I</td></tr>
</table>

and you get ⁻3. Now you take 20 away from 60. That leaves 40. Then take away 3 from the 40, and that's 37. The answer is 37.	can't subtract 5 from the 2 because 5 is bigger than 2. And then I take 20 away from 60 and get 40. Since there is nothing to take away from the 40, the answer is 40.

Figure 1. Pedagogical Task: "How would you respond to Tommy and Sally?"

Three questions were developed in order to assess preservice teachers' understanding of and teaching strategies to these two students' strategies as well as the traditional method for whole number subtraction. In the first question, preservice teachers were asked to determine whether each method worked for all whole numbers and then were asked to explain their reasoning. Preservice teachers were then asked to respond to Tommy and Sally (intervention). In the third, they were asked to provide interventions for Tommy in order to develop the traditional algorithm for subtraction.

One hundred-eleven preservice teachers participated in the study from three mathematics methods classes. Ninety-six preservice teachers were in their pre-internship year of elementary teacher preparation program and fifteen were math majors seeking middle and secondary certification at a large southwest university in the United States. Preservice teachers in elementary and secondary programs were included in order to obtain a broad range of responses to the study's task. The task went through multiple layers of development which included pilot testing with two volunteers who were interviewed in order to check for possible misunderstandings. The task was then administered as a survey to the entire class in three mathematics methods course sections towards the end of the semester. I only report on the data of the preservice teachers who signed the study's consent form.

In analyzing preservice teachers' interpretation and approaches to these student-invented strategies and the traditional method, I was guided by studies such as Rittle-Johnson and Alibali (1999), which distinguish between the use of procedural and conceptual knowledge. I also referred to three criteria articulated by Campbell, Rowan, and Suarez (1998)—validity, generalizability, and efficiency—that teachers should consider to support various student-invented strategies in teaching situations. In particular, I analyzed based on two broad categories developed from Kuhs and Ball's (1986) models of mathematics teaching—'show and tell' and 'give and ask'. After that, their teaching strategies were further analyzed by looking at the patterns apparent. More detailed examples of these categories and the criteria used will follow as I discuss the findings.

Summary of Results

1. How Do Preservice Teachers Interpret Tommy's and Sally's Method?

I mention briefly the findings from preservice teachers' interpretation of Tommy who invented a correct strategy and Sally with an incorrect strategy, as they were helpful in providing an initial framework for analyzing their responses to student-invented strategies. In the first problem, preservice teachers were asked to determine whether Tommy's method works for all whole number subtraction problems and justify why it works. All preservice secondary teachers answered that Tommy's method was correct and generalizable whereas 77% of preservice elementary teachers did so. 23% of the preservice elementary teachers (22 out of 96) answered that Tommy's answer was correct but did not work for all whole number subtraction problems. Preservice teachers' explanations to Tommy's method were further analyzed by looking at whether they pointed out the underpinning ideas of Tommy's method, e.g., place value and distributive/ commutative properties while referring to negative numbers.

Table 2 shows a four-point rubric developed based on the sophistication of preservice teachers' justification, ranging from four, being mostly conceptual, to one, being incorrect or ambiguous. More than half in both groups fell in level two, indicating that both preservice elementary and secondary teachers tend to provide procedural-oriented explanation. However, notice the different tendency between preservice elementary and secondary teachers. While more than half preservice secondary teachers fell in levels 4 and 3, similar percent of preservice elementary teachers fell in levels 1 and 2 (see Table 2).

	Description	Participant Example	Elem. (N=96)	Second. (N=15)	Total (N=111)
4	Describe both the use of property (e.g., distributive/ commutative) and flexible use of place value (decomposing) while making reference to negative numbers.	*The distributive property always works well with whole numbers. Because subtracting is really adding a negative, (60 + 2) - (20 + 5) = (60 + 2) + -1(20+5). Distributive=> 60 + 2 + -20 + -5 Because of the commutative property of the whole numbers with addition this = 60 - 20 + 2 - 5 = 40 + -3 = 37.*	2%	20%	5%
3	Demonstrates somewhat conceptual understanding with use of negative numbers instead of regrouping and/or vague understanding of place value.	*As long as he keeps his place values correct, he can subtract 60 - 20 or any of the tens numbers. Also, because he understands negatives the problem will work every time.*	33%	33%	33%
2	Mention place value, but response is mostly based on procedure or slightly ambiguous.	*I believe Tommy's strategy will work for all whole numbers because he subtracted the one's first and the ten's next.*	40%	47%	41%
1	Provide irrelevant, or incorrect information	*Because he cannot invert the numbers because it changes the value.*	25%	0%	22%

Table 2. Rubric describing the nature of explanation about Tommy's method

As for Sally's strategy, all preservice teachers answered that Sally's method was incorrect and were not generalizable. Table 3 shows a four-point rubric describing the different levels of preservice teachers' justification to Sally's method. In general, overall scoring for responses to Sally were lower than Tommy. The majority of participants scored level 2 in which preservice teachers just restate Sally's procedure without further explanation. When comparing elementary and secondary teachers, similar portions scored level one where preservice teachers were correct in their thinking of the strategy, but provide too ambiguous or incomplete explanation.

	Description	Participant Example	Elem. (N=96)	Second. (N=15)	Total (N=111)
4	Describe limited concept of place value and regrouping by Sally while describing the relationship between tens and ones in place value.	*Sally's strategy does not work because not only does she not understand the concept of place value, she does not understand the concept of regrouping. She does a good job at taking away in the 10s place. It is important for subtracting problems to take both the 1s and 10s place values into account by knowing 10 ones make 1 tens.*	1%	7%	2%
3	Mention limited concept of place value or limited concept of regrouping by Sally but do not fully explain the relationship between tens and ones in place value	*Sally doesn't have a concept of how tens relate to ones with subtraction. In this case, she would need to realize that 62-25 is not the same as 62-22. Her assumption does not work here, and it will not work for all whole number pairs.*	12%	20%	13%
2	Describes Sally's procedure by pointing out the use of '0' and vaguely mention place value or lack of understanding with negative numbers.	*Sally's strategy will not work for all whole numbers because when working with multidigit problems and the bottom number is larger than the top you cannot simply place a zero in its place and move to the next digit.*	71%	61%	70%
1	Did not provide justification, or provide irrelevant or incorrect information	*It won't work because it is logically incorrect and didn't even work when she did it for this problem.*	16%	13%	15%

Table 3. Rubric describing the nature of explanation about Sally's method

2. How Do Preservice Teachers Respond to Tommy's Method?

In the second question, preservice teachers were asked to respond to Tommy and Sally. As previously mentioned, in analyzing the responses of the participants, two categories were identified first—'show and tell' and 'give and ask'—and then their responses were further analyzed by looking at the patterns apparent under each category. In a 'show-tell' approach, the preservice teacher tells, explains or shows whether each method works and provides direct information about the two strategies. The 'give-ask' approach in this study is evident when a preservice teacher provides students opportunities to explain and justify their strategy, and would guide students as they figure out when each method is or is not efficient. Preservice teachers who concluded that Tommy's method was not generalizable were also analyzed to see how they responded to Tommy as categorized into 'incorrect'. Shown below is the various responses in each approach and the percentage of responses. The data dispersion for secondary and elementary preservice teachers was consistently about the same for all categories. A large portion of the preservice teachers was categorized into the 'show-tell' approaches by confirming or rephrasing Tommy's procedure (44% in the 'show-tell' approach) and/or showing Tommy the traditional method as Tommy was assumed to be incorrect (20% in the incorrect). Only around 40% asked Tommy to explain his thinking. About sixteen percent of the participants gave what we would consider fairly ideal responses through student involvement and connecting or building on strategies.

Show and Tell (44% in total): Conformation only (7%), Confirms and reminds of negative numbers (7%), Rephrases Tommy's strategy (17%), Confirms and shows another (traditional) method (6%), Points out the usefulness of Tommy's strategy (5%), and Recognizes Tommy's strategy is inefficient for larger numbers (2%)
Give and Ask (36% in total): Asks Tommy to explain thinking (14%), Asks Tommy to solve another problem (7%), Asks Tommy to show other students to help them understand his strategy (8%), Asks Tommy to come up with another strategy (5%), Get Tommy to recognize the inefficiency with larger numbers (1%), and Asks other students to show strategies to connect (2%)
Incorrect (20% in total): Show and tell proving counterexamples without reason (4%) or tell Tommy his strategy does not work and ask to find another way without negatives (3%), No explanation and shows another method (traditional) (7%), Asks Tommy to apply method where top number is greater (2%), Asks Tommy to explain use of negatives (1%), and Asks Tommy to explain and solve another problem (3%)

3. How Do Preservice Teachers Respond to Sally's Method?

Preservice teachers' replies to Sally were overwhelmingly "show and tell" based. About 77 percent of all participants explained Sally's errors or showed her the traditional method. Only about ten percent of the participants, all being elementary level, attempted to have Sally recognize her error without being directly told. Only around 23% asked Sally to explain her thinking and used it in correcting her mistake. Shown below is the various responses in categories of 'show and tell' and 'give and ask' and the percentage of responses.

Show and Tell (77% in total): Tell or explain errors to Sally (28%), Tell or explain errors and strengths (5%), Tell Sally to use traditional method (20%), Tell Sally to use Tommy's (with negative numbers), or other strategy (16%), and Use concrete model to show errors with place value and or address the traditional method (9%)
Give and Ask (23% in total): Point out the source of Sally's errors and have her rework the problem to correct (12%), Ask her to explain her strategy and teacher shows traditional method (1%), Ask Sally to explain her method and to try another method (4%), Ask Sally to use manipulatives to find her error and see the traditional method (3%), and Have Sally recognize her error by comparing to other methods (3%).

4. How Do Preservice Teachers Connect Tommy's Method to the Traditional Method?

The third question, referring to Tommy, asked participants to explain the traditional method by making a connection to his invented strategy. In analyzing the responses of the participants, two categories—'show and tell' and 'give and ask'—were utilized first. And then two additional categories were identified separately depending on whether preservice teachers provided the connection between Tommy's method and the traditional method. Then their responses were further analyzed by looking at the patterns apparent under each category. Below shows the various responses in each approach and the percentage of responses. Even though participants were asked to connect Tommy's method to the traditional strategy, only 28% of the preservice teachers attempted to show the connection between two methods. 42 % relied on solely explaining the traditional method without any models or using any type of questioning with Tommy.

No Connection (71% in total): Explains traditional method (without manipulatives) with regrouping (38%), Explains traditional method (without manipulatives) with compensation (5%), and Explains traditional method with manipulatives or pictorial model (28%).
Connection (28% in total): Has Tommy explain place value/his method and explore regrouping (5%), Points out inefficiency with negative numbers and explains traditional method (11%), Points out inefficiency with negative numbers and explains traditional method with manipulatives (5%), and Shows both methods and how they relate (5%).

Discussion and Implications

The findings of this study shed light upon preservice teachers' mathematical understandings and their teaching approaches to student-invented strategies for whole number subtraction. While all preservice secondary teachers in this study determined the validity and the generalizability of the student-invented method correctly, a large portion of preservice elementary teachers did not initially recognize Tommy's strategy as a legitimate method for subtracting. However, such differences became less distinct when it came to justifying the reasons behind the procedure and providing good intervention. Preservice teachers in our study, both secondary and elementary, gave mostly procedural explanations for Tommy's strategy. They also showed a tendency to use 'show and tell' approaches rather than 'give and ask' when trying to help students construct their own knowledge. In particular, procedural-oriented responses were more evident when student methods were assumed to be incorrect and when preservice teachers were asked to respond to students, in particular to Sally who provided an incorrect method. This result is consistent with the findings from the Son and Crespo' study (2009). Based on the findings from this study, I propose that preservice teachers' insufficiently developed knowledge of students and teaching strategies would be a reason for creating a procedurally oriented instruction. Despite the limitations of this study (e.g., the small ration of secondary to elementary preservice teachers), this study has implications for teacher educators and future studies. For example, teacher educators must provide opportunities for preservice teachers to explore student-invented strategies and be able to clearly explain the mathematics behind it. Additional research is needed to provide substantiation in how we can help teachers prepare for the multiple student strategies that arise in a typical mathematics classroom.

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TEACHERS' KNOWLEDGE IS BUILT AROUND THEIR GOALS

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Teachers' goals are an understudied link in the chain from teachers' knowledge, beliefs, and decisions to students' learning and opportunities to learn. Whereas the domains of teacher knowledge, beliefs, and behavior have received much research attention recently, teachers' goals—a critical link between these areas—remain understudied. The reasoning teachers engaged in related to their goals provided a window into their knowledge and beliefs and how their knowledge and beliefs developed around their goals. This information indicates that teachers' goals can be very useful to uncover teacher knowledge and highlights how influential teacher goals are in the overall learning environment.

Purpose

High on the radar of education researchers is the special amalgam of content and pedagogy known as pedagogical content knowledge [pck] (Shulman, 1986). And while much effort is being made to define and measure pck, equal effort is being made to develop it. And yet, often overlooked is how pck develops organically in the ordinary course of teachers' experiences. The research question guiding this research is: How do teachers' goals of instruction contribute to their development of pck?

The findings reported in this paper were derived from original research that broadly examined the nature of five secondary mathematics teachers' goals of instruction. Hiebert and Grouws (2007) defined teaching as consisting of "classroom interactions among teachers and students around content directed toward facilitating students' achievement of learning goals" (p. 372). Learning goals are at the core of the educational enterprise and research on teaching toward specific learning goals in intervention settings claims robust results. Hiebert (2003) wrote, "When extra attention is paid to designing classroom instruction with specific learning goals in mind, students usually improve their achievement of these goals (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb et al., 1991; Cognition & Technology Group at Vanderbilt [CTGV], 1997; Fawcett, 1938; Good, Grouws, & Ebmeier, 1983; Griffin et al., 1994; Heid, 1988; Hiebert & Wearne, 1993; Markovits & Sowder, 1994; Stein & Lane, 1996; Wearne & Hiebert, 1989; Wood & Sellers, 1996)" (p. 14). Echoing and extending Heibert's assertion here, I contend that when extra attention is paid to designing various aspects of classroom instruction with specific goals in mind, the teacher develops greater knowledge and power in that area to achieve the recognized goals.

Theoretical Considerations

Goals are integral to the cognitive aspects of teaching. In Schoenfeld's (1998) efforts to develop a theory of teaching that enables one to "explain how and why teachers do what they do *while engaged in the act of teaching*" (p. 6, emphasis in original), he generated a cognitive model of teaching, where knowledge, goals, and beliefs activated in context affect and influence teachers' decision making and actions. These components are influenced by one's historical knowledge of past experiences with the students, the mathematics they have learned, and the plans and routines that have previously been enacted. Schoenfeld's model advances the cognitive research on teaching and brings goals onto equal footing with knowledge and beliefs.

Goals are defined as "cognitive representations of what individuals are trying to accomplish and their purposes or reasons for doing the task" (Pintrich, 2000, p. 96). Goals direct action toward desired end states and come with recognition that the present state is lacking. To have a goal is to recognize the difference between an ideal state and one's perception of present reality. This recognized difference creates "a state of tension that [is] maintained until reduced by the performance or completion of the intended activity or a substitute activity" (Locke & Latham, 1990, p. 13).

Knowledge plays a special role in attaining goals. When a goal is chosen, related knowledge is activated to the attainment of the goal. For example, if a teacher chooses the goal that students understand equivalent fractions, the teacher's knowledge of representations and classroom sequences are drawn upon to meet the goal. But if a goal is chosen that one does not have the knowledge to attain, and if one is committed to the goal, the teacher will search for and build knowledge necessary to meet the goal. This explains Ball's (1993) efforts to develop meaningful representations of negative numbers for her 3rd grade students, an effort prompted by her dilemmas that arose "directly from [her] explicit goals" (p. 377). Locke and Latham (2002) reported that challenging "goals… [can lead] to the arousal, discovery, and/or use of task-relevant knowledge and strategies (Wood & Locke, 1990)" (p. 707). In other words, the motivation to develop specific knowledge arises through dilemmas related directly to our goals.

Method

Case study methodology provided direction in conducting this research. As little research has been done to link and describe the connection between teachers' goals and their pck, I chose teachers who were likely to be information rich by virtue of their experience and expertise. The two teachers in this study held master's degrees in curriculum and instruction and were recognized by their administration and university faculty as leaders in the school.

Natalie (as with all participants, a pseudonym) was in her 11th year as a teacher when I observed her. The majority of her experience was in teaching from the *Core-Plus* curriculum. However, in her ninth year she moved to a new school where she principally taught Algebra 1. The year in which the observation was made she was teaching Algebra 1 for the second time.

Sarah was in her fourth year teaching high school mathematics. Sarah has since been accepted into a PhD program in mathematics education. Sarah had taught only integrated courses using the Core-Plus curriculum until the year of observation in which she also taught Geometry using a traditional textbook.

Data Collection and Analysis

In keeping with the case study research tradition, this study drew upon "multiple sources of information such as observations, interviews, documents, and audiovisual materials" (Creswell, 1998, p. 62). Each teacher was observed at least four times and was interviewed after second, fourth, and sixth classroom observations. Classroom teaching sessions were video recorded and used for purposes of stimulated-recall interviews. These interviews were semi-structured and were designed to allow investigation of teachers' goals around significant components of their instruction.

The focus of the data analysis was to find and describe how teachers' goals contributed to their development of pck. Analysis proceeded by identifying dilemmas the teachers were struggling with, then examining how their efforts led to development of pck.

Findings

To illustrate the findings for each teacher, I describe a broad mathematical pedagogy dilemma followed by a more narrowly focused mathematical pedagogy dilemma and the resulting efforts and development of pck. I begin with Natalie. Natalie struggled with getting her Algebra 1 students to share their thinking and to keep them engaged. These two dilemmas turned into broad goals that she spent a great deal of time thinking about. This thinking led to her reasoning about the dilemmas and experimenting with solutions. Natalie recognized a strong contrast between teaching Algebra 1 and Core-Plus courses. Indeed, in her interview she lamented that students were not used to having to discuss ideas. She explained, “When I taught with the core-plus curriculum it had more of the questions there for me. The questions were just embedded in the students work….[My Algebra 1 students] haven’t had to do that. Their way of learning hasn’t been that style” (Natalie, Interview 1).

This challenge to engage students and provide them an environment in which they would feel comfortable sharing their thinking was an issue that Natalie brought up in two of our three interviews. In the last interview she said:

> One of my goals is to get closer to 100 percent of my students engaged and involved. The kids don’t want to go up to the front they don’t want to write on my board. They don’t want to answer out in class even when they probably have the right answer. I just want more of my kids to feel comfortable about sharing in class. (Natalie, Interview 3)

So Natalie experimented with different strategies searching for something that helped. She said:

> Something I’ve been thinking about this year is different ways, what can I do to get the kids engaged more? More of them on task and doing the problems. So I’ve tried the clickers, I’ve tried working in partners more. Thursday and Friday the learning specialist is going to let me use her white boards. So a lot this year I’ve been thinking about what are different ways to present my lesson so that I can get more kids involved. (Natalie, Interview 2)

In order to get students engaged she experimented with different strategies and formats for her lessons.

Natalie also wanted to create a classroom where students felt comfortable participating and discussing mathematics, but found that she needed to find and enact strategies in which students could become prepared to participate in whole-class discussion. She used strategies such as pairing, and wait-time, She said:

> I don’t like to [randomly] call on a kid. I don’t like to say ok “Mike would you do this problem?” because if Mike doesn’t know how to do it then that’s going to make him less likely to answer later. So I don’t want to call on a kid who doesn’t want to offer. So finding a way so that, if they do check with a partner, check your answer, or somebody and then be able to answer my question, then I should be able to call on anyone. (Natalie, Interview 1)

Natalie avoided calling on students unprepared to respond, to overcome this she found ways to ensure they had opportunities to think and discuss the problems. Because of her goal to get them engaged and willing to contribute to class discussion (broad mathematical pedagogy goals) she developed strategies to successfully meet these goals. She developed pck specific to the area of general mathematical instruction.

Natalie: Teaching Equation Solving—Where To Start and Why Do We Do That?

In a more topic-focused area of mathematical instruction, a goal Natalie consistently identified was to get students to engage in mathematical reasoning and justification. She was teaching equation solving. On the first day Natalie began the class with two warm-up problems. One of the problems was "$-(2/5)x - 7 = 11$." After students had a few minutes to work both problems she asked the students: "I need three reasons, why did I add sevens to both sides?" (Natalie, Video Observation 1). In the interview I asked Natalie why she asked this question. She said:

> First of all to the students that look at that problem and see a fraction and then they don't know where to go from there. They don't know where to start. And kind of an order of operations thing. Do I add seven first or do I multiply first, or do I distribute first? You know, what's my first step and why is that my first step. So to kind of give them somewhere to go, because I heard it today I think. "I don't even know where to start." "I don't know what to do." For other ones just to make them think about, because there are the ones that can just do these problems because that's the way we've done them forever. They just probably never have been asked: "Why are we doing that?" It's just Ok we're supposed to add 7 so let's add 7. "That's just how we were taught." So what's the math behind it? (Natalie, Interview 1).

Based on her knowledge of students past and present she realized they struggled to get started, afraid perhaps of making a wrong step. She was also aware that her students typically did not know why they were doing certain processes. Her goals here reflected students' learning needs and her strategies were developed to meet these goals. In addition to her students' learning needs, however, her push for reasoning and justification also came from teaching Core-Plus, and her knowledge that students would have to justify steps in Geometry. She said:

> I guess I saw that a lot in *Core-Plus*. And also leading up to proofs in geometry. I used to teach stuff with proofs, not the two-column format proof, but more of a paragraph of this is the first thing I noticed and giving reasons why. (Natalie, Interview 1)

Natalie's goals related here stemmed from her experience with another curriculum, her knowledge of students future mathematical demands, and her knowledge that students were not confident in the steps to take to solve the problems. With these goals in mind, she developed or adapted previous strategies to her current context to promote the reasoning that explains traditional equation solving. She held class discussions with many participating in sharing justifications for certain steps. She also learned to ask for different ways to begin a problem.

Sarah

Sarah taught Honors Integrated 4. When she first began teaching this class she initially used a strategy she had used previously—holding whole class discussions frequently to summarize the mathematics and clear up misconceptions. But when she held these discussions in the Honors Integrated 4 class she observed and perceived that the students already understood the mathematics. Therefore she was led to ponder how to reduce these unnecessary stops, and at the same time ensure learning. These were her broad pedagogical goals in the context of a senior-level honors, problems-based mathematics class. She said:

> I was used to, they work on the investigation for a while then we stop to kind of clear up some misconceptions some groups are having. It was very sectioned off.
>
> I realize there were times that I thought we needed to stop and talk as a class, and everyone would just kind of stare at me like we already know that.
>
> They worked so hard together, and they worked so well together, that if there's anyone that had it off, the rest of the group helped them figure it out, and they had it corrected.

> I realized that I was standing in their way by saying ok I want you to stop after this problem. I [realized I] need to get out of their way. We don't need to stop at certain points. (Sarah, Interview 1, emphasis added)

At first, she *thought* that she needed to hold frequent whole class discussions for her students' sake, but when she did, she observed student expressions that indicated they already knew the mathematics she was intending to discuss. She perceived these discussions as unnecessary. Her goal was to optimize learning and this meant she needed to get out of the way, to do away with unnecessary discussions so students could learn.

So Sarah developed a plan where the students were given dates for quizzes and tests and were instructed to develop group plans to learn the mathematics for those tests and quizzes. This was the second year she implemented this idea. She said:

> So I just started it with them this week. However, I'm really struggling with the way they're grouped. They're not working at the same pace in their group. So we're going to have a quiz next week, and I'm sure that I will put the kids who are working faster together, and the kids who work a little slower together, because I think they'll work more together.
>
> I don't want everyone to take the same amount of time of time on every problem. That's not the way they learn or have to think through problems. There has to be that point where they thought about it before I'm going to give them an answer. (Sarah, Interview 1)

After initiating the strategy to let the groups work more freely with targeted quiz and test dates she ran into a new dilemma—students working at different paces within groups—this led to her thinking about grouping students in pace-alike groups. In other words, her goal to optimize the learning organization in her classroom continued to prompt her to analyze her efforts and devise new strategies to further enhance the classroom experience. Her goal prompted pck in this context.

Sarah: Teaching Function Notation for Horizontal Translations

In this final example, Sarah held a narrower content goal—to help students learn and understand the relationship between horizontal translations of functions and the corresponding function notation. In the following example students were struggling to formulate where in the function notation this shift would be represented. She explained:

> In the book, the very first problem has them make a table for $f(x)$, $f(x + 3)$, and $f(x-3)$. And they call them $f(x)$, $g(x)$, and $h(x)$. So they actually found the y-values, and they made this table so it had four rows: the x-values then 3 sets of y-values. Then they were supposed to talk about the coordinate rule. It looks like when you looked at the table, weren't the y-values just going up each time? This is your set x-value and the y-values were what were changing. So yeah, if you look at the table, that sure is what's happening. You're x-values went up and your y-values went down. So then I thought, wow, that was almost like a bad thing to do right there because now they're thinking it's a change in the y not a change on the x-axis. So I didn't want her to look at that for a minute, so I was like let me think of some other things she could look at. So I was trying to hurry and come up with something, because I had just realized what a problem that was for two groups.
>
> And did you hear me say let's do y equals x. And I said, "Nope, let's not do that one." And I said let's look at y equals x squared. Well then I was getting the other side of the parabola's outputs. So then it was deceiving again now the left side was crossing over

> when I moved it over. I was like "Ok this is another bad example, let me come up with something where we can look at it." So I came up with $y = 2x$, so we graphed some points, and then we graphed some points on $y = 2(x - 3)$.

In this excerpt so far, it is clear that Sarah was struggling to find a way to make the notation of horizontal translations clear. She was aware that others were similarly confused. So with the goal to help these students make sense of the idea, she is searching for examples. If not the best examples, she is searching for better examples to use to make this particular content more understandable and accessible to her students. She went on to relate an idea she had found in a previous class that she intended to use.

> An Integrated 4 Honors student last year explained it to class this way that made sense to the rest of the students, so that's why I've been trying to say it this way a little bit. "If you have *x* minus 3 in the function, like you're *x*-values have to be 3 higher to get that same output you got on the original function". And he said, "So the whole graph has to happen 3 later on the *x*-axis." And that helped the students understand it more than anything that I had done. So I'm like ok I'm using that one, I wrote it down in my book and I looked at it last week and I was like ok I got to remember to explain it the way he did. That's what I was trying to get her to see. We had to put in an *x*-value that was 3 higher into this new function, than we did the original function to get that same output. So we had to move everything 3 higher on the *x*-axis. We had to go over further on the *x*-axis to get that same output.

In this example Sarah explained her challenge to provide a clarifying example to teach about horizontal translations. Her goal to teach this content led her to searching for and collecting clarifying examples and powerful phrases that would enable her students to understand. In this example her goal led to her development of pck in this area.

Natalie and Sarah were both working on engaging students and optimizing learning formats and routines in the classroom. Their goals in these areas led to their exploration and experimentation with various strategies. In this process they developed pck broadly in their contexts of facilitating mathematics classes. Then, in content specific ways their content goals prompted a search for and development of pck narrower topic areas.

Discussion and Conclusion

Hiebert and colleagues (Hiebert, Morris, Berk, & Jansen, 2007; Hiebert, Morris, & Glass, 2003) have delineated the process of learning to teach as a process of monitoring and revision. In their 2007 paper they identified four skills for teachers to hold and a process to engage in for learning to teach. Those skills are: "Skill 1: Specify the learning goal(s) for the instructional episode (What are students supposed to learn?)" (p. 51). "Skill 2: Conduct empirical observations of teaching and learning (What did students learn?)" (p. 51). "Skill 3: Construct hypotheses about the effects of teaching on students' learning (How did teaching help [or not] students learn?)" (p. 54). "Skill 4: Use analysis to propose improvements in teaching (How could teaching more effectively help students learn?)" (p. 55). Findings in this research indicate that teachers naturally engaged in this process based on the goals they held in the classroom and in doing so developed pck in specific areas.

In this study, teachers' goals led to their sensitized observations, evaluations, and reasoning about teaching mathematics that in turn led to their development of pck. It is clear that they developed pedagogical content knowledge as they learned "ways of representing and formulating subject matter that make it comprehensible to others" and "the conceptions and preconceptions that students… bring with them to the learning of… topics and lessons" (Shulman, 1986, p. 9). However, my interest here is not that teachers developed pedagogical content knowledge; rather it is how this knowledge developed by reasoning about how to achieve their goals. This process is called pedagogical reasoning, "the *process* of transforming content knowledge into forms that are pedagogically powerful and adaptive to particular groups of students" (Brown & Borko, 1992, p. 221). Elsewhere Cooney (1994) used the words adaptation and pedagogical power to describe this effort to transform "what we are able to do to what we want to do" (p. 9).

Teachers' own interests, contexts and teaching philosophies sensitize them to enhancing instruction in unique and idiosyncratic ways. Shulman and Shulman (2004) discussed this issue in their development of a framework for teacher learning. In essence, they asserted that the teachers' vision of teaching, their goals, served as a sort of gatekeeper for teacher development and learning. They said, "A highly developed and articulated vision serves as a goal toward which teacher development is directed, as well as a standard against which one's own and others' thoughts and actions are evaluated" (p. 261). Having this vision, they argued, prepares teachers to learn things necessary to enact it.

Learning because of goals is not a new idea. Locke and Latham (2002) stated, "goals affect action indirectly by leading to the arousal, discovery, and/or use of task relevant knowledge and strategies (Wood & Locke, 1990)" (p. 707). But having a goal is not the whole of the process. Goals sensitize and prompt teachers to observe, to think about, to hypothesize, and to experiment in order to meet them. Consequently, teachers' goals and interests lead them to monitor and reflect on aspects of the classroom related to these goals.

In this section I have made the assertion that teacher growth and development is built around their goals. Locke (2000) made the assertion about the links between knowledge and goals in these words:

> Once the individual decides to act to achieve a goal, spurred on or not by emotion, both conscious and subconscious knowledge come into play. Consciously one has to ask: how will I go about reaching this goal? How does it tie into my other goals: How long will it take: How much effort will be required: Can I do it: What resources will I need: Some of these questions will pull relevant subconscious knowledge into awareness and some will require more thinking and information search. (p. 412)

Still, teachers' goals have not received the attention they should in our quest to understand teacher thinking.

Implications

Teachers' goals lie at the heart of educational activities. In terms of teacher knowledge, goals become a highly accessible avenue to explore teachers' knowledge, as teachers are highly able to explain their reasoning as it relates to specific goals that they hold. In the area of teacher learning or learning to teach, goals form a basis for all that follows. Further research in both of these areas will be enhanced by increased attention to the role and meaning of teacher goals. Additionally, research on beginning teachers' goals and the sources they cite for their goals would shed additional light on the trajectory and challenges beginning teachers face as they develop their own expertise. Research on expert or veteran teachers might help us distil those goals we would desire for all teachers.

Teacher development programs and professional development programs can be strengthened and more powerfully serve teachers by targeting instruction and support to help them develop on the basis of their own goals of instruction. To the extent that teachers' goals are ignored, the learning process will be limited. Interested parties may voice concern that basing teacher development programs on teachers' goals may unnecessarily limit the potential value added by programs constrained in this way. But if teachers are likely to develop competencies according to their own goals then organizing this way can actually accelerate the pace at which teachers come to realize the limits of their typical practice. By building on teachers' deeply held teaching goals and helping them think about and achieve them, professional development schools could quicken the rate of teacher development in positive ways. This effort could help teachers more quickly develop strategies to meet their goals, or help teachers more quickly see the limitations of their strategies and conceptions of teaching. Research should explore these possibilities.

Goals serve as the lynchpin between teacher knowledge and beliefs, and behavior. While much research has been focused on these three areas, goals have been studied to a much lesser extent. I have asserted that goals support the investigation of teacher learning and knowledge. If much of learning to teach and teacher's knowledge is bundled and built around teachers' goals then research in these areas must do better to take this critical area into account. Doing so will enhance these research areas. Afterall, goals are the vehicle and path of teacher growth and the basis of students' learning opportunities.

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PATTERNS OF KNOWLEDGE INTERACTIONS DURING INSTRUCTION

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This study used stimulated recall interviews to investigate the patterns of knowledge interactions that were exhibited by two fifth-grade teachers during the act of teaching. Schematics were developed to describe the flow of knowledge interactions and were subsequently examined for patterns. During the interviews, the teachers primarily focused on their knowledge of learners and curriculum and instructional strategies, with mathematics being used mainly to justify claims about learners and instruction. In addition, several patterns of knowledge integration were observed and described.

A focal point of mathematics teacher education is to prepare preservice teachers to make effective instructional decisions. Since teachers' actions are based on their integrated knowledge of mathematics, pedagogy, and student thinking, developing integrated knowledge is an important component of mathematics teacher preparation.

Brown and Borko (1992) wrote, "an individual's knowledge structures and mental representations of the world play a central role in that individual's perceptions, thoughts, and actions" (p. 211). In order to plan effective experiences for preservice teachers, teacher educators must understand the structure and organization of the various forms of teacher knowledge, and how teacher knowledge functions in instructional settings. Teacher educators must also understand the connections teachers make between the different types of knowledge and the patterns of knowledge interactions that lead to effective instructional decisions.

The research conducted in this study focused on the interaction among three components of teacher knowledge—knowledge of mathematics, knowledge of learners, and knowledge of curriculum and instructional strategies. Because of the limited research on teacher knowledge integration, we sought to answer the following research question: What were the patterns of knowledge integration that were exhibited by two fifth-grade teachers while making decisions in the classroom?

Literature review

In recent years, research has been conducted on teachers' mathematical knowledge (Ball, Thames, and Phelps, 2008; Winsor, 2003). The aim of the research has been to classify the different types of mathematical knowledge that teachers possess or need (e.g. Hill, Rowan, and Ball, 2005). Unfortunately, there is still little known on how teachers use their mathematical knowledge in conjunction with other types of knowledge in the decision-making process.

Bishop (1976) noted, "Decision making is therefore an activity which seems to me to be at the heart of the teaching process. If I can discover how teachers go about making their decisions then I shall understand better how teachers are able to teach" (p.42). Bishop and Whitefield (1972) propose that decisions are made using a framework or schema. Marshall (1995) examined student schema with respect to problem solving in mathematics. Given that teaching is similar to problem solving, Marshall's treatment of schema theory serves as a foundation for this study. Marshall posits that:

> A schema is a vehicle of memory, allowing organization of an individual's similar experiences in such a way that the individual:
>
> - can easily recognize additional experiences that are also similar, discriminating between these and ones that are also dissimilar;
> - can access a generic framework that contains the essential elements of all of these similar experiences, including verbal and nonverbal components;
> - can draw inferences, make estimates, create goals, and develop plans using the framework; and
> - can utilize skills, procedures, or rules as needed when faced with a problem (or teaching episode) for which this particular framework is relevant. (p. 39).

Understanding the structure of schema and how teachers draw upon their schema when making decisions will aid teach educators in preparing meaningful experiences for preservice teachers. The main operation of a schema is to store knowledge through a network of connected pieces of knowledge called "elements"(Marshall, 1995, p. 43). The more connections that exist within a teacher's schema, the stronger and more useful the schema will be.

Theoretical Framework

In order to investigate the research question of this study it was necessary to construct a model that reflects how teachers use their knowledge in the act of decision-making and how they learn from their experiences. The model is based on Marshall's (1995) work on schema theory. As teachers encounter opportunities to make teaching decisions, connected schemata are activated in order for the teacher to make decisions.

The knowledge structures used for decision-making are shaped and adjusted as teachers make decisions reflect on instruction. In essence, reflection can be seen as a process that allows teachers to build and adapt connected knowledge structures that are then used in decision-making. Barrett and Green state that, "through this process of reflection, teachers transform their inert knowledge into active, classroom practice that continually evolves as they encounter new situations and reconsider past experiences in light of more recent experiences" (2009, p. 19). A researcher-developed model for the knowledge structure used in decision-making, and the role of reflection in that process is provided in Figure 1. The ellipse represents the set of knowledge that the teacher has acquired. The arrow entering the ellipse represents reflection on a teaching incident that promotes connections between different knowledge elements (the knowledge of mathematics (M), of learners (L), and of curriculum and instructional strategies (CI)). Note the connections among the different knowledge elements that result from the process of reflection. The arrow exiting the ellipse represents the connections among various knowledge elements were needed to make an instructional decision.

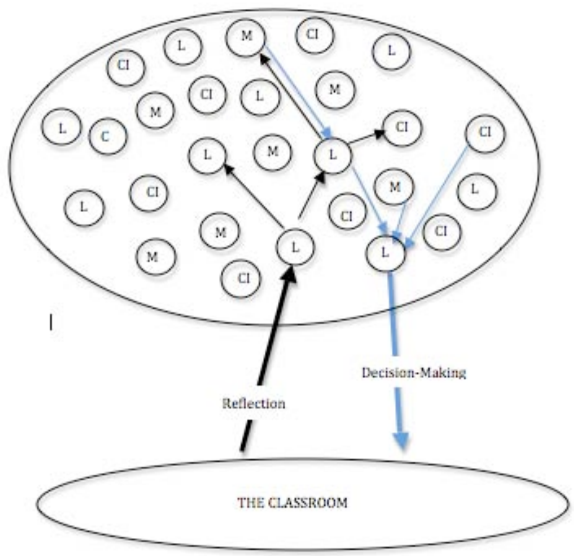

Figure 1: Model of Integrated Knowledge

Methods

A case study methodology was used to investigate how two fifth-grade teachers—Amanda and Emily--drew upon their knowledge of curriculum and instructional strategies, mathematics, and learners while teaching. Each teacher was observed and videotaped as they taught a series of three lessons from the *Patterns of Change* unit of *Investigations in Number, Data, and Space* Curriculum (Tierney, Nemirovsky, Noble, & Clements, 1998). Following each observation, stimulated recall interviews were conducted to obtain the teachers' reactions to the videotaped lessons. As they watched the videos, the teachers were prompted to reflect on what occurred in the classroom and to comment on the thought processes used to make their instructional decisions. The transcripts from these interviews served as the primary data source for analysis.

Following data collection and transcription, the data were retrospectively analyzed using a data reduction approach (Miles & Huberman, 1994). The analysis occurred in several phases. The transcripts were initially coded in terms of the type of knowledge discussed: mathematics (M), learners (L), or curriculum and instructional strategies (CI). At this point, the research team coded portions of the transcripts to develop and test the viability of these definitions. Discrepancies in coding were discussed, and definitions adjusted, until all disagreements were resolved.

The data for each interview were divided into individual episodes, defined to be sections of the transcript where the teacher was discussing a single thought. An initial analysis of these episodes found that one knowledge type was often the primary focus of the episode (primary knowledge type), with a second or third knowledge form (secondary knowledge type) being used in a supportive role. This support was found to occur in one of two ways: (a) the second knowledge type was used to support or justify what was being stated about the primary knowledge type, or (b) a discussion of the primary knowledge type led to an implication or investigation of the secondary knowledge type. In other words, there was a temporary shift in focus to the secondary form of knowledge. Given these initial findings the data were translated into schematics that illustrated knowledge interactions, including which knowledge form was primary and the purpose of interaction. These mappings were then analyzed to characterize

L 2 CICI
L
1 M

patterns among the interactions and are the basis of the findings that follow. An example mapping is provided below.

In the following excerpt Amanda used her knowledge of curriculum and instructional strategies to support a comment she made about one of her students.

> *Amanda:* I was surprised that Kate raised her hand and was giving answers. She usually does not participate. It was like a reversal.
> *Researcher:* Why do you think she was participating?
> *Amanda*: I think it is because we are using a context, we are using blocks, we are showing it, we are showing it in two different formats so we are including two totally different types of thinkers.

In this case, the knowledge of learners (L) is the primary knowledge form, represented by the bold box, and the knowledge of curriculum and instructional strategies is the secondary knowledge type. In this episode Amanda used a statement involving curriculum and instructional strategies to justify her original comment about one of her students. An arrow was drawn from the secondary knowledge type to the primary knowledge type to signify that the secondary knowledge type was used to justify the primary. Figure 2 illustrates this notion.

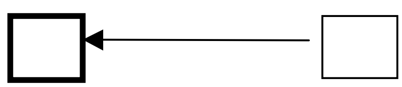

***Figure 2*: CI used to clarify L**

Each episode was then mapped out using this notation; an example mapping is provided in Figure 3.

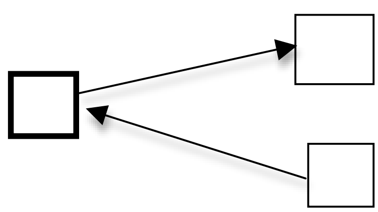

***Figure 3*: Example Mapping**

In this example, the arrow from the primary knowledge type (L) to the secondary knowledge type (CI) represents the second type of interaction, where a discussion of the primary knowledge type led to an implication for the secondary knowledge type. In addition, numbers were added to the arrows to indicate the order in which the interactions occurred. So in this example M was initially used to justify a statement about L, which then led to an implication regarding CI.

Findings

Mathematics' Place in Teacher Knowledge Integration

Researchers such as Ma (2000) and Ball (e.g. Ball, D.L., Thames, M.H., & Phelps, G., 2008) have focused on the type of mathematics teachers need to know to be successful. Kahan, Cooper, and Bethea, (2003) report that knowledge of subject matter alone does not ensure effective teaching performance. This study reveals how mathematics knowledge is used in conjunction with the knowledge of learners and the knowledge of curriculum and instruction in order to make instructional decisions. Table 1 depicts the ways in which Amanda and Emily used their knowledge of mathematics. The direction of the arrow depicts the role of interaction as described earlier.

Summary of Amanda and Emily's Knowledge Use					
	Primary Focus	Support/ Implication	Amanda	Emily	Total
M & L	M →	L	0	6	6
	M ←	L	1	1	2
	L →	M	5	5	10
	L ←	M	38	10	48
M & CI	M →	CI	1	2	3
	M ←	CI	0	0	0
	CI →	M	0	1	1
	CI ←	M	8	8	16

Table 1: The role of Mathematics

Of the 86 interactions that involved mathematics, only 11 had mathematics as the primary focus of conversation. Of these 11 interactions, 9 of them led to an implication or investigation involving mathematics, and only two used CI or L to support a claim about mathematics. In other words, mathematics was not the main focus of conversation for Amanda and Emily as they were making decisions in the classroom. There were 75 interactions that had mathematics as the secondary knowledge type; 11 of these interactions involved a shift of focus from knowledge of learners or knowledge of curriculum and instruction to an implication about mathematics. In other words as the participants talked about their knowledge of learners or curriculum and instruction, their attention was re-directed to investigate or think about mathematics. The main use of mathematics was to either justify a claim about learners (48 interactions) or to justify a claim about curriculum and instructional strategies (16 interactions). From the data, mathematics appears to primarily play a supportive role in teaching for Amanda and Emily, with more than half of the interactions utilizing mathematics to justify a claim about learners. In addition, of the 86 interactions involving mathematics, 66 integrated knowledge of mathematics with learners, and only 20 integrated knowledge of mathematics with curriculum and instructional strategies.

Structures of Teacher Knowledge Interactions

The structure of the knowledge interactions also sheds light on how teachers' use their knowledge while teaching. The following table describes the frequency of interactions where each type of knowledge was the primary focus of conversation (see table 2).

Primary Focus	Amanda	Emily	Total
M	2 (1.6%)	9 (10.2%)	11 (5.1%)
L	88 (68.8%)	43 (48.9%)	131 (60.6%)
CI	38 (29.7%)	36 (40.9%)	74 (34.3%)
Total	128	88	216

Table 2: Frequency of M, L or CI as primary knowledge

Knowledge of learners received the most attention from Amanda and Emily. Emily was more balanced between knowledge of learners and knowledge of curriculum and instruction. Amanda, on the other hand, relied more heavily on her knowledge of learners.

When knowledge of learners was the primary knowledge type, two prominent patterns of interactions occurred. The first pattern observed was that both CI and M were used to justify L or M→**L**←CI (note that the letter in bold represents the primary knowledge type). This pattern occurred in 14 of the 24 episodes that involved L as the primary knowledge type. The other pattern that occurred was M→**L**→CI, which was observed in 11 of the 24 episodes that involved L as the primary knowledge type. In this pattern, mathematics was used to support the knowledge of learners, which in turn had implications for knowledge of curriculum and instruction. There was one episode that simultaneously contained both patterns. One might say that the knowledge of learners was a linking knowledge between the knowledge of mathematics and the knowledge of curriculum and instructional strategies. More complicated connections involving L as the primary focus were not observed.

When knowledge of curriculum and instruction was the primary knowledge type, there are several different patterns that occur. There are two patterns that are similar to the patterns for when L is the main focus. The first pattern is M→**CI**→L and the second pattern is M→**CI**←L, both of which occurred in 4 of the 12 episodes where CI was the primary focus. In other words, CI can serve as a linking element between M and L, but not as frequently as L serving as a linking element. There are also more complicated interactions when CI is the main focus. An example is L→**CI**↔L←M. It seems that in the previous example, starting with CI as the primary focus can promote many connections between L and M.

As noted previously, based on the data, mathematics seems to be used more to justify or support CI or L; therefore, mathematics was rarely the primary focus of conversation. For the two episodes where M was the main focus of conversation the interactions are as follows: CI←**M**←L and **M**→**L**↔CI. In the second mapping, mathematics is the focus at the beginning of the episode but the focus then shifts to the knowledge of learners.

Discussion

From the findings, it is evident that Amanda and Emily employed their knowledge of learners frequently. Moreover, their knowledge of learners appeared to play a linking role between their knowledge of mathematics and their knowledge of curriculum and instructional strategies. In addition, Amanda and Emily rarely had mathematics as a primary focus; instead they used Mathematics primarily in support of the other types of knowledge. However, it must be noted that the results are based on two teachers that were high regarded by colleagues and administration and may not be representative of all teachers.

Mathematics role as a support knowledge seems to be counterintuitive to the emphasis in recent research. This study does not discount the importance of mathematical knowledge, because mathematical knowledge is an important foundation for the other forms of knowledge. However, it raises the issue of how much focus should be placed on acquiring mathematical knowledge without the opportunity to make connections to knowledge of learners and knowledge of curriculum and instruction. This study raises questions regarding the role of the knowledge of mathematics in teaching elementary mathematics, and subsequently how the knowledge of mathematics, learners, and curriculum and instructional strategies are provided during teacher preparation. In addition, the common patterns of knowledge interaction demonstrated by Amanda and Emily may provide insight into how we can structure learning opportunities for pre-service teachers to better encode the connections necessary to become successful teachers.

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○ PRESERVICE MATHEMATICS TEACHERS' VISUALIZATION OF POLYNOMIAL-RATIONAL INEQUALITIES, EXPO-LOG FUNCTIONS & DERIVATIVE-INTEGRAL FUNCTIONS ON GEOGEBRA

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This study investigates mathematics secondary preservice teachers' visual representations of selected precalculus and calculus tasks (polynomial and rational inequalities, graphs of exponential and logarithmic functions, derivative and integral functions) using GeoGebra. Seven students who already completed precalculus and calculus classes were asked to use GeoGebra to solve selected precalculus and calculus problems. For each problem, they were asked to indicate their problem solving procedure on GeoGebra, and to write and explain their reasoning for each step in detail with reference to GeoGebra work. I situate my research within a framework of visual representations in calculus. The main result is that though teachers were successful in obtaining an algebraic solution (using an interval notation, correctly executing derivative and integral calculations), in none of them was there a clear indication of a solution set (intervals of x) constructed where it should be – on the x-axis. Another result is that geometry plays a crucial role as a prerequisite not only in calculus, but in the sense-making of precalculus explorations as well.

Theoretical Background

Visual representations play a crucial role in understanding and making sense of mathematics. Students' favorite mathematics textbooks and teachers are often the ones that use a variety of colorful figures, diagrams, pictures and graphs. Visualizing is also a very important step in understanding and attempting to solve a problem (Polya, 1957). Even when a visual representation is vague or missing, producing a meaningful one is an indispensable route leading to a meaningful solution to the problem (Larkin, 1989; Polya, 1957; Schoenfeld, 1985; Simon & Larkin, 1987).

Just as is valuable to create an algebraic formalism along with its symbols and notation, so is the case for creating a visual representation and connecting the two models (Caglayan & Olive, 2010). As stated by the Principles and Standards for School Mathematics (NCTM, 2000): *Representation is central to the study of mathematics. Students can develop and deepen their understanding of mathematical concepts and relationships as they create, compare, and use various representations. Representations – such as physical objects, drawings, charts, graphs, and symbols – also help students communicate their thinking* (p. 280). Being able to represent mathematical ideas in various ways is as important as relating those representations and connecting them meaningfully.

The theoretical framework of this study is inspired from Zimmerman, who postulated that successful solutions to calculus problems are the ones that are accompanied by meaningful visual representations and graphs (1991). He also reported excellence in geometry as one of the main prerequisites of visual thinking in calculus. Though a calculus instructor may take it for granted, college students may not always be as proficient in visualizing as they are assumed to be (Eisenberg & Dreyfus, 1991). In what follows, I focus on mathematics preservice teachers' visual representations of selected precalculus and calculus tasks modeled on GeoGebra, and on their abilities to connect their visual and algebraic formalism.

○ Context and Methodology

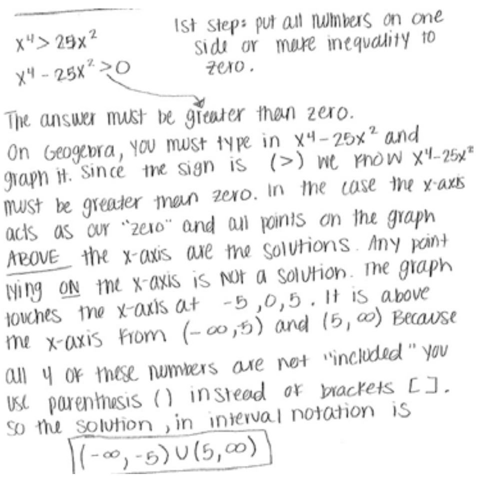
This study investigates mathematics secondary preservice teachers' investigation of selected precalculus and calculus tasks (polynomial and rational inequalities, graphs of exponential and logarithmic functions, derivative and integral functions) using GeoGebra, a free mathematics software that intertwines geometry, algebra, and spreadsheets. Seven students who already completed precalculus and calculus classes were asked to use GeoGebra to solve selected precalculus and calculus problems. For each problem, they were asked to indicate their problem solving procedure on GeoGebra, and to write and explain their reasoning for each step in detail with reference to their work on GeoGebra.

This project included seven mathematics secondary preservice teachers whom I met weekly for three weeks in three-hour sessions. My data consists of these research participants' detailed written algebraic solutions and comments, and work saved on GeoGebra. I then created scanned version of their work and conducted an in-retrospect preliminary thematic analysis of the whole data to generate possible themes for a more detailed analysis. The dataset has then been revisited multiple times in order to generate a thematic analysis (with constant comparison) from which the following results emerged.

Results

The first set of tasks to be modeled using GeoGebra was the following: *Use GeoGebra to solve the polynomial and rational inequalities given at Pearson's MyMathLab tech environment. For each problem, indicate the solution set on GeoGebra (Find a way to make your solution set visible on GeoGebra). Also enter your answer to Pearson's MyMathLab tech environment. For each problem, explain your reasoning for each step in detail with reference to your work on GeoGebra*.

For the first set of tasks, all students except Ashley wrote their comments in outline form. In this format, students basically described each inequality leading to the solution set of the original inequality via a set of phrases, rather than prose format. Only one student, Ashley, was able to relate her reasoning with reference to her work in GeoGebra completely. The other students seemingly treated the algebraic and graphical solutions as independent tasks. Figure 1 depicts Ashley's solution on GeoGebra and written work for the polynomial inequality $x^4 > 25x^2$.



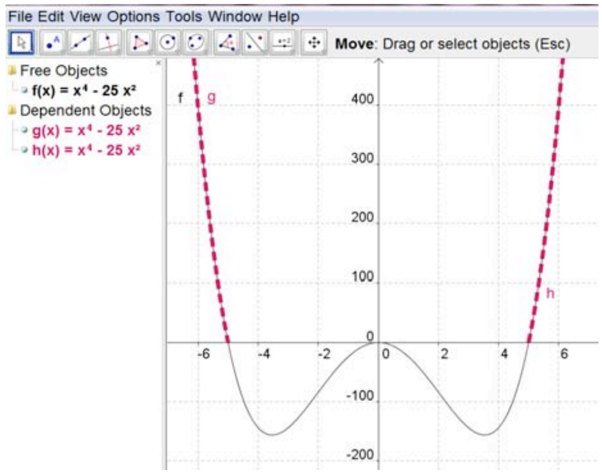


Figure 1 - Ashley's Written Work (left) and GeoGebra (right) Solutions to Polynomial Inequality $x^4 > 25x^2$

Ashley clearly understands the polynomial inequality problem and addresses problem solving steps meaningfully (Polya, 1957). She also finds her own way of representing the "solution set" on her GeoGebra work, using the two pink dashed subsets of the induced function $f(x) = x^4 - 25x^2$ she defines and makes use of[1]. Her comment "all points on the graph above the *x*-axis are the solutions" corroborates the fact that her interpretation of a solution set to a polynomial inequality is actually a set of points on the graph of the induced function; rather than the union of the two rays $(-\infty,-5)$ and $(5,\infty)$ on the *x*-axis.

In his written work for the rational inequality problem $\frac{4x-5}{x+2} \le 3$, Norman first "*breaks down the equation*," by which he obtains the equivalent inequality $\frac{x-11}{x+2} \le 0$. He then goes on to state "*There is a vertical asymptote at* $x = -2$. *I proved this in GeoGebra by graphing* $x = -2$ *along with* $\frac{x-11}{x+2}$ *and the result was undefined meaning that it cannot exist due to the denominator being equal to 0*." Figure 2 depicts Norman's solution on GeoGebra and the concluding parts of his written work for the rational inequality $\frac{4x-5}{x+2} \le 3$.

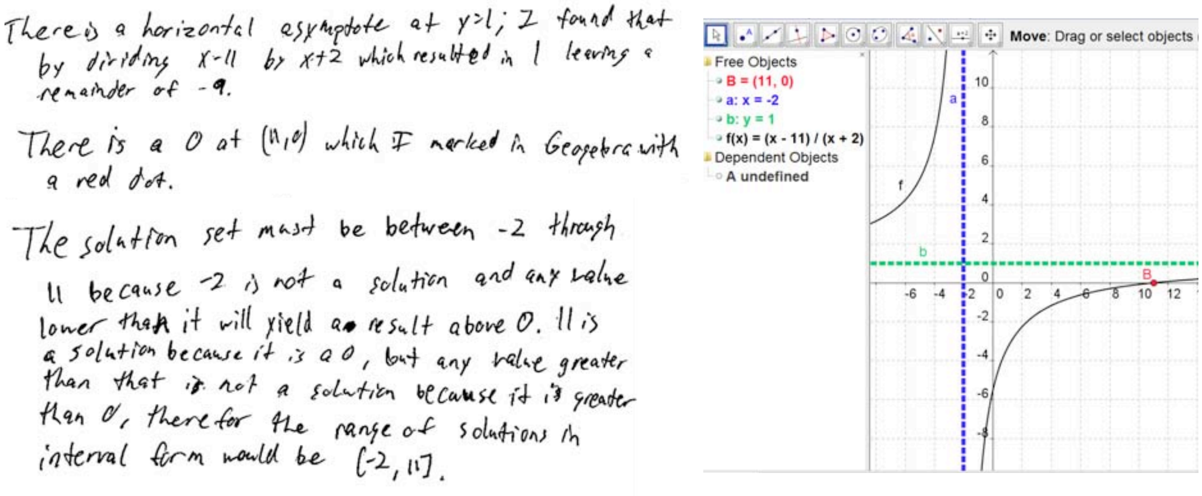


Figure 2 – Norman's Written Work (left) and GeoGebra (right) Solutions to Rational Inequality $\frac{4x-5}{x+2} \leq 3$

Norman clearly uses his GeoGebra work as a check to his algebraic solutions, as evident from his written comments. Rather than indicating the solution set on the graph, he relies on the traditional rational inequality solving procedure algebraically. Though he graphs the vertical asymptote $x = -2$ and point $B(11,0)$ on GeoGebra, he fails to construct the line segment $(-2,11]$, which is the actual visual representation of his solution in "*interval form*."

The second set of tasks to be modeled was about graphing an exponential function and constructing its inverse (the logarithmic function) on GeoGebra. Students were also asked to find a way to make everything (e.g., the domain and the range of the functions, asymptotes, visual proof that the two functions are inverse functions of each other) visible on GeoGebra. The instructions once again asked for a detailed explanation with reference to work on GeoGebra. Mary first graphed the given exponential equation, $f(x) = e^{x-3} + 4$, on GeoGebra and stated that the domain of *f* would be the interval $(-\infty, \infty)$, which she showed as *black dashes* in function graph (Figure 3a). She obtained the horizontal asymptote of *f*, $y = 4$, by "*just looking at the graph. If you zoom in on the graph in geogebra you would see that the line y=4, horizontal asymptote, does not touch the function* $e^{x-3} + 4$. *This is shown as the blue dashed line*."

Mary then used the conversion formula to obtain the equation of the inverse function as $f^{-1}(x) = \ln(x-4) + 3$ algebraically. She also states that she found the range of *f* by looking at the graph. "*The y-values run from 4 to ∞. In interval notation the range is* $(4, \infty)$, *which is shown as black dots in the function graph of f.*" She then explains that she graphed the inverse function by typing its equation, which she obtained algebraically, in GeoGebra's input bar (Figure 3a).

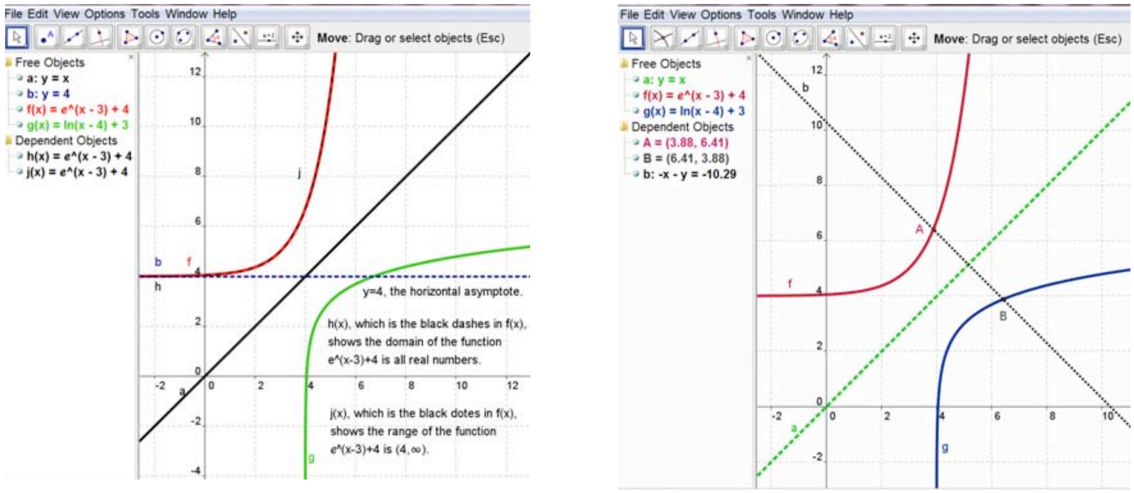


Figure 3 – Mary's (left) and Norman's (right) Exponential and Logarithmic Function Graphs on GeoGebra

We can say that Mary follows a reasoning, similar to Ashley's, in that she represents both the domain and the range of the exponential function as subsets of the exponential function[2]. Mary's interpretation of domain and range is actually a set of points on the graph of the original function *f*; rather than the line $(-\infty,\infty)$ and the ray $(4,\infty)$, respectively, on the *x*-axis. Mary is satisfied with the fact that the graphs of the two functions are reflections of each other with respect to the $y = x$ axis, visually. Norman, on the other hand, not only graphs the two functions, but he verifies that the two functions are inverse functions of each other by constructing a line perpendicular to the $y = x$ axis of reflection. For that purpose, he first constructs an arbitrary point, *A*, on the exponential function graph. He then selects Point *A* and the line $y = x$, which then enables him to construct the perpendicular line. Consequently, he constructs Point *B*, the intersection of line *b* and the logarithmic function graph, using GeoGebra's built-in *Intersect Two Objects* tool. He explains that *"Point A of the exponential function graph becomes Point B of the logarithmic function graph… They simply interchange their x and y coordinates*." The findings stated in this paragraph could be considered as evidence of how geometry plays a crucial role as a prerequisite not only in calculus (Zimmerman, 1991), but in the sense-making of precalculus tasks as well.

The final set of tasks to be modeled was about graphing a function along with its derivative and integral functions. The instructions were similar to the ones for the previous tasks (detailed explanations, request for construction of geometric objects to be determined by the research participants). All participants came up with almost identical constructions on GeoGebra, so in what follows, I focus on Sally's work on the derivative and integral function constructions, without loss of generality.

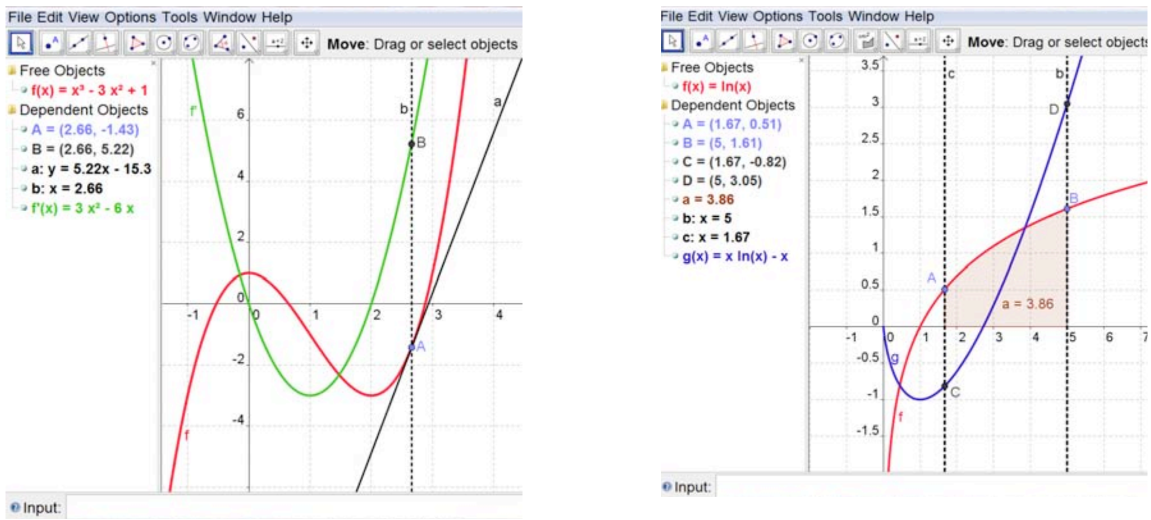


Figure 4 – Sally's Sketches for the Derivative Function (left) and Integral Function (right)

Sally starts by writing that she finds GeoGebra easy to manipulate and very helpful in understanding the derivative of a function. She first created the graph of the polynomial function $f(x) = x^3 - 3x^2 + 1$ by simply typing the function into the input bar. She then used GeoGebra's built-in *Tangents* tool to create the line tangent to the curve *f* at Point *A*. She states "*Being able to see the equation of this tangent line helped me evaluate the slope of the line very quickly*." She then used the syntax Derivative[f] to graph the derivative function. The next task, constructing a geometric object in order to create Point *B* on the graph of f' that should always have the same *x*- coordinate as Point *A*, was challenging for everyone. Sally immediately thought of using a rectangle because the vertical line from Point *A* would intersect f' at some Point *B*, and they would have the same *x*-coordinate. She then thought that if she drags Point *A* down a point where the slope of the tangent line was 0, she could easily create a perpendicular line to the tangent through Point *A* and it would intersect f' at the same *x*- coordinate. This approach did not work because any movement of *A* changed the slope of the tangent line and the resulting perpendicular line was not guaranteed to intersect f' at the same *x*- coordinate. She then decided to use GeoGebra to construct a vertical line through A by typing the equation of the vertical line, $x = x(A)$, into the input box. By using the *Intersect Two Objects* tool, she then was able to construct Point *B* from the intersection of the vertical line and f' (Figure 4a).

Although Sally claimed that GeoGebra was very useful and easy to understand while doing this problem, there was no clear indication of how Point *B* was related to the equation of the tangent line. Sally just mentioned that the slope of the tangent line was very easy to evaluate, but there was no mention of the fact that the *y*- coordinate of Point *B* would be equal to the slope of the tangent line. In fact, a similar disconnect happened in the task on the exploration of a function and its integral function. Sally first constructed the graph of the given function $f(x) = \ln x$ and two arbitrary points *A* and *B* on the curve. She then used the syntax Integral[f, x(A), x(B)] to evaluate the integral $\int_{x(A)}^{x(B)} f(x)dx$. She commented that she was able to see the area under the curve between these two *x*-values had the same value as the expression $\int_{x(A)}^{x(B)} f(x)dx$. She then constructed the graph of the integral function using the syntax Integral[f]. The next task was to construct a geometric object in order to create Points *C* and *D* on the graph of *g* that should always have the same *x*- coordinates as Points *A* and *B*, respectively. This time, everyone easily figured out that what needed to be constructed were two vertical lines (Figure 4b). Sally then evaluated the difference $y(D)-y(C)$ algebraically, using the traditional integration procedure, instead of working it out just by typing the expression into the input bar. Participants failed to relate the difference $y(D)-y(C)$ to the area under the curve between $x(A)$ and $x(B)$.

Conclusions and Discussion

This study aimed to depict visual representations of polynomial and rational inequalities, graphs of exponential and logarithmic functions, derivative and integral functions exposed by secondary mathematics preservice teachers on GeoGebra. Analysis of these participants' visual representations, accompanied by their algebraic formalism and thorough reasoning, has a crucial impact to determine important insights into their sense-making of the mathematics they are exploring. This has direct implications for the teaching of precalculus and calculus in a technology based environment. Mathematics teachers should be more conscious and explicit in modeling problems because otherwise this may be prone to a misinterpretation of the problem situation, or even the solution to the problem, as depicted in this present study. For example, although students were confident with their algebraic solutions (in interval notation) to the polynomial and rational inequality problems, their interpretation of the "solution set" concept differed ostensibly from their algebraic formalism when they were asked to represent this "solution set" on GeoGebra: A function graph, or a subset of the function graph is not the same thing as an interval on the *x*-axis, therefore, an appropriate distinction of these two fundamentally different objects is necessary.

Constant comparison of Mary and Norman's work for the problem of creating an exponential function and its inverse function and Sally's work on the derivative and integral functions on GeoGebra yielded that proficiency in a variety of geometric entities (e.g., transformations, symmetry, constructions of points, lines, line segments, rays, parallel and perpendicular lines) is an undeniable prerequisite in the sense making of not only calculus (Zimmerman, 1991) but precalculus tasks as well. College geometry is often offered as an upper division class that many mathematics and mathematics education majors enroll in long time after they complete the precalculus and calculus sequence. In many universities, Calculus II or III are listed as prerequisites for College Geometry. In fact, therein lies the dilemma: How could we expect our students to master precalculus and calculus while these two being listed as prerequisites for college geometry? Some of the participants of this study overcame this obstacle thanks to their overexcellence in manipulating the GeoGebra software, however, excellence in a geometry software such as Geometer's Sketchpad or GeoGebra does not necessarily implies excellence in geometry.

This study focused directly on the visual representations and written comments and algebraic formalism that the research participants produced. Ferrini-Mundy (1987) found that calculus students made more mistakes in the process of just writing a definite integral than in the process of calculating the definite integral itself. This finding corroborates my theories concerning Sally, for instance, who failed to relate the difference $y(D)-y(C)$ to the area under the curve between $x(A)$ and $x(B)$ in dealing with integral function task (Figure 4b). While she was able to carry out the derivation and integration procedures, and the construction of the derivative and integral functions on GeoGebra perfectly, she failed to *relate* the slope of tangent line to the value of the derivative function at the corresponding *x*-value. She also failed to relate the area under the curve of the original function to the difference of the *y*-values of the integral function on GeoGebra. Bremigan (2005) postulated that mathematically strong students often modified and reconstructed their visual representations in solving calculus problems. The participants of this present study showed a similar behavior. For instance, Sally was very creative and determined to modify and make changes (e.g., in the task on derivative function, for the construction of Point *B* on the graph of f' that should always have the same *x*- coordinate as Point *A*) as opposed to claims by Eisenberg and Dreyfus (1991), who reported a general reluctance on the part of calculus students to visualize in mathematics.

Endnotes

1. She creates those using the piecewise defined function syntax of GeoGebra.
2. Mary too creates those using the piecewise defined function syntax of GeoGebra

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CHARACTERIZING DISCOURSE IN TWO TECHNOLOGY-INTENSIVE HIGH SCHOOL GEOMETRY CLASSROOMS

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Two different teachers implementing The Geometer's Sketchpad in high school geometry classes were analyzed to examine how the patterns, types, and modes of mathematical discourse differed when technology was used. The patterns, types, and modes of mathematical discourse differed between the two teachers when they were and were not using technology. For each teacher, the mathematical discourse modes appeared to shift when technology was used but the types of mathematical discourse did not.

In 1991, the National Council of Teachers of Mathematics (NCTM) published the *Professional Standards for Teaching Mathematics* in which they emphasized the importance of discourse in the teaching and learning of mathematics. They state:

> Like a piece of music, the classroom discourse has themes that pull together to create a whole that has meaning. The teacher has a central role in orchestrating the oral and written discourse in ways that contribute to students' understanding of mathematics.

While many acknowledge the important role discourse plays in the teaching and learning of mathematics, transforming the classroom to one in which students are active participants in classroom discussions does not occur automatically (Hufferd-Ackles, Fuson, & Sherin, 2004; Nathan & Knuth, 2003). The teacher needs to choose worthwhile mathematical tasks, identify appropriate tools to use, anticipate, monitor, select and sequence student work on those tasks, and then bring the class to an appropriate close (e.g., Stein, Engle, Smith, & Hughes, 2008). They need to be able to orchestrate the communicational processes (who is speaking, when, and how) as well as focus on the mathematical content of the conversations.

One method of examining discourse processes is to consider the sequence of activities involved in an interaction. Cazden (2001) defines "traditional" classrooms as classrooms in which discourse follows a typical initiation, response, evaluation/feedback pattern (I-R-E/F) pattern. In contrast, discourse in "non-traditional" classrooms more closely resemble a conversation two people might have. Discourse in "non-traditional" classrooms do not typically follow the I-R-E/F pattern.

Although it is important to know how discourse transpires in a classroom, to support mathematics learning, the content of discourse is also critical to consider. Students providing a single answer response to a low-cognitive demand question is different from students discussing different solution strategies to a high-cognitive demand mathematical task (Stein, Smith, Henningsen & Silver, 2000). How students participate in conversations as well as the types of responses they may provide may both be influenced by the norms established in the classroom (Yackel & Cobb, 1996).

Because technology supports students' conjecturing, problem solving, and exploration of mathematical tasks that can be of high cognitive-demand, and may also serve as an agent disruptive of current norms and practice (Rochelle, Pea, Hoadley, Gordin, & Means, 2000), this may affect the nature of the content of the discourse in which students engage. Also, since students are interacting with a computer as well as one or more students as they work on problems in pairs or small groups they may pose more questions to each other and to the teacher. A question that guided this study was, "In what ways does the inclusion of technology in a mathematics class influence the nature of mathematical discourse?" The purpose of this study was to characterize the nature of mathematical discourse when students were using technology and to contrast this to discourse that was typical in the class when technology was not used.

Methods

During the summer of 2010, twenty teachers from four different schools districts attended a one-week professional development workshop. The school districts were in different stages of implementing 1:1 laptop initiatives. One school district had been using laptops for seven years while another was planning on distributing laptops to their students for the first time during the 2010-2011 school year. The professional development workshop focused on assisting teachers in using a dynamic geometry software program, *The Geometer's Sketchpad* ver. 5.0, to teach particular topics in the high school geometry curriculum (e.g., triangles, quadrilaterals, transformations). The summer workshop was followed by an online course that was taught during the 2010-2011 school year. These experiences are part of a larger 2-year professional development project. The data reported here were collected during the early stages of the teachers' participation in the professional development experience[1].

During the first semester of the school year, twelve of the twenty teachers from the summer institute were assigned to teach a high school geometry class. All four school districts were on a 4 by 4 block schedule. Students took four classes in the first semester and four different classes in the second semester. Each class met for 90 minutes. Thus, the typical year long geometry course was taught in one semester. Each of the twelve teachers was observed and video-recorded three times during the semester.

The Oregon Mathematics Leadership Initiative (OMLI) Classroom Observation Protocol (RMC Research Corporation) was modified and used to gather more detailed information about discourse in the classroom when technology was and was not used. This instrument identifies the modes and types of mathematical discourse and the tools that were used. However, the purpose of this instrument was to focus on student discourse and it did not attend to discourse that was initiated by the teacher. Thus, the instrument was modified to allow researchers to code both student discourse as well as teacher discourse. The mode was used to identify who was participating in the discourse (e.g., teacher to student, student to student) and the type of discourse was used to identify the nature of the response or question during the interaction (e.g., explanation, prediction, justification, answer). The OMLI project assigned a number to each of the different types of discourse to suggest a ranking similar to the categorization of lower and higher-level questions identified by other researchers (e.g., Hollebrands & Heid, 2005; Moyer & Milewicz, 2002). Especially for this study, the tools that were used for the discourse were important to also note. Computers and calculators were assigned different codes rather than the same code as was indicated in the original observation protocol. Tools refer to what was used during the discursive interaction (e..g, actions, written or spoken words, computers/calculators, graphs). The codes that were used for the analysis are shown in Figures 1, 2, and 3.

TS	Teacher-Student		SS	Student-Student
ST	Student-Teacher		SIR	Student Individual Reflection
TG	Teacher-Group		TIR	Teacher Individual Reflection
SG	Student – Group		TW	Teacher to Whole Class
			SW	Student to Whole Class

Figure 1. Modes of Mathematical Discourse

1	A	Answering		5	C	Challenging
2	S	Making a Statement		6	R	Relating
3	E	Explaining		7	P	Predicting
4	Q	Questioning		8	J	Justifying
				9	G	Generalizing

Figure 2. Types of Mathematical Discourse

V	Verbal		S	Symbolization
A	Gesturing/Acting		N	Notation
W	Written		C	Computers
G	Graphs, Charts, Sketches		Ca	Calculators
M	Manipulative		O	Other

Figure 3. Tools for Discourse.

From the twelve teachers, two teachers were initially selected for more in-depth analysis based on teaching experience and technology usage. Mrs. Adams and Mr. Smith represented teachers at both ends of the teaching continuum and demonstrated facile use of various types of technology. The researchers assumed their technological aptitude would eliminate many obstacles related to implementing technology in their classrooms; therefore the analytic focus would shift mainly to discourse. A training session was conducted with a consultant who was part of the original OMLI team. Then a training session with three coders was conducted to check for agreement on the number of episodes and the coding of those episodes. An iterative process was carried out on two of the six video-recordings until an agreement rate greater than 70% on the codes was reached. Then two coders analyzed the remaining four video-recordings individually. The codes were then tabulated to identify themes and trends in the data.

Mrs. Adams

Mrs. Adams has taught high school mathematics for 4-7 years at a small rural high school. There are 728 students enrolled in this school: 59% of the students are classified as economically disadvantaged, 64% of the students are African American, and 6% are Hispanic. Mrs. Adams reported using *The Geometer's Sketchpad* for about two years, and she rated her confidence in using the software as a 2 on a 5-point scale (5 being very confident). She reported using laptops for 1.5 years and demonstrated facile use of technology. She taught an honors level geometry course with 30 freshman students.

Patterns of Mathematical Discourse

A statement, question, answer, statement cycle was predominant in the analysis of Mrs. Adams' teaching when facilitating classroom discourse both with and without technology. Generally, she would initiate discourse with a statement that lead into a question for either the whole class or a particular student. Student(s) would answer the question posed at which time Mrs. Adams would confirm/revise their response with another statement, often initiating the pattern again. The following episode provides an example.

Mrs. Adams: Today we are going to look at the other two pieces of our quadrilateral tree. (S)
Mrs. Adams: What were our other two pieces of the quadrilateral tree that we talked about? (Q)
Student A: Trapezoids and uh…(A)
Mrs. Adams: Trapezoids and…(S)
Student B: Rhombuses…(A)
Mrs. Adams: Rhombus we talked about with parallelograms (S)
Student C: Kites (A)
Mrs. Adams: Kites. We are going to look at our trapezoid and our kites today and those properties that you need to know. (S)

This pattern is similar to the I-R-E/F pattern typical of "traditional" classrooms. In this instance, Mrs. Adams initiates with a question about the quadrilateral tree, students respond and the teacher provides feedback about their responses to her question by correcting or restating what the student said. While this example is taken from a portion of class in which students did not use technology. Similar instances were noted when technology was used.

Modes of Mathematical Discourse

In addition to examining the patterns of discourse, knowing who is participating in classroom discussions is important to consider. By analyzing the modes of discourse, the actors involved in the discussion become more explicit. For Mrs. Adams, a significant shift in discourse can be seen in the reduction of discourse initiated by or directed to the teacher in favor of more student-to-student discourse when technology was employed (Figure 4a & 4b).

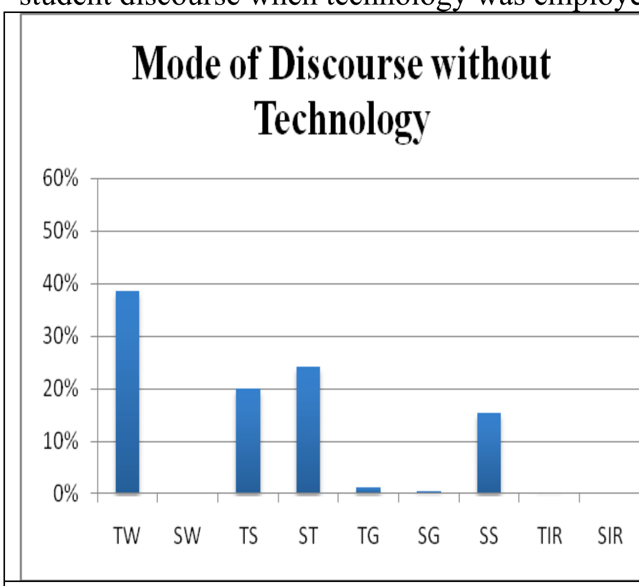

Figure 4a. A chart summarizing the percent of

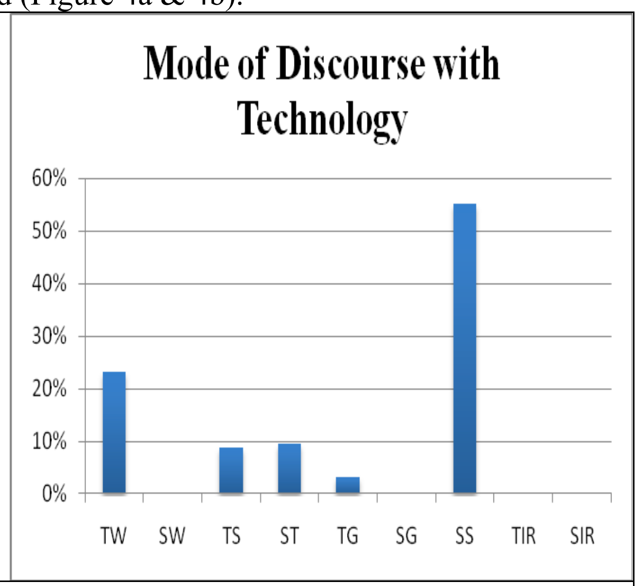

Figure 4b. A chart summarizing the percent of

each mode of discourse recorded in Mrs. Adams' classes when technology was not used.	each mode of discourse recorded in Mrs. Adams' classes when technology was used.

It should be noted that when students were working with the technology they were generally in pairs or small groups and the teacher often allowed them to work for extended periods of time on their own. Although she circulated around the room while students were working, she generally did not interrupt their work and encouraged students to direct questions toward their partner or group.

Types of Mathematical Discourse

Although a marked shift was noted in the mode of discourse, changes in the type of mathematical discourse when technology was used were minimal. The types of discursive exchanges between teachers and students remained approximately the same whether students were using technology or not (Figure 5a and 5b).

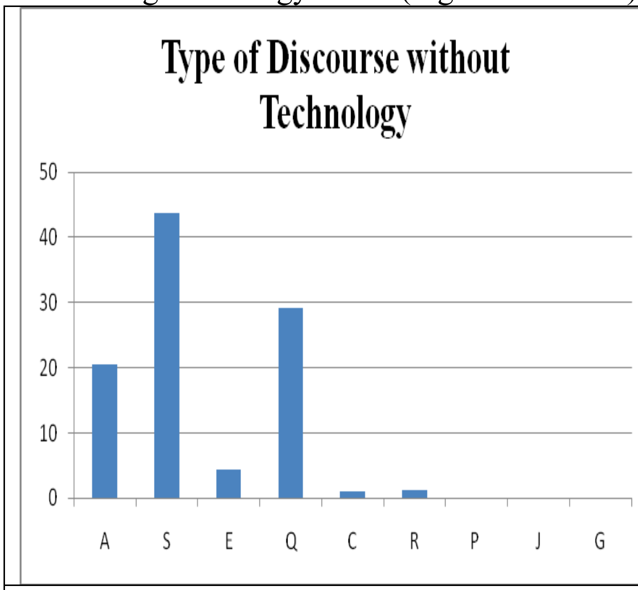

Figure 5a. The percent of each type of discourse recorded in Mrs. Adams' classes when technology was not used.

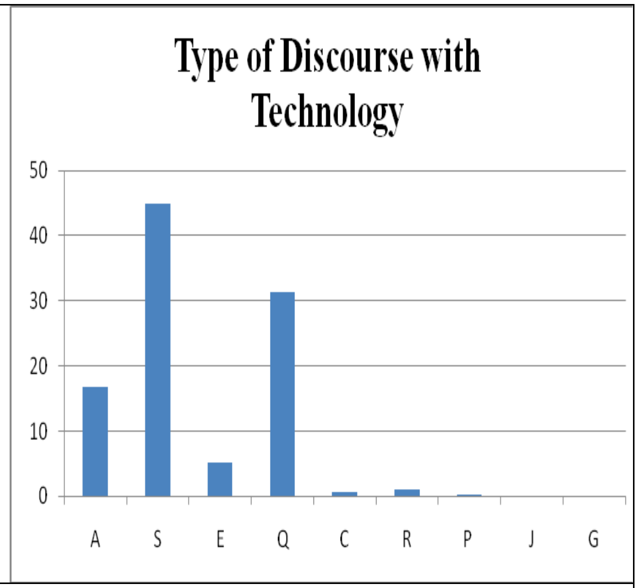

Figure 5b. The percent of each type of discourse recorded in Mrs. Adams' classes when technology was not used.

One might expect that if the participants in the discourse change then the nature of that discourse change also. However, this was not noted in our observations of Mrs. Adams.

Mr. Jones

Mr. Jones has taught high school mathematics for 11-14 years and is the chairperson of the mathematics department at a medium-sized suburban high school. There are 948 students enrolled in this school: 23% of the students are classified as economically disadvantaged, 18.8% of the students are African American, and 8.9% are Hispanic. Mr. Jones reported using *The Geometer's Sketchpad* for about 6 years and he rated his confidence in using the software as a 4 on a 5-point scale (5 being very confident). He was named "Teacher of the Year" for the school district seven years ago. The class he was observed teaching was a regular geometry class that contained 22 students. This school district was beginning the first year of a 1:1 laptop initiative.

Patterns of Mathematical Discourse

Mr. Jones made a determined effort to foster discourse in his classroom. He posed many questions to students and provided an appropriate amount of time for students to think about the question before requesting an answer. Some of the patterns that were observed when technology was and was not used could be described as following a typical I-R-E/F sequence. However, there was another discourse pattern that was prevalent when technology was and was not used that differed from this traditional pattern. Often the pattern would begin with the posing of a question from a student, the teacher would respond with a question, a student would answer, and the teacher would follow with a statement. For example, during the first observation students were working on the computer on an activity, which required them to identify rotational symmetry for a pattern that was provided in a pre-constructed sketch.

Student:	I don't get it. (S)
Mr. Jones:	Okay. Look at the small purple triangles on the outside. These ones. What is the smallest angle of rotation? (Q) Did you rotate that one? (Q) I see that you have it highlighted. Did you rotate that one? (Q)
Student:	Yes. (A)
Mr. Jones:	What are your answer choices? (Q)
Student:	(student reads answer choices) (S)
Teacher:	Okay. Hold on a second. Did you tell it where the center of rotation was? (Q) You always have to tell it this is the center of rotation is and then this is the thing that I am trying to rotate. You've got to put a center of rotation on the picture. Use the arrow tool to double click. (S) Good. (Student performs a rotation) Does that look exactly like the other one? (Q) (Teacher walks toward another group in the class)

In this interchange the teacher posed a sequence of questions to assist the student in using the technology so that she is able to respond to the question on her own. These series of questions are different from the typical I-R-E/F pattern. In particular, the question about whether the image under the rotation looks exactly like the original focused the student on what it means for two figures to be the same under a rotation. This enabled the student to use the technology to determine the rotational symmetry of the original pattern.

Modes of Mathematical Discourse

During the observations when technology was not used, Mr. Jones engaged the class in whole-class discussions during which time he generally addressed the entire class (TW). Students also responded to the teacher (ST) and the teacher addressed particular students either during a whole class discussion or when students were working individually or in small groups (TS, TG). There were also instances of students speaking to their group members (SG). The percent of episodes that were coded for each of the modes when technology was not used is shown in Figure 6a.

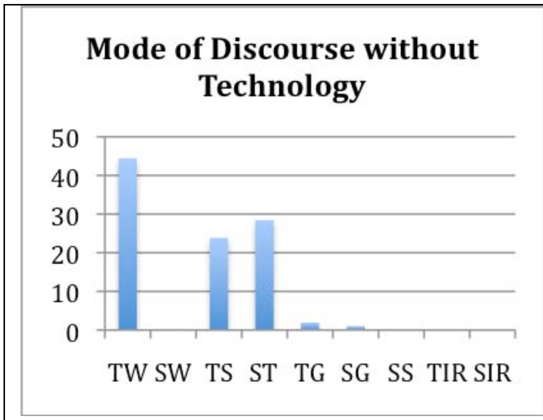	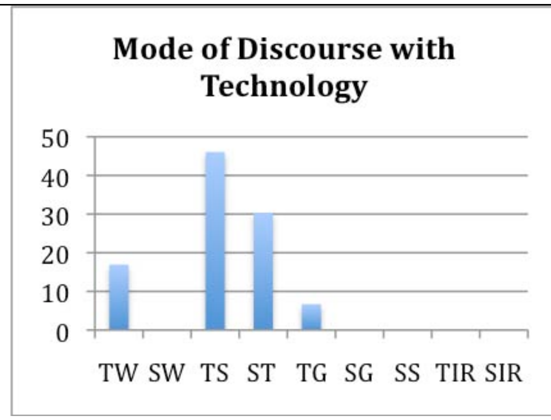
Figure 6a. The percent of each mode of discourse recorded in Mr. Jones' classes when technology was not used.	Figure 6b. The percent of each mode of discourse recorded in Mr. Jones' classes when technology was used.

When technology was used, the percent of discursive interactions between the teacher and whole-class decreased (TW) while the teacher-individual student interactions increased (TS). The number of interactions with the teacher that were initiated by the student (ST) remained about the same and there was a slight increase in the number of interactions the teacher initiated with small groups (TG). There were no instances of student-group interactions or student-student interactions.

The lack of student-to-student interactions was a bit surprising. One might expect that when students worked in groups with or without technology there would be an increase in the mathematical discourse that was exchanged. However, it is important to note that non-mathematical discourse was not coded. So there were times when students would ask each other about directions or how to complete a task with the technology, but these instances were not counted. Also, when students were working in pairs or small groups, Mr. Jones would often interject questions to individuals and groups about the task on which they were working and students would frequently ask questions of him. This was different from the interactions that were observed in Mrs. Adams classroom.

Types of Mathematical Discourse

While the mode of discourse allows examination of who is participating and initiating discourse in the classroom, it does not provide a clear picture of the content of the discourse. To better understand what was discussed when technology was and was not used, the type of discourse was analyzed. Figures 7a and 7b depict a summary of the different types of discourse identified from all three video-recorded observations of Mr. Jones' class.

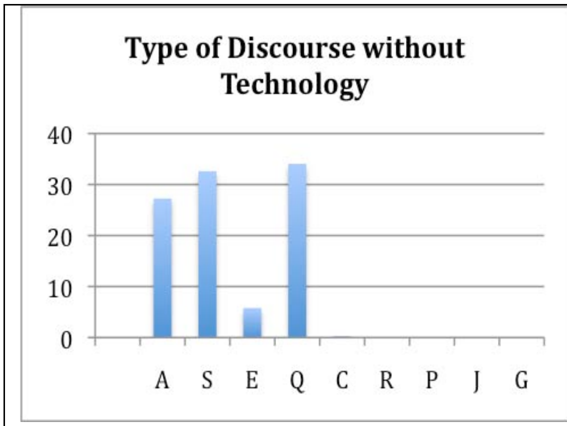

Figure 7a. The percent of each type of discourse recorded in Mr. Jones' classes when technology was not used.

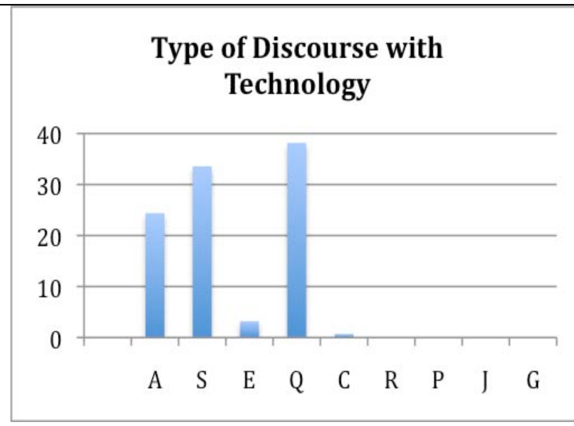

Figure 7b. The percent of each type of discourse recorded in Mr. Jones' classes when technology was not used.

There were approximately the same percentage of answers and statements provided with and without technology. There was a slight decrease in the number of explanations provided when using technology and a slight increase in the number of questions and challenges noted when technology was used.

Conclusions

The purpose of this study was to characterize the nature of mathematical discourse when students were using technology and to contrast this to discourse that was typical in the class when technology was not used. The analysis of the patterns, modes, and types of discourse that occurred in three video-recorded classes of two teachers provides us with a glimpse of those differences. For both Mrs. Adams and Mr. Jones the mode of discourse shifted when technology was used. For Mr. Jones there was less teacher-to-whole class discourse and more teacher-student and student-teacher discourse. For Mrs. Adams there was a striking increase in the number of student-to-student discursive interactions when technology was used. However, it is important to note that while students were using technology they were also working in small groups or in pairs. It is possible that similar changes in the modes of discourse would be observed in non-technological settings when students are working in pairs or small groups. This is a question that could be further examined. And while changes in discourse modes were noted, changes in discourse types were not. While many have suggested that technology has the potential to support reasoning and sense-making activities that could foster justification and generalization discourse types, this was not observed. Research in non-technological learning environments suggests that teachers first build an environment that students feel comfortable interacting (modes) and then turn their focus to mathematics (types) (Silver & Smith, 1996; Wood, Cobb, & Yackel, 1991). It is also important to note that the first year of the professional development did not focus teachers on classroom discourse. As teachers transition into the second year and this becomes a more explicit focus for them it would be interesting to see if more dramatic changes become evident in the type of discourse that occurs within their classrooms.

Endnote

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CLASSROOM COMMUNICATION SYSTEMS IN THE MATHEMATICS CLASSROOMS: STUDENT LEARNING AND ENGAGEMENT

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This study is designed to measure students' perceptions of classrooms using a Classroom Communication Systems (CCS). Drawing on the work of Bransford, Brown, and Cocking (1999), the study is designed to investigate the extent to which CCS classrooms facilitate environments which students experience as learner-centered, knowledge-centered, assessment-centered, and community-centered. Results suggest that students perceive that CCS technology gives teachers more information on what they are thinking, gives them more information on what other students are thinking, gives them more information on their own progress, and supports sharing of information to facilitate collaborative learning.

Introduction

More than two decades and of research and experience supports the idea that advanced digital technologies can have an important role to play in supporting student learning (Heid, 1988; Kaput, 1992; Kutzler, 1996; Papert, 1980; Waits and Demana, 1999). The development of Classroom Communication Systems (CCS) is providing new possibilities for the role technologies play in creating and supporting effective learning environments. The TI-Navigator™, from Texas Instruments, is a wireless CCS designed for K-12 classrooms. The pedagogical potential of CCS technology is still in its early stages but research suggests benefits to active student participation in class and collaborative inquiry in the classroom (Abrahamson, Davidian & Lippai, 2000; Bransford, Brophy, & Williams, 2000; Dufresne, Gerace, Leonard, Mestre, & Wenk, 1996; Pape, Irving, Owens & Abrahamson 2005; Wenk, Dufresne, Gerace, Leonard, & Mestre, 1997). This study is designed to measure students' perceptions of the extent to which CCS classrooms facilitate effective teaching and learning environments.

The research question driving the analysis presented below was:

Does the use of CCS result in an environment which students experience as more learner-centered, knowledge-centered, assessment-centered, and community-centered?

Theoretical Framework

Bransford, Cocking and Brown (1999) developed a framework for designing effective learning environments. The design of such environments is based on three principles:

"To develop competence in an area of inquiry, students must:

- (a) have a deep foundation of factual knowledge,
- (b) understand facts and ideas in the context of a conceptual framework, and
- (c) organise knowledge in ways that facilitate retrieval and application." (p 18)

Furthermore the framework proceeds from the premise that “Students come to the classroom with preconceptions about how the world works. If their initial understanding is not engaged, they may fail to grasp the new concepts and information that are taught, or they may learn them for the purposes of a test but revert to their preconceptions outside the classroom.” (p.14).

These principles form the basis for a model for describing learning environments with specific implications for teaching called How People Learn (HPL). Consequent to these principles Bransford et al. propose that effective learning environments should be learner-centered, knowledge-centered, assessment-centered, and community-centered.

Aspects of a teacher taking a learner-centered approach include using questions, tasks, and activities to show existing conceptions that students bring to the classroom, and exerting an appropriate amount of demand on students to think through issues and establish positions. A knowledge-centered approach manifests itself in a focus on conceptual understanding, and the diagnosis and remedy of misconceptions. Assessment-centered instruction concentrates on formative assessment to provide feedback to students and to teachers on student conceptions. Finally, a community-centered approach is reflected in, for example, class discussion, peer interaction, and non-confrontational competition. (Bransford et al., 1999).

Description of the Technology

Since the late-80s CCS have been an evolving technology. This genre of classroom tools began with hard-wired multiple-choice “response' systems, but is now being transformed by high speed wireless networks, handheld devices, and programmable control by a computer.

Modern CCS consist of five parts: (a) a computer, operated by a teacher, running a software package displaying information to only the teacher; (b) a projection panel which displays public information; (c) student devices which may be calculators, laptops, etc.; (d) a network which connects student devices to the teacher's computer, interprets communication protocols, and sends tasks to and from students and the teacher; and, (e) software which allows the whole system to function as an integrated classroom tool. Using CCS students can send in answers to multiple choice questions, alpha numeric answers, or lists of numbers based on measurements. The TI-Navigator™ is a modern CCS which possesses the unique characteristic of being almost totally programmable. That is, a Navigator™ “activity” or “task” may be designed to accept teacher-authored questions or other specific curricular materials, and also whole new activities written by third-party developers or technically sophisticated teachers.

The following is an example of how a Quick Poll activity might work: The teacher sends students a set of multiple choice questions to be answered including the question “What is –3^2?” with possible answers “A. 3; B. 9; C. –9; D. –3; E. 0”. Students’ progress on the questions can be tracked on the teacher console as they send in their answers. The particular question “What is –3^2?” is a question that students answer incorrectly due to misunderstanding or misremembering the order of operations necessary to get a correct answer. When all students have sent in answers to the set of questions, the teacher can display histograms of the students’ answers. In a class of 25 students the histogram for the question might look like Figure 1:

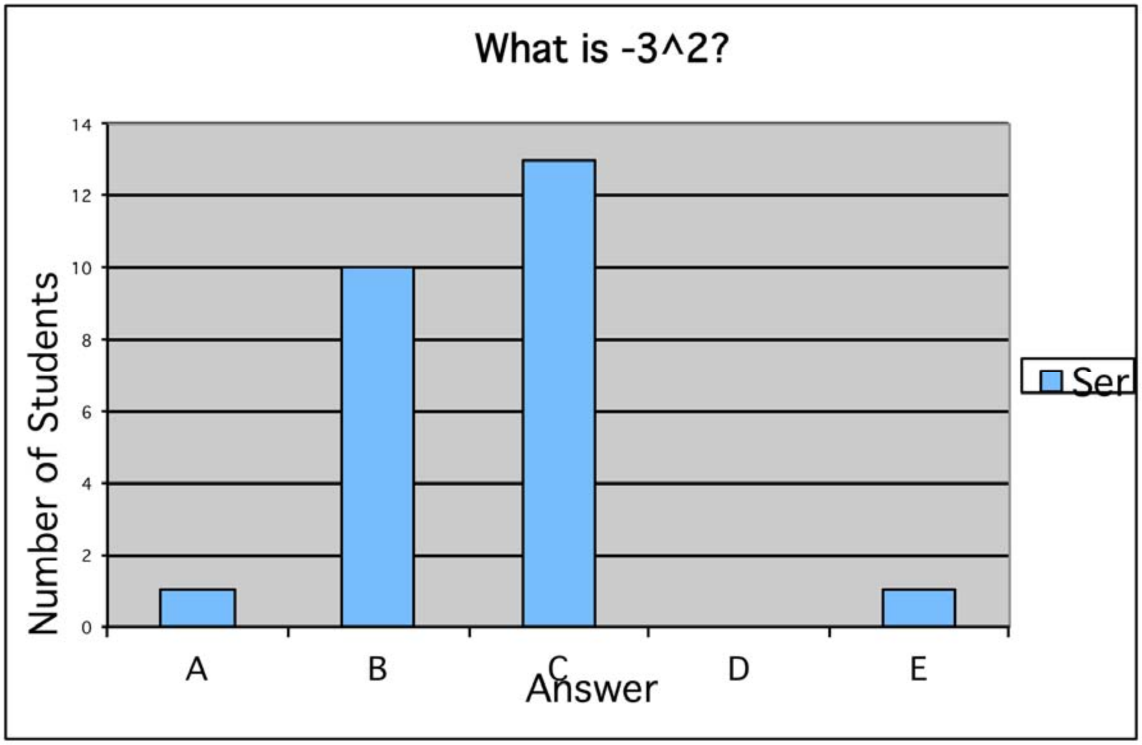

Fig 1 An example of CCS use for polling a class

There are several consequences of using the TI-Navigator™ worth noting here: (a) every student has had to choose and send in an answer to the question (they cannot just wait for someone else to answer), (b) the teacher knows that almost half of the students have a misconception as to how to simplify this arithmetic expression, (c) students know that probably at least half of them answered incorrectly, (d) students know that if they were one of those who answered incorrectly they are not alone, (e) misconceptions about how to answer this question have come into the open without any student having to risk the embarrassment of declaring an incorrect answer (although the teacher can know which students answered correctly).

Methods and Methodology

The participants in this study were eight high school mathematics and science teachers and their students (n=389). The teachers were experienced in teaching with graphics display calculators and took part in a professional development on the technical aspects of operating the TI-Navigator™ system as well as pedagogical techniques and possibilities in a networked classroom. Before the training teachers were surveyed in open question format about their pedagogical practices and were surveyed at the end of the school year about possible changes in their practice. A series of visits was undertaken to their classrooms where the TI-Navigator™ system was being used.

Shortly before classroom visits, teachers and students took Likert-style surveys designed to elicit their views on the extent to which TI-Navigator™ classrooms reflect the HPL model i.e. the extent to which CCS classrooms are learner-centered, knowledge-centered, assessment-centered and community-centered. Responses to the Likert-style surveys were used as the basis for teacher interviews focus group interviews with students.

The Focus Group interviews were transcribed and coded independently by at least two of the research team for instances of each of the centrednesses. The relevant passages were then organised by centredness and further analysed to provide both exemplary and consistent instances of manifestations of students' experiences of the centredness in the CCS environment. Finally the data were coded for themes beyond the centerednesses.

Student Survey Items and HPL Centeredness

The Likert-style student surveys we wrote two statements representing key aspects of each centeredness. Each statement was then also expressed in a negative sense, to give a total of four statements per centeredness. Thus, for the student survey, we presented statements such as:

Learner Centeredness

- Doing activities with the TI-Navigator™ helps me relate new material to things I already know
- I am equally on task in TI-Navigator™ classes and other classes

Knowledge Centeredness

- Doing activities with the TI-Navigator™ helps me get a better understanding of concepts
- Some TI-Navigator™ questions make me try really hard to answer them

Assessment Centeredness

- Using the TI-Navigator™ I can quickly tell whether or not I am right or wrong
- The teacher knows just as much about my understanding without the TI-Navigator™ as with it.

Community Centeredness

- Class dynamics are not affected by the use of the TI-Navigator™
- There is a greater sense of community in a TI-Navigator™ class than in other classes

Student Focus Group Interviews

Student focus group interviews took place after the observed classes. The starting point of the interviews was analysis of the student surveys which had been filled out prior to the visit. Thus, a typical question was: One of the survey questions asked you respond to the statement "The TI-Navigator™ helps me tell if the students understand a concept." Most students said they Strongly Agreed (SA). Why did you think they responded in this way? The purpose of this approach was to retain the focus on the centerednesses in order to directly address the research question.

Data Analysis

The analysis presented in this paper is focuses on the student surveys and student interviews (with occasional quotes from Teacher interviews) in order to answer the research question:

Does the use of a CCS result in an environment which students experience as more learner-centered, knowledge-centered, assessment-centered, and community-centered?

Student Surveys

The student surveys were constructed from eight pairs of confirming and disconfirming statements designed to reflect aspects of each of the four centerednesses. These results are based surveys of all twenty-two classes from eight schools.

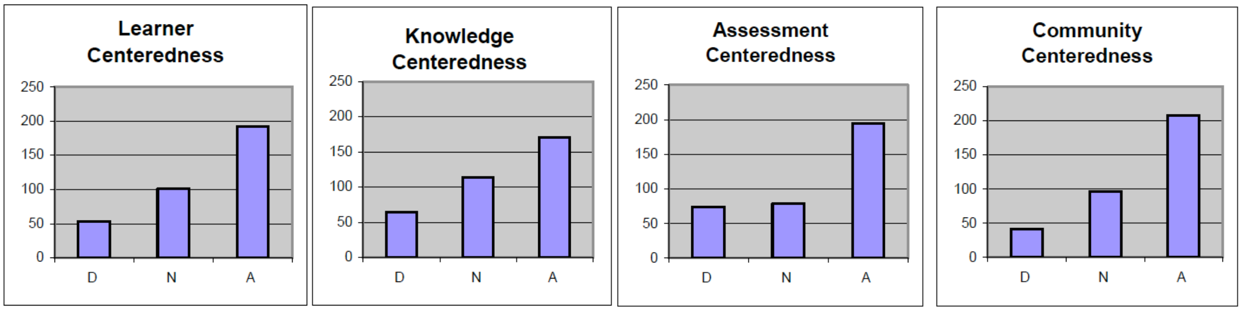

Figure 2. Composite Results from Student Surveys of Classes Listed

Each of the four centeredness plots (in Figure 2) above are composite tallies of responses to four statements. (Student responses were tallied with 'D' and 'A' merged for each survey statement to gives totals for SD + D, N, A + SA. The resultant four component items for each centeredness were summed and normalize (i.e. divide by 4) so totals reflect composite numbers of students).

The results are generally positive especially in the area of assessment-centeredness and community-centredness suggesting students are experiencing the benefit of formative assessment made possible in a CCS classroom, and that class dynamics are improved in a CCS classroom. In the interviews reported below we see some of the reasons behind these student perceptions.

Interviews

Focus Group interviews took place at each school after the surveys had been analysed and a class at that school observed. The analysis, in keeping with addressing the research question, deals separately with each of the centrednesses in turn.

Learner Centeredness

"The learner-centered perspective focuses on the knowledge, skills, goals, and cultural beliefs that each person brings to the learning situation" (Bransford, et al., 2000, p.72). In a TI-Navigator™ class this is facilitated through the possibility of the work that students are doing being available for discussion much more readily than in a non-CCS classroom. The display of results, while it is anonymous, has the effect on students that "because they know it's going to be showing on the [panel], they're trying a little bit harder to get a right answer." (Teacher 1) and "the fact that everybody got it and I didn't get it makes me think, 'Ok, I'm doing something wrong.' and it makes me want to go back and see what am I doing, because everybody else got it right and I didn't do it right." (Focus Group 1, Student 1).

From the students' point of view it seems to be apparent that the TI-Navigator™ helps them to build constructively on prior knowledge. They attribute this partly to the fact that it also helps the teacher to know if they understand knowledge needed to gain further understanding:

Int: *"Doing activities in class with TI Navigator™ helps me relate new material to the things that I already know"?*

FG2 Student 4: I think it's because … we like to see how … where we stand in the class; by doing it they get to know if they understand past material. So you have past material and then you lead up into the new material when we use the TI Navigator™.

Int.: Do you think that's the way her lesson worked today?

FG2 S4: Yeah, I think it was because she gave us a problem on finding just the derivative first, of the equation, and then we worked on the anti-derivative. So it was like going back to make sure if we know

In many interviews students brought up the topic of active engagement. In the following, students discuss the group conversations that are generated by TI-Navigator™ activities:

FG3 S1: If other people are attentive that keeps me attentive as well.

FG3 S2: No one is really goofing off and stuff.

FG3 S4: I think it keeps your mind moving because when one person asks the question it can spark another person's mind and they're gonna have a question and it just keeps going and going. I think you're learning a lot. Always thinking.

Students also feel a more direct connection with the teacher:

FG3 S5: Especially with classes getting bigger like they are now. It gives you more of a feeling of you're more like one on one with the teacher.

The quotes above illustrate that many students experienced the CCS classroom as more learner-centred than their regular classes confirming the possibility, in the CCS, class of an experience where the knowledge each student brings to class can be used in a lesson.

Knowledge Centeredness

A knowledge-centered environment is one in which transferable skills and knowledge in context are both developed (Bransford et al., 1999). A major aspect of a knowledge-centered approach is, therefore, the building of connections between concepts, between different representations of the same concept, and between previously learned concepts and new concepts.

In Teacher 1's classroom, visualization and a focus on the essential characteristics of graphs allowed students "to see how the equation was similar to the other equations that we had seen and where we might start with that picking up with what the vertex was" (Teacher 1).

In the following excerpt, the discussion began with "grades" but quickly turned to issues of effort, understanding, and receiving feedback in a timely manner:

Int.: If you had been using it all year would it have made a difference to your grades?

FG3 S2: I would have understood a lot more. I mean, it couldn't hurt my grade, obviously, because I would have stayed awake, and really paid attention.

FG3 S3: So many times, I would think I had the concept at the end of the chapter, and I would take the test and completely bomb it. So I had *no idea* that I didn't get it. And, if I had [the Navigator™] all year I would probably like been able to tell if I had.

FG3 S4: I do too. It allows you to know at the time when you're studying the topic, what you know and what you don't know, and why, and he can explain it to you better.

However, the process of sense-making and understanding requires sustained effort. Using the TI-Navigator™ system, students are seen to be excited and to want to get things right. The fact that on some activities they can get immediate feedback "gets them more to the level of thinking, 'What was it that I did wrong?' instead of just glazing over the fact that, 'Oh. It's not correct and I don't want to ask about it.' " (Teacher 1). The public nature of information and answers in a TI-Navigator™ classroom leads to them try harder to get a right answer, "I think it makes you try harder because you don't want everybody else to see that you don't know what you're doing" (FG5, S2). Also, the element of immediacy seems to play an important part for some students and this has positive effects on the effort that they apply:

Int.: Do you think having to do something with the calculator makes you think through things that you wouldn't have put the effort into thinking through?

FG4 S2: Yeah and you think, "Oh my gosh, am I ready to send?" and, …

FG4 S4: It makes you think twice about your answer.

However, it appears that this hesitation, does not result in adverse feelings about the system:

FG2 S2: It's like a hands-on activity kind of thing, and it makes it easier.

FG2 S3: And it makes it more interesting.
FG2 S1: Yeah, if we do worksheets, and go to the board. I don't find that much fun.
Int.: So you think generally you're working a bit harder doing it like this?
FG2 S 1, 2, 3, 4: Yeah. … Yeah...

Using the system also seems to provide a level of demand on students to think more and try harder. However, they tend to view this in a positive light:

FG2 S1: Yeah, because on homework, a lot of us will leave it blank if we don't understand, but by having to enter an answer it makes you really try and figure it out because you want to see if you were right or wrong compared with the rest of the class.
FG2 S3: Yeah, it makes you try to think on the answer harder.

Sometimes students had didn't make connections or see the value of an activity: "I think [the simulation] was meant to be more of a fun activity than teaching us anything" (FG1, S3).

The fact that the TI-Navigator™ system can run simulations and allow multiple sets of data generated by students to be compiled and displayed, allowed students the opportunity to make connections between the experiments they have done themselves and aggregated results of the whole class's experiments. Data gathered from many students can be compiled, visualized and understood as a physical process. "Plugging in numbers and seeing what happens … and see[ing] why what you plugged in does what it does or means helps me," (FG1, S3).

Assessment Centeredness

"Effective learning environments also require frequent opportunities to make students' thinking visible to see what they are learning" (Bransford et al., 2000, p.69). It is vital to the learning process that students have feedback on whether they are learning concepts, and vital for teachers to have feedback on how well students are learning concepts. Certain TI-Navigator™ activities allow students to find out immediately if they have an answer correct; whereas, without the technology "if they're doing their regular homework, they didn't necessarily know that they didn't get it correct, so they're still just doing the same thing and not really asking you, because they're not really thinking that they're making a mistake." (Teacher 1). The advantages of this feedback are clear to the students: "… it was quicker, more efficient … we like that you automatically find out what you made on the quiz." (FG1, S5).

The fact that students send answers to a computer means the teacher can see whether they are active or not and whether they are successful or not. Teachers noted that "you get more honest answers with the Navigator™. It is anonymous so it's not threatening." (Teacher 5).

Advantages of both types of feedback (i.e. to students *and* teacher) are clear to the students:
I: "Using the TI Navigator™ does not help in letting me know where I stand on a question?"
FG3 S3: I disagree.
FG3 S2: I would disagree a lot on that, because you get instant results right up on the board, and it tells you if you got it wrong or right, and if enough people got it wrong, he'll know that he needs to go over that, and you'll know you need to go over that - instantly.
FG3 S1: And I think it's easier on the teacher too, because instead of passing out the papers, having to pass them back and him go through it, and then passing them back out and going over it he knows right away as a class where everyone stands, instantaneously.
FG3 S5: And you're able to fix things at the moment. You know, instead of 2 or 3 days.
FG3 S1: By that time you forgot even more. You forgot what the question even *was*.

Similarly, in the following, from a different teacher's classroom:

FG2 S3: If we're all wrong, or the majority is wrong, she'll go over the concept again.

FG2 S1: I think it gives her an idea of where everybody stands in the class, so she knows what we need to review more, … or you know, we can move on, or not.

A problematic area with this feedback is that, for some activities, it can focus on right/wrong answers and doesn't give a physical record of students making their thinking processes visible.

It can be argued that the presence of new types of feedback from systems like TI-Navigator™, means that teachers have to learn to listen in ways that they did not have to before. Teachers have a great deal more information but still need to investigate the cause of misconceptions, and devise ways to rectify them.

Community Centeredness

The effect of the TI-Navigator™ system on the classroom environment and dynamics we visited was quite dramatic. There was a strong sense of students working harder and working towards a common goal: "It's more closely knit, the class. Because in a normal class you have people that are sleeping and here you don't have room for that." (FG5, S3) and, "When we used it, everyone was doing it and everybody was interacting and it made everybody work and I think the class participation went up a lot" (FG1, S4). The public nature of the knowledge state of the class, visible in results on screen, gave rise to discussions of the provenance of answers and provided many "teachable moments" because misconceptions are made visible: "we're all connected to the same system so we can see each other's mistakes, where before we would just do [our work] individually," (FG1, S2). Furthermore, students are less afraid to admit mistakes because of the more free flowing nature of information in the class. Some students felt that the sense of community arose from the novelty of the system and did not impact their learning: "I think the sense of community comes just from being something that's experimental but I haven't felt like I've learned very much" (FG1, S3). Others felt that technical problems with the system and efforts to overcome those were the source of the community feeling, "Everyone was helping everyone else figure out what to do but if it worked properly, then I don't think that everyone's community would be going on." (FG1, S3).

The TI-Navigator™ classroom is a very interactive environment. Students are presented with multiple representations such as symbolic, visual, and simulated. The TI-Navigator™ classroom is one in which there are class discussions arising out of shared information as well as collaboration and a feeling that something novel has happened: "It's more student based. You're not just looking at chalk boards" (FG1, S2)

Int: Class dynamics are not affected by the use of TI Navigator™?

FG2 S4: I guess that's why everyone disagreed, cause I think it does change the dynamics of how our class is. More interaction …we're all connected … and if we don't have the TI Navigator™ we're just sort of in our separate note-taking little world, but with this … we're actually working together. It's just a different way of learning. This is something where we can talk about it and understand it.

Conclusion

This study sought to address the research question:

Does the use of CCS result in an environment which students experience as more learner-centered, knowledge-centered, assessment-centered, and community-centered?

The analysis of the surveys and focus group interviews presented above present compelling evidence that a significant level of transformation occurs in students' experiences of classes when CCS technology is deployed. The data are self-reported by the students but reflect accurately and consistently their experience of the CCS environment and their perceptions are consistent with the teacher and researcher observations of the classrooms.

Clearly much research remains to be done on the impact of CCS on classrooms but we believe that the work done on this project shows that:

1. The role that CCS technology has to play in implementing HPL principles in the design of an effective learning environment centres on the free flow of information in the classroom. That is, CCS technology gives teachers more information on what students are thinking, gives students more information on what other students are thinking, gives students more information on their progress, and supports sharing of information to facilitate collaborative learning.

2.The accumulation of effects of the use of the TI-Navigator™ on student feeling, motivation, behaviour and classroom dynamics suggest that the TI-Navigator™ can have a positive influence on student learning.

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REPRESENTATIONS OF LINEAR FUNCTIONS IN THE PRESENCE OF CONNECTED CLASSROOM TECHNOLOGY

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We examined the representations used by the teachers in connected classrooms and whether changes occurred in their discourse about multiple representations of functions over time and as the result of access to technology. Analysis of videotapes of 6 teachers' practices when teaching a unit of instruction on functions revealed three major patterns. First, teachers' approaches to teaching functions and their representations were influenced by whether they maintained a calculational or conceptual orientation to teaching. Second, their choices of representations were closely linked to their view of functions as either a process or an object. Third, as the teachers' technical knowledge of technology increased they illustrated various representations more often, however their discourse did not address connections among different representations.

Introduction

Principles and Standards for School Mathematics (NCTM 2000) place tremendous emphasis on the importance of children learning about and using representations when solving problems, communicating mathematics, and representing real life situations. Representations can be taken to be the external representations produced to help think, communicate and interpret mathematical concepts. They may also be considered the internal representations a student might possess. Representation(s) refer to both the act of representing as well as the external, noun form of representation. In mathematics, internal representations can be considered to be abstractions of mathematical ideas or a structured way of thinking that are developed by and are internal to a learner (Pape & Tchoshanov, 2001). External representations typically have a sign or configuration of signs, characters, or objects that can stand for something else, such as graphs, pictures, diagrams, words, symbols, and manipulatives. Internal and external representations are believed to mutually influence each other. However, the literature (Cunningham, 2005; Even, 1998; Galbraith & Haines, 2000; Hitt, 1998; Knuth, 2000; Sfard, 1991) points to the difficulty for students to transfer among the various types of representations while preserving meaning. In addressing this challenge, scholars have argued that the use of technology can facilitate both the development of knowledge of multiple representations and the ability to transition among them (Manouchehri, 2004). A particularly promising tool for doing so is TI-Navigator™. Several features of the TI-Navigator™ have the potential to increase discussion or student use of representations. Quick Poll (QP), Activity Center (AC), Screen Capture (SC), and Learning Check (LC) all can invite all students to participate in activities and practice using representations. Through AC and SC, students can see their own representations as well as other students' representations allowing for reflection on and discussion of the representations. AC has the ability to move among three major representations of the same object, algebraic, graphical. With several students participating, alternative representations are likely to occur. Pursuing explanations and justifications of why are or are not particular representations reasonable can aid students to see representations as tools.

The purpose of the current study was to investigate the nature of teachers' use of representations in classrooms where connected classroom technology (CCT) was used. This report is a part of a much larger research project in which we investigated the relationship between teacher use of CCT

Wiest, L. R., & Lamberg, T. (Eds.). (2011). Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Reno, NV: University of Nevada, Reno.

in algebra classrooms and the growth of teacher and student choice of representations of linear functions as manifested in the classroom discourse were investigated directed by Classroom Connectivity in Promoting Mathematics and Science Achievement (CCMS) research project (Irving et al., 2010). In focusing the current study, we considered the topic of linear equations as the context for analysis of teachers' use of and discourse about representations of functions and their connections. Two specific questions guided the data collection and analysis:

a. What representational modes are used by teachers when discussing linear functions?
b. How is the quality of teachers' discourse about representations of functions influenced by the presence of technology?

Theoretical framework

In studying classroom use of representations of functions we grounded our work in the three major perspectives on representations of functions as described by Moschovich, Schoenfeld, and Arcavi (1993) including algebraic, graphical, and tabular. Functions have a dual nature. Functions can be seen as a process, acting on a set of inputs to obtain outputs. They can also be seen as objects that can be manipulated as a whole. Although not specifically discussed by Moschovich and colleagues, the process perspective and object perspective of functions can be thought of as part of the internal representation of functions, since they are the primary ways to think about functions. This framework excludes a major representation particularly vital in classroom interactions, the verbal representation. For example, the function f(x) = 3x+5 can be verbally represented as "the function that multiplies the input by 3 and then adds 5," which is describing a process. The same function could be represented as "a line with a slope of 3 and a y intercept of 5," which describes an object. The different combinations of external representations and perspectives are captured in table 1. Here, the four main representations can be used in conjunction with either the process or object perspective.

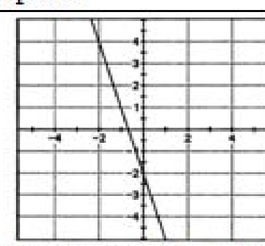
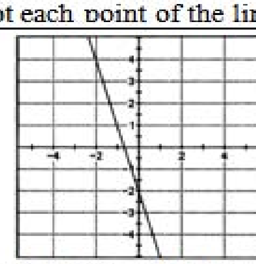
	Tabular	Verbal	Algebraic	Graphical
Process	<table><tr><td>x</td><td>y=f(x)</td></tr><tr><td>-2</td><td>4</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>0</td><td>-2</td></tr><tr><td>1</td><td>-5</td></tr><tr><td>2</td><td>-8</td></tr></table> Given these x's find the corresponding y's	The function that multiplies the input by -3 and then subtracts 2.	f(x)= -3x-2 The function multiplies a number by -3 and subtracts 2.	 Plot each point of the line.
Object	<table><tr><td>x</td><td>y=f(x)</td></tr><tr><td>-2</td><td>4</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>0</td><td>-2</td></tr><tr><td>1</td><td>-5</td></tr><tr><td>2</td><td>-8</td></tr></table> The function is the set of ordered pairs.	A line with a slope of -3 and a y intercept of -2.	f(x)= -3x-2. The function is a static relationship between x and f(x).	 The function is the line.

Table 1. Different representations and perspectives of lines.

The process-object framework (Moschkovich, Schoenfeld, & Arcavi, 1993) merged with the external-internal representations interaction framework (Pape & Tchoshanov, 2001) allowed us to suggest a model for how representations may interact within a learning environment. The four main representations of a line, algebraic, graphical, tabular, and verbal representations (Moschkovich et al., 1993) can be considered to interact with a student's internal representations and perspectives via practice, negotiation, reflection, interaction (Pape & Tchoshanov, 2001).

Developing student representational thinking requires not only the use of multiple representations but also discussing the connections among them. This relates closely to how multiple representations are addressed and elaborated in the social context of the classroom, as evidenced in teachers' and students' discourse. Certainly, the more elaborate the discussions of connections among representations, the greater opportunities for students' development occur (Manouchehri, 2004). To capture the quality of teachers' discussions of representations we used the Complex Mathematical Discourse Network CMDN (Manouchehri, 2004) to rank the level of complexity of teacher talk surrounding the usage of different representations of functions.

Methodology

Participants

In the larger project, Algebra 1 teachers from 28 states and 2 Canadian provinces and their students participated. Initially, "a total 127 teachers (66 control, 61 treatment; 74% female) were randomly assigned to either control or treatment groups" (Irving et al., 2010, p. 8). The student participants were students of the teachers from both cohorts during year 1 to year 4 of the study. Fifty-five of the teachers in the CCMS study had videotaped classroom observations in at least one of the four years of the study. Since the purpose of the study was to describe the types of linear representations used, only teachers who were videotaped giving lessons on linear functions for more than one year of the study were considered for investigation. The six teachers who met these criteria were used for analysis in the current report.

Data Sources

Quantitative and qualitative data were gathered by the CCMS project. This study uses the video data. Classroom observation videos were typically two consecutive days of class. Linear function topics ranged from slopes of lines, properties of lines, systems of equations to finding best fit lines. Only the video data and verbatim transcripts of these six teachers were used in the study reported here.

Data Analysis

Observational notes were taken during the first viewing of the video data. The videos, transcripts and observation notes were uploaded into Nvivo, a data analysis software.

Classroom instruction was coded using three different techniques. First, teachers' utterances in each lesson were catalogued according to 4 criteria: the level of cognitive load of questions asked and their mathematical significance (mathematics), the mode of response elicited from the students (closed/dichotomous vs. open/explanatory), the quality of explanations offered to the group (conceptual vs. algorithmic), and the quality of discussions regarding the representational choice used or analyzed. Additionally, we coded the number of times technology generated representations were used by the teacher and whether this choice seemed appropriate to the goals of the lesson.

To describe the types of representations used in the classroom and the quality of discourse surrounding the use of representations by teachers and students in CCT classrooms, we traced the number of each representation used or elicited in class, as well as the number of connecting statements that teachers made to link these representations in their instruction.

The quality of the discourse either about or using representation were coded using the Complex Mathematical Discourse Network (CMDN) (Manouchehri, 2004). Manoucherhi (2004) created a holistic scoring procedure that assigned scores from 0 to 4 depending on how complex the representational discourse might be. *Simple representational discourse* can be characterized by an individual using one aspect of a single representation as a solution without support or multiple representations or aspects of one representation are present but are not connected. Discourse involving only one aspect of one representation without support could be scored 0 and discourse involving multiple representations or aspects but no connections or elaborations could be scored 1. *Intermediate representational discourse* can be characterized by individuals explaining, justifying or reasoning with or about representation(s) of lines or aspect(s) of one representation. Discourse explaining, justifying, or reasoning with or about one representation or multiple representations or aspects of one representation without statements connecting them could be scored a 2. Discourse with these characteristics but where the connections among multiple representations are discussed could be scored 3. Other evidence of intermediate discourse would include reflective behavior such as modifying one's own representation because of the presence of another representation. *Complex representational discourse* can be categorized by intermediate discourse that is used to make generalizations about the connections among different representations. Complex representational discourse could be scored a 4.

Individual case studies are created for each of the six teachers describing their practices. Combining these individual cases we searched for themes that described common and unique instructional approaches among data.

Findings

The primary goals of our research project was to examine the type of representations used by the teachers in connected classrooms and whether changes occurred in their discourse about multiple representation over time and as the result of access to facilities provided by technology which could have enhanced children's thinking. Analysis of videotapes of teachers revealed three major patterns.

First, teachers' particular approaches to teaching, what they chose to include in instruction and their expectations for mathematical work of children in the context of representations of functions, were influenced by their particular orientations to teaching (A. G. Thompson, Philipp, T. W. Thompson, & Boyd, 1994). Teachers' orientations motivated how they used technology and which of its capabilities they chose to capitalize in instruction. A majority of the teachers exhibited an algorithmic orientation to teaching mathematics; instructional emphasis when talking about functions and representation was on production of these models as opposed to connections among them. Technology was used primarily for completion of calculational and procedural tasks and not to advance children's understanding of how to allow access to a deep understanding of multiple representations of functions and their connections.

Second, the teachers' choices of representations in instruction were closely linked to their view of functions as either a process or an object. The questions they asked in class, comments and illustrations they used and ways in which they utilized technology were influenced by this basic mathematical understanding. Data indicated that teachers used and motivated the view of functions and processes and rarely attempted to draw from functions as objects to help children develop an understanding of connections among various representations. The preferred representational mode when talking about functions was symbolic.

Third, as the teachers' technical knowledge of technology increased, they tended to use it to illustrate various representations, even simultaneously at times. However their discourse did not address connections among different representations and ways in which information obtained from these various representations could be used to extract additional information about the behaviors of functions. This disconnect was most prominent in places where the primary focus of instruction was on symbolic competency.

Teachers' orientation and teaching of function concept

Thompson and colleagues (1994) offered two sharply contrasting orientations to teaching characterized referred to as *calculation* and *conceptual*. In elaborating on the substantive differences between these two orientations, authors characterized a teacher with a conceptual orientation to teaching as one who focuses students' attention away from application of procedures and towards a rich conception of ideas, conceptual coherence. In further elaborating on this construct the authors suggested that the conceptually oriented teachers tend to move students towards non-calculational contexts and their actions are driven by a system of ideas and ways of thinking that she intends for students to develop; an image of how these ideas and ways of thinking can develop; … a productive way of thinking that generates a method that generalizes to other situations (A. G. Thompson et al., 1994, p. 86). In contrast, the calcualtionally oriented teachers show, among many: "A disposition to cast solving problems as producing a numerical solution; An emphasis on identifying and performing procedures; a tendency to do calculations whenever an occasion to calculate presents itself; … A narrow view of mathematical patterns as limited to finding patterns in numerical sequences" (pg. 87). The impact of these orientations became visible in the instructional practices of the teachers, what they emphasized in their lessons, tasks they assigned in class and ways in which they used technology to discuss functions and their multiple representations. That is, the teachers with greater tendency towards a conceptual orientation to teaching tended to refrain from being the locus of authority in class and allowed students to generate different representations of functions and insisted on children using technology to examine patterns and relationships. Those with a calculational/algorithmic orientation to teaching, even in the presence of technology, continued to emphasize procedural productions. Consider the following episode as an illustration of such practices. Prior to this particular episode, the teacher had plotted 5 points of the line y=-x/4 and asked students to send in their predictions of other possible points using the CCT.

T: Look at the point (-26,7). It looks really close, but is it correct? What's neg. 26/4? Aren't we dividing by 4 each time?

S: No. Oh, yes.

T: Does that divide evenly?

S: No.

T: No, so it can't come out to 7. So that one was very close, but not quite. And you can tell it's just slightly off, isn't it? And what about this one? Neg. 24, 6. If you divide neg. 24/4 you will get…

S: 6.

T: 6. That one's correct. How about this one? A couple people did that one. Neg. 20,5. neg. 20/4 is…

S: 5.

T: All these are correct. What about this one? It's 22, 6.

S: No.

T: What's 22/4? Adele? What's 22/6? Is it a nice number?

SS: No. No, it's not.

T: No. It's 22/4; I'm sorry. It's not going to be 6, is it?

A similar vignette from another class is presented below. In this vignette the teacher left the graph of y=x+1 displayed on the screen, while students calculated and sent in points for the function y=2x. The teacher then prompted a discussion asking students to compare the two lines, one green, and one red.

S: They meet at (1, 1). They intersect. SS: (1, 2)

T: (1, 2); you're right, it is, (1, 2). So let me just talk about that for one second. Look, that means that the first one which was X + 1; right? 1 + 1 is 2, and the second one is 2 times 1 and 2 times 1 is 2, and that's why because they both make that equation true. Something else? Anything else you want to say?

S: There's a positive trend. T: There's a positive trend, awesome. Anything else? Something else? S: They're all real numbers? T: Yeah, they do, they're all real numbers, um hum.

S: It's a straight line.

T: It is a straight line. Anything else? And that's important because pretty soon we might not see a straight line. Anything else? …

S: They all increase.

T: They all increase, and I think somebody said positive.

S: The green line is like closer in to the Y.

T: Yeah, it is, it's closer in to the Y. How can you say the same thing? How else can we talk about that?

S: The Y axis, the angle is smaller?

The teacher interprets and verifies what the student is saying by placing her fingers on the two lines and measuring the angles. She agrees with the student and moves on to the next problem.

Note that in the first vignette the calculational orientation of the teacher is evidenced in criteria and methods she used and suggested for testing points. Relying on computational procedures for testing values remained a consistent part of her lessons. The teacher in the second vignette, which might appear less calculational, maintains the same orientation. This is apparent partially in the quality of explanation she offered students on why the two lines intersect at (1,2). With computation not being the primary focus, different questions can be asked, such as having students to compare the graphs of lines. The resulting discourse led naturally to the notion of slope, even though it was not taken advantage of at this time. The teacher *appears* to not to be the locus of authority as she uses statements such as, "I'm looking for observations about those two lines; anything you want to say about them." However, the resulting discourse is limited to short answers directed back to the teacher for verification. Students are not pressed for elaboration, explanations, or to comment on peers' statements.

Teachers' Knowledge of Functions: Functions as Objects vs. Processes

As previously discussed by Moschkovich, Schoenfeld, & Arcavi (1993), functions can be seen as a process acting on a set of inputs to obtain outputs. Functions can also be seen as objects that can be manipulated as a whole. In mathematics classrooms, when functions are viewed as processes, the primary focus is on computational production of their representations. For instance, children may be provided with a symbolic representation of a function and asked to generate a table of values, graph the function findings specific points, and moving from one to other in a one directional motion. In contrast, when the view of functions as objects is advocated, the instructional tendency is to focus students' attention to the general behaviors of the functions as a whole. As such, learners might explore for instance, families of functions, the impact of various parameters on the behavior of parent functions, and describe movements of functions as transformations relying on graphs, numerical data, and symbolic forms simultaneously.

Data indicated that the primary view emphasized in classrooms by teachers was that of function as a process. The teachers' discourse regarding the representations of the function and what they expected of children to do thus was influenced by this view. Teachers spent a majority of their instructional time on production of multiple representations, as the main goal of activity, without elaborating on connection among them and ways in which these different representations could be used to make and verify conjectures regarding the behaviors of functions. Such discussions were absent even when ample opportunities were present due to either un-expected results provided by technology or children's own questions and comments. This is most illustrative in the episodes described below.

During one lesson, a teacher had given 5 points that were on the line y=6x. The students were asked to send using the CCT their predictions of another point on the line. When the result of the collection was displayed almost all the points followed the pattern of the line, except for the point (6,6) that was far away from any of the other points. An interesting discussion could have occurred if the teacher had asked, "Is it still possible to have a line, or a function with all the points displayed including (6,6)?" reinforcing the notion of what makes a line a line or a function a function. What if the student legitimately thought (6,6) was a viable prediction of the pattern? Can a line go from (5,30) to (6,6) then to (7, 42)? Rather, the teacher immediately dismissed the point as being wrong, "That's the point (6,6). What do you think happened there? Just a typo? If you plug in 6 are you going to get out 6?" The teacher asked the class what may have been wrong with the point, but the question was answered by the teacher herself who concluded the most likely error was a typo. To identify the correct point, the teacher appealed to the process definition of the line to state the point should have been (6,36).

Technical Confidence and Content Presentation

Previous studies regarding the teachers' use of technology present the view that with increased technical knowledge about technology teachers tended to show the tendency to use it more frequently in teaching. The same was true among the range of data we examined. It also became evident, that while teachers successfully offered children a greater number of illustrations of various representations in their instruction, the quality of their discourse regarding functions or representations was not substantially different. Typical practice is described below.

A teacher put torn pieces of paper with a number written on each and passed them to the students and asked the students to evaluate several functions, such as y=-3x, y=x+1, y=2x, and $y=x^2+1$ using the number they were given. Students submitted their single point on the curve to the teacher using the CCT. The CCT allowed the teacher to collect and display all the students' points. This activity allowed students to see the tabular, algebraic and graphical representations of three different functions as well as a small sample of other students' representations. After collecting the data points for the quadratic, the teacher quickly identified two points as being wrong. By appealing to computation, the teacher asked the two students to correct their points. A student noticed that the quadratic was "crooked. Not crooked, but curvy," and another student mentioned it looked like a skateboard half pipe when asked to compare the four functions. A discussion after the teacher prompted with "What was alike in the other three that's different on this one? I mean because something's making that one curve where all the others are going straight," hinted at the notion of the square being the reason but was not fully realized and explored. Three main representations of functions were present as well as the notion of function as process due to the teachers appeal to computation to correct errors. The notion of function of object did not explicitly appear. However, it seemed students noticed object like properties of functions, i.e. this one is curvy, the others are straight. Two years earlier this teacher frequently used a calculator program to generate graphical and numerical representation of different functions and asked students to guess the slope and y-intercept. The program did not include a connection back to the algebraic representation of the line; neither did the teacher mention the connection. A maximum of two different representations from the process perspective of function were used during the lessons this year.

Final comments

The study of multiple representations in general and those particular to functions are among the most notable goals of current visions for Algebra I curriculum. The assumptions surrounding the utility of graphing technologies within connected learning environments for advancing the learning and teaching of these objectives are certainly valid. However, as our data indicated using technology in ways to substantially impact learning depends largely on teachers' knowledge about the mathematical goals, content specific knowledge regarding concepts taught and their orientations to teaching. These are in line with findings of other studies which have examined the interactions between teachers' resources, instructional motivations and choices they make during instruction (Silverman & Thompson, 2008), confirming Thompson's (2002) view that teachers' pedagogical actions must be viewed as manifestations of their current understandings and visions for and of student learning. Placing new technologies in classrooms without providing them the space to engage in reflective abstractions about learning and teaching, which can lead to the institution of key developmental understandings (Silverman & P. W. Thompson, 2008) central to transformative practice, would have little utility for enhancement of knowledge. Professional development programs designed for teachers need to focus not just on increasing teachers' technical knowledge about the tools but also alternative ways that mathematical ideas might be interpreted or taught using them; that is, using technology as a vehicle for establishing deep personal and pedagogical understanding of the subject matter.

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VIEWS OF TURKISH AND U.S. PRESEVICE MATHEMATICS TEACHERS ON THE USE OF GRAPHING CALCULATORS

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This study utilized two conceptual frameworks to investigate the views of Turkish and American preservice mathematics teachers on the use of graphing calculators in mathematics instruction. An open-ended survey questionnaire and group interviews were the main data collection methods. Results revealed that both Turkish and U.S. participants worried about calculator dependency but they both thought that calculators could transform the teaching and learning environment positively. However, some differences in their views were also detected.

Introduction

In the age of technology, mathematics teachers must decide whether to use technology in their classes. If so, more decisions must be made, such as which technology to use, and how, in order to create an effective teaching and learning environment. While making these decisions, teachers' knowledge and beliefs influence their decisions. (Ball et al. 2001; Stipek, Givvin, Salmon, & MacGyvers, 2001). Teachers' differing views and beliefs about the use of technology in mathematics instruction could lead to different levels of use of graphing calculators in mathematics instruction. Researchers encourage conducting studies about teachers' beliefs about the benefits/weaknesses of using graphing calculators in mathematics instruction to learn why and how teachers come to adopt or reject the use of graphing calculators (Burrill et al. 2002; Kastberg & Leatham, 2005). Moreover, the beliefs of teachers about the use and roles of graphing calculators influence the use and roles of those calculators in mathematics classrooms (Burrill et al., 2002; Doerr & Zangor, 2000; Fleener, 1995; Kastberg & Leatham, 2005).

There is limited research comparing the views of teachers from different countries about the use of technology in mathematics instruction. Wagner, Lee and Özgün-Koca (1999) compared American, Korean and Turkish preservice teachers' views on many different subjects, including the use of technology. They concluded that American teachers were more open to the use of both computer and especially calculator technologies in mathematics classes. Turkish preservice teachers had more worries about the negative effects of the use of graphing calculators on teaching and learning. Other such comparative studies mainly refer to international studies such as TIMSS (Antonijevic, 2007; Özgün-Koca & Şen, 2002) with considerably varying calculator use in different countries, including an especially dramatic difference between the U.S. and Turkey.

Research Questions

The current study investigates views on the use of graphing calculators in mathematics instruction held by Turkish preservice mathematics teachers with very limited experience with graphing calculators and views held by more experienced American preservice mathematics teachers. The research questions are:

- What are their views, and underlying reasoning, on using graphing calculators in mathematics instruction?
- What are the similarities and differences between the views of Turkish and American participants?
-

Conceptual Framework

This study employs two conceptual frameworks —the possible roles of graphing calculators in mathematics teaching and learning, as well as technological, pedagogical, and content knowledge (TPACK). The aim is to see the general perspective with the TPACK framework and then tease out the specific perspective on the use of graphing calculators with the framework for the possible roles of graphing calculators in mathematics education.

The Possible Roles of Calculators

The first conceptual framework emphasizes the five overlapping roles of graphing calculators described by Doerr and Zangor (2000): (1) computational tool, (2) transformational tool, (3) data collection and analysis tool, (4) visualizing tool, and (5) checking tool. These roles are interconnected and several roles could occur simultaneously (see Figure 1).

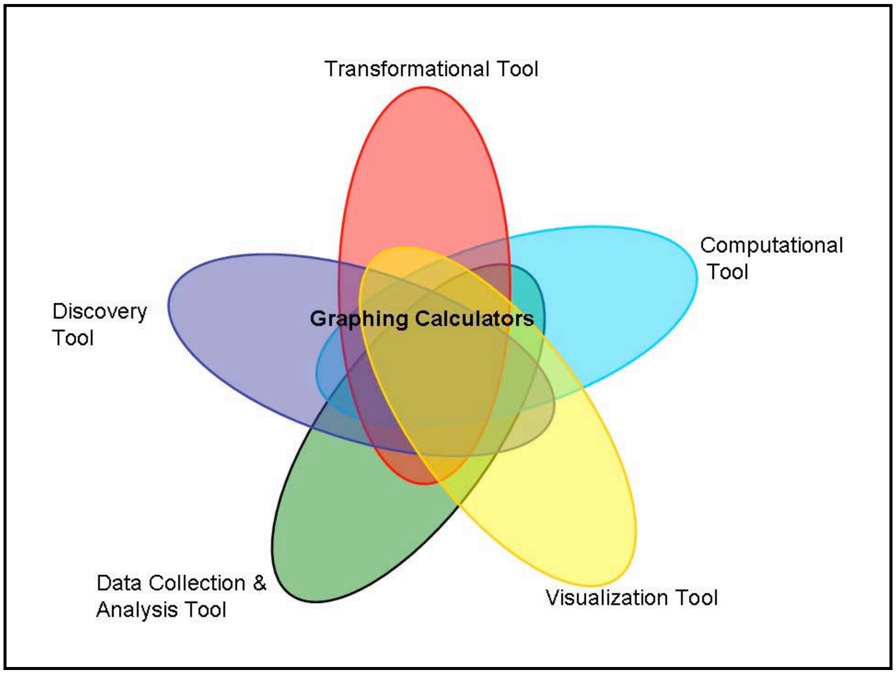

Figure 1. The Use of Graphing Calculators in Mathematics Classrooms

Among these different uses, calculators as a visualization tool and as a computational tool have been emphasized in many mathematics classrooms. Doerr and Zangor described the role of graphing calculators as a checking tool "when it was used to check conjectures made by students as they engaged with the problem investigation" (p. 156). Because checking could be used when students are checking conjectures and/or computational work, this role of graphing calculators is renamed as a discovery tool when calculators are used to support inquiry. When calculators are used as data collection and analysis tools, they make it possible for students to collect and analyze data, and when they are used as transformational tools they change the nature of the teaching and learning environment.

Technological Pedagogical Content Knowledge

The introduction of pedagogical content knowledge by Shulman (1986) was ground-breaking and the research on teachers' knowledge thereby gained a framework. With Mishra and Koehler's (2006, Koehler & Mishra, 2005) and Niess' (2005) introduction of the concept of the teachers' TPACK (see Figure 2), technology-related research in the teacher education and professional development has gained a conceptual framework. TPACK involves content knowledge (CK), pedagogical knowledge (PK), technological knowledge (TK), and their combinations. Looking at these combinations, Pedagogical Content Knowledge (PCK) focuses on the mutual relationship between content and pedagogy, Technological Content Knowledge (TCK), focuses on how the content can be affected as a result of use of technology, and Technological Pedagogical Knowledge (TPK) focuses on how teaching could be affected by the use of technology. TPACK is at the heart of all of this knowledge and takes all of these factors and combinations into consideration.

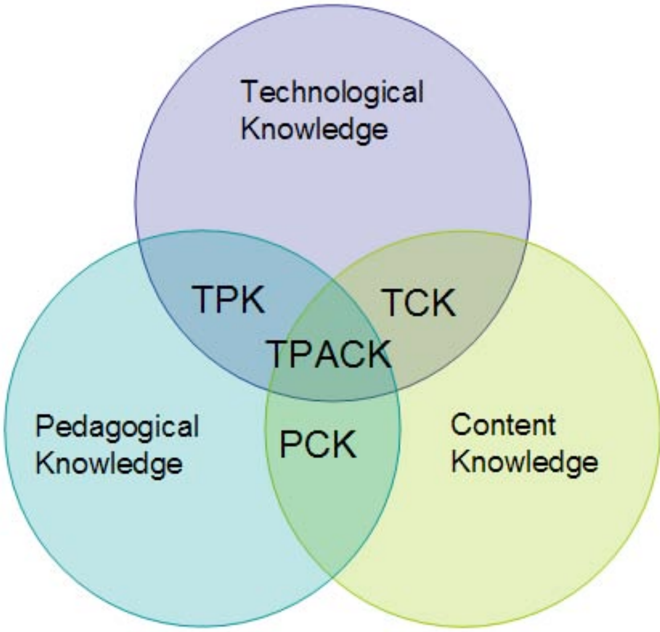

Figure 2. Re-creation of Mishra and Koehler's TPACK Model

Data Collection Methods and Analysis

There were two settings for this study. One was a 5- year teacher education program in Ankara, Turkey. The other was in a university in the Midwestern United States. Turkish participants in this study were 27 preservice secondary mathematics teachers. Even though these participants had no prior experience with graphing calculators in their K-12 schooling, they experienced some graphing activities with TI-92 calculators in their methods courses. They did not have a chance to use these calculators in their field studies or student teaching. One can wonder that if these participants did not have any extensive experience with these technologies, why their opinions would be worth investigating. As mentioned above, some of the instructional decisions that we make are based on the views and beliefs that we hold. Even though one does not have an extensive knowledge about the use of calculators in mathematics instruction, one still has an initial view of graphing calculators. U.S. participants included 23 preservice teachers. Most of the U.S. participants categorized themselves as intermediate level users of graphing calculators. Many of them mentioned that they used graphing calculators in their 9-12 or college education as learners of mathematics.

Data collection involved group interviews and a survey that included open-ended questions. The participants were asked to write about their beliefs on the use of graphing calculators in mathematics instruction in general. Some of the questions were of a general nature to see what different parts of their TPACK participants would bring in to their discussion. However, other questions aimed to reveal different parts of the TPACK model. For instance, one asked the effects of calculators on the teaching and learning environment and process with the aim of revealing participants' TPK. All participants engaged in taped group interviews in which their writings were discussed. The researcher facilitated the group interviews. In these semi-structured group interviews, the same questions from the writing portion were discussed with follow up questions: How should graphing calculators be used for better learning in mathematics classrooms? When are the appropriate times? [As a response to a participant saying that the teacher needs to decide appropriate times.]

Due to the nature of qualitative data, the analysis was based on categorizing in order to investigate emerging themes. First, participants' individual writings were analyzed for common codes to create patterns. At this point, the two conceptual frameworks guided code and pattern recognition in the data. Common codes and patterns were tallied in checklist matrices (Miles & Huberman, 1994). Checklist matrices were created in order to analyze participants' views on the use of calculators in mathematics instruction. According to Miles and Huberman (1994), "the basic principle is that the matrix includes several components of a single, coherent variable" (p.105). Therefore, several elements of the participants' views on the advantages, disadvantages, and possible roles of calculators were analyzed more thoroughly. Percentages of participants sharing similar views were calculated from those tallies. Taking these codes and patterns into account, group interviews were analyzed. Data triangulation and member checks were used to ensure the trustworthiness of the study. Although writings guaranteed that the participants had sufficient private time to respond without pressure, interviews were added to provide an environment where group communication and interaction could be observed. Thus, the two forms of data collection with different encounter levels provided rich sources of data for this study. Member check questions during the interviews were used in order to ensure correct understanding of what the participants meant.

Results

The Roles of Graphing Calculators

When Doerr and Zangor's (2000) framework for the roles for graphing calculators in mathematics classrooms was used to analyze the data, four roles of graphing calculators were mentioned by the participants—computational, visual, transformational, and discovery tools. In general, graphing calculators as computational, visualization or transformational tools and their two and three combinations were mentioned by a majority of the participants. Eighty-eight percent of Turkish and 74% of U.S. participants see graphing calculators as a way to divert time from drudgery and cumbersome work. The time that can be saved from graphing or calculating by hand was one of the points brought up by many participants: "[Graphing calculators] could save time and energy for the graphs that are hard to draw and require detailed calculations" (Writing—Turkish participant). However, most did not mention what they would do with the time saved. That is why their data was coded as computational use of graphing calculators. Nevertheless, one could still argue that these comments might be a sign of the beginning of a change in mathematics classrooms and a signal of the birth of transformational use of graphing calculators by these participants (Özgün-Koca, 2009). However, in this analysis, an emphasis on change was sought in order to code the data as transformational tool. Approximately half of the Turkish and U.S. participants clearly discussed the role of graphing calculators in transforming the nature of the teaching and learning environment: "Teachers are able to spend more time on concepts not facts so students are actually learning, not just memorizing" (Writing—U.S. Participant). Seventy percent of the Turkish and 48% of the U.S. participants stated that one of the advantageous uses of graphing calculators was being able to help students to visualize mathematical concepts. One-third of the Turkish and 22% of the U.S. participants emphasized that graphing calculators could be used as discovery tools: "Calculators, especially in the learning environments that are designed according to the discovery approach or constructivist theory, could be used to enable students to practice [using their previous knowledge], relate different concepts, discover or construct the knowledge [by themselves]" (Writing—Turkish participant).

No participant in this study mentioned the use of graphing calculators as data experimental tools. Since Turkish participants did not experience or observe the use of graphing calculators with Calculator Based Laboratories (CBL) or similar technology, it is not surprising that they would not discuss this use of graphing calculators. The U.S. participants, on the other hand, did not discuss this use, either. During the interviews, the researcher did not intentionally bring up each use of graphing calculators with the aim of observing the roles that participants discuss freely. This situation emphasizes the importance of the experiences that teachers obtain (or not) in their teacher education programs.

Technological Pedagogical and Content Knowledge

When many participants discussed their views about graphing calculators, they mainly focused on the teaching and learning issues of TPACK. Participants' main discussions focused on how graphing calculators might help or hinder students' learning. A major issue in this category of TPACK was the fear of calculator dependency where participants were using their TPK by discussing how the availability of technology might affect students' learning. Approximately sixty percent of the U.S. participants and half of the Turkish participants shared their worries about students becoming dependent on calculators: "Some students may rely on the calculator to an extreme level. This may cause them to be handicapped when doing things by hand" (Writing—U.S. participant). One interesting discussion by U.S. participants was whether to use calculators with a group of students who lack basic skills. One group said that they would allow such students to use calculators to catch up. The other group emphasized that students should be able to compute by hand. At the end of the discussion, participants came to the conclusion that they would like their students to have both computational proficiency and higher-level skills. Another main topic in this category, especially for Turkish participants, was being able to get students' attention and interest with graphing calculators. Seventy-five percent of the Turkish and 48% of U.S. participants stated that teachers can engage students when graphing calculators are used, which could be a positive benefit for students' learning: "The main superiority of using graphing calculators in teaching is that they would attract the students' attention and students get interested more in the lesson" (Writing—Turkish participant).

When participants discussed their TPK from a teaching perspective, according to 15% of Turkish and 56% of U.S. participants, instructional use of graphing calculators in mathematics classrooms should come after students master the skills: "Advanced calculators should start to be used by the students after the topics are well understood and exercises are done with paper-and-pencil" (Writing—Turkish participant) and "I've always been not a proponent of calculators until they [students] have totally mastered it" (U.S. Interview). Almost half of the Turkish participants but only one U.S. participant discussed their worries about classroom management: "I believe there is the possibility that a student could sit at a corner during the whole class session without doing anything" (Turkish Interview).

Turkish participants had no experience using graphing calculators as students, nor had they any experience teaching with them during student teaching. Their technological knowledge was very limited compared to their U.S. counterparts. Thus, 44% of the Turkish participants said that the teacher needs to have technical knowledge of graphing calculators: "[Another disadvantage] could be that the teacher would not be competent in, i.e. s/he doesn't know how to use the calculator" (Writing—Turkish participant). Forty percent of Turkish and 35% U.S. participants discussed that teachers should have necessary TPK about how to use graphing calculators effectively in the classroom: "It is the teacher's role to use discretion for when calculators should be used. Calculators are most beneficial, any kind, when used for learning not as an answer-giving black box" (Writing—U.S. participant).

Content related discussions in participants' writings were very scarce. Participants were referring to their TCK when discussing appropriate subjects to address with graphing calculators in mathematics education. Algebra and graphs of functions were mentioned often in these discussions. Geometry was the next most-mentioned topic for Turkish participants and statistics, data analysis and probability was for U.S. participants. Mainly advanced topics were listed as appropriate to use graphing calculators, possibly due to the fear of calculator dependency.

Relationship between the Two Frameworks

When the two frameworks—the roles of the graphing calculators and TPACK—were taken into consideration concurrently, an emergent theme evolved. The main part that was discussed by all participants in the TPACK framework was TPK. While doing that, different roles of the calculators came into play. When some participants discussed the role of calculators as discovery tools, then tended to discuss how this role helps students' understanding: "It helps students internalize the 'why' behind a process, rather than what to do…They can also be used to verify experimental ideas. 'What do you think would happen if…?'" (Writing—U.S. participant). Conversely, when calculators are used as computational tools, some participants mentioned that this might hinder students' learning: "I do not think that students should use calculators for basic computations/operations…the student should do that mentally in order to gain more basic computational practice" (Writing—Turkish participant).

Summary of the Views of U.S. and Turkish Participants

Turkish and U.S. participants had similar views regarding the use of graphing calculators in mathematics education. Both worried about calculator dependency but thought that calculators could transform the teaching and learning environment, carrying the mathematics experienced in classrooms to new levels. Both groups saw graphing calculators as computational tools. Turkish participants brought the visualization aspect more into play, while calculator dependency was the main discussion topic in U.S. interviews. More U.S. participants than Turkish participants thought graphing calculator use was more appropriate after students have mastered basic skills, and they focused on more practical issues. More Turkish than U.S. participants worried about classroom management and cited their need for technical knowledge. Significantly more Turkish participants discussed the motivational aspect of graphing calculators as an advantage. This could be the result of the novelty of graphing calculators for Turkish participants.

Discussion and Conclusions

Even though Turkish and U.S. participants had different experience levels with graphing calculators, the main role given to them by both groups was doing fast and accurate computations and creating complicated graphs in a short period of time without drudgery—a computational tool according to the Doerr and Zangor (2000) typology—without discussing how they would use the time saved. This is a superficial use of TPK. Some of the roles that participants assigned to the graphing calculators were due to many advantages they saw in using them. Here the use of two frameworks was beneficial, because when participants reflected on the advantages/disadvantages, they were referring back to their TPACK. One implication of these results is to provide environments where preservice teachers experience the potential and effective use of different technologies in mathematics education in order to help them develop more informed views. That Turkish and American participants held similar views about the use of graphing calculators is both encouraging and questionable. How were Turkish participants able to have more positive views in some categories compared to U.S. participants? When half of both Turkish and U.S. participants agree that the use of graphing calculators could transform the nature of their classrooms so that concepts are more valued, we can see a universal trend where understanding is more valued than computation, but one can still wonder what U.S. participants have learned from having graphing calculators in use for the last decades (e.g. seeing students reaching for calculators for basic calculations) and how this might have affected their TPACK as teachers.

One implication of the results of this study is the need to help teachers develop and reflect on a cohesive TPACK (Thompson & Mishra, 2007), rather than a set of disconnected knowledge types (TK, PK, CK, TPK, or TCK). The activities and practices that teachers experience frame the knowledge and beliefs they construct and develop. For example, if the participants had more experience with calculators as data collection tools, they might have discussed it in their writings or in the group discussions. Moreover, group interviews helped participants reflect on, confront, and defend their own beliefs. Listening to others' views might have helped many of them to reconsider not only their beliefs, but also their knowledge, or to expand their viewpoints to include other possibilities. Teacher education offers teachers the knowledge and experience they will require. During their education, teachers not only acquire knowledge, but also form their beliefs about pedagogy which may affect their future instructional decisions (Ball et al. 2001; Stipek, Givvin, Salmon, & MacGyvers, 2001). Therefore, it is critical to identify and study teachers' beliefs in order to advance teacher education to an evolving entity presenting opportunities for teachers to reflect on those beliefs.

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LIGHTS, CAMERA, MATH! THE F PATTERN NEWS

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We define mathematical performance as the process of communicating mathematics using the performance arts. By highlighting the role of technologies in shaping mathematical thinking, we present two episodes of a qualitative case study. First, considering the use of songs, lyrics, video clip, manipulative blocks, and online applet, we illustrate how thinking collectives of students-teacher-media investigated the L Patterns (sequence and series of odd and even numbers) and conjectured a new sequence of numbers called F pattern. Then, we describe how these collectives created a digital math performance for the Math Performance Festival based on their conjecture. We believe that digital media and the performance arts are helping to create transformative means of mathematical communication and learning.

Traditionally, "mathematical performance" is conceptualized as pertaining to the domain of assessment and evaluation. However, we see *mathematical performance* as the process of communicating mathematics using the performance arts (Gadanidis & Borba, 2008). "Exploring mathematical ideas through performance can offer a way to challenge some of the most limiting stereotypes around mathematics learning" (Gadanidis, Gerofsky & Hughes, 2008 p. 19). By creating music or movies, math performers can provide surprises to audiences and communicate mathematical ideas in an imaginative way. Moreover, *digital mathematical performances* are digital media (e.g., video and audio files, flash animations, and virtual objects) used to communicate math through music, cinema, theatre, poetry, storytelling, etc. We do not suggest all mathematics should be taught using performance arts. However, thinking about mathematics as performance may lead to new ways for (a) student to produce knowledge with technologies and (b) sharing mathematics beyond classrooms (Gadanidis, Gerofsky & Hughes, 2008).

Since 2008, students and teachers have been submitting digital mathematical performances to the Math Performance Festival (MPF) (Gadanidis, Borba, Gerofsky & Jardine, 2008). The MPF is based on a virtual environment where digital performances are published. Every year, Canadian celebrities (e.g., musicians, poets, TV presenters) and mathematicians select their favourite performances in terms of: (a) depth of the mathematical ideas; (b) creativity and imagination; and (c) quality of the performance. The participants of these performances are awarded with medals. Gadanidis and Geiger (2010) have referred to the MPF as "one example that helps bring the mathematical ideas of students into public forums where it can be shared and critiqued and which then provides opportunity for the continued development of knowledge and understanding within a supportive community of learners" (p. 102).

In this paper, we present a qualitative case study to illustrate how Brazilian students investigated and created a digital mathematical performance for the MPF. We suggest math performance can offer ways for transformative teaching and learning mathematics.

Theoretical Framework

Vygotsky (1978) investigated children's development and how development is conditioned by the role of culture and language. According to Vygotsky, higher mental functions are historically developed within particular cultural groups, through social interactions with significant people of children's life, particularly parents and teachers. Through these interactions, children learn the habits of the culture, including patterns of speech, verbal and written language, and other symbolic representations. Thus, Vygotsky emphasized (a) the social interaction with more knowledgeable others in the zone of proximal development and (b) the role of culturally developed sign systems and languages as psychological tools of thinking (Cobb, 1994). In this sense, socioculturalism highlights the role of activity in mathematical learning and development, by linking activity to participation in cultural practices, taking the *individual-in-social-action* as their unit of analysis (Cobb, 1994). Thinking develops from practical and object-oriented activity, that is, "human action is mediated by cultural tools and is fundamentally transformed in the process" (Goos, Galbraith, Renshaw & Geiger, 2000, p. 306).

Tikhomirov (1981) claims that computers reorganize human activities in diverse levels. Tikhomirov suggests computers do not replace, substitute, or merely complement humans in their intellectual activities. Processes mediated by computers *reorganize* thinking. According to Tikhomirov, computers play a mediating role as language does in Vygotsky's theory. Considering the nature of human-computer interaction in terms of feedback, the dimensions involving computational mediation provide new insights for learning and development.

Levy (1998) claims "as humans we never think alone or without tools. Institutions, languages, sign systems, technologies of communication, representation, and recording all form our cognitive activities in a profound manner" (p. 121). Levy uses the term *thinking collectives* to discuss the collaboration between human and non-human actors in the cognitive ecology.

Borba and Villarreal (2005) argue the notion of regulation by computers is qualitatively different when compared to the mediation by verbal and written languages. Borba and Villarreal argue (a) humans-with-media are thinking collectives in constant reorganization of mathematical thinking and (b) media transform mathematics. That is, mathematics produced with paper and pencil is qualitatively different from the mathematics produced with computers.

Therefore, we notice that our research interest is focused on how students-teacher-media form thinking collectives when they create digital math performances. Regarding notions within socioculturalism, we are interested in understanding the role of technologies in shaping students' thinking and learning when they communicate mathematics using the performance arts.

Methodology

Stake (2003) argues qualitative case studies are specific and bounded, they have patterns, and the focus is on the understanding of the complexity of the case. Yin (2006) claims "the strength of the case study method is its ability to examine, in-depth, a case, within its 'real-life' context" (p. 111). Case study "is best applied when research addresses descriptive or exploratory questions and aims to produce a firsthand understanding of people and events" (p. 112).

In this paper, we analyze data produced through the Students as Performance Mathematicians project (Gadanidis, Borba, Gerofsky, Hoogland & Hughes, 2008). Over a period of two years, part of the research team of this project worked in partnership with a state public school in Rio Claro, Sao Paulo, Brazil. These activities engaged middle school students (and some teachers) in a number of activities with the goal of producing digital mathematical performances. These performances were submitted and published at the MPF. Thus, our data consists of several resources produced from these activities. These resources are: (a) field notes; (b) students' writings; (c) video recordings of semi-structured interviews with students and teachers; (d) video recordings of teaching and learning sessions with students; and (e) digital mathematical performances. We use data resources from 4 learning sessions (4 hours each). Eight students participated in the first, second, and third sessions. Four students participated in the fourth session. The first author of this paper played a role as a teacher. In this paper, we first highlight how students-teacher-media conjectured their own sequence of numbers, based on an initial investigation. Then, we describe how this thinking collective created a new digital mathematical performance for the MPF.

Results

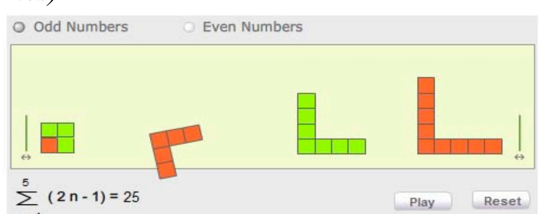
Barton (2008) states "mathematics is created in the act of communication … An educational perspective asks whether this implies that mathematics is learned through communication. This perspective also focuses on the nature of the communication, and the role played by different people in it" (p. 144). That is our focus in describing the episodes, that is, our focus on mathematical thinking emerges through our analysis of how communication through performance happened and how media were non-human actors in this scenario.

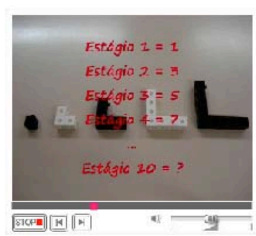
Episode One

At the beginning of the first learning session, the teacher engaged all students in investigating the "L patterns" using several materials as such as songs, lyrics, video clip, manipulative blocks, and online applets. The L patterns (see Figure 1) refer to the study of sequence and series of odd and even numbers. L patterns have the potential to offer surprise in generalizing patterns. Working with manipulative and applets, students can construct sequences of numbers and they can connect blocks (or simulate the connection) to have a geometrical representation for the series which works as a visual proof. Playing with L patterns becomes a surprising and visceral experience when students connect L's forming squares (odd numbers) and rectangles (even numbers).

Figure 1. *L Patterns (Applet, video clip, and lyrics)*

L-Patterns
Playing with blocks, making L-Patterns.
What is going to happen?
I started with a single block,
And I was adding two at each stage.
Playing with blocks, making L-Patterns.
Odd numbers at each stage
1, 3, 5, 7, 9, 11 ... I pose the question:
How many blocks do we need at the tenth stage?
My mind pops up, I understand everything suddenly.
Fitting each L then another pattern emerges.
Playing with blocks, making L-Patterns.
4, 2 by 2, 9. 3 by 3
Adding at each stage, one at a time
Playing with blocks, making L-Patterns.
10, 10 x 10, 100, then I see.
The sum each stage is a perfect square!
Playing with blocks, making L-Patterns
The N x N pattern seems consistent
N odd, N squared.
Playing with blocks, making L-Patterns
What is going to happen?





From the humans-with-media perspective (Borba & Villarreal, 2005), experimentation-with-technologies and visualization play fundamental roles in mathematical thinking. Students-teacher-media were involved in figuring out a generalization for $S_n = 1 + 3 + 5 + \ldots + (2n - 1)$. By investigating the algebraic sum at each stage (writing, singing, and watching a video) and articulating it to the connections of Ls forming a square (constructing with blocks and simulating with an applet), the students-teachers-media identified a new pattern together, as thinking collectives. They also worked with algebraic representations.

At the second session, students conducted an investigation in pairs. They were focused on the use of both manipulative blocks and online applet, but they were also regarding links with other materials as such as the lyrics. By instructing the pairs of students, the teacher was interested in hearing conjectures from them. The teacher proposed to students to speak their ideas and/or write them all on their paper-notes. From the analysis of the video recording and field notes, we highlight that Student 3 came up with an interesting conjecture asking: "*But, if we think about it, there are even numbers in rectangles, are there not?*" The teacher said: "*Perfect. What is the shape related to the series of even numbers?*" Student 3 then argued: *"A rectangle, right? I can see it with the applet."* With the intention of negotiating the meaning of this idea with all students, the teacher brought Student 3's conjecture to the whole class. However, no further discussion happened at this moment involving all students collaboratively. Students were focused on their investigation in pairs. Even though students did not engage in a collective meaning negotiation, we suggest Student 3 did an important articulation involving several representations and media in investigating the L Patterns. Thinking-with-Applet-song-lyric-video-blocks-and-other-media, Student 3 was able to conjecture a visual representation for both the sequences and series of odd and even numbers. Students did not have developed an investigation about even numbers until that moment, but Student 3 was able to visualize and manipulate the applet and the blocks, relate it to the lyrics, images and sounds, recognize an approach on even numbers, relate it to the investigation of odd numbers, and communicate that the series of even numbers can be geometrically represented by rectangles. This moment revealed a significant role of information technology in shaping student's thinking and learning. Other students came up with interesting ideas along the first session as well, but these students' conjectures and ideas toward odd and even numbers will be addressed in other papers.

At the third session, students were still working in pairs. Based on the investigation of the L patterns, Student 1 and Student 8 decided to create a new sequence of numbers using the manipulative blocks. These students created a first F using 6 blocks. They discussed the growth of the sequence and they decided to add 3 blocks each stage, forming the sequence (6, 9, 12, 15,…) (see Figure 2). The teacher brought up this discussion of F pattern to all students. He praised students' conjecture and imagination in creating a new pattern. He proposed that they could develop a similar investigation on the F pattern as they had made to *L patterns*. "*What could be a generalization for the sequence? What would be the series? Would be possible to connect the blocks and create a visual proof to the series? Could we create a performance about F pattern?*" Student 8 said they were trying to figure out the series, but they did not find a regular shape (as a rectangle) to express an algebraic formula for the series. Figure 3 shows students' tentative of creating a visual proof for the *F pattern* series. The teacher asked them if they had already found a formula to the sequence. After fifteen minutes of dialogue between Students 1 and Student 8, where they used blocks, and paper and pencil, Student 1 said: "*I found it! It is n times 3 plus 3*." The teacher said the generalization $3n + 3$ seemed to be a good candidate. Focusing on negotiating that meaning collectively, the teacher proposed to the class to test it for several stages, and they confirmed students' conjecture. Student 1 and Student 8 celebrated the confirmation of their conjectures greeting each other. This moment revealed students' pleasure in thinking mathematics. It shows the beauty of mathematics as an emotional, visceral, and surprising experience in learning (Gadanidis, Hughes & Borba, 2008).

Student 1: The F pattern is composed by 6, 9, 12, 15, 18, and so on. We are adding three blocks each stage. So, what would be the n stage? Here, at the first stage, we have 1 times 3 plus 3 equals 6. At the second stage, 2 times 3 plus 3 equals 9. At the third stage, 3 times 3 plus 3 equals 9. And so on. So, at the n^{th} stage, we have n times 3 plus 3.

Teacher: Great! Congratulations for your discovery!

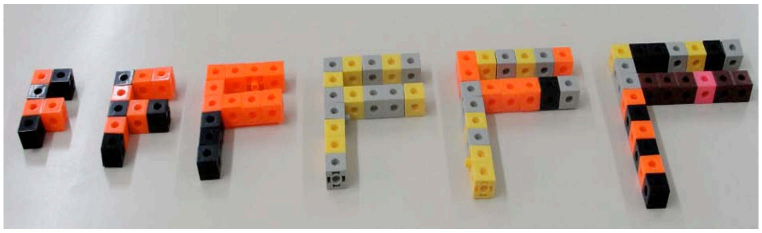


Figure 2: The F-Pattern constructed with blocks by the students

Figure 3: Students' visual proof with blocks.

Episode Two

Four students participated in the fourth session. After a teacher's suggestion, Student 1 and Student 8 came up with the idea of creating a new performance based on the theme of TV News. Their idea was: to present an interview with a mathematician who had discovered the *F pattern*. That was nicely accepted by the other two students, who indicated important suggestions to the script as such as provide humour, use specific songs and sound effects, what kind of images we should portray about reporters and mathematicians, how to explain the series of numbers, and so forth. "Biased" by his interests as a researcher, the teacher tried to create a parallel between the students' conjecture on *F pattern* and a performance about a new math discovery. This parallel was intentionally related on the performative dimension of students creating identities as performance mathematicians (Gadanidis, Hughes & Borba, 2008). Student 1 and Student 8 wrote the words they would like to say. Student 3 wrote a description of each scene as a draft of a script. Student 5 collaborated with script suggestions and technical support. Students-teacher-media recorded the scenes at the same order they would like to present it. The process of playing and recording involved students' speech repetition and improvisation through many takes of scenes. These actions revealed a significant aspect of improving communication and imagination in learning mathematics trough creating a digital skit performance.

After the recordings, based on other students' suggestions, the teacher edited the video called *The F Pattern News* (see Figure 4). A theoretical lens on cinema was fundamental to create the video. It refers to Boorstin's (1990) categories about the pleasures we have when we watch a film. Here, there is an emphasis on the audiences' feelings, which in fundamental in terms of narrative and identity. Boorstin argues about three different categories: *voyeur*, *vicarious*, and *visceral*. The *voyeur pleasure* refers to the mind's eye and not the heart. It is a more rational aspect about the reality or the consistency of the world that is portrayed, but it involves surprise and *making-sense-of-something*. The *vicarious pleasure* puts our heart into the body of the actors. It is when we feel what the actors feel. While the voyeur's eye requires a more opened-close, the vicarious requires a close-up on the actors' faces. The *visceral pleasure* is when we fail to feel what the actors are feeling and began to feel our own emotions.

The *F Pattern News* video plays a lot with voyeur's and vicarious' pleasures. The close parallel with TV News creates a good sense of reality. It involves a lot of surprise in terms of a "new mathematical discovery" and making-sense-of the mathematical ideas presented: sequence of numbers, use of manipulative materials, explanation on the board, experimentation and test of hypothesis, generalization, etc. The sense of humour, the sound effects and music used, the reporters' and mathematician's actions and speeches allow us to fell a vicarious pleasure. If you feel your own feelings in watching this video, if you feel you want to enjoy creating your own digital math performance, it means that the F pattern News also works at the visceral level.

Moreover, the teacher worked with Portuguese/English translations to create subtitles and some students were interested in learning how to speak in English the words they said in the video. Students agreed that the mathematical idea was been communicated clearly. Then, the thinking collective created a final draft at the school's Lab and one of the administrators of the school submitted the video to the MPF. It is available at www.edu.uwo.ca/mpc/mpf2010/mpf2010-134.html. We suggest that by creating a digital mathematical performance, students can talk about mathematics using multiple modes of communication and languages as such as the arts, gestures, sounds, words, visual and symbolic elements, and others (Kress, 2003).

Following, we present a transcription of the video:

Reporter 1: We are interrupting the TV show to tell you terrific news! One just discovered that beyond the L patterns, there is an F pattern!

Figure 4: The F Pattern News

Report 2: F pattern? What are you talking about?

Reporter 1: Our reporter has more information about it.

Reporter 2: Hey, can you hear us?

Reporter 3: Yes. I am here with Hypotenuse. She knows everything about math.

Reporter 2: What is this story about F pattern?

Hypotenuse: The F pattern is here represented by blocks

Reporter 1: Could you please ask her what would be the sequences?

Reporter 3: Sure. What would be the sequence for the F pattern?

Hypotenuse: As we can see, the sequences increase three blocks each stage.

Reporter 3: Look, at the first stage you used 6 six blocks. At the second stage you used nine blocks. At the third, twelve blocks. And so on. What about the hundredth stage? How many blocks do you need to construct the hundredth stage?

Hypotenuse [using the whiteboard and the blocks]: We have a simple formula to figure it out. You can notice the result each stage is equal the index of the stage times three plus three. That is, 3N + 3. At the one hundredth stage we have one hundred times three plus three. It is equal three hundred and three.

Report 2 [with a surprising sound on the background]: I am getting the information that the series was discovered. What would it be?

Reporter 1: In other words, what would be six plus nine plus twelve and so on?

Reporter 3: Hypotenuse, what would be the series?

Hypotenuse [in fast motion and using the whiteboard]: I am going to use Gauss's ideas. The series would be the sum of these numbers. Gauss did it twice. Once from the beginning to the end, that is, $S = 6 + 9 + 12 + \ldots + 3n\text{-}3 + 3n + 3n\text{+}3$. *And once from the end to the beginning,*

> *that is, S = 3n+3 + 3n + 3n-3 + … + 12 + 9 +6. Then, we can add these two expressions. So, we have 2S = (3n+9) + (3n+9) + (3n+9) + … + (3n+9). Therefore, we have 2S = n times (3n+9). It gives us the following formula… [Showing it on the board]. With this formula we can calculate the total number of blocks we need until the stage we wish.*
> *Reporter 1: Last time we talked about the L patterns. Today, we talked about the F pattern. What about you? Which one is your favourite pattern?*
> *Reporter 2: Which one will be our next pattern?*
> *Report 1: Your local show is coming up next.*
> *Reporter 2: Good night… have a good day… good afternoon.*

Conclusions

This paper deals with the learning of mathematics and how students can use both digital technologies and the performance to communicate mathematics. We suggest mathematical performance offers ways for transformative teaching and learning mathematics. Students interacted with could be called a new ‘language”, that is, by using multiple modes of communication that emerge from the digital world in which videos, pictures, drawings, songs and animations are linked to usual printed-based text. In discussing this, we believe that our paper is relevant for some of the main topic of the PME-NA 2011 discussion.

Noss, Hoyles, Geraniou, Gutierrez-Santos, and Pearce (2009) clarify that the difficulties that mathematical generalization and algebraic expression pose for students have been thoroughly studied. For these authors, the difficulties students face when dealing with generalization activities are in some measure due to the way in which they are presented and the constraints of the teaching approaches used. Usually, teachers tend to teach the techniques isolated from all context to help their students find the rule. There is a need to introduce students to different approaches involving generalization of patterns (Noss et al., 2009).

We do see mathematical performance as one more pedagogical tool to address this need. Two students who participated in the creation of the *F patterns News* were interviewed. They said:

> *Student 1: I was learning doing something that I love: theatre. To create a performance we have to know the content, you have to understand it. If you don’t, it doesn’t work very well. It also makes things different. You have to know what you are talking about to create something natural. There are interesting things and with performance I realized it is not so hard. I love the performance about F pattern. To figure out the formula is something big. It is cool because you visualize something and you have to figure out the generalization to know the number for any stage, and what is going to happen. You have a set of numbers and the formula shows how it is growing up. There is a pattern in this process and I realized that by using blocks and seeing the applet. It was a way to learn enjoying. We can use performance to make this transformation.*

> *Student 8: Many colleagues asked me what math performance is or what we do to perform math. First, they said they did not what to do that, because they hated math. They already have too much math classes, and do a drama about math sounded wired. But I started to explain to them exactly what we were doing and the purpose of it. Then, they started to change their opinion, seeing math performance as something different. I tried to convince and encourage them saying that I was playing what I was learning.*

Finally, we highlight that, even though meddle-grade students explored different representations for the sequence and series of numbers they created (e.g., $\sum_{i=1}^{n}(3i+3)=\frac{3n^2+9n}{2}$), we do recognize the mathematics explored in the learning sessions could have gone further. For example, the F pattern (6,9,12,15...) can also be seen as the sum of natural numbers (3,4,5,6,...) and odd numbers (3,5,7,9,...) (that is, 3+3, 4+5, 5+7, 6+9, ...). This could be represented as two separate visual patterns to get a visual picture of the series in a similar way as the L-patterns. On the other hand, by *creating* a digital math performance, we did see interesting ways for mathematical exploration. "Mathematical exploration and play is always possible, at any level. Within the environment of existing mathematics there are (have always been) educational resources full of wonderful open questions … It is nearly always possible to change some basic assumption of mathematics, and to genuinely explore or play in a new environment" (Barton, 2008, p. 150).

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HOW TEACHERS' BELIEFS ABOUT TEACHING AND LEARNING IMPACT IMPLEMENTATION OF FORMATIVE ASSESSMENT IN CONNECTED CLASSROOMS

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We report on research conducted with six seventh-grade mathematics teachers who participated in a two-year professional development research study on implementing formative assessment in networked classrooms. While the full study used a variety of both quantitative and qualitative data sources, this report focuses on data from the semi-structured interviews conducted at the end of the project. We describe three of the categories that emerged from the coding (Strauss & Corbin, 1998) and relate these to teachers' use of the technology.

There are few studies that examine teacher learning and practice for more than a year, especially looking at the impact of technology-focused professional development (Mouza, 2009). In this paper, the impact of two years of professional development on teachers' implementation of formative assessment in a connected classroom is analyzed through the lens of the interactive relationship between practices and beliefs. Analysis is based on case study data collected from six of 30 teachers who participated in Project FANC[1], a research study of implementing formative assessment in a networked classroom using the TI-Navigator System[2] and graphing calculators. The goal of the research in Project FANC was to investigate the use of formative assessment in a networked classroom as it affects middle grades student learning of algebra concepts. In particular, Project FANC studied the effects of one-half of the 30 teachers using formative assessment with the TI-Navigator System for two years and compared them to the effects of the other half using formative assessment with the TI-Navigator System for one year after receiving professional development in formative assessment the first year. Detailed descriptions of the two different models of professional development can be found in Olson et al. (2010).

In *How People Learn* (NRC, 1999), classroom networks were suggested as one of the most promising technology-based education innovations for transforming the classroom environment. Wiliam's (2007) description of a pedagogy of contingency, in which the essence of formative assessment is instruction contingent on what students have learned, can be accomplished through the use of technology that has potential to overcome the major hurdle to utilizing formative classroom assessment: the collection, management and analysis of data. While feedback loops in the regular classroom are very slow, classroom networked technology has the capability to provide rapid cycles of feedback to improve ongoing activity in real time (Roschelle, Penuel, & Abrahamson, 2004). Using the TI-Navigator System, what students know and can do is easily assessed and anonymously displayed. Students can enter and send their responses to the teacher computer and teachers can easily send questions, and receive, organize, and display students' answers, so that the interaction between the teacher and students and among students is greatly facilitated. Four functions of TI-Navigator System particularly helpful for formative assessment implementation are: (a) Quick Poll—allowing teachers to immediately collect and display all students' responses to a question; (b) Screen Capture—allowing teachers to monitor individual students' progress at anytime; (c) Learn Check—allowing teachers to administer quick and frequent formative assessments and provide timely feedback; and (d) Activity Center—allowing students to work collaboratively to contribute individual data to a class activity.

Although teachers may obtain information about student's knowledge, they do not necessarily change their instructional procedure based on the information obtained. Researchers at Ohio State found stronger evidence for technology implementation than for change in instruction. Even with technology tools available to assist with implementation of formative assessment instructional strategies, not all teachers who make significant changes in using of technology necessarily make full use of the potential of the connected classroom for formative assessment (Owens, Pape, Irving, Sanalan, Boscardin, & Abrahamson, 2008). Each teacher will have unique ways of incorporating effective formative assessment practices into her or his classroom using connected classroom technology.

Substantial evidence suggests there is a complex relationship among teachers' classroom practices and the interconnected knowledge and perceptions of mathematics, technology, learning, and teaching. The integration of technology in classroom practices is influenced by teachers' background knowledge and experiences, and conceptions of technology, mathematics, and learning (Zbiek & Hollebrands, 2008). Teachers' disposition towards mathematics and its pedagogies have been highlighted as a major factor in determining the way teachers make sense of technological tools and integrate technology in the classrooms (Ruthven, Deaney, & Hennessy, 2004; Mousley, Lambdin, & Koc, 2003). Mouza (2009) found that teacher change was highly dependent on the continual interaction between practices and beliefs and that teachers went back and forth between implementation of practices from professional development experiences and previous practices. If teachers are to effectively use technology, changes in their knowledge, practice and beliefs are needed. Mouza also reported that teachers' beliefs about their students constituted a critical variable influencing their decision to use technology.

Research shows that technology may change the nature of mathematical activity in the classroom and consequently the teaching and learning of mathematics (Laborde, 2007). Laborde described two levels of teachers using technology: 1) teachers who master the use of technology to do an activity and 2) teachers who used the technology for organizing instruction and learning to take advantage of the specificities of the technology in relation to the meaning of the mathematical content (p.88).

Methodology

The 30 teachers in Project FANC were equally divided into two groups, called FA and NAV, to study the effects of two different models of professional development on student learning of algebraic concepts. All teachers were provided with laptop computers, LCD projectors, Elmos, and a classroom set of TI-73 calculators at the beginning of the project. During the first summer, the participants in the NAV group received TI-Navigator Systems and received professional development using the TI-Navigator for formative assessment while the FA participants received professional development in formative assessment only. Each group participated in five follow-up sessions during the first school year as well as received coaching visits from project staff. During the second summer, the NAV participants received continued professional development in using TI-Navigator with more of a focus on formative assessment, while the FA participants received TI-Navigator Systems and professional development on using the TI-Navigator for formative assessment. Five follow-up sessions for the combined groups were conducted during the second year and coaching continued as well.

It was expected the implementation of the ideas presented in the professional development would vary by teacher and the teacher's beliefs and perceptions about mathematics teaching and learning, efficacy in using formative assessment and networked technology, and support within the school community. To study the implementation process, we conducted case studies on ten of the teachers, five in each group. Data sources for the case studies were background information provided by teachers, classroom observation field notes, notes from coaching visits, focus group and individual interview videos, writing prompts, and the Learning Mathematics for Teaching assessment.

While case study data were collected for 10 participants, this paper focuses on six of the teachers. The focus is on issues of implementation of formative assessment in a networked classroom and not on individual teachers. A brief description of each teacher, by pseudonym, is included.

Kate was in her late 20s and sixth year of teaching when she entered Project FANC. She has secondary mathematics certification and received her degree in mathematics education from Northern Colorado. Kate was interested in Project FANC because of the technology. She teaches in a middle school that has a significant variety of technology available for student and teacher use and was somewhat disappointed that she was not selected for the NAV group. The middle school where she teaches has approximately 900 students. Ethnicities of the diverse student body include 29% Filipino, 22% part Hawaiian, 15% White, and 11% Japanese. Thirty-five per cent of the students receive free and reduced lunch, 6% have limited English proficiency, and 9% are in special education.

Sharon is in her mid 40s and was in her 9th year of teaching when she entered Project FANC, although she served as a long-term substitute for several years prior to that. She has secondary mathematics certification and a major in mathematics education. Sharon does not consider herself a 'genius' in mathematics but loves working hard. She teaches at a combined intermediate and high school in a rural area with about 1650 students. Ethnicities of the diverse student body include 40% part Hawaiian, 21% White, 17% Filipino, and 13% Samoan. Forty-five per cent of the students receive free and reduced lunch, 5% have limited English proficiency, and 12% are in special education.

Yaz is in his 30s and had three years of teaching experience in middle grades mathematics when he became a participant in Project FANC. Both of his parents were teachers. He attended K-12 schools in the Pacific islands but received his undergraduate degree, with certification in elementary and middle school education, in Hawai'i. Prior to participation in Project FANC, he was in his current teaching position for one year. His colleagues view him as the 'tech' person. Yaz teaches at the same school as Kate.

Denby is in her 30s and was in her eleventh year of teaching when she entered Project FANC. She received a bachelor's degree in mathematics and one year later received her secondary mathematics certification. She recently received National Board Certification. Always looking to learn more, she has been a voracious participant in professional development experiences to which she has had access. Denby teaches at an intermediate school in a rural area with about 925 students. Ethnicities of the diverse student body include 40% part Hawaiian, 16% Hawaiian, 10% Filipino, and 7% Samoan. Seventy-five per cent of the students receive free and reduced lunch, 5% have limited English proficiency, and 17% are in special education. Although a participant for two years, she was not able to participate in the activities of the first summer.

Clarise is in her 30s and had one year of teaching experience in middle grades mathematics when she became a participant in Project FANC. She began teaching after Teach for America training to teach third grade. With no elementary teaching positions available in Hawai'i, she was asked if she wanted to teach mathematics or English. Clarise chose mathematics and became an eighth grade teacher at a middle school five days before the start of school. She moved to seventh grade her second year because the school loops with teachers following students from seventh grade to eighth grade. She teaches in a middle school with approximately 900 students. Ethnicities of the diverse student body include 64%

Filipino, 8% part Hawaiian, and 6% Samoan. Sixty-three per cent of the students receive free and reduced lunch, 25% have limited English proficiency, and 10% are in special education.

Iris is in her 30s and was in her 10th year of teaching when she entered Project FANC. She has a degree in mathematics education and both middle school and secondary mathematics certification. She teaches at a large middle school with about 1750 students where only three-fourths of the students are on campus at any one time. Ethnicities of the diverse student body include 26% Japanese, 21% White, 16% part Hawaiian, and 6% Filipino. Thirteen per cent of the students receive free and reduced lunch, 2% have limited English proficiency, and 11% are in special education. While she has considerable experience with technology, she felt she was volunteered for Project FANC.

Results

In this paper, we report on interviews with six of ten teachers who were part of the larger case study. These six were chosen because they represented a wide range of uses of technology, differed in their definitions and implementation of formative assessment, have a range of years of teaching experience, and were from schools representing a wide-range of ethnic diversity and background. While other data were collected, this report focuses on data from the semi-structured interviews conducted at the end of the project. Our report concentrates on three of the categories that emerged from the data analysis (see Olson et al., 2011). We focus on categories that involve teachers' views of formative assessment, pedagogy, and the role of students. In the complexity of classroom life, we believe that these aspects are interrelated in numerous ways. However, we discuss each category separately before looking at how they are manifested in teachers' use of technology in the classroom.

Formative Assessment

All teachers in the case study had been familiar with the term formative assessment before entering the FANC project. Some had participated in professional development sessions about formative assessment given by the state department of education. Two phenomena are of interest here: 1) the variation in teachers' views about formative assessment and 2) whether teachers reported on having changed their views of formative assessment as a result of their participation in FANC.

While all teachers believed formative assessment yielded valuable information, the range of views extended from conceiving of formative assessment as a series of mini-summative assessments to a view of formative assessment tasks indistinguishable from tasks of the ongoing lesson. At one end of the spectrum, teachers used questions and check-ups as one would use quizzes to monitor students. The information from students' responses helped them to find out who was following the lesson, which students 'got it,' and it provided guidance for pacing the lesson. These teachers tended to use Quick Poll and Learn Check more than any other of the Navigator features. Moreover, they interpreted students' responses in terms of correct or incorrect answers. Denby's comment represents this viewpoint. "I use it more just…to assess, Where are they? How many of them got this? Can I move on? Or do I have to still wait and go back and check?"

Other teachers used students' responses to Quick Polls and Learn Checks to focus on misconceptions. Displaying these results provided opportunities for the whole class to discuss misunderstandings. Such discussions helped teachers better understand student thinking, and by making incorrect answers public, students and teacher were able to jointly learn from and address incorrect responses.

In Clarise's classroom, formative assessment was woven into all teaching and learning activities. Clarise had made a major change in her grading policy by no longer grading assignments, including homework. "Homework…is more like a discussion point rather than something to grade." (Clarise) In this way, the homework responses, collected on Quick Poll, became part of the dynamic of teaching and learning, so that both students and teacher could make formative assessments in the course of the lesson.

Almost all teachers reported that their participation in the project had broadened their view of formative assessment. Most teachers had previously thought that formative assessments were those based on a smaller amount of material, such as a quiz, and summative assessments were those that covered a larger body of material, such as a test. They now realized that there are many formats for conducting formative assessment, and it can be a daily occurrence. Yaz expressed his new understanding,

> …formative assessment is more than just a quiz and seeing where kids are and what I need to re-teach. It goes into depth on misconceptions and why kids think this way, how can we address it….

Pedagogy

Among the pedagogical issues that arose in the interviews, two key topics were questioning and planning. Both of these are important aspects of formative assessment (Ayala & Brandon, 2008) and both are integral to using the Navigator features. In their interviews, most teachers focused on the importance of questioning, but not all used questions in the same way. Iris was representative of teachers who used questions to guide students.

> I try to direct them in the way I want them to go, so to speak. With the equations, where they had to put their own equations, I ask them, "What is the coefficient?" And using that vocabulary, trying to get them used to the vocabulary….

Other teachers tried to learn more about students' thinking through questioning. They focused on how their students might solve a problem and created questions to expose misconceptions and provide interventions as needed. They spoke about "thinking as a student yourself" (Yaz) and the importance of knowing the students when introducing new concepts, "what might they have a hard time with, what might they misunderstand or misinterpret…" (Kate). Kate characterizes herself as a 'questioner' and not as a 'teller.'

> I like to ask kids questions, rather than direct them to an answer or tell them their answer is correct. I like to know more about why they think they are right or wrong….

Teachers varied in how they planned lessons. The teachers who were focused on student thinking tried to anticipate what problems and misunderstandings students would have. Yaz and Kate, who collaborate in their planning like to have students' misconceptions in mind as they plan, "…then we are ready for the discussion…." (Kate). Clarise also focused on how her students are thinking about the mathematics. Since participating in the FANC project, she has reorganized her classroom, including getting different furniture, to enable students to work in groups. In her efforts to be responsive to student thinking, her approach to teaching has become more spontaneous. "…Sometimes it's [referring to a Quick Poll question] planned ahead of time, but most of the time it's just right in the moment…." (Clarise)

The goal of Denby's planning was to create a lesson that maximized students successfully completing the tasks she intended for them. Here she carefully broke tasks down into smaller pieces so she could assess if students understood before moving on. She used this approach when introducing the Navigator technology and for introducing mathematics content. Using this approach, Denby felt she was both addressing the mathematics goals of her lessons as well as how effectively the students were learning what she hoped they would learn.

Student Role

The Navigator system is connected to a projector enabling everyone to see student responses. While it is possible for these responses to be anonymous, students usually know and identify which response belongs to them. Teachers commented that the public nature of the display made the students more accountable for their participation in the lesson activities.

> With the Navigator, it holds them a little more accountable. 'Cause they know if they are the one student who is not responding, they know they are the one the class is waiting for…. With the "Are You On the Line," they know if they are the one on or off the line because they weren't paying attention. You know, so, it holds them a lot more accountable to participate. To be engaged. (Kate)

> …having the Navigator…they put in the equation and [they] could immediately see who is getting it right or wrong, and we could help them make the corrections right away. (Iris)

Besides contributing their own responses, the Navigator features enable students to provide feedback for others. In this way, students become sources of knowledge for the class and thereby assume some of the responsibility for the group's learning, both to prompt thinking about the tasks and to assess understanding. "My kids have picked up on my questioning… they make their peers think about it…." (Kate) Not all teachers promoted a shared locus of control to the same degree, but all teachers reported on an increase in student discourse and using that discourse to guide their teaching.

> …it's a whole lot better when they're talking with each other. They teach each other stuff, and then there [are] ten teachers in the room instead of just one. (Clarise)

Discussion: Integration of Technology

Over the course of the FANC project, teachers used the Navigator system in a variety of ways in their lessons. Some examples include using Quick Poll or Learn Check at beginning of a lesson as a warm up; posing problems in Activity Center to which all students contributed data; and displaying student work on Screen Capture so students could compare work in progress. All teachers used the various Navigator features to get feedback about students' progress at critical points during the lesson. Most teachers used Quick Poll and Learn Check more often than activity Center or Screen Capture. They reported taking longer to become comfortable with Activity Center, and they viewed its applicability mainly to algebra topics, especially graphing of data or exploring features of equations.

Teachers believed that the immediate feedback from all of the Navigator features significantly supported teaching and learning in their classrooms. Yet, there was a subtle difference in viewpoint about how that support was utilized that relates to their approaches to formative assessment, pedagogy and the role of their students. Some teachers primarily used the feedback as assessment for how the students were doing. These tended to be the same teachers who viewed formative assessment as measures of correctness or incorrectness, whose questioning was designed to direct the lesson, and who viewed students primarily as receivers of information.

> Class, why do you think so-and-so got this answer? Oh, because they did the wrong step…they multiply first instead of doing power first, whatever…So that's very valuable…you have instantaneous, it's formative assessment and I can check it and see who's got [it] and who doesn't; that's huge. (Sharon)

Another group of teachers viewed the use of the Navigator system as affording greater learning opportunities for their students. These teachers also utilized the feedback from student responses to inform their teaching, but this assessment occurs via the learning activity.

> When…we are going over the answers and I see two different set-up proportions…as being correct, I would use that as a discussion on why these two look different…and are still being considered as correct answers. And then we had a discussion with the kids. (Yaz)

In summary, not only were we able to determine different levels of use of technology and formative assessment, we found this use corresponded to teachers' beliefs and perceptions related to their views of formative assessment, classroom pedagogy, and students' role in the teaching and learning process. These views then had an impact on how they used technology for implementing formative assessment in a networked classroom.

Endnotes

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2. TI-Navigator™ is a networking system developed by Texas Instruments that wirelessly connects each student's graphing calculator to a classroom computer.

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TECHNOLOGICAL, PEDAGOGICAL, AND CONTENT KNOWLEDGE (TPACK) FOR MATHEMATICS TEACHERS

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This study explores the Technological, Pedagogical, and Content Knowledge (TPACK) of three experienced secondary mathematics teachers and describes ways of using technology. Analyzing their TPACK, we noticed that teachers integrated technology to: a) give students opportunities to learn and experiment with their mathematical knowledge; b) help pass the content to the students in the process of teaching mathematics; and c) assess and evaluate student work, and give them feedback.

Introduction and Methods

This research aims to explore the integration of technology in secondary school mathematics classrooms by using the Technological Pedagogical and Content framework (Mishra & Koehler, 2008)**.** The goals are: to understand existent pedagogical ideas in the context of integration of technology into mathematics education by experienced teachers; and to document and analyze secondary school mathematics teachers' choices in integrating technology. The research question was: How do mathematics teachers describe their ways of integrating technology? Three teachers from a Toronto school agreed to participate. Data was collected from observations, document analysis, and interviews. Data was grouped according to the categories proposed by Niess (2008). We structured the findings of this paper based on Niess's four layers of integrating technology for the TPACK framework: 1) overall conceptions of integrating technology, 2) knowledge of instructional strategies, 3) knowledge of students, and 4) knowledge of curriculum and curricular materials.

Findings

First Layer

Overall, the three teachers displayed high competence in teaching the mathematics curriculum. They knew the mathematics content required for the secondary school curriculum, the students and the context of the school. They were able to acknowledge the difficulties that might appear when their students started to learn these mathematical curricular units and were willing to use technology in order to let students have their own ways of exploring mathematics.

Second Layer

Teachers used technology for various purposes in their teaching and for assessments. Therefore, we might conclude that the teachers were individuals with a sound Technological, Pedagogical, and Content Knowledge, and the administrators supported them to implement computer technology in mathematics instruction.

Third Layer

Each teacher had knowledge about the students' background, academic performance, and possible difficulties. Based on students' skills, the teacher devised his or her methods that would be deemed successful.

Fourth Layer

The teachers had a good knowledge of mathematics curriculum and knowledge of how to adapt their curriculum material for their students. For computer technology, they gave students the opportunity to experiment with their mathematical ideas and test their knowledge through computer technology.

Final Comments

The teachers were able to make technology available as a support for investigation and to establish connections with pedagogy and content. By collaborating and receiving technical support from each other helped the teachers to cope with difficulties of integrating technology in classrooms. The technology was also used for assessing student knowledge, a fact that tremendously reduced the time for feedback of the teachers.

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EMERGENT ONTOLOGY IN EMBODIED INTERACTION: AUTOMATED FEEDBACK AS CONCEPTUAL PLACEHOLDER

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Recent theories of cognition model human reasoning as tacit simulated action. Implications for the philosophy, design, and practice of mathematics instruction may be momentous. We report on findings from a pioneering design-based research study into the embodied roots of proportional reasoning that also explored the pedagogical potential of embodied-interaction (EI), a form of technology-enabled immersive activity. 22 Grade 4-6 individual/paired interviewees remote-controlled virtual objects in a non-symbolic space to solve a problem, then progressively mathematized their strategy using symbolic artifacts interpolated into the space. Drawing on qualitative analyses of the filmed work, we build a sociocognitive account of the role of automated feedback in the mediated construction of perceptuomotor schemes that undergird conceptual development, and we offer a heuristic EI design framework.

Introduction

Over the recent decades, cognitive scientists, psychologists, and philosophers have begun to increasingly question theories of cognition that model the mind as a symbol processor. Alternative "embodied" or "enactive" theories suggest that sensorimotor interaction in the natural and sociocultural ecology deeply shapes the mind—even thinking with or about "abstract" ideas is in fact the mental simulation and coordination of multimodal schematic image schemas (for a recent survey, see Barsalou, 2010). In particular, embodied cognition has been presented as a useful framework for both theorizing mathematical reasoning and designing pedagogically effective learning environments (Abrahamson, 2009; Nemirovsky & Ferrara, 2009; Núñez, Edwards, & Matos, 1999).

Parallel to the rise in popularity of theories of embodiment is the dramatic recent progress in technological affordances for embodied interaction (e.g., Nintendo Wii and Playstation Move, iPhone 4, and Xbox Kinect). Innovative designers tuned to this progress are constantly devising ways of utilizing this commercial technology in novel ways that serve a diverse audience of researchers and practitioners (e.g., Lee, 2008). As such, media that only recently appeared as esoteric instructional equipment will imminently be at the fingertips of billions of prospective learners. And yet, *What forms should "embodied" learning take? How should we theorize such learning? What are best design principles for fostering embodied interaction?*

In what follows, we discuss embodied interaction (EI) as bearing unique affordances for mathematics teaching and learning as well as research on this process. We then demonstrate these affordances by presenting an EI design for proportions as well as vignettes from implementing this design. The vignettes were selected so as to contextualize a proposed sociocognitive view on EI design: educators use EI first to foster student development of a targeted perceptuomotor scheme, then to guide student appropriation of mathematical forms as means of redescribing this scheme in accord with disciplinary practice and parlance. Finally, we offer an emerging heuristic design framework for EI mathematics instruction.

Embodied Interaction in the Research and Practice of Mathematical Learning

EI is a form of technology-supported training activity created, implemented, and researched by scholars interested in investigating multimodal learning. Through engaging in EI activities, users build schematic perceptuomotor structures consisting of mental connections between, on the one hand, physical actions they perform as they attempt to solve problems or respond to cues and, on the other hand, automated sensory feedback on these actions. One objective of EI design is for users to develop or enhance targeted schemes that undergird specialized forms of human practice, such as mathematical reasoning. As is true of all simulation-based training, EI is particularly powerful when everyday authentic opportunities to develop the targeted schemes are too infrequent, complex, expensive, or risky. Emblematic of EI activities, and what distinguishes EI from “hands on” educational activities in general, whether involving concrete or virtual objects, is that EI users’ physical actions are intrinsic, and not just logistically instrumental, to obtaining information (cf. Marshall, Cheng, & Luckin, 2010). That is, the learner is to some degree physically immersed in the microworld, so that finger, limb, torso, or even whole-body movements are not only in the service of acting *upon* objects but rather the motions themselves become part of the perceptuomotor structures learned. Thus, all EI gestures are perceived as epistemic actions, even if they are initiated as pragmatic actions (cf. Kirsh, 2006). In EI, the learner’s body—its structure and action—becomes concrete instructional material. EI is “hands *in*.”

EI activities typically emphasize explorative perceptuomotor tasks and draw less on propositional or domain-specific reasoning (e.g., Antle, Corness, & Droumeva, 2009). Notwithstanding, EI activities may include standard symbolic elements, such as alphanumeric notation, diagrams, and graphs (e.g., Cress, Fischer, Moeller, Sauter, & Nuerk, 2010; Nemirovsky, Tierney, & Wright, 1998). Indeed, content-oriented EI activities are often designed explicitly to foster the guided emergence of domain-specific conceptual structures from domain-neutral perceptuomotor schemes.

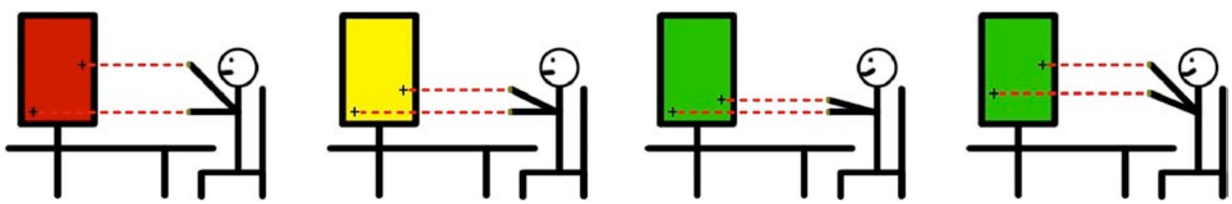
We consider EI activities as creating useful empirical settings for research on guided mathematical ontogenesis. In particular, because EI begins with *a*mathematical hands-in problem solving, data from these activities bring out in relief micro-phases of a learning trajectory that may simulate and generalize to all mathematical development: transitioning from unreflective orientation in a multimodal instrumented space to reflective mastery over the disciplinary re-description of this acquired competence. We thus propose to merge enactive and sociocultural theory to investigate how social interaction steers individuals to leverage perceptuomotor competence in appropriating mathematical forms of reasoning; more broadly, how learners come to embody, inhabit, and signify epistemic practice mediated through guided participation in cognitively demanding social activity (cf. Roth, 2009).

Our paper re-analyzes data from a recent study, in which we investigated an instructional methodology for scaffolding the emergence of proportional reasoning from EI problem-solving activities. Our analyses implicate the vital role of natural discursive modalities, such as verbal and gestural utterance, as well as mathematical semiotic artifacts, such as a virtual Cartesian grid and numerals, as the means by which students re-describe their entrained perceptuomotor enactment in disciplinary form (cf. Edwards, Radford, & Arzarello, 2009). Moreover, we found that these re-descriptions can take surprising, pedagogically useful directions, as students discover *in situ* and *ex tempore* better ways of using the symbolic artifacts so as to enact, explain, or evaluate their task strategy (Abrahamson, Trninic, Gutiérrez, Huth, & Lee, in press). As such, we see our work as expanding on neo-Vygotskian conceptualizations of appropriation (e.g., Bartolini Bussi & Mariotti, 2008; Sfard, 2002). Namely, we implicate learners' creative *re*appropriation of symbolic artifacts in ways that are *not* modeled by the instructor yet students nonetheless discover as semiotic–enactive affordances.

Finally, as reflective designers we also wish to contribute to the theory and practice of EI-based mathematics instruction, which we view as bearing promise. We therefore conclude this report with a summary of our current heuristic design framework for mathematics learning activities in learning environments availing of affordances unique to EI technology.

Design and Implementation of the Mathematical Imagery Trainer

Our design conjecture, which built on the embodied/enactive approach discussed above, was that some mathematical concepts are difficult to learn because mundane life does not occasion opportunities to embody and rehearse particular schemes that constitute the requisite cognitive substrate for meaningfully appropriating these concepts' numerical procedures. Specifically, we conjectured that students' canonically incorrect solutions for rational-number problems—"fixed difference" solutions (e.g., "2/3 = 4/5" - Lamon, 2007)—indicate students' lack of multimodal action images to ground proportion-related concepts (Pirie & Kieren, 1994). Accordingly, we engineered an EI inquiry activity for students to discover, rehearse, and thus embody presymbolic dynamics pertaining to the mathematics of proportional transformation. At the center of our instructional design is the Mathematical Imagery Trainer, which we introduce below (MIT - see Figures 1&2, below, and for detailed descriptions of the device's rationale and technical properties, see Abrahamson et al., in press; and Howison, Trninic, Reinholz, & Abrahamson, 2011, respectively).



a. b. c. d.

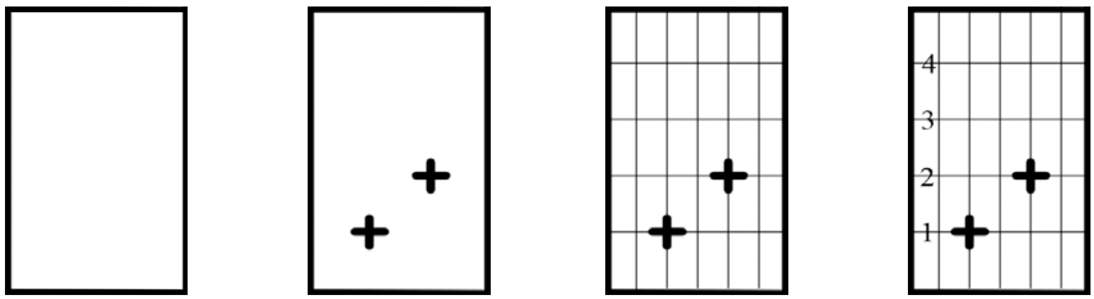
Figure 1. MIT interaction schematics, with the device set at a 1:2 ratio, so that the right hand needs to be twice as high than the left hand: (a) incorrect performance (red feedback on exploratory gestures); (b) almost correct performance (yellow feedback); (c) correct performance (green feedback); and (d) another correct performance.

a. b.

Figure 2. MIT in action: (a) "incorrect" enactment turns the screen red; and (b) "correct" enactment turns the screen green. See www.tinyurl.com/edrl-mit for a 5 minute video clip showing the MIT in action.

The MIT measures the height of the users' hands above the desk. When these heights (e.g., 10'' & 20'') match the unknown ratio set on the interviewer's console (e.g., 1:2), the screen is green. So if the user then raises her hands proportionate distances (e.g., to 15'' & 30''), the screen will remain green. Otherwise, it will turn red (e.g., raising equal distances to 15'' & 25''). As such, this MIT is designed to hone pre-numerical struggle around the additive/multiplicative tension commonly implicated in the literature as underlying student challenges in moving into rational numbers (Lamon, 2007). Study participants were tasked first to find green then to maintain it while moving their hands. The protocol included layering a set of mathematical artifacts onto the display, such as an adaptable Cartesian grid (see Figure 3c, below), to stimulate progressive mathematization of emergent strategies.

Participants included 22 students from a private K–8 suburban school in the greater San Francisco Bay Area (33% on financial aid; 10% minority students). Care was taken to balance for students of both genders from low-, middle-, and high-achieving groups as ranked by their teachers. Students participated either individually or paired in a semi-structured clinical interview (duration of mean 70 min.; SD 20 min.). Interviews consisted primarily of working with the MIT. At first, the condition for green was set at a 1:2 ratio, and no feedback other than background color was given (see Figure 3a; we used this challenging condition only in the last six interviews). Then, crosshairs were introduced (see Figure 3b): these virtual objects mirrored the location of participants' *hands* in space yet, so doing, became the objects users acted *on*, then *through*. Next, a grid was overlain on the display (see Figure 3c) to help students plan, execute, and interpret their manipulations and, so doing, begin to articulate quantitative verbal assertions. In time, numerical labels "1, 2, 3,…" were overlain along the grid's *y*-axis (see Figure 3d): these enabled students to construct further meanings by more readily recruiting arithmetic knowledge and skill so as to distribute the problem-solving task.



a. b. c. d.

Figure 3. Display configuration sketches: (a) continuous-space mode; (b) continuous-space mode with crosshairs, i.e. virtual objects users manipulate; (c) crosshairs with grid overlay; (d) crosshairs with grid overlay and *y*-axis numerals.

For the purposes of this paper, we focus on two brief episodes demonstrating milestones in students' instrumenting their perceptuomotor solution strategies then redescribing them.

Results and Discussion

All students ultimately succeeded in devising and articulating strategies for making the screen green, and these strategies were aligned with the mathematical content of proportionality. We observed minor variation in individual participants' initial interpretation of the task as well as consequent variation in their subsequent trajectory through the protocol. However, by and large the students progressed through similar problem-solving stages, with the more mathematically competent students generating more strategies and coordinating more among quantitative properties, relations, and patterns they noticed.

Each student began either by working with only one hand at a time, waving both hands up and down in opposite directions, or lifting both hands up at the same pace, possibly in abrupt gestures. They soon realized that the actions of both hands are necessary to achieve green and that the vertical distance between their hands was a critical factor. Importantly, all children initially moved their hands at a fixed difference, certainly a legitimate, reasonable strategy.

The following data excerpts will sketch how we used the MIT-based design first to foster student development of a perceptuomotor scheme centered on obtaining "green" feedback and then to leverage their skill in mediating its mathematically instrumented re-descriptions.

We begin with a 6th-grade male student, Penuel, who took longer than others in realizing that the relation between the hands' respective positions is a critical task-relevant quality.

Penuel: So it looks like... they have to be a certain distance away from each other for it to turn green….and if it's not a certain distance, it's not green, it's yellow or red.

Penuel then identified that the positions of the hand should be reinterpreted as magnitudes. That is, he re-saw the location of each hand in space in terms of how high that hand is above the desk, so that empty space below each hand took on the palpability of virtual substance.

Penuel: Well, they obviously can't be at the same distance [above the desk]. But if I start here, and if the right one is moving, like, a little faster, and it's going farther and farther away from the left hand, it will still stay green.

Note references to velocities. Later still, once the grid and numerals were introduced, Penuel quantified his sense of "moving… a little faster" as a proto-ratio by noticing a distinctly mathematical pattern emerging from green locations. Prompted to sum his discovery, he said:

Penuel: You start from the ground [indicates desk], you try to get to the first green… you have to have one… left hand on "1," right hand on "2." Exactly. And you start from there, and you keep doubling it.

Like all students, Penuel was prompted to "make the screen green." As he interacted with the MIT, and through our interview prompts, "green" transformed from an *objective* to *feedback*, as seen by his observations about the "correctness" of types of movements that elicit green. Finally, this feedback enabled him to discern a *mathematical* notion ("doubling") from the set of hand locations eliciting green. This pattern of emergence of mathematical meaning was common to all the students interviewed. Here we can provide only one more illustrative case.

Liat, a 5th-grade female student, exhibited a telltale indication of conceptual transition: mismatch between gestured action and verbal explanation (Church & Goldin-Meadow, 1986). She consistently moved her hands up in a fixed-distance motion, received red feedback, and adjusted the left hand down for green, yet she stated a fixed-distance strategy.

Liat: I think if I keep them apart and keep going up, it stays the same...
Int: If you keep them apart and you keep going up it stays the same?
Liat: It's not becoming red, but...
Int: So... how are you thinking about keeping them apart?
Liat: Oh maybe it's more. If it's farther up, then it has to be…they have to be more apart.

Later, upon the introduction of the grid and numerals, Liat was asked to predict green locations without moving her hands. She noted "one row" in between the crosshairs when the left hand is at 1 and the right hand is at 2, making green. She extended the thought:

Liat: And if you go… 10… if you go up to 10, there's gonna be like 4 or 5 rows. [i.e., if the right hand is at 10, the left should be 4-5 rows lower so as to make green.]

Thus, Liat was able to instrumentalize the grid to enhance her previous qualitative strategy for green, namely that the "farther up" her hands are, the "more apart" they ought to be. However, the 1:2 ratio was yet to become articulated, as her guess indicated she was still thinking in terms of approximate magnitude, "4 or 5," rather than relying upon more powerful mathematics (e.g., half of ten is five). Yet here precisely came the moment of guided transition to the more powerful mode of reasoning, multiplicative relations, as seen from the following exchange, where the interviewer asked her to decide between 4 and 5:

Liat: No… five!
Int: Five? How did you do that so quickly? How did you know it was five?
Liat: Half of ten is five.

Later, during the post-interview debriefing, Liat reflected on the activity of finding green.

Liat: It's not just moving hands… it's… [Liat moves her hands up and down, grasping for words]… it's… you're trying to do something and get the number.

In the ensuing discussion, Liat said that, at first, the activity was "not easy" yet that "actually, now it's easier" because she figured out how it works mathematically. Thus Liat, like Penuel, *comes to re-describe her newly developed skill of "green finding" via mathematics*, concurrently using green as an objective, feedback, and "conservation" for an ontogenesis of proportion. This pattern, common to all our study participants, suggests certain stable affordances of EI design for mathematics learning, as we discuss below in the closing section.

Conclusions and an Outline for a Heuristic Embodied-Interaction Design Framework

Even as learning scientists are increasingly accepting a view of mathematical reasoning as multimodal spatial–temporal activity, technological advances and free-market forces bode an impending ubiquity of personal devices capable of utilizing remote-embodied input. Poised between theory and industry, design-based researchers are only beginning to wrap their minds around the protean marriage of embodied cognition and remote action. Embodied interaction, a form of physically immersive instrumented activity, is geared to augment everyday perceptuomotor learning by fostering cognitive structures that leverage homo sapiens' evolved capacity to orient and navigate in a three-dimensional space, wherein the brain developed *by* and *for* action. We currently have far more questions than answers respecting both the prospects of EI and principles for best design and facilitation of these innovations. Our strategy has been to engage in conjecture-driven cycles of building, testing, and reflecting. In this spirit, the current paper aimed to share our excitement with EI and offer some early observations and caveats. Yet, as often occurs when new media are encountered, we learn as much about *pan*-media practice as about the new media per se. As such, the "ontological innovations" we have stumbled upon through our design-based research appear to bear more generally on how people do and could learn (cf. diSessa & Cobb, 2004).

Our analyses depicted learning as the evolution of users' subjective meaning for the automated feedback and, in particular, the role of perceptuomotor scheme as a vehicle or platform for the mediation of conceptual development. EI automated feedback (such as "green" in our design) evolved in the functional and cognitive roles it played. That is, green: (a) began as the task *objective*; (b) soon became the perceptuomotor *feedback*, as the users attempted to complete the task objective; and (c) came to hold together a set of otherwise unrelated hand-location pairs sharing a common effect and begging a name. Feedback on perceptuomotor performance thus came to demarcate the set of "green" number pairs as all belonging to an emerging *ontology*—a phenomenal class connecting seemingly disparate conceptions, observations, and hunches. To the expert, this emergence is centered around the concept of proportion; the student, however, is merely trying to "make the screen green" and explain exactly what makes it green. It is in this sense that green functioned as more than an objective or feedback—it served as an ontological scaffold or conceptual placeholder. Through appropriate facilitation, the scaffold ultimately collapses, or the placeholder is filled, once users determine the activity's mathematical rule and recognize the rule's power for anticipating, recording, and communicating the MIT's solution procedure. As one child gleefully quipped, upon determining the multiplicative relation of an unknown ratio, "I hacked the system!" Similar, when Penuel referred to "green" as the objective of his strategy (see his last excerpt, above), it is evident in his response that he has populated the notion of green with appropriate mathematical machinery needed to explain and predict "green."

EI thus creates arenas for launching mathematical learning trajectories from body-based qualitative notions. As the students engage in problem-solving our MIT mystery device, their physical actions inscribe a "choreographed" form with increasing deftness—forms that are very difficult or perhaps impossible to mediate outside of EI design. Whereas the student views these forms as physical solution procedures, the educator—who views the forms from the vantage point of an expert's disciplinary perspective—conceptualizes these forms as the multimodal image schema underlying the cognition of the targeted concept. Using representational resources and discursive guidance, the instructor may then steer students to progressively signify these image schema into what become concept images of the emerging mathematical ideas. That is, even as the gestured forms lend meaning to mathematical propositions, they take on the epistemological role of metaphorical simulations (concept-specific "math kata," if you will). In practice, the mathematical concept emerges when students utilize new mathematical symbolic artifacts, which the instructor introduces into the problem-solving space, as means of enacting, explaining, enhancing, and/or evaluating their solution procedure. As such, mathematical knowledge emerges through recognizing a particular cultural form (e.g., the Cartesian grid) as contextually useful tools. Specifically, students utilize these available forms to articulate their physical solution procedures, first multi-modally (verbally and gesturally) and then also symbolically (by utilizing numerical inscriptions). Initially, these articulations are naive and qualitative, but they progressively adhere to mathematical forms via situated ascension from the physical to the mathematical.

In sum, we view EI as bearing the capacity of supporting transformative teaching and learning. Specifically, EI enhances the implementation of visionary design frameworks, by which students should begin inquiry into complex mathematical concepts from presymbolic action-based quantitative reasoning (Forman, 1988; Thompson, 1993). More generally, we submit, EI activities constitute rewarding empirical contexts for research aiming to deepen and expand our field's understanding of how instructors discipline learners' perception of a shared domain of scrutiny (Stevens & Hall, 1998). Whereas our work is in its early stages and our conclusions tentative, we hope to have conveyed some enthusiasm over EI's unique instructional and theoretical affordances. Our future work will compare our results with non-EI interventions and continue to seek improvements in both theory and design, availing of recent hands-free EI development.

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SPATIAL MATHEMATICS TEACHING AND LEARNING THROUGH IMPLEMENTATION OF A TECHNOLOGY-RICH, PROJECT-BASED UNIT WITH 21ST CENTURY EDUCATORS

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This study documented means by which STEM educators experienced the mathematics and science associated with understanding lunar phenomena. The paper reports how well STEM Education graduate students interacted with project-based materials as they engaged in transdisciplinary teaching and learning. A mixed-methods approach was used with this semester-long design study to expose how focused project work facilitated spatial understanding. Significant gains were made on lunar-related concepts that had been previously resistant to conceptual change. An implication of this study suggests that technologies, such as Stellarium and Geometer's Sketchpad, facilitated project progress and aided understanding on these particular domains including spatial mathematics.

This paper describes a project-based environment created in a graduate course designed for 21st Century STEM (science, technology, engineering, mathematics) educators. The modeled environment was designed with a project-based framework aimed to allow educators the opportunity to experience the Principles and Standards for School Mathematics (NCTM, 2000), the National Science Education Standards (NRC, 1996), and the National Educational Technology Standards (ISTE, 2008) in a transdisciplinary manner. Through project work, students formulated questions, engineered research methods, collected and analyzed data, and communicated their findings. Project investigations often situated mathematics in an astronomical context and employed the use of various technologies. Use of innovative technologies embedded within a STEM project-based environment provided opportunities for future 21st Century educators to realize and apply "outside of the box thinking to create solutions that will lead to results for students" (U.S. DOE, 2010, p. 13). Four groups of students were compared as they conducted project work that required the mathematization of the Earth, Moon and sky. Students' pre/post spatial mathematical understandings were also analyzed for change.

Theoretical Framework

Project-based instruction allows students to represent, model, and apply their content knowledge in novel ways. Necessary criteria for a project classroom include driving research questions (Krajcik & Blumenfeld, 2006), benchmark lessons and innovative technologies to scaffold understanding (Singer, Marx, & Krajcik, 2000), and milestones (Polman, 2000) to give students feedback and time for revisions. Project-based instruction that embraces these key features can provide opportunities for students to improve their understanding of both mathematical and scientific practices by problematizing various situations, placing a demand for knowledge, discovering new principles, refining preexisting understanding, and applying understanding in the pursuit of research questions (Edelson, Gordin, & Pea, 1999, p. 394).

Previous research (Wilhelm, 2009) on the effectiveness of a project-based lunar unit has shown middle level students and preservice teachers significantly improved their understanding on the mathematics related to *periodic patterns* (e.g. periodicity of lunar phases), *geometric* *spatial visualization* of a moving three-bodied system, and *cardinal direction* (e.g. documenting an object's vector direction from a given position). Significant improvement on understanding these topics is noteworthy since students have had historically difficult times comprehending the spatial mathematics needed to understand physical phenomena such as lunar phases and have often harbored numerous misconceptions (Baxter, 1989; Zeilik & Bisard, 2000). Although research studies on project effectiveness (Wilhelm, 2009) have shown that participating students' mathematical skills were significantly developed in areas of periodicity, cardinal direction, and geometric spatial visualization, some spatial concepts have been more difficult to teach and resistant to change. These concepts include spatially visualizing the Moon's motion; how Moon phase, sky location, and time of day are related; and how the change in Earthly location affects the observed lunar phase. The following research reports significant gains in understanding some of these spatial topics that have been resistant to change. Some implications of this design study suggest that technologies [such as *Stellarium* (2010), free planetarium software, and *Geometer's Sketchpad* (KCP, 2009), dynamic mathematics visualization software] played an important role toward engaging students and facilitating understanding.

Research Questions and Methods

Eleven students (from an eastern south-central university in the United States) participated in the graduate course, *Project-enhanced Environments in STEM Education*, which met for 16 weeks (2.5 hours/week). Of these eleven students (three male, eight female), two were practicing teachers – one kindergarten teacher and one secondary mathematics teacher. The rest were obtaining their Masters of Education Initial Certification degrees in the areas of math or science.

In this course, the Moon was used as a context for students to experience project work since it was inherently interdisciplinary and demanded integration of mathematics and science. Students were to consider the mathematics and science involved with understanding the Moon and its phases. Approximately 20% of class meetings (on average) involved a Moon related activity or discussion. Research questions pursued were: 1) *How will students' spatial mathematics and scientific understandings develop as they engage in a contextualized, technology-rich, project-based environment?* 2) *How will the students' group projects affect their understanding of lunar related concepts?*

	Scaffolding Benchmark Lessons and Project Work
1.	Moon observations (both nature and Stellarium) and journaling.
2.	Earth/Moon/Mars scaling lesson.
3.	Initial practice project investigation: Collecting data to debunk claims of skeptics who believe the lunar landings were a hoax.
4.	Investigation of what affects a crater's size.
5.	Exploration of craters across the Solar System.
6.	Two and Three-Dimensional Modeling of Earth/Moon/Sun geometry required for various lunar phases.
7.	Asynchronous Internet discussion groups with Australian preservice teachers.
8.	Formation of groups and generation of driving questions.
9.	Multiple sharing and feedback sessions of group project data collections and representations.

Table 1. Benchmark lessons and project work task sequence

This design study was a semester-long investigation of classroom interactions provoked by the use of carefully designed project tasks (Confrey, 2006). Throughout the term, the instructor (author) refined the design task sequence (see table 1) based on the students' progress and their need for scaffolding via benchmark lessons. During the enactment of benchmark lessons and project work, students had the opportunity to work cooperatively in small groups, communicate their project status on several occasions to the whole class, and revise their work based on peer and teacher feedback.

A mixed methods approach was used with this design study to expose how project work facilitated understanding. Students' lunar-related mathematical and scientific understandings were documented before, during, and after project implementation. The mixed methods approach was employed with both quantitative and qualitative data sources and data analysis methods. Multiple research methods used in tandem allowed for triangulation and strengthened the study in greater ways than could have been done with either the qualitative or quantitative research alone (Creswell & Clark, 2007). The quantitative data sources used to assess students' pre and post understandings were a Lunar Phases Concept Inventory (LPCI; Lindell & Olsen, 2002) and a Geometric Spatial Assessment (GSA; Wilhelm, Ganesh, Sherrod, & Ji, 2007). The LPCI is a 20-item multiple-choice instrument that assesses eight science domains with four embedded mathematics domains and the GSA is a 16-item multiple-choice survey that assesses the same four mathematics domains not posed in a lunar context (see table 2). Qualitative data sources included students' Moon journals, on-line discussions with Australians, and project artifacts.

LPCI Scientific Domains	GSA Mathematics Domains
A - Period of Moon's orbit around Earth **B** - Period of Moon's cycle of phases	**Periodic Patterns** (*occurring at regular intervals of time and/or space*)
C - Direction of the Moon's orbit around Earth **E** - Phase due to Sun/Earth/Moon positions **G** - Cause of lunar phases	**Geometric Spatial Visualization** (*visualizing the geometric spatial features of a given system as it appears in space above/below/within the system's plane)*
D - Moon Motion from Earthly Perspective **F** - Phase-location in sky-time of observation	**Cardinal Directions** (*documenting an object's vector direction in space as a function of time from a given position*)
H - Effect of lunar phase with change in Earthly location	**Spatial Projection** (*projecting one's self into a different location, confined to the surface of an object, and visualizing from that global perspective)*

Table 2. Concept Domains: LPCI science domains and corresponding GSA math domains

Students began their preliminary project work by exploring the question: *Why does the Moon's appearance always seem to change?* Over a five-week period, students sketched the Moon and sky, recorded the Moon's altitude and azimuth angles, and documented patterns. One of the first design changes occurred due to inclement weather. The instructor employed the use of free planetarium software, Stellarium, in order for students to continue observations without interruption. Another design modification (an addition of a mini-project) was implemented to scaffold struggling students towards understanding how to conduct project work. The mini-project concerned an investigation of "Moon Hoax" claims. After the lunar journaling, students participated in asynchronous Internet discussion groups in a *Blackboard* blog with Australian preservice teachers (five additional weeks). Each week, students discussed an assigned topic and compared and contrasted their observation patterns.

Engaging students in inquiry benchmark investigations, Moon journaling, and on-line discussions with Australians helped to direct their follow-up Moon project work and to assist them in the generation of their own driving questions. Table 3 displays the questions that emerged from the students' preliminary investigations of the Moon and sky. Four groups transpired with topics concerning a) an evaluation of star visibility and location throughout a lunar month; b) an exploration of cratering distribution rates on Earth's surface; c) a comparison of orbital paths of Earth's Moon and other planetary moons; and d) an investigation of seasonal day length from the perspective of observers in northern and southern hemispheres.

Groups	Driving Questions
Stargazers	*Are stars in the same location each night, or do they change position?*
Craters	*What factors determine the distribution of impact craters in Earth's hemispheres?*
Orbiters	*How is our moon's path similar to other moons'?*
Anglers	*In the Northern Hemisphere, why are the days shorter in the winter than in the summer?*

Table 3. Group Driving Research Questions

Data and Analysis

The students' Moon journals contained daily lunar sketches, date and time of observations, the Moon's azimuth/altitude angles, and questions that caused them to wonder as they performed their viewings. Early journal entries served as snapshots of participants' understandings prior to the commencement of their projects. For example, approximately one-third of the journals contained reference to "a shadow" covering the Moon revealing the Earth's shadow misconception as a cause for lunar phases. An example of a shadow reference is shown in figure 1. Although it appeared this student had a shadow misconception, she did correctly note an eastward movement of the Moon throughout the month; namely, she observed the orbital direction of the Moon which denoted a mathematical development of geometric spatial visualization understanding (i.e. visualizing the geometric spatial features of a 3-bodied system as it appears in space above/below/with the system's plane). Another student wrote correctly in her journal that "*I am thinking and learning about the moon. So, since tonight is a full moon, the sun, earth and moon are in that order, in other words, the moon is on opposite side of the earth as the sun.*" This entry is another evidence of a student's geometric spatial visualization ability.

On the LPCI pre-test, eight of the eleven students (73%) chose an "Earth's shadow" cause for the lunar phases while two students selected a "Sun's shadow" explanation. Only one student chose the correct response of the Moon's position relative to the Earth and Sun as the cause. By the time of the post-test, only two students stated a shadow explanation.

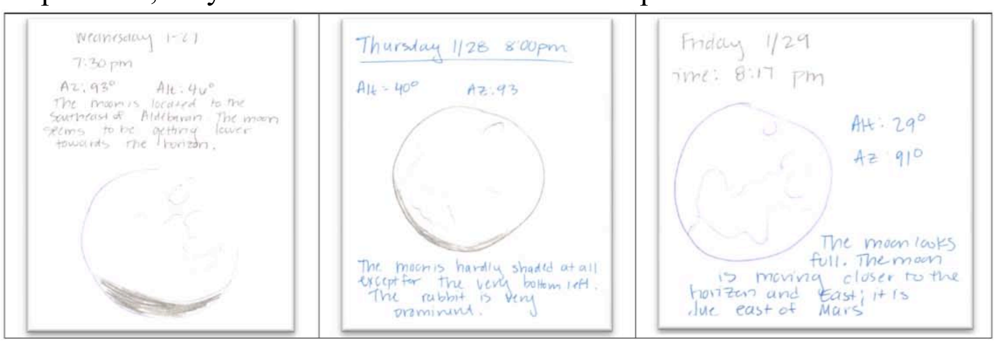

Figure 1. Student's moon observation journal entries with reference to a 'shadow' and a correct reference to the Moon's eastward movement throughout the month

In addition to the students' journals, they also participated in an Australian discussion blog where each group of students (U.S. and Australian) described patterns in their data as well as differences in their observations. For example, one student described how he noticed that when the U.S students witnessed a Moon lit on the left hand side, their Aussie counterparts observed a right hand lit lunar surface. This student attached the Stellarium screenshots (see figure 2) within his blog entry. Later, students made the realization that the reason for this was due to the fact that Australians were looking to the North to see the Moon while U.S. students were looking South.

Such a realization is evidence of a mathematical development in spatial projection understanding (visualizing from multiple global locations).

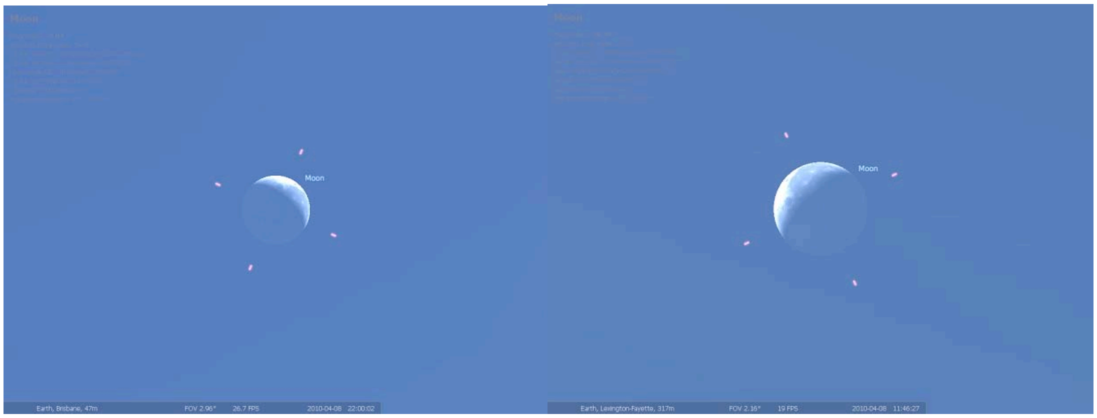

Figure 2. Stellarium screenshots from Brisbane, Australia and Kentucky, respectively.

The LPCI was given to the eleven students both pre and post project implementation. A repeated-measures ANOVA was conducted with overall test scores and by concept domain to test for significant change in mean scores. Overall LPCI results showed a mean pre-test score of 36.8% correct and a mean post-test score of 64.1%. A repeated-measures ANOVA revealed a significant increase in the mean values from pre to post-test, $F(1, 10) = 32.491$, $p<<0.001$, partial $\eta^2 = 0.765$. Final LPCI results (see table 4) showed students making significant gains on topics concerning period of Moon's phase cycle (Domain-B; Periodic Patterns Math Domain), direction of Moon's orbit (Domain-C; Geometric Spatial Visualization Math Domain), and the cause of lunar phases (Domain-G; Geometric Spatial Visualization Math Domain). Gains on these particular domains have been observed in prior studies; however, new significant gains were observed on Domain D – Moon Motion (Cardinal Direction Math Domain) and on Domain H – Effect of lunar phase with change in Earthly location (Spatial Projection Math Domain).

Table 4 also shows the LPCI results separated by gender to compare gender differences in concept domain development. Prior research showed significant gender differences on test performance (Wilhelm, 2009). Realizing that our n number is small, significant gender differences were difficult to observe; however, trends were noted. For example, male students made a 40% significant gain score on the overall LPCI where as the female students made a 22.5% significant gain score. No significant gains by individual science domains were observed for the males due to an n=3; but significant gains on domains D, G, and H were observed for the female students.

Group Project Comparisons

In order to assess whether group project work affected lunar-related conceptual understanding, a repeated-measures ANOVA was conducted with the factor being group and the dependent variables being the pre and post scores. A significant time*group interaction effect was observed, $F(1,3) = 7.528$, $p < 0.05$, partial $\eta^2 = 0.763$. Figure 3 displays the estimated marginal LPCI means by group. The two groups that achieved the greatest significant change from pre to post were the Orbiters and Craters. The gender breakdown of each group was as follows: Orbiters (one male, two females); Craters (two males, one female); Stargazers (two females); and Anglers (three females). It is interesting to note that the Orbiters and Craters groups contained the three males of the class.

Math Domain	Science Domain	All % Correct Pre (SD)	All % Correct Post (SD)	All % Gain	Male % Correct Pre (SD)	Male % Correct Post (SD)	Male % Gain	Female % Correct Pre (SD)	Female % Correct Post (SD)	Female % Gain
Overall	Overall	36.8 (11.5)	64.1 (12.6)	27.3***	36.7 (11.5)	76.7 (5.75)	40*	36.9 (12.3)	59.4 (11.2)	22.5**
Periodic Patterns	A Period of Moon's orbit around Earth	59.1 (43.7)	81.8 (33.7)	22.7	50.0 (50.0)	100 (0.00)	50	62.5 (44.3)	75 (37.8)	12.5
Periodic Patterns	B Period of Moon's cycle of phases	48.5 (22.9)	72.7 (20.1)	24.2*	33.3 (0.00)	77.8 38.4)	44.5	54.2 (24.8)	70.8 (11.8)	16.6
Geom. Spatial Visual.	C Direction of the Moon's orbit around Earth	54.6 (35.0)	86.4 (32.3)	31.8*	66.7 (28.9)	83.3 (28.9)	16.6	50.0 (37.8)	87.5 (35.4)	37.5
Card. Direct.	D Motion of the Moon	31.8 (33.7)	63.6 (39.3)	31.8*	66.7 (28.9)	83.3 (28.9)	16.6	18.8 (25.9)	56.3 (41.7)	37.5*
Geom. Spatial Visual.	E Phase and Sun/Earth/ Moon positions	42.4 (21.6)	60.6 (44.3)	18.2	33.3 (0.00)	(100) (0.00)	66.6	45.8 (24.8)	45.8 (43.4)	0.00
Card. Direct.	F Phase-Location in sky/time of observation	3.0 (10.0)	9.1 (15.6)	6.1	0.00 (0.00)	11.1 (19.3)	11.1	4.2 (11.8)	8.3 (15.4)	4.1
Geom. Spatial Visual.	G Cause of lunar phases	13.6 (32.3)	68.2 (40.2)	54.6**	16.7 (28.9)	83.3 (28.9)	66.6	12.5 (35.4)	62.5 (44.3)	50.0*
Spatial Project.	H Effect of lunar phase with change in Earth location	40.9 (43.7)	90.9 (20.2)	50.0**	50.0 (50.0)	83.3 (28.9)	33.3	37.5 (44.3)	93.8 (17.7)	56.3**

Table 4. Percentage correct on pre and post Lunar Phases Concept Inventory by mathematical and scientific domain and gender group (*p < 0.05; **p < 0.01; ***p < 0.001)

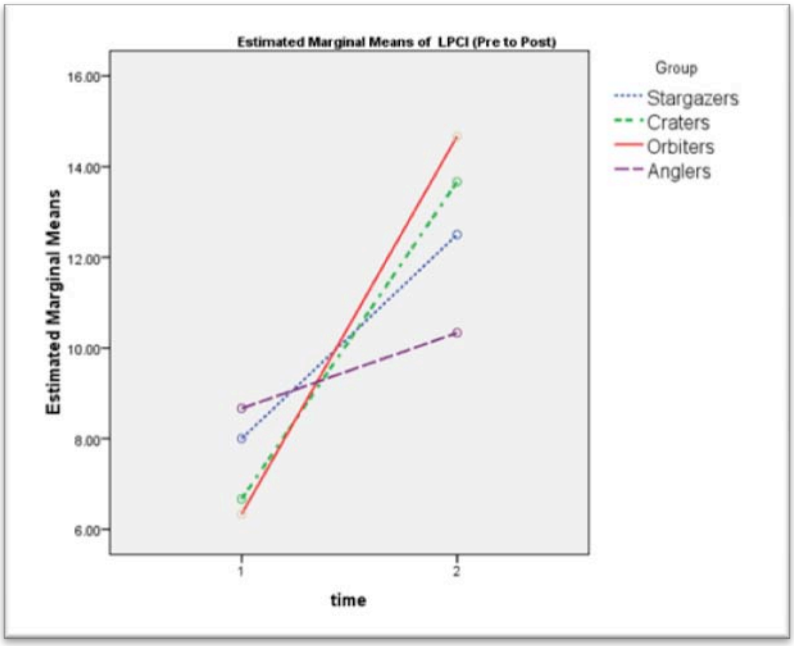

Figure 3. Estimated marginal group means of LPCI (pre to post) with a significant time*group interaction effect observed

Three of the four project groups made significant gains from pre to post on overall LPCI scores; however, due to small n, significant increases by domain were difficult to observe. Only the Orbiters had a significant gain in Domain-G (Cause of lunar phases). Embedded within this domain is geometric spatial visualization. The Orbiters had to pay special attention to the Earth/Moon/Sun geometry in order to model the lunar path for their project. The Orbiters utilized the software program *Geometer's Sketchpad* for their orbital modeling. Similarly, the Anglers achieved the greatest pre to post score increase (66.7%) in Domain H (effect of lunar phase with change in Earthly location; Spatial Projection); however, this increase was not statistically significant. The Anglers' project research involved examination of celestial observations from both the northern and southern hemisphere perspectives creating spatial projection experiences. Both cases are evidences of how students' project work affected their spatial understandings.

Like the LPCI, the GSA was given to the eleven students both pre and post project implementation. A repeated-measures ANOVA was conducted with overall test scores and by domain to test for significant change in mean scores. The only significant gain observed was with the cardinal direction domain. Cardinal direction (CD) results showed a mean pre-test score of 77.3% correct and a mean post-test score of 95.5%. A repeated-measures ANOVA revealed a significant increase in the mean values from pre to post on cardinal direction scores, $F(1, 10) = 26.667$, $p<<0.001$, partial $\eta^2 = 0.727$. Further analysis showed female students with an 18.8% significant gain score on the CD domain, $F(1, 7) = 21.00$, $p = 0.003$, partial $\eta^2 = 0.750$.

Conclusion and Importance

This study documented means by which 21st STEM educators experienced the spatial mathematics associated with understanding lunar phenomenon and how technology became a transdisciplinary partner in this project-rich environment. Stellarium was used to supplement lunar observations when weather issued problems. Lunar blogging with Australians caused students to wonder what others observed across the world—so much so, Anglers made this their project focus. Orbiters used Geometers' Sketchpad to model their lunar paths giving them the largest group LPCI gain score. Significant gains were observed on LPCI domains:

B, C, and G. Even more intriguing, were the significant gains made on LPCI domains D (Moon motion; Cardinal Direction) and H (effect of lunar phase with location change; Spatial Projection) which have been previously resistant to conceptual change. Domain D's corresponding mathematics domain, cardinal direction, also showed a significant gain with GSA results; however, no other GSA gains were observed. An implication of this design study suggests that the use of technology aided spatial understanding.

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EMERGING FRAMEWORK FOR MATHEMATICS AND SCIENCE INTEGRATION

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In this presentation we will explore the emerging framework for classification of types of mathematics and science integration focusing on cognitive level of mathematical task involved. Examples from several longitudinal Professional Development programs for in-service teachers will be used for exploring and investigation of the various integration levels.

The integration of disciplines of mathematics and science helps students develop a better understanding of the concepts and establish more connections between the subject areas. One of the benefits of such integration is that science can provide students with concrete examples of abstract mathematical ideas, and thus can provide relevance to math concepts and a motivation to learn. Mathematics, in turn, gives students a means for quantitative exploration and analysis of scientific processes.

In this presentation we will explore the emerging framework for classification of types of mathematics and science integration focusing on cognitive level of mathematical task involved. For example, the most popular type of integration occurs when a science investigation is conducted, and data is collected through some simple measurements, briefly discussed, and then displayed in tables or graphs. This type of integration is at the first level, since it only involves the simplest mathematical analysis, the procedural.

Integration at level two is when science investigation is conducted, and then mathematical exploration follows with a focus on understanding the mathematical concepts.

Integration at level three is when science investigation occurs, and then mathematical exploration follows with a focus on generalizations and deriving new kinds of mathematical relationships. We strongly believe that this classification of mathematics and science integration is necessary in order for curriculum to include higher order and critical thinking in both content areas.

Examples from a several longitudinal Professional Development programs for in-service teachers will be used for exploring and investigation of the various integration levels.

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HUMOR IN MATHEMATICS – IS THERE SUCH A PHENOMENON?

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Mathematics and humor are not perceived as being in line with each other. Mathematics is perceived as an esoteric and alienated body of knowledge without any humanistic elements. Humor constitutes one of the forms of human expression and it has numerous uses in society as well as in daily life. Mathematical contents studied at school mostly lack word messages which are frequently associated with humor as in most other subjects. The mathematician Littlewood wrote that a good mathematical joke is better and better mathematics than a dozen mediocre papers. In a subject like mathematics, considered one of the most difficult to comprehend, it is highly important to integrate humor. Humor serves as relieving tension, removing anxiety, improving self-image and motivating learning. The present paper aims to present the functions of humor in general and the possibility to integrate it in the teaching of mathematics in particular. The following example may serve as the first course. A math teacher is riding his bicycles. On the way he meets one of his students driving a deluxe Volvo. The teacher turns to the young man, who he recalled as the student with the lowest grades in class, and asks him: "How did you become so successful with your attainments in mathematics?" The young man replies: "I owe it all to you, my teacher. I buy pens for $2.00; sell them for $4.00 earning only 10%..." More mathematical jokes examples will be included.

TEACHER KNOWLEDGE OF STUDENTS AND ENACTMENT OF MOTIVATIONAL STRATEGIES IN TEACHING THE CONCEPT OF FUNCTION

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This research linked educational psychology and mathematics education to investigate how a teacher ("Mr. Algebra") used his knowledge of students in designing and implementing mathematical tasks related to piecewise functions and composition of functions. The study revealed that Mr. Algebra faced many challenges in the implementation of mathematical tasks because students had not mastered early algebra concepts. Additionally, students carried with them incomplete formal learning about domain and range, function evaluation, and constant functions, which made learning piecewise functions and composition of functions more difficult. The study employed various frameworks of mathematical tasks, Self-Determination Theory, and Keller's (2010) ARCS Model.

This research linked educational psychology and mathematics education to investigate how a teacher ("Mr. Algebra") used his knowledge of students to design and implement mathematical tasks related to piecewise function and composition of functions. To guide the study, we applied several theoretical frameworks of teacher knowledge (Ernest, 1989), mathematical tasks (Stein, Grover, & Henningsen, 1996), Self-Determination Theory (Ryan & Deci, 2000), and the ARCS Model (Keller, 2010).

Our study revealed many students in Mr. Algebra's course had lost confidence, but he had the power to restore that confidence with his flexible teaching strategies and making the course materials relevant to their lives. However, he faced many challenges in the task implementation because students had not mastered previous course content. Their understanding of function was limited in terms of their abilities to (1) identify pre-images and images; (2) transfer from one representation to another; or (3) identify functions satisfying some given constraints. Students' limited understanding contributed to their incomplete formal learning when working with function concepts. Hence, recognizing the origins of misunderstanding (incomplete formal learning) is necessary to improve the teaching of algebra (Kieran, 2007).

The study also called for a deeper understanding of students from motivational perspectives. As McLeod (1992) suggested, "research in mathematics education can be strengthened if researchers will integrate affective issues into studies of cognition and instruction" (p. 575). In other words, increased attention to motivational variables can play a vital role in improving mathematics teaching and learning. Therefore, the analysis of students' understanding of function concept discussed in our study contributes to the literature on how to engage students in mathematics learning related to function concepts.

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INTERPRETING ZONE OF PROXIMAL DEVELOPMENT THROUGH THE ONTOLOGICAL QUADRIVIUM

Definition: *Zone of Proximal Development* (ZDP) "is the distance between the actual developmental level as determined by the independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Traditionally, it is interpreted as (a) the difference between what a learner can do without help and what the learner can do with help, or (b) the range of abilities that a person can perform with assistance, but cannot perform independently.

Proposition: ZDP can be interpreted through *metric*. There are many issues on Russian/English translations involving Vygotsky's works (Wertsch, 1985), including those on ZDP (Junior, 2010). If distance is an appropriate word to define ZDP, then it is possible to interpret it through topology and metric spaces (Lewin, 1936). Thus, it is convenient for educational psychologists (or any individual interested in developmental psychology) to investigate mathematical problems as such as "the taxicab problem or taxicab geometry." Let $M \neq \varnothing$. $d: M \times M \rightarrow R_+$ is a metric in M if: (1) $d(x, y) = 0 \leftrightarrow x = y$, (2) $d(x, y) = d(y, x)$, $\forall$ x, y $\in$ M and (3) $d(x, z) \leq d(x, y) + d(y, z)$, $\forall$ x, y, z $\in$ M. What are the representations of circles when $d(x, y) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$ and when $d(x, y) = |x_2 - x_1| + |y_2 - y_1|$?

Theorem: ZDP can be interpreted through the *ontological quadrivium* (Levy, 1998). ZDP is defined under the traditional Aristotle's notion of metaphysics, where the dialectical relation between actual/potential defines reality. However, regarding Deleuzian perspectives (Deleuze, 1994; Deleuze & Guattari, 1994; Deleuze & Parnet, 2002), Levy (1998) highlights four states of being. They are: real-possible and actual-virtual. Levy (1998) uses the expression *ontological quadrivium* to discuss the notion of *becoming virtual* as the nature of collective intelligence. Thus, ZDP can be interpreted as *a path connected by humans-technologies through virtual, actual, real, and possible levels of development*.

Corollary: Mathematical learning can be understood through this alternative notion of ZDP.

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BASE-x: BRIDGING THE COGNITIVE GAP BETWEEN ARITHMETIC AND ALGEBRA

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A teaching intervention study demonstrates the effective bridging of arithmetic to algebra through the linear algebra concept of 'basis.' After a unit on base-5 emphasizing the isomorphic structural relationships between number bases, students learn to work in 'base-x,' applying their ordinary notions of arithmetic to compute algebraic arithmetic 'basis-free,' without variables. Students exposed to base-x performed significantly better at computing algebraic arithmetic than a non-treatment control group; and moreover, base-x students demonstrated significant retention of knowledge, as compared to control, upon taking a post-post test one month later.

Several studies on algebra pedagogy indicate that the variable concept presents significant challenges for many beginning algebra students. Herscovics and Linchevski (1994) describe difficulties with the variable concept as a 'cognitive gap' between arithmetic and algebra, defined as an 'inability for students to operate spontaneously with or on the unknown (p.59).' While much of previous research highlights the 'differences' between arithmetic and algebraic thinking, this poster describes a teaching intervention focusing on key structural similarities using the concept of isomorphism to link algebraic formalism to familiar arithmetic. After generalization of the number base concept by exploring base 5 and other bases, the instructor introduces a generalized 'base-x.' Base-x teaches students to perform subtraction, multiplication, and division by first converting polynomials to base-x 'numbers,' perform the familiar typical algorithms, and then convert back from base-x numbers to their algebraic representations, as seen in the following example of polynomial multiplication (see Fig.1).

x^5	x^4	x^3	x^2	x^1	$x^0 = 1$
			1	-11	13
		×	1	0	-1
			-1	11	-13
		0	0	0	
+	1	-11	13		
	1	-11	12	11	-13

Figure 1. $(x^2 - 1) \cdot (x^2 - 11x + 13) = x^4 - 11x^3 + 12x^2 + 11x - 13$ **in base-x.**

The significant results of the study support the hypothesis that isomorphic understanding relating arithmetic and algebra enhances transfer of inert arithmetic ability to polynomial arithmetic. The results appear particularly interesting due to the apparent retention demonstrated by the treatment group, which the author believes to be evidence of the possible anchoring of new isomorphic algebraic thinking to the students' stable well-known knowledge of arithmetic.

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EXAMINING STUDENTS' UNDERSTANDINGS OF CHANGE IN AN ALGEBRA CLASS

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The Algebra Standard (NCTM 2000) emphasizes that one of the essential elements in the teaching and learning of Algebra is the analysis of change (p. 36). However, many students still struggle with developing an understanding of rates of change (Bell & Janvier 1981; McDermott et al. 1987). This poster contributes to work that has been done to describe how students develop an understanding of rates and speed (e.g. Thompson 1994; Thompson & Thompson 1994, 1996). We present an analysis of students' work on a problem where they made use of the concept of change to study piecewise linear functions. We draw upon Sfard's (1991) work describing two different ways in which mathematical concepts can be conceived of: *structurally*—as objects, or *operationally*—as procedures. We argue that students' work on the problem provides evidence that they were able to move beyond an operational knowledge of change, to a structural knowledge.

We looked for evidence in students' work of a three-phase transition from an operational conception to a structural conception of speed: interiorization, condensation, and reification (Sfard 1991, p. 18). Interiorization is when the learner becomes acquainted with the processes that give rise to the concept of speed; for example the learner can compute the difference between starting and ending points, starting and ending times, and the quotient of the two. Condensation is when the learner is able to think of the process as a whole, without needing to consider separately every step in the procedure. Reification is the point when the idea of speed becomes a static object, for example a property of a graph.

We analyzed 44 students' written work on the problem. We found that 32 students showed evidence of interiorization, 27 students showed evidence of condensation, and 20 students showed evidence of reification. We discuss possible elements in the design of the task that could have supported students' transition through these phases. For example, the lesson provided students with a piecewise linear graph to analyze; and students used this graph as an opportunity to consider how speed is related to a graph. An examination of students' understanding of change is helpful for practitioners because teachers can become aware of the resources that students may deploy when learning this mathematical concept.

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HOW TEXTBOOKS INFLUENCE STUDENTS' ALGEBRA LEARNING: A COMPARATIVE STUDY ON THE INITIAL TREATMENT OF THE CONCEPT OF FUNCTION

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To illustrate cross national differences in schooling, this study analyzed content and problems in three curricula: a U.S. conventional text, a U.S. standards-based text, and a Chinese reform oriented text on the topic of the initial treatment of the concept of function. Analysis results revealed that content presentation in the Chinese text is aligned with that in the U.S. conventional text; they are designed for teacher-centered instruction. Problems in the Chinese text are aligned with those in the U.S. standards-based text, aiming at cultivating students' critical abilities.

To give insights into cross national differences in schooling this study analyzed the initial treatment of the concept of function in three curricula: a U.S. standards-based text--*Connected Mathematic 2: Variables and Patterns*, a U.S. conventional text--*Glencoe: Mathematics Applications and Concepts: Course 2*, and a Chinese reformed text--*Shu Xue: Grade 8, first volume*. The concept of function is the fundamental concept of algebra. NCTM (2000) stressed that the concept of function should be placed as one of the cornerstones of middle and high school mathematics curricula. However, the results from the international assessment studies showed that U.S. students fall behind their counterparts from high-achieving countries on this topic. Although various factors influence student learning, many researchers generally agree that textbooks, as potentially implemented curricula, have a large influence on learning and teaching (e.g., Son & Senk, 2010). This study may raise hypotheses about factors affecting achievement of students in the U.S. and China, and reveal insights into improving mathematics education practice in those countries.

This study analyzed the corresponding content and problems in three textbooks. For content analysis, we explored how the concept of function was introduced, defined, and developed. Problems were then analyzed extensively with respect to three criteria: (1) contextual feature, (2) response expectation, and (3) cognitive expectation to investigate the kind and level of learning opportunities provided by different textbooks.

Analysis results showed that *Connected Mathematics* is designed for student-centered instruction while Glencoe and the Chinese text are more teacher-centered. However, in terms of problems, *Glencoe* presented mostly procedure-based examples and problems without illustrative contexts. Chinese reform-oriented textbook and *Connected Mathematics* aimed at cultivating students' critical mathematical abilities, such as mathematical reasoning and problem solving. Different from the U.S. texts, the Chinese text provided enough explanations and illuminations to facilitate students' understanding; rather than just presenting worked-out examples, it also provided explanations of why and how.

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HOW TEXTBOOKS INFLUENCE STUDENTS' ALGEBRA LEARNING: A COMPARATIVE STUDY ON THE INITIAL TREATMENT OF THE CONCEPT OF FUNCTION

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AN INTIAL LOOK AT COLLEGE INSTRUCTORS' CONCEPTUALIZATIONS OF SLOPE

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This study provides an initial investigation into college instructors' conceptualizations of slope. Data were collected from 26 instructors of 11 two- and 3 four-year colleges. Participation was optional, and each participant had 15 minutes to complete five items. Working independently the members of the research team coded each response using the 11 conceptualizations of slope identified by Moore-Russo and colleagues (Moore-Russo, Conner, & Rugg, 2011; Stanton & Moore-Russo, in press) and adapted from the work of Stump (1999, 2001a, 2001b). After discussing findings and reaching a consensus on each response, the researchers reviewed the frequency of use for each conceptualization.

The number of conceptualizations used by a single instructor on the five items varied from three to nine, with a majority of instructors using five, six, or seven conceptualizations. While all 11 conceptualizations were used by the instructors, some conceptualizations occurred more frequently than others. Based on responses across all five items, *functional property* was the most common conceptualization of slope, followed in order by *calculus conception*, *geometric ratio*, and *real world situation*. However, the researchers recognized that some of the assessment items might elicit particular conceptualizations while limiting others. Thus, further analysis was done to determine the number of instructors who used each conceptualization at least one time on the five-item assessment. Based on this analysis, *geometric ratio* was the most common conceptualization, followed by *functional property* and *real world situation* (tied) and *calculus conception*. Stump (1999) similarly reported that *geometric ratio* was the most common conceptualization of slope used by secondary teachers.

While nearly 85% of the instructors used the *geometric ratio* conceptualization at least one time, *functional property* and *calculus conception* were more robust conceptualizations in terms of being used on multiple items. The strong presence of these two conceptualizations among college instructors provides some evidence that teachers' dominant conceptualizations of slope may be linked to the emphasis of the content they teach. Additional research is required to fully understand college instructors' conceptualizations of slope and how their conceptualizations compare with those of college students.

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COGNITIVE AMBLYOPIA

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The purpose of this work is to identify the transformative effects that traditional teaching can have on student cognition and learning, and to posit a new theoretical framework to connect research about basic neurological development and function to phenomena associated with mathematics learning.

This framework identifies basic functions of the brain that produce certain natural phenomena that may be observed both at a cellular level and at, what this author will term, the "conscious/subconscious" level. The conscious/subconscious level is the area of conscious functioning under study that is impacted by subconscious mechanisms independent of conscious will. Further, this author proposes a name, cognitive amblyopia, and a justification for the existence of these phenomena.

Amblyopia is a partial or complete loss of eyesight that is not caused by abnormalities in the eye (Hubel, 1988). *Cognitive amblyopia* is the partial or complete loss of the ability to see and understand concepts that is not caused by abnormalities in the brain. One aspect of this phenomenon may be explicated by discussion of the research underlying the development of the theory of cognitive amblyopia.

While attempting to induce visual amblyopia during a neurobiological research project it was found that mechanically inducing discordant visual input between the mammalian subjects' eyes resulted in damage to the subjects' still plastic eye brain connections. Specifically connections between the eyes and the brain that served both eyes developed into connections that were dominated by the connections' dominant eye (Hubel, 1988). In short the non dominant eye brain connection was effectively lost. This loss of vision indicates that the brain rejects confusing input at the most fundamental level.

Research conducted by Dr. Jerome Bruner and Leo Postman provides an extension of this phenomenon into the conscious/subconscious level. In their research, they identified what they termed a "dominance" reaction and an extended denial of incongruous stimuli in adult humans who were asked to identify flash cards (Bruner & Postman, 1949). This subconscious denial of comprehension was the result of a dominant anticipation and represented a mild and easily reversible cognitive amblyopia.

The theoretical framework surrounding the phenomenon of *cognitive amblyopia* proffers that the brain naturally rejects contradictory and confusing information at a conscious/subconscious and cellular level. Additionally, it asserts that extended exposure to confusing or contradictory input will cause structural changes to plastic regions of the brain.

Some student thinking is subject to change and is plastic by nature. Therefore, ineffective methods of teaching mathematics including covering material, inducing potentially irresolvable perturbations and presenting material, without respect for student thinking, in a contradictory or confusing manner may not just fail to foster learning but may also engender a *cognitive amblyopia* and damage students' future ability to see meaning in mathematics.

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IMPROVING MATHEMATICAL EQUIVALENCE PERFORMANCE IN GRADES 2 AND 4

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A great majority of children in Canada and the United States from Grades 2-6 fail to solve equivalence problems (e.g., 2 + 4 + 5 = 3 + __) despite having the requisite addition and subtraction skills. The goal of the present study was to determine the relative influence various instructional variables in improving performance. Instructions emphasizing procedural and conceptual understanding both were very effective in improving problem solving, but conceptual instruction was superior in facilitating performance on other measures to understanding equivalence. Manipulatives had no effect. These findings provide insights into optimizing children's understanding of equivalence.

A great majority of children in Canada and the United States from Grades 2-6 fail to solve equivalence problems (e.g., 2 + 4 + 5 = 3 + __) despite having the requisite addition and subtraction skills (e.g., Falkner, Levi, & Carpenter, 1999; Hattikudur & Alibali, 2010; McNeil and Alibali, 2005b). The goal of the present study was to determine the relative influence of two variables, instructional focus (procedural or conceptual) and use of manipulatives (with or without), in helping children learn to solve equivalence problems and develop an appropriate understanding of the equal sign. Instruction was provided in four conditions consisting of the combination of these two variables.

Students in Grade 2 (n = 122) and Grade 4 (n = 151) participated in four sessions designed to assess the effectiveness of four instructional methods for learning and retention. Session 1 included a pretest of equivalence problem solving and three indicators of understanding of the equal sign. In Sessions 2 and 3 instruction was provided in one of the four instructional conditions or a control condition. Students were tested for their skill at solving equivalence problems immediately following instruction and at the beginning of Session 3 to assess what they had retained from Session 2. In Session 4, one month later, children were re-tested on all of the tasks presented in Session 1 to assess whether instruction had a lasting effect.

All four instructional groups greatly outperformed the control group in solving equivalence problems, but differences among instructional groups were minimal. Performance on indicators of understanding, however, favoured students who received conceptually focused instruction. Preliminary evidence was found that children's understanding of problem structure and attentional skill may be associated with the ability to benefit from instruction on equivalence problems. Children clustered into four groups based on their performance across tasks that are consistent with the view that children's understanding of the equal sign develops gradually, beginning with learning the definition.

These findings suggest that a relatively simple intervention can markedly improve student performance in the area of mathematical equivalence, and that these improvements can be maintained over a period of time and show some limited generality to other indicators that children understand equivalence.

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THE ROLE OF QUANTITATIVE REASONING IN DEVELOPING MEANINGFUL FORMULAS

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Overview

In this poster, we describe an investigation of precalculus students' behaviors as they attempted to make sense of novel word problems. For this report we focus on the processes by which students developed meaningful formulas to relate the relevant quantities in applied problems. The study revealed that formulas are only meaningful to students if they have first conceptualized the quantities and how they are related, and see the development of expressions and formulas as a means for representing how the quantities are related and change together.

Research Questions

Our primary research focus was to understand the role of quantitative and covariational reasoning in a student's ability to make sense of novel word problems. We identified two research goals.

1) Describe the ways of thinking about solving novel mathematics problems students have prior to receiving instruction in a redesigned precalculus course.
2) Describe, explain and theorize about the role construction of quantity and covariation of quantity play in students' orienting to novel applied precalculus problems.

Results

A few major aspects of orienting to novel problems emerged in these analyses: (1) constructing an image of the situation by relating the attributes (quantities) of the box and bottle problems and (2) imagining those attributes simultaneously changing (variable quantities) and (3) modeling these dynamic situations with functions defined algebraically and graphically. We propose that students' make sense of complex, novel problems by creating diagrams of the measurable attributes of the situation, and these diagrams become useful to the student as a tool for thinking about how quantities vary and how they are related. We believe that the students' creation of these diagrams depicting relationships between the measurable attributes of a situation provided the students an image with which they could reason about a solution to the problem. We think that students' images of quantities and their relationships, and covariation of those measurable attributes allow a student to predict and explain patterns in the formulas that represent the relationships between those attributes by generating a specific algebraic function.

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CREATING CROSS-GRADE ASSESSMENTS OF THE DEVELOPMENT OF CORE ALGEBRAIC CONSTRUCTS

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Learning progressions hold promise as a basis for formative assessment. We describe the first phase of a four year project intended to develop and pilot two forms of formative assessment based upon mathematical learning progressions for equations and expressions, linear functions, and proportional reasoning.

This poster describes the first phase of a four year project focused on improving teaching and learning in the classroom by creating and evaluating a set of formative assessments that are built around developmental models of student learning. The models guide the assessment design and provide support for teachers in understanding the terrain of student learning. The assessments focus on four core algebraic constructs — equality, notion of a variable, proportional reasoning, and linear functions— and build on what the project team has learned in the context of other research projects regarding developmental models (reference removed). The evaluation plan addresses the technical quality of the assessments, with a multi-faceted approach to collecting validity evidence.

The poster will focus our approach in the project to characterizing patterns of learning in middle school students, called learning networks (LNs), Recently, learning progressions (LPs) (Heritage, 2009) have gained popularity among researchers as a way of describing increasingly sophisticated ways that students understand and use concepts. LPs typically take the form of a linear sequence of levels with respect to one concept. Characterizing students with respect to a learning progression then means collecting evidence of their mathematical behavior to make an inference to categorize them as being at a particular level. But the learning of one concept may be connected to the learning of other concepts. Lesh, Lamon, Gong, and Post (1993) explored this notion of connectivity within the idea of Learning Progress Maps (LPMs), which provide a view of the achievement of students with respect to “big ideas” within a curriculum. LPMs provide information about students’ depth of understanding of each idea and the connections between the ideas.

In our work, we use research-based LPs as a starting point to develop a learning network (LN) that describes likely pathways students can take as they learn multiple concepts in mathematics. In assessment, the LN becomes a predictive model of the likely states of understanding and pathways through these states. This approach can then draw upon the relative simplicity of assessing students with respect to individual LPs while reporting on and supporting learning of multiple concepts and the connections among them as in LPMs.

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HIGH SCHOOL TEACHERS' DISCOURSE-BASED FORMATIVE ASSESSMENT PRACTICES IN MATH INSTRUCTION

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The effective use of formative assessment methods during instruction improves students' learning, particularly low-performing students' achievement (Black & Wiliam, 1998). Though the achievement gap is a national concern embodied in the US's No Child Left Behind Act, researchers note that many teachers are not engaged in such practice. The current situation prompted three broad questions: First, do teachers believe they are engaged in formative assessment practices as they teach? Second, could we develop a tool to document formative assessment practice? Last, if so, are teachers' beliefs about their assessment practices similar or different from our classroom observations? Thus, the purpose of this illuminative study was (a) to develop an analytic framework that captured teachers' instructionally-embedded formative assessment practices (discourse-based assessment practice or DAP), (b) to identify the nature of teachers' questions and forms of feedback in extended exchanges with their students and (c) to discuss teachers' interpretations of their own practices and to consider them in light of what we observed.

In this study, we take DAP to be teachers' provision of questions and feedback prompted by their students' thinking manifested in classroom discourse as they make moment-by moment instructional decisions. Literatures on questioning, feedback and classroom discourse inform the analysis (Cazden & Beck, 2003). We borrow the construct "extended sequence" from Mehan's (1979) ethnomethodological approach for structuring classroom lessons, which is the primary unit of analysis. The extended sequence conmprises teachers' extended exchanges with students, providing an opportunity to examine the nature of these exchanges in terms of their questions and feedback, which constitute formative assessment practices as teachers deliver instruction.

The preliminary examination of 17 lesson videotapes and teacher interview transcripts suggests that (a) the analytic framework, though parsimonious, can differentiate teachers' formative assessment practices in terms of questions and feedback and (b) that some aspects of teachers' beliefs about their practices were in alignment with what we observed. If formative assessment is critical for productive teaching and learning, then intensive professional development is needed to bring beliefs and practices in line.

JOURNAL WRITING IN COLLEGE MATHEMATICS

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The effect of journal writing on achievement in and attitudes toward mathematics has been generally inconclusive (Jurdak & Zein, 1998). Results differ based on the age of the students and the type of journal writing employed. However, at the college level, results indicate that there are strong relationships between journal writing and conceptual understanding (Grossman, Smith, & Miller, 1993).

By writing in journals, students make use of writing as a learning tool in the context of mathematics; by reading students' journals, teachers gain insight into their students' understanding in unique and valuable ways (Borasi & Rose, 1989). Furthermore, by responding to students' journals, teachers open up lines of communication with students that traditionally are not open. This in-progress study seeks to explore the benefits of journal writing. Particularly, three different types of benefits: benefits for students, benefits for teachers, and benefits for communication between students and teachers.

College students enrolled in beginning calculus classes will write homework journals throughout the semester. Each homework journal will be a two-column homework assignment based on a mathematical problem of each student's choosing from the course textbook (Powell & Ramnauth, 1992). The left-hand column is the mathematical problem worked out in detail. The right-hand column is a written reflection of the problem. The written reflection is guided by a number of different prompts. For example, "What connections do you see between this problem and other topics from this course or other mathematics courses?" The teacher responds to what each student writes by posing additional problems, challenging thinking, and drawing out conclusions. Students then have the opportunity to respond to the teacher's comments, continuing the communication.

At the end of the semester, students will be asked to give an evaluation of homework journals by responding to open-ended questions. Therefore, data will be collected in these two ways: homework journals collected throughout the semester as well as end of the semester evaluations.

Analysis of the data will hopefully provide rich descriptions of the journal writing experience and the benefits of the journal writing experience. The results of this study could be useful to college mathematics instructors. Possible implications of this study are for the teaching and learning of college mathematics, especially when it comes to communication between instructors and students.

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USING NEURAL-NETWORKS TO PREDICT AP-CALCULUS TEST SCORES FROM PCA AND ACT MATHEMATICS TEST SCORES

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Neural-Networks are a powerful alternative to regression especially for prediction and forecasting but not widely used in educational research. This study explores how AP-Calculus AB and BC scores can be predicted from the Precalculus Concept Assessment (PCA) and American College Testing (ACT) mathematics scores employing two commonly used Neural-Networks models of MLP (Multilayer Perceptron) and RBF (Radial Basis Function). Strong positive correlations between the actual and predicted values of the AP-Calculus exam scores confirm that Neural Networks are an efficient toolset for prediction. This can help identify the students who are at the risk of not passing the AP-Calculus exams and help students, parents and teachers take remedial measures in a timely manner.

Neural-Networks are a powerful alternative to linear and nonlinear regression especially for predicting and forecasting but not widely used in the field of education. A neural-network is an interconnected group of artificial neurons, processing information through a series of connections as a means of computing. The modern neural networks are used for non-linear modeling of statistical data revealing the complex nonlinear relationships between the inputs and outputs better than nonlinear regression methods for discovering the patterns in the data.

On the other hand, the PCA was developed by faculty from the Department of Mathematics at Arizona State University and was designed to reflect core content and common misconceptions students have about functions (Carlson et al., 2010). Carlson et al. (2010) administered the PCA to 902 college precalculus students and found the "Cronbach's alpha of 0.73 indicating a high degree of overall coherence" (p. 137).

Overall, this study demonstrates how AP Calculus AB and BC scores can be predicted from the Precalculus Concept Assessment (PCA) results (Carlson et.al. 2010) and ACT mathematics test scores (ACT, 2011) employing two commonly used Neural-Networks models: the MLP and the RBF Models (Kutner et.al. 2005). The simulation revealed Pearson correlation values, between the actual and predicted values of the AP Calculus AB and BC, in the range of 0.50 to 0.95 and statistically significant at the 0.01 level. This confirms that Neural Networks are an efficient toolset in predicting students' AP Calculus AB or BC scores from the PCA and the ACT mathematics scores which can help identify the students at the risk of not passing the AP Calculus AB or BC exams and help students, parents and teachers take the necessary measures in a timely manner.

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TECHNOLOGY TOOLS IN HIGH SCHOOL MATHEMATICS: PROMOTION, PERCEPTIONS AND USE

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This poster is a preliminary report for a research project that aims to build an understanding of the complexity of teachers' technology tool promotion in classrooms as it relates to students' adoption of such tools in their independent mathematical work. The project focuses not only on the mathematical aspects of technology tool promotion and use, but the affective aspects as well. We examine how these aspects impact the *how* and *why* of students' choices to use technology tools when working independently. Given the expectation that today's students must use technology tools to engage in mathematical problem solving (CCSSI, 2010; NCTM, 2000) both in and outside of the classroom (i.e. on homework, teacher created and standardized assessments, the application of mathematical concepts to other contexts) this is an important sequence of events to understand.

This investigation is a mixed method multi-case study of six high school mathematics classes. We are investigating six classes, two classes each from three high school teachers who work in three types of schools (rural, suburban, and urban). Teachers that regularly use technology tools and who teach at least one section of algebra I and one of algebra II, pre-calculus or calculus were invited to participate. Since the purpose is to gain insight from all members of the classroom community about how technology use is being promoted by the teacher and actually used by the students, data will be collected from both the teacher and the students before, during, and after designated units of study. Data will include video recording of two instructional units for each class, teacher interviews, student surveys, student task-based journal artifacts, post assessment questionnaires and focus students' interviews. The quantitative data will be analyzed using descriptive statistics. Qualitative data will be analyzed using rubrics developed based on an adaptation of Pierce and Stacey's (2004) framework for aspects of effective technology use. Ultimately this study will develop an empirically refined set of conjectures about teacher-related factors that impact students' independent technology use when doing mathematics.

The study will be in its early stages at the time of PME-NA 2011 and we will use the poster session to report preliminary results and to engage in critical conversations about the project design with fellow researchers.

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THE COLLABORATIVE NATURE OF GROUP DISCUSSIONS AND THEIR ROLE IN MATHEMATICAL REASONING

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The purpose of this study was to investigate the nature of group collaboration as student's reasoned about a pattern task. The fifth grade participants attended a science, mathematics and technology magnet school. The participants were asked to complete a series of three isomorphic pattern finding tasks using pattern block squares, triangles, and hexagons to generalize a rule. During the teaching experiment students were encouraged to collaborate with their partners to find patterns and write either recursive or explicit rules. Once the participants had generalized their rules a whole class discussion was held so that participants could share their ideas and justify their reasoning to the entire group.

Mueller, Maher and Yankelewitz (2009) developed a framework for analyzing the ways that collaboration influences the construction of mathematical arguments and its promotion of mathematical understanding. We used this framework to answer the research questions: What types of collaboration emerged during the teaching experiment and which types influenced the reasoning of the group? This framework, which is composed of three modes of student collaboration, includes: *co-construction, integration and modification.* With the *co-construction* mode of collaboration, students build their argument in a negotiary or back and forth type of dialogue. *Integration* occurs when another student strengthens another student's ideas. Finally, *modification* occurs when one student finds another student's argument faulty or unclear and attempts to help that student make sense of the argument.

This study focused on one pair of students, Jennifer and Felix, as they reasoned about the tasks. It is important to note that the students typically did not work in collaborative pairs. During day one of the teaching experiment, the pair did not collaborate at all. Each worked on the task separately and only Jennifer was able to find both a recursive and explicit rule for the task. On day two, the pair began to collaborate with each other; however, their collaborative effort switched back and forth between modification and integration in an argumentative manner. Felix found a rule for the triangle tables, but Jennifer did not agree. Her participation consisted of trying to understand his argument and pointing out his mistakes. During day two the pairs' collaborative effort did not progress to an explicit rule. On day three, as the pair worked on a rule for hexagon tables, their collaborative effort took the form of co-construction, with each student contributing to the effort. The final results of day three were that the pair found both a recursive and explicit rule for hexagon tables.

Over the three day experiment the collaborative efforts of these students progressed from no-collaboration, collaboration based on defending one's strategy to productive collaboration on the final day. During day one only one student was able to find an explicit rule, during day two neither was able to find any rule and on day three both found an explicit rule.

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SUPPORTING ELEMENTARY MATHEMATICS TEACHERS IN FACILITATING CLASSROOM COMMUNICATION THROUGH THE USE OF CURRICULUM MATERIALS

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Recent reforms in math education have outlined particular processes students should use to develop meaningful mathematical understandings (NCTM, 2000). Extant literature indicates that teachers' responses to student mathematical thinking shapes the ways in which students communicate about their thinking and can influence levels of mathematical understanding (Franke, Webb, Chan, Ing, Freund & Battey, 2009). The purpose of this study is to examine how curricula provide teachers with pedagogical support as they build and promote students' capacities in communicating mathematically. Although curriculum materials can be considered a "representation of the content for instruction" (Ball & Cohen, 1996, p. 7), a body of research conceptualizes it as a transformative educative tool for teachers (Collopy, 2003; Remillard, 2000). Specifically, curricula may be valuable artifacts that support teachers in introducing students to the practices and language of the mathematical community (Herbel-Eisenmann, 2007). Past studies have examined texts as an *objectively given structure*, in which "the structure and discourse of the written unit – not what happens when an individual (i.e., the teacher or student) interacts with it – is the focus" (Herbel-Eisenmann, 2007, p. 346). I analyzed the written language of the textbook's pedagogical guidance to see how it actively guides teachers to create opportunities for students to make their thinking visible and facilitate communication of mathematical ideas. The theoretical basis for the framework was developed from interpretations of mathematical communication drawn from existing empirical studies (Brendefur & Frykholm, 2000; Cai, Jakabcsin, & Lane, 1996; NCTM, 2000). Five lessons from three NSF funded elementary mathematics texts were analyzed to illuminate how teacher learning opportunities are instantiated. The results suggest the possibility of a divergence between the conceptualized ways in which the texts may intend for teachers to interpret the pedagogical guidance and the ways teachers may be "reading the textbook" (Remillard, 2000). Some pedagogical guidance provided could potentially be read in ways that lead to the implementation of discussions where students rely on the authority of the teacher to validate the "correctness" of an answer.

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LEARNING MATHEMATICS THROUGH PLAY: THE EMERGENCE OF MATHEMATICS DURING PLAYTIME IN PRESCHOOL

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This poster focuses on the quality of mathematics that emerged during playtime in a prekindergarten classroom. Studies have shown that play contributes to math development in the early ages (e.g., van Oers, 2010); however, there is still relatively little research on the "naturally occurring mathematics activities in which children engage" (Tudge & Doncet, 2004, p.34). To address this need, my study focused on two main questions: (1.) How does mathematics emerge during play? and (2.) Under what conditions does play promote deep thinking in mathematics?

Theoretical Frameworks and Methods

In this study, I drew on socio-cultural theories of learning that see children's mathematical learning as shaped by interactions among children, with teachers, and with materials in the classroom (Vygotsky, 1978). The study was situated in the preschool of Taylor County Public School, a small rural school with a largely African-American student body. I collected fieldnotes from August until mid-November, observing two focal children: Carter and Jamal. I observed the children six times during weekly visits to the school. In total, I collected around 12 hours of data, including written observations, audio records and video clips. Following traditional methods of ethnographic analysis (Erickson, 1986), I coded my data for formal and informal learning settings, mathematical topics, activity type, and materials used.

Results

In this classroom, I found that students engaged with more significant mathematical concepts during their play when they were allowed to work independently than when their play was guided by teachers. For example, when Carter built a tall Dulpo structure of two towers that stood side-by-side, he had an opportunity to experiment with symmetry, composing and decomposing shapes, equivalence and comparison of size. He searched for only 4-by-2 blocks or 8-by-2 blocks to build his tower, rejecting all others, and worked to make sure that his two towers stayed the same size. In contrast to episodes like this one, interactions that involved teachers often were limited to basic mathematics, like counting and shape identification.

Discussion

I am not arguing that all teachers focus on low-level mathematics during play interactions; however, at a time when we are pushing toward more formal mathematics curricula for PK children, it is good to remember that there is still educational value in play. Giving children the ability to explore on their own without adults monitoring, guiding, and interrupting them is important. Playing without adult interruption can give children a comfortable environment where they do not have to worry about right or wrong answers, or think about the expected answer that adults want to hear. At the same time, teachers need professional development on how to push children toward more significant mathematics in their interactions with students during play.

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BEYOND THE "PUZZLED PENGUIN": ENGAGEMENT WITH ERRORS IN ELEMENTARY MATHEMATICS TEXTBOOKS

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Analysis of mathematical errors can serve as learning opportunities for elementary school students. Whether they function as a "springboard for inquiry" (Borasi, 1996) or a turning point in improving a solution method (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver, Human, 1997), by discussing and analyzing errors, students can learn how to productively work with errors when they encounter them in their work. This study draws on error analysis models in mathematics education research (Lannin et al., 2006, Oser (1999) as cited by Heinze, 2005, Schleppenbach, et al., 2007), to develop a new, three-part framework for analyzing text and tasks that deal with errors in elementary mathematics textbooks. The framework illustrates characteristics of (1) error *analysis* tasks, and also expands our understanding to include text and tasks that can support students in (2) *preventing* and (3) *recognizing* errors in their own work.

In this study, I applied the framework to analyze a fractions chapter in two fourth-grade *Standards*-based textbooks, *Math Expressions* and *Math Trailblazers*. An interesting contrast resulted from applying this framework. Through suggestions of tools or strategies to use in *Math Trailblazers*, and example problems for students to view (prior to completing a set of similar problems) in *Math Expressions*, both units offered suggestions and hints for students that could result in *preventing errors* in their own work. There were also multiple problems in both units where students were asked to explain their thinking, however, only *Math Expressions* required students to show work for every story problem given, which could potentially result in students *recognizing their own errors* on a more regular basis. In regards to the nature of error analysis problems, the *Math Expressions* chapter was nearly evenly split between conceptual and procedural, while the *Math Trailblazers* chapter focused exclusively on concepts, with one exception bringing in non-mathematical content. The content of error analysis tasks in both units was similar, primarily focusing on comparing and ordering fractions, as well as analyzing different sized wholes. Overall, results from the application of this framework illustrate the variety of opportunities available to students to engage with errors through curriculum materials.

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CONNECTING PLACE VALUE AND FRACTIONS

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Place value and fractions are two of the major concepts in the elementary mathematics curriculum. While there is no doubt about the importance of teachers' understandings of these concepts, there is little focus in mathematics education or mathematics teacher education on the mathematical structures that underlie these concepts. As a result, when these concepts are taught in each grade level, it creates the illusion that there are many independent ideas to learn. However, when teachers and students become aware of the invariants across a variety of ideas and connections that exist among these topics, they develop a deeper understanding of mathematics that allows them to see the sameness and the ways in which various mathematical ideas can build upon each other (Ma, 1999; Silverman and Clay, 2009).

In this poster presentation I will highlight the existing research on students' understandings of fractions and place value and describe a framework that includes three key ideas that can be used to highlight the similarities between conceptual understandings of fractions and place value. For example, research has documented that understanding place value involves two interrelated concepts: (1) understanding the rate of change between units and (2) concurrently recognizing a number in terms of different units (Ma, 1999; McClain, 2003; Thanheiser, 2009). Similarly research has identified conceptual structures that underlie an understanding of fractions, including multiplicative relationships and also being able to compose a fraction in terms of different units (Lamon, 2007; Thompson and Saldanha, 2003). Building on these ideas, I will discuss a framework for understanding the similarities between fractions and place value that highlights unitizing, composition, and decomposition. I will also present my ideas for an online course for preservice teachers that seeks to help them develop mathematical understandings that includes underlying similarities between fractions and place value and to understand the mathematical and instructional affordances of developing an understanding of the connections between place value and fractions.

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TEACHING MATHEMATICS AS AGAPE: ARTICULATING A RELATIONSHIP BETWEEN STUDENTS AND MATHEMATICS

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In this poster, I illustrate a desired relationship between all students and mathematics within a classroom community of practice that challenges the status quo. The premise for how to teach mathematics begins with a guiding principle of agape, or universal love, and organizes the literature according to that principle of "intentionally promoting success in others".

The realm of mathematics education has a few positions that most would not dispute: 1) mathematical practices are necessary for access to academic and economic opportunities and 2) not all students are given an equal chance at learning those practices (Diversity in Mathematics Education, 2007). Given the importance of providing students with access to mathematical practices, and the influential role mathematics educators can have in promoting or denying access (Berry, 2008), this poster illustrates a pedagogical ideal for how to teach mathematics. The guiding principle of agape, or unconditional love, is used to organize equitable approaches to teaching (e.g. culturally relevant pedagogy (Ladson-Billings, 1994), inclusive pedagogy (Udvari-Solner, Villa, & Thousand, 2005)) and teaching mathematics (e.g. teaching mathematics for social justice (Gutstein, 2006)) within a classroom community of practice (Lave & Wenger, 1991), to construct *teaching mathematics as agape*. *Teaching mathematics as agape* is partitioned into four facets that are defined and necessitated through the literature, with the end product being a relationship between students and mathematics that promotes the access to mathematical practices that is necessary for participation in society (Moses & Cobb, 2001).

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A LONGITUDINAL STUDY OF THE VIEWS AND COGNITION OF AFRICAN-AMERICAN STUDENTS FROM URBAN COMMUNITIES

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Inequitable access to mathematics instruction and, consequently, low mathematical achievement among students from urban communities have been identified as major contributors to societal inequalities (Moses & Cobb, 2001) and economic deprivation (Gutstein & Peterson, 2005). Nationally, students from poor communities struggle through mathematics courses and become less engaged in the subject as grade level increases. On the one hand, it has been suggested that school experiences typically fail to provide early adolescents with opportunities to learn the subject matter in meaningful ways, thus de-emphasizing the subject's importance and perpetuating adverse attitudes toward enrolling in elective mathematics courses beyond those required for graduation. On the other hand, academic expectations set for children in such settings are often too low, resulting in gaps in achievement. Many teachers' attitudes toward children from poor communities remain negative, as they believe them to be incapable of academic success (Gutierrez, Baquedano-López, & Tejeda, 1999). Indeed, scholars have revealed that even when children in urban schools do succeed, their success is often attributed to factors other than ability (Tiezzi & Cross, 1997). In short, formal schooling not only fails to enhance children's education; often it both implicitly and explicitly hinders their motivation, desire to learn, and their sense of efficacy towards mathematics (Nasir & Cobb, 2007). Despite this, empirical data which documents how school experiences might impact children's mathematical thinking, their views about the nature of mathematics, and sense of efficacy are rare. Indeed, it is not clear what aspects of children's mathematical thinking might be influenced by their school experiences over time. Our research project addresses this gap.

Our longitudinal research project traced the growth of mathematical thinking and practices of a cohort of 20 children from urban communities as they progressed from 7th to 9th grade. Each child was interviewed once a year for three years using a focused clinical interviewing format. Each interview consisted of two parts. The first part of the interview elicited children's views about mathematics and their level of confidence towards mathematics. The second part of the interview aimed to capture their internalized mathematical practices as exhibited on tasks chosen from various mathematical areas. Featuring members of this cohort, our poster offers an analysis of areas in which children have shown greatest growth and decline over time.

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TEACHING FRACTION EQUIVALENCY THROUGH RATIO INTERPRETATIONS: PERFORMANCE OF STUDENTS WITH MLD

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Students with mathematics learning disabilities (MLD) have been shown to have a weak understanding of fraction concepts that underlie success in fraction computation, problem solving, and estimation (Mazzacco & Devlin, 2008). One such concept is fraction equivalency. A multiple baseline across participants design was employed to evaluate the effects of a fraction intervention based in ratio interpretations on performance on a test of fraction equivalence. A functional relationship was demonstrated between the instruction and performance through replication of skill increase across three students. Results as well as possible implications and future research are discussed.

Learning fraction equivalency through ratio interpretations removes the act of partitioning used in traditional part-whole approaches that have been demonstrated as problematic for many students with MLD (Grobecker, 2000; Hecht, et al., 2006; Lewis, 2010). This study evaluated the effects of a fraction intervention rooted in ratio interpretations on performance in fraction equivalency for middle school students with MLD. A multiple baseline across participants design was used. The experimental design demonstrated a functional relationship between the intervention and understanding of fraction equivalence by repeatedly increasing performance of correct responses after the introduction of treatment (Horner et al., 2005; Kazdin, 1982). Two additional tests, a conventional independent t-test and a split-middle technique, were employed to further evaluate the results of the intervention.

Visual analysis suggests relatively flat trends in data points collected during baseline as opposed to positive, steeper slopes in intervention for all three students. The combined baseline mean was 4.47 (SD = 3.17; range = 1-11); during instruction the mean increased to 18.38 (SD = 4.95; range = 7-26); at the end of instruction, the mean number of points earned was 25.00 (SD = 1.00; range = 24-26). Treatment effects were also calculated using percentage of non-overlapping data (PND) (Kazdin, 1982), a non-parametric means of finding effect size. A moderate PND of 70% was revealed for the intervention effects.

Using elements of Lamon’s (1993) ideas on ratio-backed unitization and expansion of concepts of equivalency, the instructional sequence seemed to have built an increased understanding of fraction equivalency through the ratio interpretation. The students’ initial inability to generate equivalent fractions from their part whole based instruction from school was evident before instruction; after instruction, students increased their performance in fraction equivalency.

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A LONGITUDINAL STUDY OF THE RELATIONSHIP BETWEEN MATHEMATICAL AND LINGUISTIC GROWTH OF ELL STUDENTS

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The fastest growing student population in the United States is children of immigrant families. In 2000, 5 million school-age children, totaling more than 10 percent of all K–12 students, were English language learners (ELL) (National Clearinghouse for English Language Acquisition, 2006). While 63% of all U.S. schools have students enrolled who are considered limited-English-proficient (National Center for Education Statistics, 2004), that statistic grows to 77% for urban communities. In 2007-2008, according to NCES, total populations of ELL students were below 10% for suburban, 9% for town, and 6% for rural, while urban was nearly 20%. Indeed, the U.S. Latino population is expected to grow to 100 million by 2050, which will result in ELL students in most city, suburb, and rural schools and classrooms (Moschkovich, 2007). As the ELL school population continues to grow, and with the importance of mathematical achievement and literacy at an all-time high, the relationship between mathematics and English language learning is becoming important. Yet little is known about how these children transition in their mathematical development when immersed in prominently English language environments.

Our poster features the profiles of two ELL students, one male and one female, with differing limited-English-proficiency classifications, from our longitudinal research cohort of 20 children from urban communities. Our data, captured through three years of annual clinical interviews, traces the growth of each child's mathematical thinking using Pirie and Kieren's (1994) theoretical model. The interviews elicited the children's view about mathematics and their level of confidence towards mathematics, as well as captured their internalized mathematical practices as exhibited on tasks chosen from various mathematical areas. Gee's (1996) critical discourse theory was used to analyze the interview data. The poster features an analysis of areas in which children showed greatest growth and decline over time. Data indicated that with greater linguistic diversity reliance on multiple representations diminished; students with limited linguistic ability rely more frequently on pictorial representations to communication their understanding. Over time, the ELL students have increased participation, are more resistant to agreeing to different views of mathematical thought. These students also displayed more standard and conventional ways of doing mathematics as their English language fluency increased.

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CHALLENGES IN TEACHING IN APPLIED MATHEMATICS CLASSES IN URBAN SCHOOLS

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Secondary school mathematics teachers, mathematics department heads, curriculum leaders, and administrators from 11 schools, with support from a local university, participated in the project Collaborative Teacher Inquiry. In this presentation, we will focus on challenges that teachers face when they teach grade 9 applied mathematics.

INTRODUCTION

The transition from Grade 8 to Grade 9 is particularly challenging in many school systems (Galton, 2009). In Ontario, students are promoted to the next grade regardless of their level of performance until Grade 8. In addition, Grade 9 students in Ontario are subject to streaming, being categorized into academic, applied, or essential levels (Ontario Ministry of Education, 2007). As a result, Grade 9 is a critical year for students and teachers. This is especially challenging for urban populations such as the Greater Toronto Area (GTA), where there is a large multicultural population with the highest percentage of foreign-born students in Canada (Statistics Canada, 2005). In fact, 44% of the student population of the GTA has a first language other than English. In this paper, we report the findings to the research question: What challenges do Grade 9 Applied Mathematics teachers face? By identifying the difficulties encountered by mathematics teachers, this research promises to move forward teaching mathematics in applied classrooms, by energizing teams of teachers within schools to activate and guide the teacher improvement process.

FINDINGS AND DISCUSSIONS

We will discuss the various challenges faced by teachers in teaching Grade 9 Applied Mathematics. Due to time limitation, we selected only few categories. There are a number of challenges in urban mathematics education in Greater Toronto schools. There were a number of situations when challenges identified were multifold. Sometimes teachers did not believe in their capability to achieve success under these circumstances. Teachers said that these negative attitudes along with the students' lack of mathematical background were major barriers to foster student success. Teachers mentioned that they can not be successful alone if their students and their parents do not believe themselves in succeeding.

A mix of behavioral issues and lack of attitudes toward learning was another area of concern. Family situations of students differ considerably and these contexts also brought many issues for several schools. Teachers emphasized the complexity of students being in the same time with behavioral issues and having individual education plans. Lack of time for teachers to prepare their lessons, students need to learn the basics first, concerns about student behavior; and the concerns of preparing students for provincial tests were common themes in both studies. At times, teachers' professional development programs had their own limitations. Implementing reforms that the new curriculum required was challenging by many teachers. In addition, some new teachers had problems in class management. By contemplating on teachers' views in our study, some teachers faced considerable pressure when teaching for applied.

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A CASE STUDY OF A CHINESE-AMERICAN STUDENT'S MATHEMATICAL ABILITIES

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In recent decades, many cross-national studies showed that students in East Asian countries outperformed their counterparts in the United States in mathematics. However, there are also claims that East Asian students and U.S. students have different strengths in their mathematics learning. In this sense, Chinese-American students, who are raised in Chinese families and educated in U.S. schools, become a special group for mathematics education researchers to investigate the possibility of combining the strengths of both U.S. and Chinese ways of mathematics education. This study investigated a Chinese-American student's strengths and weaknesses in his mathematical abilities. The participant of this study was a fifth grade Chinese-American student. The student was interviewed one and a half hours every week for four weeks and was asked to give both written responses to the problems as well as oral explanations of his solutions. All the interviews were videotaped. The tasks used in this study included computational tasks, component problems, open-ended problems, and problem posing tasks. These tasks were adopted from Krutetskii's (1976) study on the structure of mathematical abilities and Cai's (1995) comparative studies on sixth grade Chinese students and American students. The results of the study indicate that, on the one hand, like the Chinese students in Cai's research, this student has proficient computational skills and simple-problem solving skills, and tends to use symbolic or notational representation more frequently than visual or pictorial representation in solving problems. On the other hand, like the American students in Cai's research, he shows a good ability in solving complex problems. These findings suggest that this student has obtained both Chinese students and U.S. students' strengths in mathematical abilities. The results of this study also indicate that the student' mathematical ability type can be categorized as "abstract-harmonic type" (Krutetskii, 1976, p. 327), which means that the student can depict mathematical relationships equally well by verbal-logical and visual-pictorial means but shows an inclination for mental operations without the use of visual-pictorial means. The findings of this study also suggest that the middle zone (Gu, 2003) between the U.S. and the Chinese way of teaching and learning of mathematics may exist.

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BEYOND THE "PUZZLED PENGUIN": ENGAGEMENT WITH ERRORS IN ELEMENTARY MATHEMATICS TEXTBOOKS

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Analysis of mathematical errors can serve as learning opportunities for elementary school students. Whether they function as a "springboard for inquiry" (Borasi, 1996) or a turning point in improving a solution method (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver, Human, 1997), by discussing and analyzing errors, students can learn how to productively work with errors when they encounter them in their work. This study draws on error analysis models in mathematics education research (Lannin et al., 2006, Oser (1999) as cited by Heinze, 2005, Schleppenbach, et al., 2007), to develop a new, three-part framework for analyzing text and tasks that deal with errors in elementary mathematics textbooks. The framework illustrates characteristics of (1) error *analysis* tasks, and also expands our understanding to include text and tasks that can support students in (2) *preventing* and (3) *recognizing* errors in their own work.

In this study, I applied the framework to analyze a fractions chapter in two fourth-grade *Standards*-based textbooks, *Math Expressions* and *Math Trailblazers*. An interesting contrast resulted from applying this framework. Through suggestions of tools or strategies to use in *Math Trailblazers*, and example problems for students to view (prior to completing a set of similar problems) in *Math Expressions*, both units offered suggestions and hints for students that could result in *preventing errors* in their own work. There were also multiple problems in both units where students were asked to explain their thinking, however, only *Math Expressions* required students to show work for every story problem given, which could potentially result in students *recognizing their own errors* on a more regular basis. In regards to the nature of error analysis problems, the *Math Expressions* chapter was nearly evenly split between conceptual and procedural, while the *Math Trailblazers* chapter focused exclusively on concepts, with one exception bringing in non-mathematical content. The content of error analysis tasks in both units was similar, primarily focusing on comparing and ordering fractions, as well as analyzing different sized wholes. Overall, results from the application of this framework illustrate the variety of opportunities available to students to engage with errors through curriculum materials.

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MATHEMATICS TEACHER EDUCATORS' DIFFICULTY IN ASSESSING PRE-SERVICE TEACHERS' MATHEMATICAL KNOWLEDGE

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This poster reports on the difficulties that three instructors of mathematics content courses for prospective elementary teachers had in designing assessments to test their students' understandings of fraction multiplication and division. It discusses the nature of these difficulties and some possible reasons why they occurred.

Researching and designing assessments to determine mathematical knowledge for teaching (MKT) has been an important part of educational research since the 1960s and before. Being able to determine what teachers know is important for multiple reasons, especially providing "highly qualified teachers" to our nations' students. (Hill, Sleep, Lewis, & Ball, 2007). However, designing assessments to accurately assess teacher knowledge beyond procedures is very difficult (Hill, Schilling, and Ball, 2004). Skemp, (1976) discusses "the difficulty of sound examining in mathematics" (p. 24) because knowing whether a person understands mathematics relationally or instrumentally by looking at what he or she writes on paper is very challenging.

Despite the difficulties of designing appropriate assessments, an integral part of the job of mathematics teacher educators is being able to assess what teachers know and also how these teachers are able to access that information. This poster reports on the difficulties that three instructors of mathematics content courses for prospective elementary teachers had in designing assessments to test their students' understandings of fraction multiplication and division.

This study, which is part of a larger study looking at mathematical knowledge for teacher educators looked at three mathematics teacher educators (MTEs) who were teaching mathematics content courses for prospective elementary teachers. Despite all three of the MTEs expressing that their goals were for their students to develop conceptual understanding of fraction multiplication and division, only one of the teacher educators designed assessments that helped her determine if her students were actually developing these understandings. The other two had questions that were either strictly procedural, or seemed conceptual in nature, but produced procedural responses from their students. The poster will discuss possible reasons for the difficulty the MTEs had in designing effective assessments as well as implications of these difficulties for teacher education.

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MATHEMATICS TEACHING, 20-26.INSTITUTIONALIZATION OF KNOWLEDGE IN THE MATHEMATICS CLASSROOM: A STUDY ON CLASSROOM DISCOURSE

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Taking as a basis a collection of video recordings from a mathematics classroom at the secondary level, this article presents an analysis of several forms of discourse used in the process of institutionalization of knowledge. In particular, the functions and effects of the teacher's interventions in formulating generalizations, synthesis and recapitulations of certain activities in the classroom are analyzed. This study allowed us to identify regularities in the verbal forms, which affect the development of the lesson, either by maintaining the continuity of the discourse, or supporting the institutionalization process.

What the teachers and students express during the collective construction of mathematical ideas? To answer this question we use the concept of institutionalization. This concept is associated with the definition and articulation of the mathematical ideas studied in the classroom. We analyze the institutionalization process through the identification and characterization of the verbal structures used during classroom discussions. We believe that the words used, their meaning and emphasis, have an impact on how the mathematical ideas are defined and structured; but also on how arguments are validated and justified. The methodology used in this study is based on the assumption that the discursive devices can be inferred from observable indicators in the classroom. Furthermore, such indicators have an explicit role in teacher's discourse. These indicators become variables in the institutionalization process, because as explained before, institutionalization is associated with the conclusion or synthesis of ideas in the classroom. We consider necessary to place this study in a school setting, so we could observe the everyday teacher's and students' discourse during the process of institutionalization of knowledge. We claim that this institutionalization process is associated with very specific discursive resources having their own semantics. Thus, our aim in this paper is to analyze the video recordings of the classroom discourse in order to identify empirical evidence indicating the existence of a process of institutionalization. This not only involves identifying phrases or words, but also the effects that the teacher's discourse has in the construction of knowledge.

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LISTENING MATTERS: UNDERSTANDING THE OTHER SIDE OF LANGUAGE IN MATHEMATICAL LEARNING

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Over the past decade and a half, reform efforts have called for mathematics classrooms to become discussion and inquiry based (National Council of Teachers of Mathematics [NCTM], 2000). As this reform has taken hold teachers and students have been called upon to envision and enact new and different participation patterns while engaging in the study of mathematics. The primary voice in the classroom is no longer the teacher telling students how to solve problems, but rather students sharing their ideas about their understandings and (mis)conceptions of mathematics. Thus, discussions in mathematics classes have become an important part of learning mathematics. It is ambitious work for teachers and students. Ambitious not only in terms of what teachers and students say to one another in mathematical discussions, but ambitious in how teachers and students are called to listen to one another as well.

While much work has been done to learn about and understand what teachers and students need to know and do to engage in this ambitious work (e.g. Lampert, 1992) very little effort has been directed toward understanding the other side of language (Fiumara, 1990), listening. How we listen, and what we listen for informs the ways we hear words and ideas, process and understand (or not) what has been said, and respond to others, and yet, only a handful of researchers and educators have fully considered listening and it's potential to reveal and expand what it means to teach and learn mathematics.

In this poster, we present findings from two different qualitative studies that focused on listening in elementary mathematics classrooms. Study A took place in a first and fourth grade classroom and focused specifically on identifying the mathematical and interactional demands students experience as listeners during mathematical discussion. Study B occurred in a fourth and fifth grade classroom in a school with high numbers of students who are becoming bilingual and who are from households who live in poverty. The purpose of this study was to learn about the roles of listening in learning, by studying how teachers and children listened to one another. Both studies drew from aspects of the same theoretical framework, socio-cultural theory, and both researchers collected and analyzed video data of mathematics lessons, interview data with children, and artifacts of students' work.

Our findings reveal important implications for learning and teaching, the details of which will be highlighted on our poster. First, listening, which includes not only what students hear, but also whether or not they feel heard, is an important form of participation. In addition to broadening their own listening practices, teachers can deepen students' learning by supporting students to grow their repertoires for listening. Second, different kinds of listening produce different types of mathematical learning. What teachers and students listen for has important implications for participation. Third, when students feel heard, it creates a socially supportive environment. Reciprocally, students are more willing to risk sharing their thinking in a safe environment.

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INTERSECTIONS OF MATHEMATICAL IDENTITIES

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Queer students face special challenges in their schooling, this schooling includes the mathematics classroom. This group of students has not been studied in regards to mathematics. This poster examines preliminary data pertaining to the lived experience exploring the intersection of a queer identity and mathematical identity. Because of the inclusive nature of Queer Theory there is also discussion of the impact of racial/ethnic identities and gender identity within this study.

Students with a queer identity, non-heteronomative, face special challenges in in the classroom. Harassment and discrimination lead to high rates of skipping school, and this in turn is problematic for students' educational attainment (Kosciw Et.Al, 2008). A positive attitude towards mathematics, a positive mathematical identity, results in greater engagement in mathematics (Nasir, 2002).

This Phenomenological study explored the intersection of the participant's mathematical identity with their queer identity. This poster will present preliminary findings and analysis from multiple participants and reports on the essence of the intersection. It also reports on the variance of experiences that improve resilience as well as those that cause barriers to improving mathematics performance. By using the lens of Queer Theory this study seeks to include the complicating factors of racial/ethnic identity and gender identities (Kumashiro, 2002).

The poster will begin the conversation of how queer students both perceive and respond to mathematics and the mathematics classroom. While it is not possible to separate all of the various factors involved quantitatively, this qualitative examination of the data seeks to make sense of the queer students experience while examining risk factors and factors that lead to resilience in the mathematics classroom. This data may assist teachers and teacher educators in working toward an anti-oppressive pedagogy in the mathematics classroom.

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MATHEMATICALLY PRODUCTIVE AND SOCIALLY SUPPORTIVE DISCOURSE: A TEACHER AND HER STUDENTS' PERSPECTIVES

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Recent efforts within practice-based teacher education have spurred the development of innovative pedagogies that support teachers to learn the practice of teaching as they enact practice (Grossman & McDonald, 2008). One such effort employs a set of *Instructional Activities (IA's)* that provide a purposefully-bounded space for teachers to manage complex interactions among students and content (Lampert et al., 2010). Teacher educators are using IA's that travel across contexts as teachers learn how to create interactive environments for mathematics learning that include: eliciting student thinking, pushing for mathematical reasoning, and orienting students to each others' ideas. As teachers facilitate the IA, they enact pedagogical moves that support a range of student participation, manage discursive interactions, and foreground important mathematical ideas that arise from student contributions. All of this occurs within the accessible, bounded space of the activity.

The use of IA's, then, is a way to make the demands of facilitating and participating in mathematical discourse accessible while maintaining mathematical rigor. Just as teachers navigate a host of demands as they orchestrate mathematical discourse, students simultaneously encounter a wide range of mathematical and interactional demands as well. Mathematically, there is subject matter knowledge a student must make use of, and work with, in order to engage in the IA. Interactionally, a child must be able to engage socially because s/he is a student in a classroom with other children engaging in the IA.

This poster highlights two qualitative studies that focused on a particular IA, *choral counting*. *Choral counting* is a 10-15 minute warm-up activity where students engage in chorally counting by a number as the teacher records the count, discussing both strategies and patterns within the count. Across the two research studies, we examined what the IA of *choral counting* affords for mathematics teaching and learning from two distinct angles within the same elementary mathematics classroom. Study A examined the teacher's role within *choral counting*, tracking pedagogical moves and considering the potential for the enactment of "ambitious instruction" (e.g. Lampert, 2001). Study B examined the affordances of the IA from the students' point of view, documenting the mathematical and interactional demands that students must manage while engaged in chorally counting with classmates. Uniquely drawing from the dual perspectives of a classroom teacher and her elementary students, our research makes connections to current research on teaching and learning while offering implications for practicing teachers and teacher educators who strive to create discourse that is both mathematically productive and socially supportive for students.

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TECHNOLOGY TOOLS IN HIGH SCHOOL MATHEMATICS: PROMOTION, PERCEPTIONS AND USE

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This poster is a preliminary report for a research project that aims to build an understanding of the complexity of teachers' technology tool promotion in classrooms as it relates to students' adoption of such tools in their independent mathematical work. The project focuses not only on the mathematical aspects of technology tool promotion and use, but the affective aspects as well. We examine how these aspects impact the *how* and *why* of students' choices to use technology tools when working independently. Given the expectation that today's students must use technology tools to engage in mathematical problem solving (CCSSI, 2010; NCTM, 2000) both in and outside of the classroom (i.e. on homework, teacher created and standardized assessments, the application of mathematical concepts to other contexts) this is an important sequence of events to understand.

This investigation is a mixed method multi-case study of six high school mathematics classes. We are investigating six classes, two classes each from three high school teachers who work in three types of schools (rural, suburban, and urban). Teachers that regularly use technology tools and who teach at least one section of algebra I and one of algebra II, pre-calculus or calculus were invited to participate. Since the purpose is to gain insight from all members of the classroom community about how technology use is being promoted by the teacher and actually used by the students, data will be collected from both the teacher and the students before, during, and after designated units of study. Data will include video recording of two instructional units for each class, teacher interviews, student surveys, student task-based journal artifacts, post assessment questionnaires and focus students' interviews. The quantitative data will be analyzed using descriptive statistics. Qualitative data will be analyzed using rubrics developed based on an adaptation of Pierce and Stacey's (2004) framework for aspects of effective technology use. Ultimately this study will develop an empirically refined set of conjectures about teacher-related factors that impact students' independent technology use when doing mathematics.

The study will be in its early stages at the time of PME-NA 2011 and we will use the poster session to report preliminary results and to engage in critical conversations about the project design with fellow researchers.

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THE COLLABORATIVE NATURE OF GROUP DISCUSSIONS AND THEIR ROLE IN MATHEMATICAL REASONING

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The purpose of this study was to investigate the nature of group collaboration as student's reasoned about a pattern task. The fifth grade participants attended a science, mathematics and technology magnet school. The participants were asked to complete a series of three isomorphic pattern finding tasks using pattern block squares, triangles, and hexagons to generalize a rule. During the teaching experiment students were encouraged to collaborate with their partners to find patterns and write either recursive or explicit rules. Once the participants had generalized their rules a whole class discussion was held so that participants could share their ideas and justify their reasoning to the entire group.

Mueller, Maher and Yankelewitz (2009) developed a framework for analyzing the ways that collaboration influences the construction of mathematical arguments and its promotion of mathematical understanding. We used this framework to answer the research questions: What types of collaboration emerged during the teaching experiment and which types influenced the reasoning of the group? This framework, which is composed of three modes of student collaboration, includes: *co-construction, integration and modification.* With the *co-construction* mode of collaboration, students build their argument in a negotiary or back and forth type of dialogue. *Integration* occurs when another student strengthens another student's ideas. Finally, *modification* occurs when one student finds another student's argument faulty or unclear and attempts to help that student make sense of the argument.

This study focused on one pair of students, Jennifer and Felix, as they reasoned about the tasks. It is important to note that the students typically did not work in collaborative pairs. During day one of the teaching experiment, the pair did not collaborate at all. Each worked on the task separately and only Jennifer was able to find both a recursive and explicit rule for the task. On day two, the pair began to collaborate with each other; however, their collaborative effort switched back and forth between modification and integration in an argumentative manner. Felix found a rule for the triangle tables, but Jennifer did not agree. Her participation consisted of trying to understand his argument and pointing out his mistakes. During day two the pairs' collaborative effort did not progress to an explicit rule. On day three, as the pair worked on a rule for hexagon tables, their collaborative effort took the form of co-construction, with each student contributing to the effort. The final results of day three were that the pair found both a recursive and explicit rule for hexagon tables.

Over the three day experiment the collaborative efforts of these students progressed from no-collaboration, collaboration based on defending one's strategy to productive collaboration on the final day. During day one only one student was able to find an explicit rule, during day two neither was able to find any rule and on day three both found an explicit rule.

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SUPPORTING ELEMENTARY MATHEMATICS TEACHERS IN FACILITATING CLASSROOM COMMUNICATION THROUGH THE USE OF CURRICULUM MATERIALS

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Recent reforms in math education have outlined particular processes students should use to develop meaningful mathematical understandings (NCTM, 2000). Extant literature indicates that teachers' responses to student mathematical thinking shapes the ways in which students communicate about their thinking and can influence levels of mathematical understanding (Franke, Webb, Chan, Ing, Freund & Battey, 2009). The purpose of this study is to examine how curricula provide teachers with pedagogical support as they build and promote students' capacities in communicating mathematically. Although curriculum materials can be considered a "representation of the content for instruction" (Ball & Cohen, 1996, p. 7), a body of research conceptualizes it as a transformative educative tool for teachers (Collopy, 2003; Remillard, 2000). Specifically, curricula may be valuable artifacts that support teachers in introducing students to the practices and language of the mathematical community (Herbel-Eisenmann, 2007). Past studies have examined texts as an *objectively given structure*, in which "the structure and discourse of the written unit – not what happens when an individual (i.e., the teacher or student) interacts with it – is the focus" (Herbel-Eisenmann, 2007, p. 346). I analyzed the written language of the textbook's pedagogical guidance to see how it actively guides teachers to create opportunities for students to make their thinking visible and facilitate communication of mathematical ideas. The theoretical basis for the framework was developed from interpretations of mathematical communication drawn from existing empirical studies (Brendefur & Frykholm, 2000; Cai, Jakabcsin, & Lane, 1996; NCTM, 2000). Five lessons from three NSF funded elementary mathematics texts were analyzed to illuminate how teacher learning opportunities are instantiated. The results suggest the possibility of a divergence between the conceptualized ways in which the texts may intend for teachers to interpret the pedagogical guidance and the ways teachers may be "reading the textbook" (Remillard, 2000). Some pedagogical guidance provided could potentially be read in ways that lead to the implementation of discussions where students rely on the authority of the teacher to validate the "correctness" of an answer.

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LEARNING MATHEMATICS THROUGH PLAY: THE EMERGENCE OF MATHEMATICS DURING PLAYTIME IN PRESCHOOL

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This poster focuses on the quality of mathematics that emerged during playtime in a prekindergarten classroom. Studies have shown that play contributes to math development in the early ages (e.g., van Oers, 2010); however, there is still relatively little research on the "naturally occurring mathematics activities in which children engage" (Tudge & Doncet, 2004, p.34). To address this need, my study focused on two main questions: (1.) How does mathematics emerge during play? and (2.) Under what conditions does play promote deep thinking in mathematics?

Theoretical Frameworks and Methods

In this study, I drew on socio-cultural theories of learning that see children's mathematical learning as shaped by interactions among children, with teachers, and with materials in the classroom (Vygotsky, 1978). The study was situated in the preschool of Taylor County Public School, a small rural school with a largely African-American student body. I collected fieldnotes from August until mid-November, observing two focal children: Carter and Jamal. I observed the children six times during weekly visits to the school. In total, I collected around 12 hours of data, including written observations, audio records and video clips. Following traditional methods of ethnographic analysis (Erickson, 1986), I coded my data for formal and informal learning settings, mathematical topics, activity type, and materials used.

Results

In this classroom, I found that students engaged with more significant mathematical concepts during their play when they were allowed to work independently than when their play was guided by teachers. For example, when Carter built a tall Dulpo structure of two towers that stood side-by-side, he had an opportunity to experiment with symmetry, composing and decomposing shapes, equivalence and comparison of size. He searched for only 4-by-2 blocks or 8-by-2 blocks to build his tower, rejecting all others, and worked to make sure that his two towers stayed the same size. In contrast to episodes like this one, interactions that involved teachers often were limited to basic mathematics, like counting and shape identification.

Discussion

I am not arguing that all teachers focus on low-level mathematics during play interactions; however, at a time when we are pushing toward more formal mathematics curricula for PK children, it is good to remember that there is still educational value in play. Giving children the ability to explore on their own without adults monitoring, guiding, and interrupting them is important. Playing without adult interruption can give children a comfortable environment where they do not have to worry about right or wrong answers, or think about the expected answer that adults want to hear. At the same time, teachers need professional development on how to push children toward more significant mathematics in their interactions with students during play.

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BEYOND THE "PUZZLED PENGUIN": ENGAGEMENT WITH ERRORS IN ELEMENTARY MATHEMATICS TEXTBOOKS

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Analysis of mathematical errors can serve as learning opportunities for elementary school students. Whether they function as a "springboard for inquiry" (Borasi, 1996) or a turning point in improving a solution method (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver, Human, 1997), by discussing and analyzing errors, students can learn how to productively work with errors when they encounter them in their work. This study draws on error analysis models in mathematics education research (Lannin et al., 2006, Oser (1999) as cited by Heinze, 2005, Schleppenbach, et al., 2007), to develop a new, three-part framework for analyzing text and tasks that deal with errors in elementary mathematics textbooks. The framework illustrates characteristics of (1) error *analysis* tasks, and also expands our understanding to include text and tasks that can support students in (2) *preventing* and (3) *recognizing* errors in their own work.

In this study, I applied the framework to analyze a fractions chapter in two fourth-grade *Standards*-based textbooks, *Math Expressions* and *Math Trailblazers*. An interesting contrast resulted from applying this framework. Through suggestions of tools or strategies to use in *Math Trailblazers*, and example problems for students to view (prior to completing a set of similar problems) in *Math Expressions*, both units offered suggestions and hints for students that could result in *preventing errors* in their own work. There were also multiple problems in both units where students were asked to explain their thinking, however, only *Math Expressions* required students to show work for every story problem given, which could potentially result in students *recognizing their own errors* on a more regular basis. In regards to the nature of error analysis problems, the *Math Expressions* chapter was nearly evenly split between conceptual and procedural, while the *Math Trailblazers* chapter focused exclusively on concepts, with one exception bringing in non-mathematical content. The content of error analysis tasks in both units was similar, primarily focusing on comparing and ordering fractions, as well as analyzing different sized wholes. Overall, results from the application of this framework illustrate the variety of opportunities available to students to engage with errors through curriculum materials.

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CONNECTING PLACE VALUE AND FRACTIONS

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Place value and fractions are two of the major concepts in the elementary mathematics curriculum. While there is no doubt about the importance of teachers' understandings of these concepts, there is little focus in mathematics education or mathematics teacher education on the mathematical structures that underlie these concepts. As a result, when these concepts are taught in each grade level, it creates the illusion that there are many independent ideas to learn. However, when teachers and students become aware of the invariants across a variety of ideas and connections that exist among these topics, they develop a deeper understanding of mathematics that allows them to see the sameness and the ways in which various mathematical ideas can build upon each other (Ma, 1999; Silverman and Clay, 2009).

In this poster presentation I will highlight the existing research on students' understandings of fractions and place value and describe a framework that includes three key ideas that can be used to highlight the similarities between conceptual understandings of fractions and place value. For example, research has documented that understanding place value involves two interrelated concepts: (1) understanding the rate of change between units and (2) concurrently recognizing a number in terms of different units (Ma, 1999; McClain, 2003; Thanheiser, 2009). Similarly research has identified conceptual structures that underlie an understanding of fractions, including multiplicative relationships and also being able to compose a fraction in terms of different units (Lamon, 2007; Thompson and Saldanha, 2003). Building on these ideas, I will discuss a framework for understanding the similarities between fractions and place value that highlights unitizing, composition, and decomposition. I will also present my ideas for an online course for preservice teachers that seeks to help them develop mathematical understandings that includes underlying similarities between fractions and place value and to understand the mathematical and instructional affordances of developing an understanding of the connections between place value and fractions.

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A CASE STUDY OF A CHINESE-AMERICAN STUDENT'S MATHEMATICAL ABILITIES

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In recent decades, many cross-national studies showed that students in East Asian countries outperformed their counterparts in the United States in mathematics. However, there are also claims that East Asian students and U.S. students have different strengths in their mathematics learning. In this sense, Chinese-American students, who are raised in Chinese families and educated in U.S. schools, become a special group for mathematics education researchers to investigate the possibility of combining the strengths of both U.S. and Chinese ways of mathematics education. This study investigated a Chinese-American student's strengths and weaknesses in his mathematical abilities. The participant of this study was a fifth grade Chinese-American student. The student was interviewed one and a half hours every week for four weeks and was asked to give both written responses to the problems as well as oral explanations of his solutions. All the interviews were videotaped. The tasks used in this study included computational tasks, component problems, open-ended problems, and problem posing tasks. These tasks were adopted from Krutetskii's (1976) study on the structure of mathematical abilities and Cai's (1995) comparative studies on sixth grade Chinese students and American students. The results of the study indicate that, on the one hand, like the Chinese students in Cai's research, this student has proficient computational skills and simple-problem solving skills, and tends to use symbolic or notational representation more frequently than visual or pictorial representation in solving problems. On the other hand, like the American students in Cai's research, he shows a good ability in solving complex problems. These findings suggest that this student has obtained both Chinese students and U.S. students' strengths in mathematical abilities. The results of this study also indicate that the student' mathematical ability type can be categorized as "abstract-harmonic type" (Krutetskii, 1976, p. 327), which means that the student can depict mathematical relationships equally well by verbal-logical and visual-pictorial means but shows an inclination for mental operations without the use of visual-pictorial means. The findings of this study also suggest that the middle zone (Gu, 2003) between the U.S. and the Chinese way of teaching and learning of mathematics may exist.

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TEACHING MATHEMATICS AS AGAPE: ARTICULATING A RELATIONSHIP BETWEEN STUDENTS AND MATHEMATICS

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In this poster, I illustrate a desired relationship between all students and mathematics within a classroom community of practice that challenges the status quo. The premise for how to teach mathematics begins with a guiding principle of agape, or universal love, and organizes the literature according to that principle of "intentionally promoting success in others".

The realm of mathematics education has a few positions that most would not dispute: 1) mathematical practices are necessary for access to academic and economic opportunities and 2) not all students are given an equal chance at learning those practices (Diversity in Mathematics Education, 2007). Given the importance of providing students with access to mathematical practices, and the influential role mathematics educators can have in promoting or denying access (Berry, 2008), this poster illustrates a pedagogical ideal for how to teach mathematics. The guiding principle of agape, or unconditional love, is used to organize equitable approaches to teaching (e.g. culturally relevant pedagogy (Ladson-Billings, 1994), inclusive pedagogy (Udvari-Solner, Villa, & Thousand, 2005)) and teaching mathematics (e.g. teaching mathematics for social justice (Gutstein, 2006)) within a classroom community of practice (Lave & Wenger, 1991), to construct *teaching mathematics as agape*. *Teaching mathematics as agape* is partitioned into four facets that are defined and necessitated through the literature, with the end product being a relationship between students and mathematics that promotes the access to mathematical practices that is necessary for participation in society (Moses & Cobb, 2001).

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A LONGITUDINAL STUDY OF THE VIEWS AND COGNITION OF AFRICAN-AMERICAN STUDENTS FROM URBAN COMMUNITIES

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Inequitable access to mathematics instruction and, consequently, low mathematical achievement among students from urban communities have been identified as major contributors to societal inequalities (Moses & Cobb, 2001) and economic deprivation (Gutstein & Peterson, 2005). Nationally, students from poor communities struggle through mathematics courses and become less engaged in the subject as grade level increases. On the one hand, it has been suggested that school experiences typically fail to provide early adolescents with opportunities to learn the subject matter in meaningful ways, thus de-emphasizing the subject's importance and perpetuating adverse attitudes toward enrolling in elective mathematics courses beyond those required for graduation. On the other hand, academic expectations set for children in such settings are often too low, resulting in gaps in achievement. Many teachers' attitudes toward children from poor communities remain negative, as they believe them to be incapable of academic success (Gutierrez, Baquedano-López, & Tejeda, 1999). Indeed, scholars have revealed that even when children in urban schools do succeed, their success is often attributed to factors other than ability (Tiezzi & Cross, 1997). In short, formal schooling not only fails to enhance children's education; often it both implicitly and explicitly hinders their motivation, desire to learn, and their sense of efficacy towards mathematics (Nasir & Cobb, 2007). Despite this, empirical data which documents how school experiences might impact children's mathematical thinking, their views about the nature of mathematics, and sense of efficacy are rare. Indeed, it is not clear what aspects of children's mathematical thinking might be influenced by their school experiences over time. Our research project addresses this gap.

Our longitudinal research project traced the growth of mathematical thinking and practices of a cohort of 20 children from urban communities as they progressed from 7th to 9th grade. Each child was interviewed once a year for three years using a focused clinical interviewing format. Each interview consisted of two parts. The first part of the interview elicited children's views about mathematics and their level of confidence towards mathematics. The second part of the interview aimed to capture their internalized mathematical practices as exhibited on tasks chosen from various mathematical areas. Featuring members of this cohort, our poster offers an analysis of areas in which children have shown greatest growth and decline over time.

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TEACHING FRACTION EQUIVALENCY THROUGH RATIO INTERPRETATIONS: PERFORMANCE OF STUDENTS WITH MLD

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Students with mathematics learning disabilities (MLD) have been shown to have a weak understanding of fraction concepts that underlie success in fraction computation, problem solving, and estimation (Mazzacco & Devlin, 2008). One such concept is fraction equivalency. A multiple baseline across participants design was employed to evaluate the effects of a fraction intervention based in ratio interpretations on performance on a test of fraction equivalence. A functional relationship was demonstrated between the instruction and performance through replication of skill increase across three students. Results as well as possible implications and future research are discussed.

Learning fraction equivalency through ratio interpretations removes the act of partitioning used in traditional part-whole approaches that have been demonstrated as problematic for many students with MLD (Grobecker, 2000; Hecht, et al., 2006; Lewis, 2010). This study evaluated the effects of a fraction intervention rooted in ratio interpretations on performance in fraction equivalency for middle school students with MLD. A multiple baseline across participants design was used. The experimental design demonstrated a functional relationship between the intervention and understanding of fraction equivalence by repeatedly increasing performance of correct responses after the introduction of treatment (Horner et al., 2005; Kazdin, 1982). Two additional tests, a conventional independent t-test and a split-middle technique, were employed to further evaluate the results of the intervention.

Visual analysis suggests relatively flat trends in data points collected during baseline as opposed to positive, steeper slopes in intervention for all three students. The combined baseline mean was 4.47 (SD = 3.17; range = 1-11); during instruction the mean increased to 18.38 (SD = 4.95; range = 7-26); at the end of instruction, the mean number of points earned was 25.00 (SD = 1.00; range = 24-26). Treatment effects were also calculated using percentage of non-overlapping data (PND) (Kazdin, 1982), a non-parametric means of finding effect size. A moderate PND of 70% was revealed for the intervention effects.

Using elements of Lamon's (1993) ideas on ratio-backed unitization and expansion of concepts of equivalency, the instructional sequence seemed to have built an increased understanding of fraction equivalency through the ratio interpretation. The students' initial inability to generate equivalent fractions from their part whole based instruction from school was evident before instruction; after instruction, students increased their performance in fraction equivalency.

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A LONGITUDINAL STUDY OF THE RELATIONSHIP BETWEEN MATHEMATICAL AND LINGUISTIC GROWTH OF ELL STUDENTS

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The fastest growing student population in the United States is children of immigrant families. In 2000, 5 million school-age children, totaling more than 10 percent of all K–12 students, were English language learners (ELL) (National Clearinghouse for English Language Acquisition, 2006). While 63% of all U.S. schools have students enrolled who are considered limited-English-proficient (National Center for Education Statistics, 2004), that statistic grows to 77% for urban communities. In 2007-2008, according to NCES, total populations of ELL students were below 10% for suburban, 9% for town, and 6% for rural, while urban was nearly 20%. Indeed, the U.S. Latino population is expected to grow to 100 million by 2050, which will result in ELL students in most city, suburb, and rural schools and classrooms (Moschkovich, 2007). As the ELL school population continues to grow, and with the importance of mathematical achievement and literacy at an all-time high, the relationship between mathematics and English language learning is becoming important. Yet little is known about how these children transition in their mathematical development when immersed in prominently English language environments.

Our poster features the profiles of two ELL students, one male and one female, with differing limited-English-proficiency classifications, from our longitudinal research cohort of 20 children from urban communities. Our data, captured through three years of annual clinical interviews, traces the growth of each child's mathematical thinking using Pirie and Kieren's (1994) theoretical model. The interviews elicited the children's view about mathematics and their level of confidence towards mathematics, as well as captured their internalized mathematical practices as exhibited on tasks chosen from various mathematical areas. Gee's (1996) critical discourse theory was used to analyze the interview data. The poster features an analysis of areas in which children showed greatest growth and decline over time. Data indicated that with greater linguistic diversity reliance on multiple representations diminished; students with limited linguistic ability rely more frequently on pictorial representations to communication their understanding. Over time, the ELL students have increased participation, are more resistant to agreeing to different views of mathematical thought. These students also displayed more standard and conventional ways of doing mathematics as their English language fluency increased.

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Challenges in Teaching in Applied Mathematics Classes in Urban Schools

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Secondary school mathematics teachers, mathematics department heads, curriculum leaders, and administrators from 11 schools, with support from a local university, participated in the project Collaborative Teacher Inquiry. In this presentation, we will focus on challenges that teachers face when they teach grade 9 applied mathematics.

INTRODUCTION

The transition from Grade 8 to Grade 9 is particularly challenging in many school systems (Galton, 2009). In Ontario, students are promoted to the next grade regardless of their level of performance until Grade 8. In addition, Grade 9 students in Ontario are subject to streaming, being categorized into academic, applied, or essential levels (Ontario Ministry of Education, 2007). As a result, Grade 9 is a critical year for students and teachers. This is especially challenging for urban populations such as the Greater Toronto Area (GTA), where there is a large multicultural population with the highest percentage of foreign-born students in Canada (Statistics Canada, 2005). In fact, 44% of the student population of the GTA has a first language other than English. In this paper, we report the findings to the research question: What challenges do Grade 9 Applied Mathematics teachers face? By identifying the difficulties encountered by mathematics teachers, this research promises to move forward teaching mathematics in applied classrooms, by energizing teams of teachers within schools to activate and guide the teacher improvement process.

FINDINGS AND DISCUSSIONS

We will discuss the various challenges faced by teachers in teaching Grade 9 Applied Mathematics. Due to time limitation, we selected only few categories. There are a number of challenges in urban mathematics education in Greater Toronto schools. There were a number of situations when challenges identified were multifold. Sometimes teachers did not believe in their capability to achieve success under these circumstances. Teachers said that these negative attitudes along with the students' lack of mathematical background were major barriers to foster student success. Teachers mentioned that they can not be successful alone if their students and their parents do not believe themselves in succeeding.

A mix of behavioral issues and lack of attitudes toward learning was another area of concern. Family situations of students differ considerably and these contexts also brought many issues for several schools. Teachers emphasized the complexity of students being in the same time with behavioral issues and having individual education plans. Lack of time for teachers to prepare their lessons, students need to learn the basics first, concerns about student behavior; and the concerns of preparing students for provincial tests were common themes in both studies. At times, teachers' professional development programs had their own limitations. Implementing reforms that the new curriculum required was challenging by many teachers. In addition, some new teachers had problems in class management. By contemplating on teachers' views in our study, some teachers faced considerable pressure when teaching for applied.

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PATTERNS OF MATHEMATICAL PROBLEM SOLVING AMONG FEMALES: A LONGITUDINAL STUDY

Over two decades ago, Leder (1990) reported that fewer women enroll in advanced courses in mathematics, suggesting that this trend leads to unequal participation of females in STEM related careers (cited in Geist & King, 2008). More recent reports suggest this trend has been reversed, as currently there is no enrollment bias in advanced high school math courses. According to Science Daily (2008) women earn 48 percent of mathematics bachelor's degrees in the US. Despite gender equality in number of courses taken, there is sufficient evidence that while boys and girls achieve at the same level in elementary school mathematics (4th and 6th grades), gaps in achievement widen as high school approaches (Amelink, 2009). This performance gap is most apparent on open-ended problem solving tasks. Several scholars have attributed this gender gap to affective factors such as a loss of confidence in mathematics ability and a lower valuation of the usefulness of mathematics (Zhu, 2007). These scholars offer that with less confidence, girls tend to feel less autonomous when experiencing mathematics, particularly during problem solving when they must make decisions about how to proceed and what strategies to use. Other studies posit that differences in performance might be due to the use of differing strategies and problem solving patterns (Zhu, 2007). Zhu (2007), in synthesizing a substantial body of research, advocated the need to study patterns of performance and problem solving strategies of females on authentic tasks to create a platform for discourse on gender differences. The goal to meet this need motivated the current study.

Our longitudinal research traced attitudes and growth of mathematical thinking and practices of 5 female adolescents from urban communities as they progressed from 7th to 9th grade. Each child was interviewed once a year (for three years) using a focused clinical interviewing format. Each interview consisted of two parts. The first part elicited views about mathematics, level of confidence towards mathematics, and usefulness of a concept being learned about in school. The second part aimed to capture their performance on problem solving tasks from different areas of mathematics including discrete mathematics, geometry, and probability. The contexts chosen for data collection elicited the use of differing heuristics and modes of reasoning. While subjects' attitudes towards mathematics remained positive, their performance on mathematical tasks that resembled contexts seen in school became more mechanical. In their inclination to rely on procedures learned in school, they became reluctant to aggressively tackle problems that appeared more open ended.

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ELEMENTARY STUDENTS' VIEWS OF MATHEMATICS AND MATHEMATICIANS: BOYS' AND GIRLS' PERSPECTIVES

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In North America, many people fear or dislike mathematics and fail to see its relevance to their lives (Boaler, 2008). Mathematicians are often negatively stereotyped as white, middle-aged males who are socially inept and unattractive (Mendick, Moreau, & Hollingworth, 2008). These views are perpetuated by popular media, which portray mathematics as a difficult subject only understood by a select few, who are depicted negatively (Applebaum, 1995).

This study investigates elementary students' views of and experiences with mathematics and mathematicians. It also examines how those views may be impacted by popular media, parents' views, and teachers' views. This ongoing, multi-faceted study, conducted in Ontario, Canada, involves Grade 4 and 8 students, teachers, and parents. To date, students have completed online questionnaires and drawn pictures of mathematicians, and parents and teachers have participated in interviews. Children's media will later be analyzed for representations of mathematics, and these examples will be used in focus group interviews with students.

This poster session reports on initial findings from an analysis of student questionnaire data, specifically 30 Likert-scale questions regarding views of and experiences with mathematics and mathematicians. These questions were divided into four topics – views of mathematics, views of mathematicians, parents and mathematics, and gender and mathematics – and the results were separated by the gender of the 156 participants (86 girls, 70 boys).

As with similar studies (e.g., Picker & Berry, 2001), boys and girls in my study had very comparable, generally positive views about mathematics, but expressed little interest in becoming mathematicians. My participants generally felt there were no gender differences in mathematics ability or enjoyment. These sentiments accord with contemporary views of mathematics as an increasingly gender-equitable field at the school level (Forgasz, Leder, & Kloosterman, 2004). Participants' parents were perceived as holding positive views of mathematics, which may have impacted the participants' views (Toliver, 1998).

The similarities in boys' and girls' views of mathematics, particularly in terms of confidence, enjoyment, and gender issues, are encouraging. However, students continue to adhere to some stereotypes regarding mathematics and mathematicians, and the lack of occupational interest in the field they enjoy is troubling. These and other issues will be discussed in the poster session.

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EXAMINING COGNITIVE DEMAND, FUNCTIONAL REPRESENTATIONS, AND TEACHER QUESTIONS IN CONNECTED MATHEMATICS PROJECT 2

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This study aims to analyze how two *Connected Mathematics Project 2(CMP2)* units, *Variables and Patterns* and *Moving Straight Ahead*, adhere to the reform goals of the 2000 NCTM *Standards* with respect to functional representations and high-level student thinking. In order to determine the extent to which CMP2 utilizes high-level tasks, the Investigations were analyzed using the Task Analysis Guide levels of cognitive demand (Boston and Smith, 2009). Ronda (2009) claims that teachers are not provided with the appropriate tools in a textbook to know how to facilitate student learning to a more advanced level. Therefore, questions that were specifically written in the teachers' materials as suggestions to pose to students were analyzed to see what support teachers were given for implementing high-level tasks. A framework was developed in which Boaler and Brodie's (2005) question types were ranked based on their importance in students' cognitive development of mathematical understandings. The two units claim to focus on getting students to understand and use different functional representations. Results from the analysis indicated that most tasks in the CMP2 chapters provided opportunities for students to use multiple functional representations and had higher-level demands, most frequently Procedures with Connections. Questions provided for teachers in the teachers' materials were most frequently higher-level questions that could lead to the development of deeper understandings. The most cognitively demanding tasks, Doing Mathematics tasks, usually had fewer functional representations and fewer suggested questions for teachers to pose than Procedures with Connections tasks, potentially due to the range of multiple solution strategies or their ambiguous nature. These findings suggest implications for teachers, professional developers, and curriculum writers.

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CONCEPT ANALYSIS AND THE PRESEVICE MIDDLE SCHOOL MATHEMATICS TEACHER

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Over the last several decades mathematics teachers and their knowledge about mathematics have been under close scrutiny. This new era with standardized testing has placed the spotlight on the mathematics teacher and demands teachers ensure their students master mathematical skills and develop a conceptual understanding of the mathematics. This requires that teachers gain skill and knowledge about the mathematics and skill and knowledge about teaching. The joining of mathematics knowledge and the related pedagogy that is required to teach mathematic, pedagogical content knowledge (PCK) that Shulman (1986) defines as the "understanding of what is to be learned and how it is to be taught" (p.7). We believe that a conceptual understanding of mathematics and the integration of the appropriate pedagogical knowledge as promoted by PCK can be achieved by examining mathematics teaching as an applied mathematics, as suggested by Usiskin (2001) for preservice middle school mathematics teachers.

Researchers have demonstrated the interplay between teachers' content knowledge and pedagogical content knowledge during teaching practice that influences the quality of their teaching (Ball, & Wilson, 1990; Usiskin, 2001; Hill, Rowan, & Ball, 2005). Walshaw and Anthony (2008) found that teachers, who engage their students in mathematical argumentation, critique, interpretation, generalization, and the defending of their own ideas, enhance the development of their student's mathematical thinking on key concepts. We believe the notion of mathematics teaching as an applied mathematics, like engineering, offers a new perspective for mathematics teacher educators when unpacking PCK for the middle school mathematics teacher. Considering mathematics teaching as an applied mathematics places the focus on three categories of mathematics: mathematical generalizations and extensions, problem analysis, and concept analysis (Usiskin, 2001). We believe concept analysis is the foundational component for PCK when applied to mathematics education. Concept analysis includes "alternative definitions or conceptions of mathematical ideas and their consequences, why the concepts arose and how they have changed over time and the range of applications of the concept" (Usiskin, 2001, p. 90). Usiskin (2001) draws on a four dimensional model of understanding that is found in the UCSMP texts, "Skills, Properties, Uses, and Representation" (p. 90), SPUR[1]. We applied concept analysis to geometric concepts to explore the potential benefit of addressing mathematics teaching as an applied mathematics.

Endnotes

1 "SPUR." Abbreviations.com. STANDS4 LLC, 2011. 2 March. 2011. http://www.abbreviations.com/b1.aspx?KEY=283547

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ADIDACTIC ACTIVITY FOR THE INVERSE TRIGONOMETRIC RATIO

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In this report we deal with the design of software that has been programmed in the Program of Mathematics Education (PROME) hold at the National Polytechnic Institute (named IPN for the Spanish Instituto Politécnico Nacional) of Mexico. In PROME we do research on the mathematics teaching-learning processes; PROME also offers a Master Degree and a PhD Degree in Mathematics Education, both are on line. The programmed software is one of the first results of a project co-sponsored by National Council of Science and Technology and the Government of the State of Veracruz. The project is titled “Design, development and generation of on line didactic materials for teaching mathematics in the school system of Veracruz” (Register number CONACYT108952). We used Java® language for the programming of the adidactic activity (AUTHORS, 2010).

As Java applets already are used in education our question is ¿which are the new aspects our programmed activities offer? Under constructivism approach we know that people learn with the interactions with objects (Chevallard, Bosch & Gascón, 1998), we agree with this idea and this is why the basement of the project is the interactions student-computer that applets provide. It is also necessary to determine many variables: what information should the applet show on computers’ screen, how and why information is presented, and how the applet collects information from the student. This artificial place where interactions happen is called “environment” (Chevallard, Bosch & Gascón, 1998).

Many applets available in the web and designed by other researchers give to students some means to interact with the computer. Our proposal’s novelty is the implementation of an adidactic situation in the environment; in other words, a school mathematics problem given by the teacher to the student to be solved (Brousseau, 1997). This solving procedure leads the student to build new mathematical knowledge. We call adidactic activity to the couple of a computer program and an adidactic situation.

This poster includes one adidactic situation to study inverse trigonometric ratio.

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WHAT DO PROSPECTIVE MIDDLE LEVEL MATHEMATICS TEACHERS LOOK FOR IN CLASS?

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Prospective teachers' (PTs) struggle to make connections between what they learn in their teacher education programs and "real" classroom practices has been a focus of much research (Worthy, 2005). Student teaching is generally viewed as the most valuable part of teacher preparation as it offers an avenue for prospective teachers to employ theory in real classroom settings (Feiman-Nemser, 1983). However, there is considerable research showing that student teachers tend to "survive" instead of taking the benefit of the experience to practice the theories (Korthagen & Kessels, 1999). With the intent of promoting the integration of theory with practice, we developed an assignment called "observation protocol". The purpose of this presentation is to share the PTs experiences as they developed their observation protocols and to also highlight their thoughts on the role of teachers and the structure of effective classroom tasks.

Participants in this study were 20 middle grade PTs in an undergraduate mathematics methods course. The course met once weekly for three hours, and took place during Fall 2010 at a large urban state university in the South-Eastern region of the United States. The course focused primarily on theoretical readings, field-based observations guided by those readings (as described below), and lesson plan development. All students enrolled in the course were also required to be in a field experience course, which required 2 full days per week attendance in a middle school mathematics classroom.

For this assignment, each PT was required to keep an observation and interview journal for the following two course topics: nature of classroom tasks and role of the teacher. PTs were instructed to list things they thought were critical to observe and come up with questions to ask their mentor teachers in the field. Students were also required to take observation notes for each item that they listed.

Data for this presentation consists of the 20 prospective teachers' written observation protocol assignments. To analyse the assignments we read and re-read all narratives separately in order to identify general themes. After we developed these themes individually, we worked together to re-examine each of the 20 papers more carefully to confirm themes and generate a list of quotes to support each idea.

The results of the study revealed that majority of the PTs see the major role of the teacher as facilitator who should promote scaffolding, reasoning, and student participation. However, they also see the pressure of the high stake testing, curriculum load and the classroom management issues as the three major barriers for them to carry out this role. In a similar vein, the PTs looked for classroom tasks that are "problematic", intriguing, engaging and developing mathematical insight. They argued that the classroom tasks should foster classroom discourse, reasoning and "true" understanding.

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STATISTICAL REASONING IN STANDARDS-BASED MIDDLE SCHOOL MATHEMATICS CURRICULA

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Being able to analyze data, graphs, charts, and tables is essential for students to be able to effectively engage as a citizen in a democratic society. After the NCTM *Standards* introduced statistics as a major content area, there has been a trend in recent research studies that stress the importance of shifting instruction from introducing procedural skills to developing students' abilities to reason statistically (Garfield & Ben-Zvi, 2007). The purpose of this study is to analyze two standards-based eighth grade curriculum units on data analysis from the *Connected Mathematics Project 2* (CMP2) (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006) and *Mathematics in Context* (MiC) (Encyclopaedia Britannica, 2006) to determine what opportunities are available for students to learn to exhibit statistical reasoning.

The analytic framework developed to analyze the questions presented in each unit integrated a number of different conceptualizations of statistical reasoning found in the mathematics education research literature. Mooney (2002) identified four data-handling processes that are essential for middle school students to exhibit statistical reasoning: describing data, organizing and reducing data, representing data, and analyzing and interpreting data. Curcio's (1987) graph comprehension levels served as a means to rank the data-handling processes into three levels, which were labeled as suggested by Friel et al. (2001): elementary, intermediate, and overall. The questions from each unit were coded according to sub-processes defined for each data-handling process.

Application of the analytic framework revealed that both units appear to provide students ample opportunities to engage in all of the data-handling processes that are required for students to learn to exhibit statistical reasoning, despite a majority focus on the elementary level processes. There were some notable differences between the two units, including the fact that MiC lacked questions that addressed measures of center and spread and that CMP2 had significantly fewer questions addressing the analyzing and interpreting data data-handling process. Despite these issues, both units provide ample opportunities for students to engage in questions which encourage students to learn to exhibit statistical reasoning.

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ENHANCING K-8 MATHEMATICS COACHES' KNOWLEDGE FOR TEACHING PROBABILITY

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This study is embedded in the Mathematics Coaching Program (Brosnan & Erchick, 2010), a statewide professional development program designed to enhance K-8 teachers' Mathematical Knowledge for Teaching (MKT) using children's cognition as a vehicle to situate the learning. In the current study, we examined the impact of sessions designed for improving participants' knowledge of probability. 19 mathematics coaches participated in the intervention sessions. Two types of activities were designed and used. First, samples of students' work with corresponding quantitative data on certain probability items of the state achievement tests were shared. The coaches were asked to: 1) make conjectures about students' sources of misconception, 2) judge the quality of given problems, 3) offer suggestions about improving either students' learning or the quality of the problems. Second, the coaches engaged in solving probability problems which were designed to help them realize the necessity of developing an understanding of the concept of "randomness" in probability thinking (Shaughnessy, 1992). Key concepts for discussions included: experimental probability, theoretical probability and probability assumption due to their centrality in the conceptual map of the probability knowledge as identified by NCTM.

To evaluate the effect of the intervention sessions, we administered an identical pre- and post-questionnaire to the coaches. The questionnaire asked coaches to identify: 1) the concept of probability, 2) the reason why probability should be taught to children, and 3) the appropriate probability knowledge that school children would need to learn. Participants' responses to each of the questions were studied first to determine categorical themes that emerged from them. These categories were used to trace changes in coaches' knowledge. Analysis of data indicated that on the pre-questionnaire, a majority of the coaches used general descriptions of probability and referenced reasoning, decision making, and real life application as chief rationale for teaching probability in schools. Among the post-questionnaire responses, we observed that over half of the coaches were more explicit in identifying probability concepts and referenced "importance of understanding and interpreting probability" using precise and correct language of probability; we also observed that statements that logically connected two or more probability topics (as opposed to listing concepts separately) significantly increased. To further investigate coaches' learning of these concepts, a follow-up plan using a more focused evaluation program is under development. Our poster presentation will offer a detailed account of categories of coaches' responses that reveal misconceptions about their own understanding of probability.

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USING DYNAMIC GEOMETRY SOFTWARE FOR PROBABILITY TEACHING AND LEARNING WITH AREA PROPORTIONAL VENN DIAGRAMS

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In this poster presentation, I will explore the use of dynamic geometry software as a visual dynamic tool for supporting high school students' understanding of probability, more specifically, understanding of conditional probability and Bayes' theorem. Kahneman, Slovic & Tversky (1982) claimed that intuitive errors proceed from using certain heuristic principles that often lead to erroneous probability judgments. For instance, people tend to neglect the base rates which are, according to Bayes' theorem, relevant to the calculation of probabilities. On the other hand, in the last couple of years there has been a focus on dynamic learning environments which allow us to create mathematical objects and explore them visually and dynamically. Moreno-Armella (1999) and Kaput and Hegedus (2008) describe learning environment in which students can visualize, construct and manipulate mathematical concepts. The dynamic learning environments can enable students to act mathematically, to seek relationship between object that would not be as intuitive with a static paper and pen representations.

The goal of my research is to substantiate the claim that dynamic learning environment enables student to grasp abstract mathematical concept by manipulating mathematical objects constructed within these systems. More specifically, I investigate the role of dynamic visualization using area proportional Venn diagrams in students' understanding of Bayes'theorem. Moreover, I am interested whether the introduction of visualization has an effect on student's committing the base rate and the inverse fallacy. Using a quasi-experimental design, I presented a well known breast cancer problem (Eddy,1982) in which Bayes' theorem needs to be applied to students who are introduced to Bayes' theorem via the Bayes' formula as well as to the group of students who were presented with the visualization of the Bayes' theorem created by the author.

The results show that the students who were introduced to the dynamic visualization of the Bayes' theorem were significantly more successful in solving the breast cancer problem. In this case, using the dynamic visualization provided by the animation feature of dynamic geometry software, students could easily grasp the idea that the conditional probability depends on the base rate and were also less likely to commit the inverse fallacy. This serves as the evidence that dynamic visualization do create means for the deeper analysis of mathematical concepts. In addition, my research contributes to the research on application of area-proportional Venn diagrams which have not received much attention in the mathematics education community.

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EXPLORING THE EFFECTS OF CURRICULA ON FIFTH-GRADE STUDENTS' PROBLEM SOLVING

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This study explores the impact of different curricula on students' problem solving. Effective problem solvers (1) read and understand the problem, (2) create a situation model, (3) develop a mathematical model, (4) implement a strategy and arrive at a result, (5) interpret the result, and (6) communicate the solution (Verschaffel, Greer, & De Corte, 2000). Students' using reform-oriented materials tend to use more representationally diverse strategies than peers using traditional materials (Senk & Thompson, 2006), which may impact how they solve problems.

Two fifth-grade students were selected from a larger sample based on curricula use and self-reported prior achievement. All names are pseudonyms. Kristy has been using *Everyday Mathematics* (Bell et al., 2004) materials since Kindergarten whereas Gavin has only used traditional mathematics textbooks. Word problems were selected from prior research with fifth-grade students (Bostic & Jacobbe, 2010; Verschaffel et al., 1999). Students practiced thinking aloud and then solved five tasks during an audio-recorded interview. After solving each problem, participants were asked whether they could show another strategy. The interviews were transcribed and analyzed using thematic analysis (Hatch, 2000).

Both participants employed symbolic strategies initially, used the problem's context to assist with interpreting their result, and were fairly successful. Gavin read each problem one time, carried out the problem-solving process, and reported his answer as a number. Kristy read the problem, reflected on it, and reread the problem, specifically attending to areas of text. Next, she offered a conjecture of the solution followed by creating a situation model. She continued to execute the problem-solving process and reported her solution as a number with corresponding units. Gavin was usually unable to provide another strategy; however, Kristy successfully used symbolic and pictorial strategies. Kristy also often located errors in her prior strategies. Kristy's utterances and behaviors suggest she self-evaluated her progress while problem solving whereas Gavin did not reread the problem, reexamine his situation model, or employ a pictorial strategy. Results offer directions for future research.

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THE STRANGE CASE OF THE FOURTH-AND-A-HALF FLOOR: THE WAY 3rd-6th-GRADERS COPE WITH THE SOLUTION OF A REAL-WORLD PROBLEM

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Does mathematics aim to develop abstract- logical thinking and find original and interesting relations in its various structures, or to serve as an auxiliary aid for solving problems in areas of science, society and daily life? One could say that the debate is senseless since mathematics is designed for both goals. However, teaching it in elementary school does not take the authentic daily component very seriously, in spite- of the curriculum statement of intents. The aim of this study was to investigate how 3rd to 6th grade students solve a real-world problem: What is the middle floor in a 9-floor building? The highest success percentage was at the 4th-grade and the lowest at the 6th-grade. Nevertheless, the most interesting finding of the study was the students' reference to the illogical and unrealistic answer: the fourth-and-a half floor, being the result of 9:2. About half of the 6th-graders and about one third of the 5th-graders gave such answer without any word of criticism about the obtained number. Only about 13%-15% of the 3rd-4th grade students gave such an answer. An inevitable conclusion is that the higher the elementary school age group is, the more the learnt mathematics detaches students from reality. The students do not use logical considerations, solving problems mechanically, without critical reflection. It is recommended dedicating a considerable part of mathematics lessons at elementary school to the solution of real-world day-by-day problems and to pay more attention to the processes of reflection about the received answers.

DEVELOPING STUDENTS' INFORMAL UNDERSTANDING OF THE COMMUTATIVE PROPERY OF ADDITION

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The aim of this poster is to share insights gathered on students' informal understanding of the commutative property of addition. Three lessons from the ON TRACK Learn Math pilot study were selected that provide substantial insight into the students' developing reasoning. By scaffolding progressively more challenging problem-solving activities, our enrichment program provided the students with the opportunity to make important connections.

The aim of this poster is to share insights gathered on students' informal understanding of the commutative property of addition during the pilot phase of the ON TRACK Learn Math study conducted during the fall semester of 2010. ON TRACK Learn Math is an after-school program for students in grades 3-5 that focuses on problem-solving enrichment activities. For the pilot, nine sessions were held over five weeks at a local elementary school.

Of the nine sessions, three lessons were selected that provide substantial insight into the students' developing understanding of the commutative property of addition. In the Towers Task (Maher & Martino, 1996), students used blocks of only two colors and determined the number of different towers that could be built of varying heights. The video recordings show that several students recognized that they could build both a red-blue-blue tower and a blue-blue-red tower, yet none of the students were able to notate this on their papers or generalize a rule. During the third week in a similar activity, students used Cuisenaire Rods to determine how many pairs of smaller rods would sum to a specific larger rod. As with the Towers Task, the video recordings show that several of the students recognized the distinction between a purple-yellow pair and a yellow-purple pair, for example, yet they still did not notate this in writing. In the following session, students attempted to list all of the different pairs of numbers that sum to a given value. It was noteworthy that many of the students not only wrote expressions such as 2+3 and 3+2 as distinct pairs, but also made references to the Towers and Rods Tasks. We considered this to be a significant development in the students' understanding. Though they were unable to clearly notate their thinking at the times of the first two tasks, they were now connecting all three of the activities. By scaffolding progressively more challenging problem-solving activities, our enrichment program provided the students with the opportunity to make these important connections.

Peer-reviewed articles on the development of students' understanding of commutativity are scarce and this portion of the ON TRACK study is an attempt to address that gap. As the study moves into full implementation in five local elementary schools, we plan to continue gathering more written data on the development of students' understanding of commutativity.

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A PILOT STUDY ON THE IMPACT OF A MATHEMATICAL MODELING APPROACH ON TEACHERS AND THEIR STUDENTS

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With the support from an NSF grant, twenty mathematics teachers participated in a 2-week summer Mathematical Modeling Workshop. In the fall semester after the workshop, these teachers used what they learned at the workshop (mathematical modeling ideas/skills and related pedagogy) to teach students in their own classrooms. This study was conducted to investigate the impact of the mathematical modeling approach on the teachers who participated in the summer modeling workshop and on their students (including at-risk students).

The general hypothesis of this study was that the reformed teaching featured by mathematical modeling can effectively enhance the content and pedagogical knowledge of the teachers and the performance of their students in mathematics learning. Specifically, the study sought to answer the following research questions: 1) What is the impact of the summer modeling workshop on the participating teachers? 2) Do the students taught by the teachers who participated in the summer modeling workshop perform better than the students taught by the teachers who did not?

Experimental designs were the main research method used for the study. A pre-test and a post-test were administered for all participating students. The instrument for the pre- and post-tests of the students was the test created by the California Mathematics Diagnostic Testing Program (MDTP). The pre-test was administered during the beginning weeks of the school year for students in the Algebra II, Geometry, Algebra I and Pre-Calculus classes. The post-test was administered one week before Christmas break for the same classes of students. Analyses of variance (ANOVA) and t-tests were implemented to analyze the quantitative data.

Classrooms observations were conducted, and different surveys were administered respectively for the participating teachers in the experimental group and those in the control group. The classrooms observations and the surveys were designed to collect information about what was going on in the classrooms, the teachers' teaching styles, their students, any issues and concerns, possible obstacles, what support the teachers expected from the researchers, etc. The pilot study also used in-depth interviews of six teachers (three experimental teachers and three control teachers) to collect more qualitative data, and the case study method (Yin, 2003) was used to analyze the data.

The main findings of the study included: 1) The teachers who participated in the above-mentioned modeling workshop ("experimental" teachers) developed teaching styles that were significantly more Standards-based, reform-oriented, and student-centered than those who did not have such experience ("control" teachers); and 2) The "experimental" teachers were more effective in helping students understand concepts and develop problem-solving skills than the "control" teachers, at least in most of the cases. In summary, this study has, at least initially, provided evidence to support its general hypothesis about the role of the reformed teaching featured by mathematical modeling.

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PROMOTING TEACHERS' DIVERGENT THINKING WHILE PROBLEM SOLVING IN A HYBRID GEOMETRY COURSE

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The purpose of this ongoing study is to examine practicing middle and secondary teachers' different strategies and method of solutions towards given geometry problems in a hybrid (online & face-to-face) geometry course. The course is designed to encourage divergent thinking during problem solving while reflecting on and discussing solution processes. While the larger study includes 25 middle and secondary mathematics teachers, this poster is a case study that reflects the thinking of one female teacher middle school teacher. Data collected includes online and in-class discussions, solutions to problems and the teacher's reflections problems. Guilford and Merrifield's five features of *divergent thinking are used to frame this case study. Results indicate evolving levels of divergent thinking by the teacher during the semester while using technologies such as MS Excel and Geometer's Sketchpad.*

Method

Guilford and Merrifield (1960) define five kinds of thinking abilities that involve creativity and divergent thinking:

- Sensitivity to problems: seeing defects, needs, deficiencies, seeing the odds, the unusual; seeing what must be done.
- Flexibility: ability to shift from one approach to another, one line of thinking to another, to free oneself from a previous set.
- Fluency: ability to produce large number of ideas.
- Originality: ability to produce remote, unusual, or new ideas or solutions to "embroider" or elaborate.
- Redefinition: ability to define or perceive in a way different from the usual, established, or intended way, use, etc.

The teacher explored problems to seek multiple solutions while using technologies.

Results

An analysis of this teacher's solution procedures, reflections, and discussions shows an evolution of divergent thinking and seeking creative ways to solve geometry problems throughout the course. The use of technologies, more specially internet research, MS Excel, and Geometer's Sketchpad served as key elements in sparking her divergent thinking and mathematical creativity. Examples of her solutions and evolving reflections that demonstrated increases in divergent thinking and creativity will be on display.

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PROOF AND ARGUMENTATION IN THE SOLUTION OF TRIANGLE CONGRUENCE PROBLEMS: A STUDY WITH HIGH SCHOOL STUDENTS

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In this article we report the proofs and argumentations provided by 15 high school students (16-17 years of age) when solving triangle congruence problems, in a paper and pencil environment. The proofs provided by the students, from the nine problems which were assigned, coincide with those reported in the research literature related to this subject matter. Our results suggest that students, many a time, in their proofs and argumentations, lean over empirical evidences, institutions and even on personal experiences brought about by the geometric figures which were provided in the problems.

Introduction

The appropriateness of whether or not including the topic of proof in the mathematics syllabus at basic levels (secondary school and high school) has been discussed for about 40 years. It is true that not all the mathematical contents at these educational levels need formal proofs; however, when the study of Euclidean geometry topics begins, students are required to provide arguments of why certain propositions are true or false, or how to argue that a geometry problem is adequately solved. The majority of students at these educational levels are not interested in either comprehending, or learning how to prove Euclidean geometry theorems, since, for them, the geometric properties and some propositions are often evident, due to the fact that they are supported by figures.

The research literature related to proof in mathematics, and to the academic field is vast. Due to space restrictions, in this paper we only include some bibliographic references; whose emphasis is the proof and argumentation given by the students at the different academic levels. For instance, Van Dormolen (1977, quoted by Ibañes & Ortega, 2005, p.25) refers to three levels of proof, provided by students; such levels are: a) zero level, characterized thus because the student focuses only on concrete objects, b) level one, thus characterized because the student thinks of objects as representative of one kind, c) level two; in this level the student is capable of generalizing. Bell (1976) proposed three objectives for proof; they are: verification, illumination and systematization; de Villiers (1993, quoted by Ibañes & Ortega, 2005, p.29) propose two functions of proof: discovery and communication; whereas, Balacheff (1987) mentions two kinds of proof by students, supported by examples: the pragmatics, based upon the *ostention*; in this kind of proof, students resort to the action and concrete examples, and the intellectual ones; based upon the formulation of the mathematical properties brought into play and the relationships that exist between them. In turn, pragmatic proofs are subdivided into different kinds: the naïve empiricism, the crucial experience, the generic example and the mental experience. Siñeriz and Ferraris (2005), Van Ash (1993, quoted by Ibañes & Ortega, 2005, p.26) and Hanna (1990), among others, have contributed as well to the study and development of proof, in accordance with the kind of language and the level of formality used.

Paying attention to the problems outlined above, we pose the following objective for this research: To explore the function of the task in a paper and pencil environment in the proof. That is to say, how the task suggested by the teacher, motivates and makes evident the necessity to prove propositions. In this paper, we pretend to provide an answer to the question: What kinds of proofs emerge during the solutions given by students to geometric problems of triangle congruence?

Theoretical Perspective

This research was theoretically supported by diverse kinds of argumentation and proof provided by some of the authors mentioned aforesaid. For instance, the different kinds of pragmatic proofs which are referred to by Balacheff (1987, pp. 156-157):

a) *Naïve empiricism:* is based upon a few examples; generally, selected in a random way, in order to verify the statement to prove;

b) *Crucial experience*: based upon the careful selection of an example, convinced that if the conjecture is true for that case, then it will be for all the others;

c) *Generic example*: based upon the selection of an example that acts as representative of its kind; in such a way that even though the proof refers to that particular case, it could be generalized to all that kind;

d) *Mental experience*: based upon the actions previously carried out by the student who, afterwards, internalizes (generally they are observations of examples) and then separates the particular actions to turn them into deductive abstract arguments.

Montoro (2009, pp. 5-6) identifies the following categories:

a) *Formal deductive argumentation*: when the subject makes inferences based on the hypotheses, properties or definitions and presents a logic sequence of symbolic expressions;

b) *Colloquial deductive argumentation*: when the subject makes inferences based upon the hypotheses, properties or definitions, above all, in a colloquial way;

c) *Colloquial explanation*: when the subject clarifies (or it is clarified) the situation. Generally, the subject presents an aspect of deductive argument without being real inferences;

d) *Explanation by evidence*: when the subject tries to evidence the proposition by the use of affirmation of that some property or some definition is fulfilled without arguing; only making affirmations;

e) *Explanation by means of a draw*: when the subject shows, using a draw; his affirmations lack arguments.

Hanna (1990, pp. 6-12) identifies the following kinds of proof:

a) *Formal proof*: a finite sequence of affirmations where the first one is an axiom and each one of the following ones is an axiom as well, or else it derives from the previous affirmations, by means of the correct application of inference rules. The last affirmation is the one which should be proven;

b) *Proof that proves:* it is used to see that a fact is true, but it is not necessary to explain because it is. This kind of proof can be based upon mathematical induction or even only upon syntactic considerations;

c) *Proof that explains*: it provides a series of reasons which derive from the phenomenon itself. It should provide a justification based on the mathematical ideas involved, and on mathematical properties.

Beyond the different processes of argumentation and proof used by students in order to justify their results in geometry, there exist the problems related to the diverse types of interpretation of a geometric representation, in which reasoning and visualization are aspects of paramount importance, as they are discussed by Camargo, Pery and Samper (2005) and by Palais (1999).

Methodology

Participants

This study was carried out with 15 students (16-17 years old) in the second year of a high school located in Mexico City. The students did not have experience on how to carry out demonstrations in geometry. During three working sessions (the first one of one hour and the remaining two of two hours, each one was conducted by one of the authors of these paper), students solved problems of Euclidean geometry; which included tasks as: recognition of figures, geometric constructions, use of theorems and criteria of triangle congruence, as well as the validation of their results. Due to the fact of space restrictions, in this paper we will only document the work of seven students; whose demonstrations exemplify the diverse types of demonstrations reported in the research literature.

Selection and Modification of the Problems

From the book Plane and Solid Geometry (Wentworth & Smith, 1979), nine problems of triangle congruence were chosen; these problems were modified in their original writing. The a priori solution of the problems posed to students was carried out; such solutions are aimed at foreseeing the possible proofs and argumentations by students. Due to space restrictions, such a priori solutions of the problems are not included in this paper.

Data Gathering

Between 10 and 15 minutes were allocated for the solution of each problem (individually), and some minutes were devoted to group discussion; conducted by the researcher. We have the register of the nine problems of the activity, with their respective answers provided by each one of the students, using paper and pencil, as well as the video recordings from the individual work and group discussion.

Data Analysis and Discussion of Results

Next are shown four problems (the numbering of them does not correspond to the order in which they were proposed) and the argumentations provided by the students.

Problem 1

In $\square ABCD$ (Figure 1), P is the midpoint of $\overline{AB}$. Prove that $\overline{PC} = \overline{PD}$.

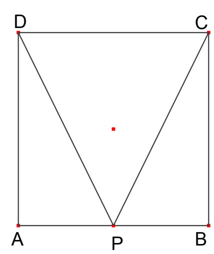


Figure 1. Problem Representation

Some solutions

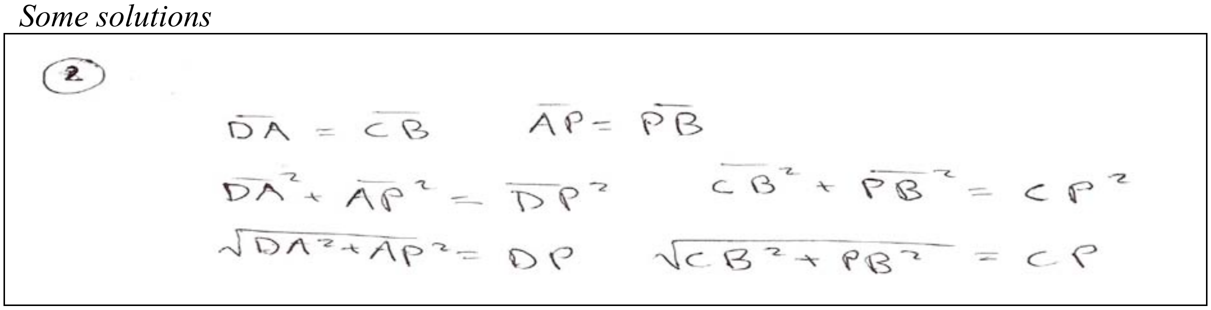


Figure 2. Student A2: Formal Deductive Argumentation; Use of the Pythagoras Theorem

Figure 2 shows the use of the problem's hypotheses: given that $\overline{DA} = \overline{CB}$ and $\overline{AP} = \overline{PB}$, then, using *The Pythagoras Theorem*, conclude that $\overline{CP} = \overline{DP}$.

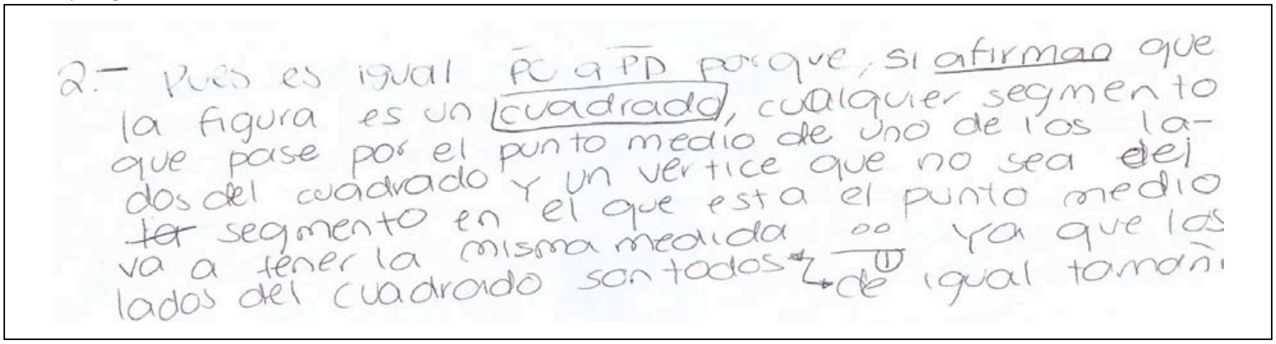


Figure 3. Student A4: Argumentation Based upon Evidences

As it is shown in Figure 3, the student asserts that the segments $\overline{PC}$ and $\overline{PD}$ measure the same, only by the fact that the figure, whose vertexes are *A*, *B*, *C* and *D*, is a square, but no argumentation is given.

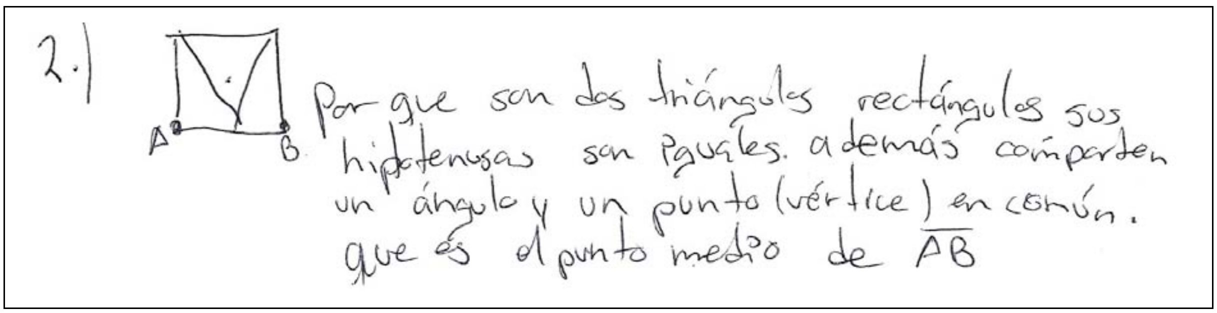


Figure 4. Student A5: Affirmations Based upon the Initial Figure

As shown in Figure 4, the student asserts that $\overline{PC} = \overline{PD}$, because he observes that *P* is the midpoint of one of the sides of the square and that right-angled triangles are formed, whose sides are the sides of the square. He only asserts without arguing.

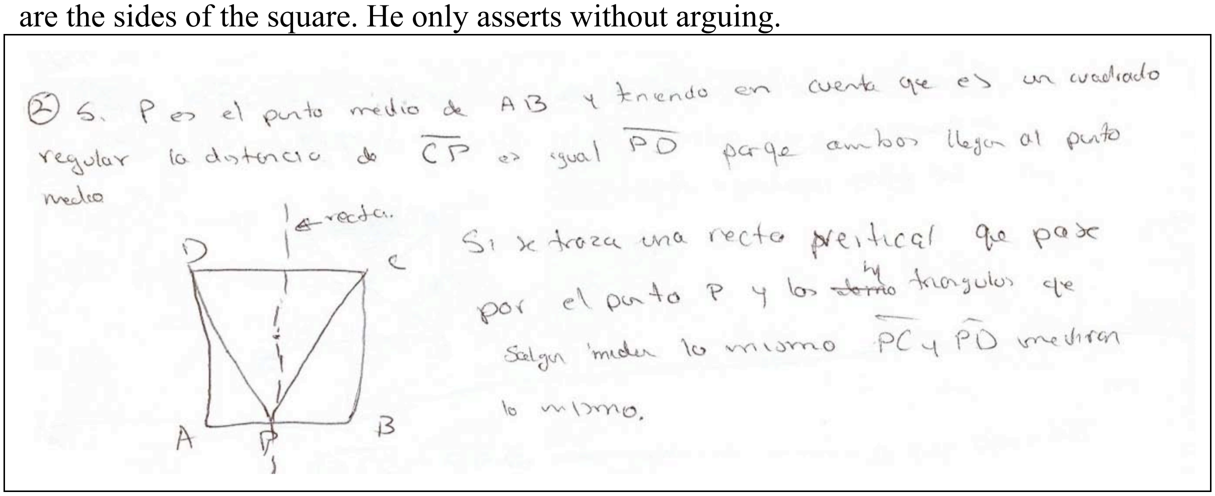


Figure 5. Student A11: Proof by Visualization

Figure 5 allows to observe that the students focuses on the elements or graphic representation, and builds the auxiliary straight line that passes *P*, in order to obtain four congruent triangles, which helps him conclude that $\overline{PC} = \overline{PD}$.

Problem 2

In Figure 6, $\overline{AC} = \overline{BC}$ and $\overline{AD} = \overline{BD}$. Prove that $\angle CBD = \angle DAC$.

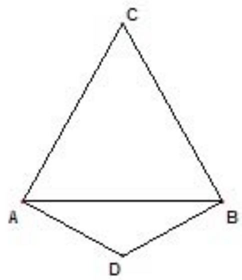


Figure 6. Representation of the Problem

Some solutions

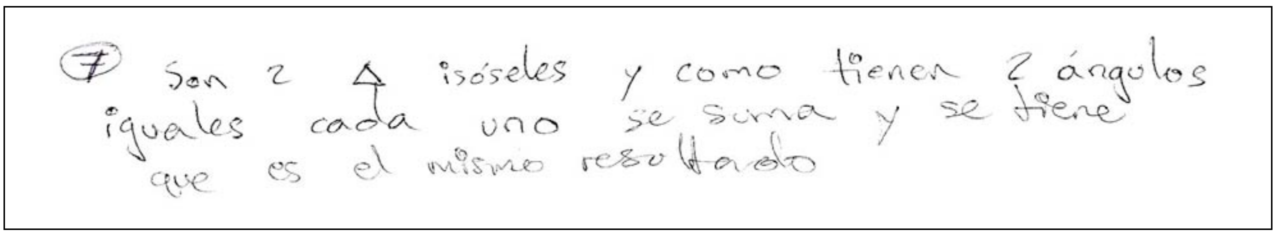


Figure 7. Student A2: Colloquial Deductive Argumentation Based on the Figure; Use of Unclear Language

In accordance with Figure 7, the student uses colloquial language in order to argue the validity of the proposition based upon the properties of isosceles triangles. Similarly, he seems to deduce that the triangles of interest are equal since he sees him as the same addition of two angles; however the way of writing his ideas is not clear.

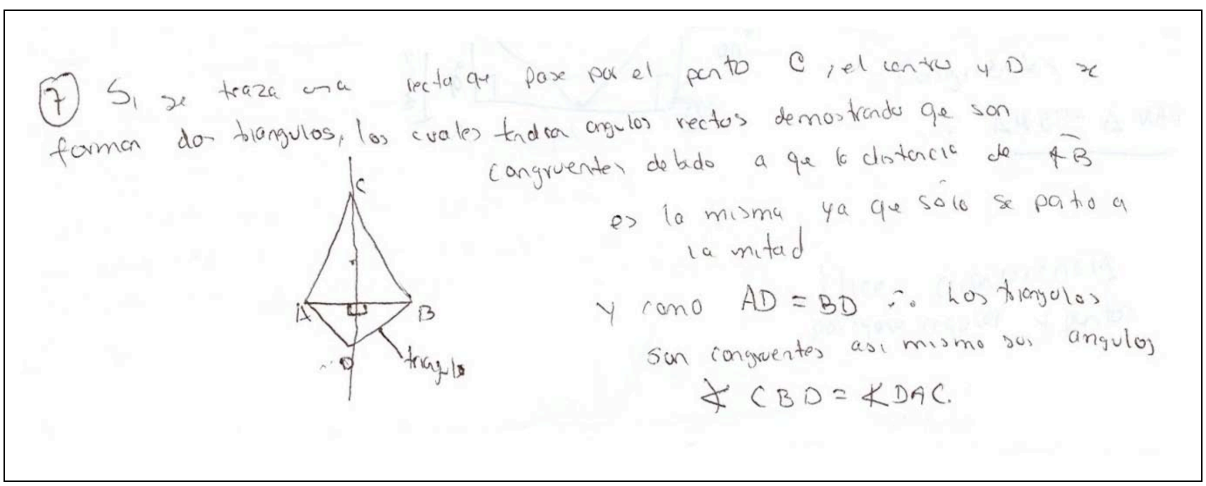


Figure 8. Student A11: Colloquial Argumentation Based upon Evident Actions

According to Figure 8, the student uses colloquial language to prove the affirmation. He traces the auxiliary straight line $\overline{CD}$ which cuts $\overline{AB}$ at its midpoint in a perpendicular way, and since $\overline{AD} = \overline{BD}$, he concludes (by the SSS criterion) that ΔADC is congruent to ΔBDC.

Problem 3

In ΔABC equilateral (Figure 9), *P* and *Q* are midpoints of $\overline{BC}$ and $\overline{AC}$, respectively. Prove that $\overline{AP} = \overline{BQ}$.

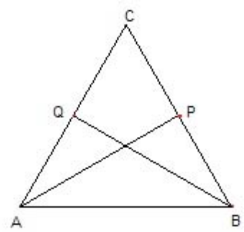


Figure 9. Problem Representation

Some Solutions

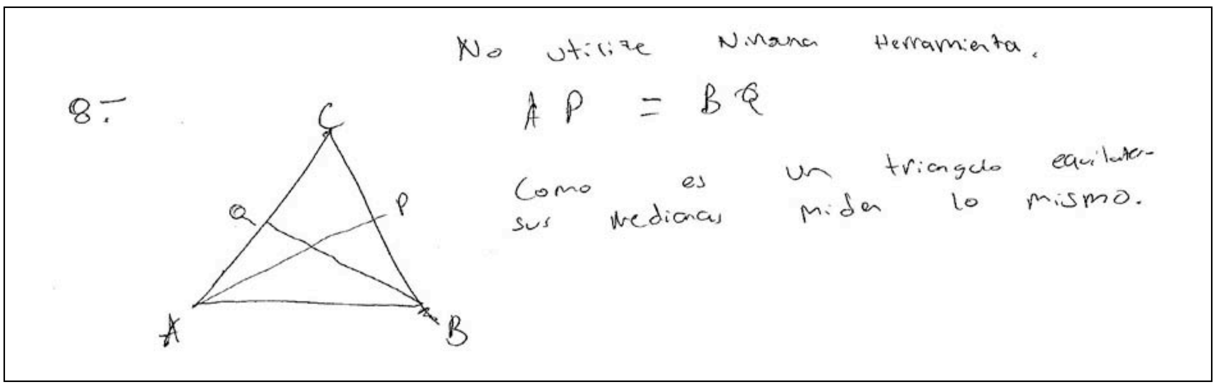


Figure 10. Student A10: Argumentation Based upon Evidences

In accordance with Figure 10, the student asserts that the measurements have the same length by the mere fact that the triangle is not equilateral. He does not provide any argumentation.

Problem 4

In W$ABCD$ (Figure 11), the midpoint of $\overline{AB}$ is *M*. With center in *M* an arc is described that cuts $\overline{AD}$ in *Q* and $\overline{BC}$ in *P*. Prove that $\Delta MBP \cong \Delta MAQ$

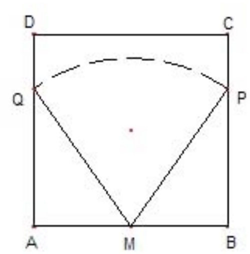
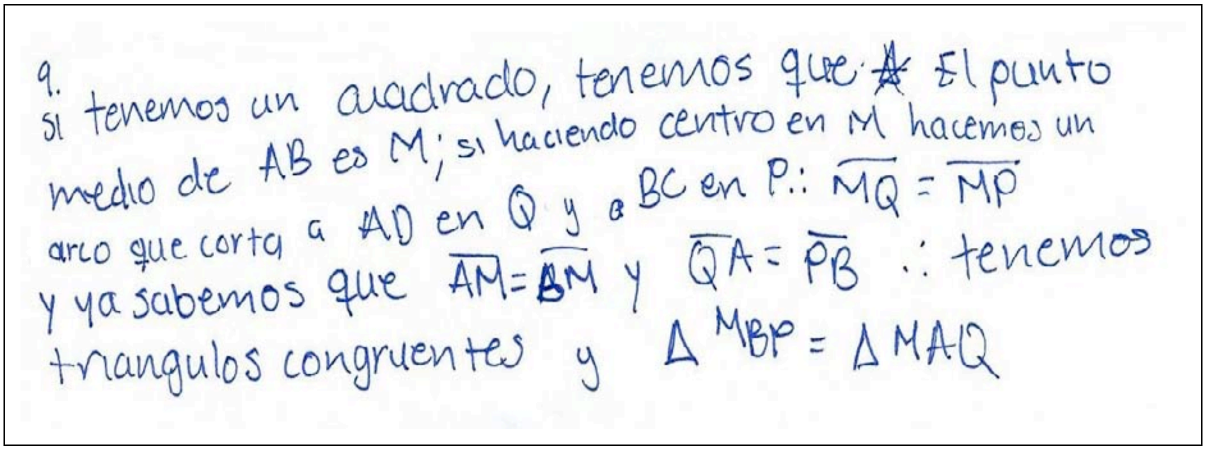


Figure 11. Problem Representation

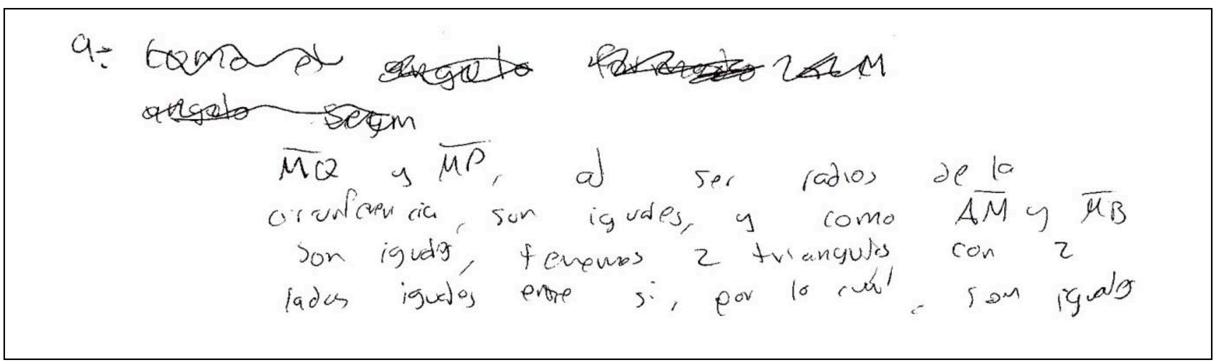
Some solutions



9.
si tenemos un cuadrado, tenemos que ~~A~~ El punto
medio de AB es M; si haciendo centro en M hacemos un
arco que corta a AD en Q y a BC en P.: $\overline{MQ} = \overline{MP}$
y ya sabemos que $\overline{AM} = \overline{BM}$ y $\overline{QA} = \overline{PB}$ ∴ tenemos
triangulos congruentes y $\Delta MBP = \Delta MAQ$

Figure 12. Student A12: Argumentation Supported by the Use of non Evident Colloquial Language

It is observed in figure 12 that for the student, two triangles are congruent is two of the sides of one are equal to two of the sides of the other, and then he concludes that the right-angled triangles are equal, even though he does not refer to the right angle.



9: ~~toma el~~ ~~angulo~~ ~~[illegible]~~ 2AM
~~angulo~~ ~~SQM~~
$\overline{MQ}$ y $\overline{MP}$, al ser radios de la
circunferencia, son iguales, y como $\overline{AM}$ y $\overline{MB}$
son iguales, tenemos 2 triangulos con 2
lados iguales entre si, por lo cual, son iguales

Figure 13. Student A7: A Proof he explains by the Use of geometric concepts

As it is observed in Figure 13, the student provides a justification based on triangle congruence (SAS criterion). Two equal corresponding sides (and the right angle which he does not mention) are enough for him to affirm that the two right-angled triangles involved are equal.

Conclusions

Generally, from the second and third working session it was observed that the majority of students justified the results using a colloquial language, and, sometimes, the processes of argumentation were imprecise or incomplete; however, in some cases, they did use a formal symbolic language with clarity. In general, students elaborated, in an acceptable way, the demonstration in which they resort to triangle congruence, or at least, this element allowed them to infer and elaborate a discourse with which they tried to justify the results.

To the light of students' answers, in this research, the kinds of proof which emerge during the solution the geometric problems of triangle congruence are the ones already mentioned in the research literature related to proof in mathematics. In our research, we could verify that there exists a great difficulty, on behalf of the students, to identify auxiliary elements which could be useful in order to ensure the veracity of the result. Students need to develop intuitive reasoning of the geometric objects in order to access both a better understanding of Euclidean geometry and a formal deductive reasoning for its study; in order to reach the lather, it is required to take into account the kind of tasks to be solved and the way in which they are guided.

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USING TRANSFORMATIVE TASKS AND TOOLS TO CONCEPTUALIZE PROSPECTIVE MATHEMATICS TEACHERS' ARGUMENTATION PRACTICES

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Introduction

Recent reforms in mathematics education have led to an increased emphasis on argumentation practices in mathematics curricula (National Council of Teachers of Mathematics, 2000). The relevance of argumentation in mathematics curricula has been demonstrated at various levels of K-16 mathematics. All findings consent that argumentation should be a central aspect of mathematics teaching and learning at all levels of schooling.

Argumentation, in this work, is defined as an act of communication intended to lend support to a claim. This definition presupposes a relationship between argumentation and proof, where proof is defined, in this work, as a logically correct argument constructed from given conditions, definitions, and theorems within an axiom system.

This exploratory work provides a description of argumentation practices among pre-service secondary mathematics teachers during an undergraduate mathematics capstone course. The work presents a possible relationship between the use of transformative mathematical tools, specifically technological tools, and pre-service teachers' argumentation practices. Additionally, this exploratory work investigates: (1) the role of pedagogically sound mathematical tools in the advancement of teaching acceptable argumentation practices, which lead to sound reasoning and proof processes; and (2) mathematical tasks useful to prospective teachers in the teaching of argumentation practices.

Results

This work revealed that the pre-service teachers' conception of argumentation directly influenced their instructional approaches in teaching. Involving them in reasoning through argumentation under different instructional practices enhanced their abilities to evaluate, integrate, bridge, and explain their mathematical knowledge. Significant developments in students' argumentation practices occurred as students used technological tools as a medium to support and define their reasoning. The integration of these transformative tools (i.e. Geometer's Sketchpad, Mathlab, etc.) with the students' content knowledge led to the production of strong mathematical products in the form of argumentation, reasoning, and proof.

With the use of similar tasks and tools, students' argumentation practices were conceptualized and categorized in four different categories: Argumentation as Confirmation, Argumentation as Validation, Argumentation as Discovery, and Argumentation as Explanation. Many aspects of the pre-service teachers' conception of argumentation are represented in work samples from each argumentation category. Models of student work from each category will be visually displayed in the full presentation.

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THE STUDY OF MOTION PHENOMENA THROUGH MULTIPLE REPRESENTATIONS

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This paper reports the results of research with the objective of analyzing the ways students related two or more representations to make sense of motion situations. In the study of movement phenomena some researchers have identified and documented the students' difficulties to read and interpret distance-time graphs. In line with this observation Bowers and Doerr (2001) point the tendency for students to interpret the distance time graph as a picture of the path of the motion phenomena. In the same sense Monk (1992) documents that when the students work with distance time graph, they can determine correctly the distance traveled by a car at different time intervals, although they can´t identify if the car speed increased or decreased. Goldin and Kaput (1996) said that when students are in problem solving activities they use different representations to work with the information provided, each representation bring forward different aspects related to the same concept. Previous researches make clear how important it is the underlying information in each representation.

The conceptual framework is based on complementary external representations to support the construction of deeper understanding. Ainsworth and Van Labeke (2004), pointed out that when learners interact with appropriate representations, their performance is enhanced.

We report three students' work which was selected after us identifying that they showed a representative work of the group, also they wrote their reports in more detail than their peers.

The activity reported was undertaken during two sessions of one hour and one half each. In this task we analyzed the relationship that students established between verbal and graph representations to make sense of motion situations; particularly we were interested in two points: a)Identify if they relate, in a distance-time graph, the slope whit the speed value; b) Identify how they represent in a distance-time graph the velocity changes at different time intervals

The findings obtained make it possible to assert that when students work on tasks that include complementary external representations they improve their understanding of motion situations.

Acknowledgements

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THE CLOTHESLINE METHOD FOR MEASURING MATHEMATICS SELF-EFFICACY

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Many researchers have used Albert Bandura's work on self-efficacy as a theoretical framework in analyzing how beliefs affect educational performance. He defines it as "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments (Bandura 1997)." Investigators have found evidence that support his theory when applied to school mathematics. Studies among college students (Hackett and Betz 1989), 7th grade (Chen 2002), and high school students (Randhawa, Beamer et al. 1993) provide data that show positive correlations between self-efficacy measurements and mathematics performance. Likewise, a qualitative research documenting the beliefs and performance of middle school children present congruent information (Usher 2009).

This research was conducted as part of a study that investigated the relationship of self-efficacy beliefs to mathematics achievement among Filipino high school students selected from a homogenous, rural, and low-SES community. Utilizing observations, interviews, and focus group discussions, self-efficacy was related to academic behavior, peer group activities, and parental support, and to mathematics performance.

Despite the emphasis on qualitative techniques, there was still a need to quickly assess self-efficacy for the whole student population of the school. The Clothesline Method is one instrument that allowed the respondents to rank and rate beliefs at the same time. Based on test-retest statistics, it was shown that the data produced were more reliable compared to a typical belief rating scale. Nonetheless, there were disadvantages in using this method. It can produce confusion among students and may result to many invalid data. The effectiveness of the instrument was highly dependent on how well the administrator explained the instructions. Finally, it was more time consuming. If perfected however, this may serve as an alternative to Likert or Bandura's scales especially for populations who are not familiar to conventional survey techniques.

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Teaching Science and Mathematics to the E-Generation Introduction

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Students of tomorrow not only need basic ideas of science but also need to have an understanding of why science is so important to societies financial, social, political and cultural being (COSEPUP, 2007). Consequently, educators and researchers have begun to carefully evaluate how today's students learn and it is generally accepted that students learn best by "doing" especially in science classrooms (Dalton et al., 1997). However, besides hands-on instruction, current researchers and educators have also recognized the impact of technology on today's learners. In the new millennium, literacy takes on a new meaning. For example, the current generation of teenagers often referred to as the E-Generation have grown up in the era of Internet, cell phones, I-pods, and Nintendo which some believe has not only changed the way students think but also the way that they learn. (Jones-Kavalier, 2006). Therefore, some argue that that the use of multimedia technologies such as virtual laboratories, simulations and computer models are better suited for today's science students (Roschelle et al., 2000).

Purpose of the Study

The study will compare the effectiveness of student learning in a virtual computer laboratory experience to that of a traditional hands-on laboratory experience. Student attitudes and opinions of virtual labs will also be considered. Students will be asked to partake in a survey where they share individual feelings of the overall effectiveness and future usefulness of computer models.

Significance of Study

The significance of this study is to gain a better perspective on today's technological learners. In planning lessons, teachers can give more consideration to computer models and virtual lab simulations if this study and future studies show that such applications are more effective and motivating to present learners.

Methods Research Design

This study randomly divided four high school chemistry classes into two control groups and two experimental groups. The control groups conducted an in-class experiment under the supervision of the teacher and the experimental groups conducted an experiment using a computer module. The objective for all groups was the same; students were to determine the limiting reactant in a chemical reaction.

The experimental groups were assigned a virtual computer simulation. Students met at the school's computer lab and were assigned a lab on limiting reactants. The computer module asked students to make a comparison between building a cheese sandwich and chemical reactions. For example, students determined the limiting reactant in making a cheese sandwich when they were given eleven pieces of bread but only four pieces of cheese.

The control groups participated in a very similar experiment which took place in the classroom with the science instructor. Again, the key concept was asking students to deduce the limiting factor in a chemical reaction. Similar to the virtual lab in which students were making a relationship between recipes and chemical reactions, students were given the ingredients and the recipe for rice crispy treats or otherwise referred to in the experiment as carbohydrate chewies. Students were instructed to make chewies following the recipe they were given and then determine which ingredient limited the product.

Prior to both experiments, all students were given identical pretests concerning their attitudes and opinions towards virtual labs and hands-on experiences. Both experimental and control groups were given identical quizzes on limiting reactants and scores were compared.

Finally, approximately three months after carrying out the study, all students who participated in the study will be given a follow up or post quiz so as to gain a better understanding of student retention. The follow up quiz scores will then be compared to the original quiz scores of all students.

Results

T-Test will be used to compare means of treatment and control group subjects. Attitudes of the experimental and control groups toward virtual and hands-on experiments will also be compared. Results and conclusions with a discussion of results will be presented at the October PME conference.

AP COHORT MODEL: A STRUCTURE FOR TRANSFORMING HIGH SCHOOL MATHEMATICS PEER CULTURE

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The Algebra Project's Cohort Model (APCM) offers a structure for high school mathematics classrooms that supports the development of an effective peer culture for developing mathematical understanding and mathematics learning efficacy. The APCM is built on 15 years of experience. The purpose of this study was to observe the peer culture transformation in one APCM classroom.

Poster Summary

The Algebra Project's Cohort Model (APCM) offers a structure that supports the development of an effective peer culture for developing mathematical understanding and mathematics learning efficacy. The APCM is an innovative approach for high school mathematics for students who have been consistently low performing on state or national achievement tests to "catch up" by graduating in four years and qualify for college mathematics courses, if they choose to attend (West, 2011). The APCM is built on 15 years of experience in middle schools and high school pilot programs that included instructional materials development funded by the National Science Foundation.

The purpose of this study is to observe within one Algebra Project Cohort Model classroom's peer culture transformation from "'I dare you to teach me' to 'I need and want you to teach me'" (Bob Moses, Summer 2010). The idea of harnessing the peer culture for mathematics learning is anticipated to occur during the traditional four years of high school in response to an APCM implementation. However, after a little over one year evidence of the transformation has manifested in the site that I have been studying.

The target Algebra Project Cohort classroom is located in a small urban community and is comprised of 19 students. My poster will present preliminary evidence of the peer culture transition using a multimedia and print display and include an overview of the Algebra Project's Cohort Model including an introduction of the cohort class and teacher, an APCM structural description, and community outreach through the Young People's Project.

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TEACHER AND PROFESSOR PERSPECTIVES ON CONTRIBUTING FACTORS FOR COLLEGE MATHEMATICS SUCCESS

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Throughout the United States, high school students who take a college preparatory curriculum are often surprised when they are placed into a remedial mathematics course in college as they assume their high school courses have adequately prepared them to enroll in courses such as calculus. Although a college preparatory curriculum is necessary for college readiness, it alone is not sufficient to assure success. Indeed, in 2004 Achieve conducted a study which confirmed a gap between what students knew leaving high school and the actual knowledge that they needed to know coming into college (2010). Differences in the skills gained in high school and the expectations of knowledge needed in college may be a key component in this issue (Kirst & Venezia, 2001). Research has also suggested that the lack of coordination between high school and post secondary institutions could be part of the problem (Kirst & Venezia, 2001). Most scholars agree that this "expectations gap" exists, yet how to address this gap and provide an easy transition from high school to college has yet to be found.

Research done by the US Department of Education shows that the single best predictor of college success is not test scores, class rank or GPA, but the intensity and quality of a student's secondary school curriculum (Adelman, 1999). Too often the college preparatory curriculum in high school does not get a student through the mathematics placement test in college. The skills and knowledge actually necessary for success in post secondary mathematics courses needs to be clarified (Haycock, 2003). High school and college courses may not use the same criteria to determine what constitutes "success" (Berry, 2003).

What should readiness standards include? What knowledge or skills are needed and expected at the college level for success? Do both high school teachers and college professors agree on these factors? Exploring these questions this study looked at the perspectives of high school teachers and college professors surrounding factors which contribute to college mathematics success. Interview data from both groups was analyzed to determine if perspectives on contributing factors were uniquely or collaboratively viewed as important for success in college mathematics courses. Initial findings indicate both overlapping and disjoint views on what contributes to student success.

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FOUR STAGE MODEL FOR DESCRIBING TEACHER IDENTITY FORMATION IN POST-SECONDARY MATHEMATICS

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In an effort to understand the development of teaching practices of future post-secondary teachers of mathematics, individual and group interviews with mathematics graduate students were undertaken to explore their experiences and perspectives of mathematics teaching. These conversations probed their personal histories in studying mathematics, current experiences as mathematics graduate students, views of mathematics, and their possible futures in academia. In observing their lives during the academic year, it was noted that as they learned of the relative importance (or unimportance) of different aspects of a life in mathematics, the research participants observed that post-secondary mathematics teaching took on a certain form and that mathematicians adopted particular identities. Framed by the idea of legitimate peripheral participation (Lave and Wenger, 1991), in their transitions from graduate students to professors, the participants in this study seemed to move from a marginal position to a central place in the community as their identities, actions, and teaching more closely resembled those of the mathematicians in their department.

Four stages through which the participants passed and by which this transition process can be described are: replication, resignation, despondence, and repetition. First, in their process of becoming mathematicians, the participants first felt a need to replicate – not only mathematical content, but also who they could be as post-secondary teachers of mathematics. They were limited in what they could do as future teachers by the structures of the department and by the advice, or lack of it, from their faculty advisors. The participants were thus left to creating meaning amongst themselves, relying solely on the replication of the teaching they observed. Beyond the sense of replication came feelings of resignation, where the participants felt as though they must resign their aspirations and expectations for their teaching practices in order to be considered legitimate within the department. However, in attending to what was legitimate in the department, there seemed to be a considerable cost, bringing about a sense of despondence, where the participants began to feel hopeless about what they could do and how they could be as post-secondary teachers of mathematics. Finally, Caputo's (1987) notion of repetition, where an individual "forges his personality out of the chaos of events" despite the "incessant 'dispersal' of the self" (p. 21), helps to understand the elements of their identities and practices the participants held onto as they transitioned to becoming professors. These themes – replication, resignation, despondence, repetition – will be illustrated on this poster presentation, along with the transitions and connections between them. Also, the details and structure of the research project, as well the data that relates to each theme, will be presented.

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TRANSFORMING PRACTICES BY UNDERSTANDING CONNECTIONS BETWEEN NEW MATHEMATICS TEACHER EDUCATORS AND NEW K-12 MATHEMATICS TEACHERS

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New K-12 mathematics teachers enter classrooms with “high expectations for themselves and for their students,” where the “early years of teaching are often characterized by a ‘sink-or-swim’ or ‘survival’ mentality because we have often failed to provide for careful support and thoughtful development of teaching expertise over time” (Bartell, 2005, p. 3). Similarly, new mathematics teacher educators must learn to navigate the often rocky terrain of their institutions, wherein they seek support for finding ways to strike a balance between teaching, research, and service. As Reys, Cox, Dingman, and Newton (2009) point out “important aspects to be monitored are the challenges that lie ahead for these new Ph.D.s as they begin their careers in higher education” (p. 9). While others (e.g., Golde & Walker, 2006) have looked to these challenges to help understand how doctoral education could provide better preparation for careers in academe, this research project sought to look at the similar challenges faced by two groups of novice teachers (with less than three years of experience): those teaching mathematics education courses at universities and those teaching mathematics in K-12 classrooms. In particular, this research study focused its attention on addressing the research questions: How are the experiences of new mathematics teacher educators similar to those of new K-12 mathematics teachers? In what ways could the knowledge of these similarities help to inform and possibly transform new mathematics teacher educators’ practices in working with pre- and in-service teachers?

To explore these questions, we asked a cohort of new mathematics teacher educators to answer questions related to their new academic positions. This poster will present: 1) the preliminary results of a survey of new mathematics teacher educators, 2) results of a meta-analysis of recent research that explores the experiences and challenges of new K-12 mathematics teachers; 3) a comparison of the two groups’ experiences and challenges; and 4) how this research has the potential to inform and transform our approaches to mathematics teacher education at all levels.

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NEW "TIP" FOR ENHANCING MATHEMATICS INSTRUCTION

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Grounded in constructionism, socio-cultural theories, and a cognitive apprenticeship model of learning, this study investigated the dynamics of the teacher-intern-professor model as a support to effective mathematics instruction and how they used the teacher work sample as a tool for instruction and performance assessment to enhance the teaching and learning process conducted by pre-service intern teachers as they implement standards-based lessons.

Although teacher preparation programs offer the strategies needed for pre-service intern teachers to be successful in their first years of teaching, the retention rates for new teachers, especially those in urban schools and high needs areas, are declining. As a result, student success is compromised with the transiency of qualified individuals (Wright & Nassar, 1991). In spring 2011, at a research I university in the southeastern region of the United States, the *TIP Anchor Action Research Project* was implemented to address the needs of teachers through their preparation phase and examine the effect on student achievement. The Teacher-Intern-Professor (TIP) model consists of a mentor or supervising classroom teacher, a pre-service intern teacher, and a university professor.

Through the use of the teacher work sample (TWS) as a tool for instruction and performance assessment, each of ten TIP groups and its comparison class examined a unit of instruction administering similar pre- and post-tests. Reflective of the knowledge gained from the pre-test, each intern use the results to hone the instructional design and to build more meaningful lessons aligned with the desired learning goals. During the study, the university professor ensured that: (1) each group received needed supplies, (2) school visits were made to support and conduct discussions with the TIP group, and (3) the research activities were in progress. Earlier research suggested that interns' teaching experiences benefit by having support that bridges both university and school experiences (Zeichner, 2010).

The TIP group participates in discussions, interviews, and records data in journals. This serves as the qualitative technique to support the quasi-quantitative data being collected from the pre- and post-tests. This study, which is grounded in constructionism, socio-cultural theories, and a cognitive apprenticeship model of learning seeks to answer the following research questions: (1) In what ways did the TWS enhance the teaching and learning process conducted by pre-service intern teachers as they implement standards-based lessons in the TIP model and (2) what are the dynamics of the TIP model as a support to effective mathematics instruction?

Preliminary results suggest that: (1) the intern teachers felt comfortable and more open to the groups' decisions on pedagogical and classroom management concerns, (2) the collaborative approach in meetings, observations, and research was useful and effective, and (3) the use of the TWS was recognized as a constructive and meaningful way in be-coming an effective mathematics teacher who consistently reflects on his/her practice.

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PRE-SERVICE TEACHERS OF COLOR INCORPORATING A CHILD'S HOME AND COMMUNITY MATH KNOWLEDGE

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The purpose of this study was to analyze the experiences of four Pre-Service Teachers (PSTs) of Color as they crafted math lessons that utilized a child's home and community knowledge. Diverse, underrepresented students benefit from such lessons because learning is more meaningful and relevant. Some PSTs have personal experiences as diverse learners that help them to effectively teach students from similar backgrounds. Data collected included lesson plans, individual interviews, and written reflections. Findings showed that while PSTs of Color eagerly incorporated a diverse child's home and community knowledge into their lessons, they did not feel confident in sustaining this practice.

As part of a larger dissertation study aimed at describing four pre-service elementary teachers' beliefs and experiences related to teaching mathematics for conceptual understanding (Fennema & Romberg, 1999) and with cultural relevance (Ladson-Billings, 1995), this poster will focus specifically on how the Pre-Service Teachers (PSTs) of Color come to teach elementary mathematics to diverse learners by connecting to the child's home and community knowledge. Children possess a home and community knowledge that may be rooted in their culture (Ladson-Billings, 1997), in their Funds of Knowledge (González, N., Moll, L. C., & Amanti, C. 2005), and/or is based on their experiences within the local community (Turner, E., Gutierrez, M. V., Simic-Muller, K., & Diez-Palomar, J., 2009). It is widely understood that PSTs come to the teacher preparation program with prior beliefs and experiences in education and mathematics (Featherstone 1992; Lidstone and Hollingsworth, 1992) and what it means to teach diverse learners (Meaney et al., 2008). Throughout the mathematics methods coursework and elementary field experiences, research has suggested that PSTs should be challenged and/or supported in their prior beliefs and experiences on teaching diverse learners (Cabello & Burnstein; 1995; Aguirre, 2009). Work in this area is particularly important for PSTs who are diverse learners and/or are from underrepresented backgrounds (Montecinos, 2004). The prior experiences of PSTs of Color may shed some insight into how all teachers (not just those from the demographic majority) might better educate diverse students in the field of mathematics—a content area that has historically marginalized students based on culture, language, ability, and class.

The three female and one male PSTs in the study (all self-identified as Latina/o) were a part of a larger elementary methods cohort in the Fall 2010 and were placed in field experience classrooms at three local elementary and middle school, made up largely of Latina/o and low socioeconomic populations. My poster will analyze how PSTs' of Color acquired and integrated the knowledge of the local community and of their students in order to write more meaningful, relevant math lessons. The experiences were recorded in a variety of ways: three interviews individual reflections on the School and Community Math Project, reflections about relevant course readings, and reflections on a case study with a young child.

The findings showed that all of the PSTs of Color considered a child's home and community knowledge as being important in how he/she learns mathematics, yet they still wanted more experiences in making meaningful and sustainable connections to their daily practice.

Additionally, nearly all of the PSTs of Color framed a child's home and community knowledge around consumption (e.g. buying a toy or consuming food). The adage of "getting the most bang for your buck" ran predominantly throughout their lessons. Bilingual PSTs of Color saw a student's native language as another way to connect to a child's home and community knowledge. Other themes and implications for teacher education will be included in the analysis.

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"I AM NOT SEEING WHAT I READ ABOUT": A MATHEMATICS STUDENT TEACHER'S STRUGGLE WITH DISCONNECTED TEACHING EXPERIENCES

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In the U.S. and most parts of the world, the student-teaching internship is the culminating experience of initial teacher education programs (Guyton & McIntyre, 1990). Most teachers view this internship as the most valuable part of their preparation, claiming that most of what they know comes from first-hand teaching experience (Feiman-Nemser & Buchmann, 1985). The purpose of this presentation is to share the story of Julie as she navigates this experience.

Julie is a middle level prospective mathematics teacher enrolled in an undergraduate teacher education program at a large urban state university in the South-Eastern region of the United States. In Fall 2010, she participated in a 2-day per week field-based practicum experience and a university-based mathematics methods course. The course and field experiences in the fall focused primarily on theoretical readings, field-based observations guided by those readings, and lesson plan development. In Spring 2011, Julie completed her full-time student teaching in the same 6th grade mathematics classroom where she completed her practicum hours in the fall.

Data for this paper consist of Julie's written assignments from the Fall 2010 mathematics methods course, observation field notes from student teaching, and transcribed interviews from both fall and spring semester. To analyse the assignments and interviews, the researchers read and re-read all narratives separately and collectively to identify general themes. Common themes from Julie's case include the following: philosophical differences between mentor and student teacher, the disconnect between theory and practice, and the question of whether this is the correct career choice. These themes are highlighted below.

From the beginning of the fall practicum experience, Julie struggled with the disconnect between her newly developing teaching philosophy as shaped by her university coursework, and the quite different mathematics teaching actions of her mentor teacher. She states,

> I wanted so badly to be with the picture perfect teacher, but I am not. I think what is frustrating for me is how she teaches math. The goal is not to understand the concept and the why but be able to do the steps and get the answers. I also know she was frustrated with me because I was spending so much time on things. We're just very different. (Interview 1)

In addition to the struggle between coursework and field experiences, Julie also appeared to be struggling with her decision to become a teacher. She states,

> I've been down and questioning whether or not I'm really going to teach and go forward. It has been difficult. It is a lot of work, which I don't think I was prepared for. (Interview 2)

These, and other quotes from Julie's written work and interviews, will be shared during the presentation to explore issues of disconnect. In particular, we focus on the lack of autonomy and flexibility experienced in this setting and possible implications for teacher education.

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INVESTIGATING PRE-SERVICE CANDIDATES' IMAGES OF MATHEMATICAL REASONING

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Subject Matter Knowledge (SMK) and Pedagogical Content knowledge (PCK) are seen as diverse, multidimensional, and vital to a teachers' knowledge base for teaching. Understanding pre-service candidates' (PSC) organizational knowledge, connections among ideas, proof and inquiry, and knowledge growth within mathematics are important factors to explore when examining their images of mathematical reasoning. This research was conducted during a methods course.

Research Focus

Borko and Putnam (1996) suggested that learning opportunities for teachers be grounded in the teaching of subject matter and "provide opportunities for teachers to enhance their own subject matter knowledge and beliefs" (p. 702). In response to this assertion and Schoenfeld's (1999) call for theoretical research that is focused on practical and relevant applications throughout education, the focus of this research is to explore PSCs' images of mathematical reasoning.

Conceptual Framework

The conceptual framework for the research helps explain PSCs' images of mathematical reasoning when examining responses to several related questions about reasoning and proof. The conceptual framework uses components of Shulman's (1986) knowledge base in teaching and incorporates Pirie-Kieren's (1994) notions of primitive knowledge and images from their model for growth in mathematical understanding.

Methods of Inquiry

The investigation of PSC' knowledge of mathematical reasoning was part of a study investigating prospective teachers' knowledge of *what* and *how* to teach concepts in an elementary mathematics methods course. The study was done over a 15-week semester, involving several PSCs.

Results

In examining the PSCs' responses to the questions there is a need to help them define terminology when explaining their images. As the PSCs draw upon their images of mathematical reasoning, the data has strong implications for their propositional knowledge while examples of case knowledge and strategic knowledge are combined throughout to exemplify either observations or the personal experiences they have had as a student in a mathematics classroom. In addition, the data reveals that it becomes evident that the PSCs have strong images with regards to propositional knowledge.

Discussion

With current reform efforts such as the Core Common Standards guiding curriculum decisions, it is imperative that teacher education programs look forward to the Standards for Mathematical Practice, so that instruction is aligned with these efforts. These practices are built on established processes and proficiencies for mathematics education that rely heavily on PSCs' knowledge and beliefs of mathematical reasoning. Therefore, investigations into PSCs' beliefs on mathematical practices need to be continually explored and best practices further defined for all mathematics teachers.

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NOVICE TEACHERS' PERSPECTIVES OF TEACHER PREPARATION

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I report here on research that examined novice secondary mathematics teachers' perceptions of their preparation and their perceptions of ideal teacher preparation. The teachers expressed more positive views of teacher preparation than those typically found in the literature. Furthermore, the teachers provided insights into how teacher preparation could be made more meaningful and effective. These suggestions were grounded in their teaching experience, their beliefs about teaching, and their experiences as prospective teachers.

Most interested groups have a less than favorable view of teacher preparation (Labaree, 2006). To examine how teachers perceive their preparation, I interviewed relatively recent graduates of a large, well-respected secondary mathematics teacher preparation program. Contrary to what might be expected, the teachers were generally satisfied with their preparation. The suggestions they made for future teacher preparation were phrased as improvements, rather than reinventions, of the existing program. Moreover, their views of their preparation were heavily rooted in their beliefs about what teaching is and should be, so that as this latter belief shifted or stabilized during their preparation and initial years of teaching, so too did their view of their preparation experience. This phenomenon aligns with Green's (1971) notions of primary and derivative beliefs.

In particular, the teachers held a view of teaching that was more social and less cognitive than many teacher educators, which sometimes led to teacher educators and prospective teachers having conflicting goals during the teacher preparation program. This in turn led to missed opportunities that directly affected the teachers' level of preparedness. For example, the teachers acknowledged a tendency to disengage from coursework they perceived as irrelevant or inefficient. From their current point of view as experienced teachers, this disengagement deprived them of possible learning experiences that would have benefited them in their later teaching. Consequently, the teachers suggested direct actions teacher educators and teacher preparation programs could take that would prevent these episodes of disengagement, thereby improving teacher education by encouraging more continuous and authentic participation from prospective teachers.

The teachers' views of ideal teacher preparation are also revealing, and surprisingly similar. One alteration every participant proposed was to either increase or foreground (or both) the field experience component of their preparation wherein they observed and taught in middle or secondary schools. This recommendation is connected to the prior point about disengagement. Field experiences, the teachers felt, had a significant impact on their beliefs about what teaching is. Experiencing classrooms from the point of view of teachers rather than students, often for the first time, affected their thinking about the nature of teaching. The experiences also provided concrete knowledge for them to draw upon in later mathematics and methods courses. Thus, foregrounding and expanding the field experience component would prompt prospective teachers to seriously attend to later instruction in mathematics and pedagogy and also allow them to better make sense of that instruction because they would have personal experiences to connect it to.

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WHAT DOES "REAL-LIFE" REALLY MEAN TO ELEMENTARY PRESERVICE MATHEMATICS TEACHERS?

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Many mathematics teacher educators and teachers emphasize the importance of connecting mathematics to real-life in order to engage students in learning mathematics. Several researchers have also explored the benefits of involving students in real life contexts (e.g. Civil, 2007 and Gutstein, 2003). For example, Luis Moll and his colleagues (Moll, Amanti, Neff, & Gonzalez, 1992) suggest that connections to household funds of knowledge can motivate students to learn mathematics in the classroom. While researchers have extensively explored the outcomes of engaging students in real-life contexts, what remains unexplored is how teachers understand and use ideas about real-life in their classrooms (Author 2 and others, 2011). This poster begins the exploration of teacher understandings by examining how one preservice teacher (Francis) made sense of real-life in her school experiences.

Francis participated in an elementary teacher preparation program at a large university in the southwest. She was also part of a research project examining the relationship between the math methods classroom and her field placement. The research project collected a variety of data from Francis including transcripts from 2 interviews and copies of all written work for the course. In total, the written data on Francis comprises over 53 pages of text. The findings below arise from examination of each of these documents.

Francis' work shows three different ways in which she connected real-life to mathematics teaching and learning: (a) *Using students' "real-lives"* – their already lived experiences (i.e. their home, language, and cultural experiences) – to engage them in mathematics. For example, Francis tried to make mathematics more accessible for one of her students from Mexico by discussing the mathematics problem in Spanish with her. (b) *Creating "real-life"* experiences for students to help them understand mathematics problems. Francis helped a student understand "round-trip" (which was an important component of a mathematics word problem) by asking the student to walk across the classroom back and forth. (c) *Using mathematics to make sense of real world* situations. Francis stated that she would use mathematical explorations of her students' community to help her students become more aware of the social issues around them. Making sense of teachers' use of "real-life" provides insights into teachers' mathematics pedagogical thinking including their constructions of students' lives and how to connect students to content. This poster is an initial exploration into this sense-making.

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GATHERING EVIDENCE AS A SUPPORT FOR NOTICING

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The purpose of the poster is to present preliminary findings from an exploration of gathering evidence as a support to develop preservice elementary teacher's (PST's) ability to notice to children's mathematical thinking. This research focuses on two primary questions: (1) How do PSTs report evidence of children's mathematical thinking? (2) How do the processes of gathering evidence and reporting evidence support PST to notice the specifics in children's mathematics thinking? Recent research provides evidence that, in-service teachers, if given opportunities to discuss children's mathematical thinking in the form of video clips, can learn to notice more specific aspects of children's mathematical thinking as opposed to noticing aspects about teaching (e.g., Sherin & Van Es, 2006). However, the trajectory of PSTs developing their ability of noticing children's mathematical thinking is different from that of in-service teachers in that they have had fewer opportunities to work with children as compared to in-service teachers, and thus have fewer resources from which to draw from. Little attention has been paid to the design of activities that can help PSTs notice children's thinking. Building on this work, we use gathering evidence as a scaffold to develop PSTs' noticing of children's mathematical thinking and examined how the activity of gathering evidence supported PSTs' ability to notice children's mathematical thinking in terms of the evidence PSTs provided for their noticing.

Data from this study is taken from the Videocases for Preservice Elementary Mathematics (VPEM) Project, which involved the implementation of 8 videocases in a mathematics content course for PSTs. Videocases were implemented at regular intervals across one semester. During the implementation of the videocases in the content course, PSTs first worked on the mathematics problems discussed in the video, and then viewed the videos through the lens of the focus questions. PSTs were encouraged to gather evidence, as part of their responses to the focus questions, to support their noticing. We analyzed PSTs' collective discussions of what they noticed and the types of evidence they provided employing a grounded theory approach (Strauss & Corbin, 2008). We first identified the discussion of notice (DON), after which we examined the ways PSTs reported their evidence to support their noticing. We then examined the patterns of evidence reported and its relation to the quality of noticing, i.e. general noticing or specific noticing.

Four types of reported evidence emerged from the data: (a) citing line numbers from the transcript, (b) quoting what children said in the transcripts or videos, (c) describing or summarizing the events that happens in the video, and (d) making interpretations. More than one third of the DONs share the following pattern: starting with a noticing by providing evidence using inferences, then being prompted by the facilitator, more specific noticing with line numbers or quotes or general descriptions are shared. As the way of reporting notice increase, the more the specific details that are provided.

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PRESERVICE TEACHERS' KNOWLEDGE OF KEY GEOMETRY TERMS

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Data from assessments such as TIMSS and NAEP indicate that students are weak in the area of measurement (Thompson & Preston, 2004). Bearing this in mind, it is important to make sure that preservice teachers are prepared to teach measurement topics to their future students. As such, in the fall of 2010, fifty-two preservice teachers were given a measurement application test as part of their mathematics methods course. About 94 % of the group were female, ranging in age from 20-59. The largest ethnic group was Anglo followed by Hispanics and then African American.

The measurement application test consisted of 12 multiple-choice questions that were modeled after the released 5th grade state tests. The scores on this section were approximately 33% mastery (scoring 90-100%), 33% passed (scoring between 75-90%) and 33% failed (scoring below 75%). The results of this test were used to develop skills modules to help the students master these concepts. In addition, a majority of the missed questions appeared to be due to a misapplication of key measurement and geometry terms and concepts. Some researchers believe curriculum should not focus solely on terminology (e.g., National Council of Teachers of Mathematics, 2000). However, there is a place for vocabulary instruction in the content areas (Gunning, 2003). Since our students will become teachers some day, our purpose was not to assess their knowledge of vocabulary, but to determine if the students, perhaps, were not able to apply this knowledge properly. As a follow-up, in the Spring 2011 semester, the students were given a vocabulary test which included 24 words that were listed by several textbook sources as geometry words expected to be mastered by 6th graders. It was given in a matching format. There were three groups of eight words where each group had one or two distracters (i.e. a triangle has three sides). While the distracter was true it could have fit several of the choices but there was only one correct answer for the other questions in the set. The distracter had to be used after the one and only correct answers were selected for the other questions. Approximately 60 % of the participants scored mastery (90-100%) on the vocabulary section. Less than 5% had a failing score (below 75%) and the remaining 35% scored passing but not mastery.

The tests or inventory of skills were given within the first month of the semester. The inventories were given in order that skill modules can be developed to meet the student specific needs. These will be developed and given to the students during the Spring 2011 semester. A large majority of these students will be in the classroom either for student teaching or as teacher of record within a year. Therefore it is imperative that the teacher candidates are prepared to teach these geometry skills effectively.

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LEARNING CREATIVELY WITH GIANT TRIANGLES

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The challenges of simultaneously developing deep content knowledge and pedagogical content knowledge (e.g., Shulman, 1986; Ball & Bass, 2000) for both in-service teachers and pre-service teacher candidates have been expressed for some time. Utilizing the power of the hidden mathematics framework as suggested by Abramovich and Brouwer (2006) requires finding, creating and working with mathematical problems that connect across the mathematics curriculum. We have met these challenges, and achieved new levels of success through a set of unique inquiry experiences, which develop visualization and deep, analytical content knowledge as well as pedagogical knowledge within an urban pre-teacher mathematics methods course. Of particular interest in this approach are the unique objects that are used - Giant Triangles. These triangles are equilateral, 1-meter in edge length, made of highly resilient, lightweight and brightly colored plastic (similar to that used in many popular kites), and can be quickly assembled and reassembled to create a wide range of polyhedra. The giant triangles, as constructed, have powerful inbuilt mathematics that naturally emerge as learners interact with them using very carefully designed activities. Furthermore, the triangles relate physically to learners through their kinesthetic character and aesthetic appeal. The learners' use of the materials changes as their knowledge develops observably shifting from concrete to abstract through interaction, in groups, with the materials.

The university where this work was done is one of the most ethnically diverse liberal arts institutions in the mid-south-western United States and is a federally designated Minority Serving Institution. It provides 4-year degree programs and has an open enrolment policy. A large percentage of its undergraduate students are the first college attendees in their families, and work full-time while attending college. We share a learning trajectory that was successfully enacted in ten mathematics methods classes for elementary and for middle-grades pre-service teachers in single 160-minute class sessions during two successive semesters.

The learning trajectory consists of constructing and comparing the pyramids formed from equilateral triangles; constructing the triangle-faced Platonic solids and reasoning about the limited number of Platonic solids through newly developed visualizations; enumeration of patterns relating the numbers of vertices, edges and faces of a variety of polyhedra; doubling the edge lengths of the tetrahedron to determine the scaling effect on surface area and volume, and associated tessellation of space using tetrahedra; and, duality among the Platonic solids. Reflective comments from learners attest to the power of this learning trajectory in changing attitudes among pre-service teacher candidates about doing and teaching mathematics.

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HELPING PROSPECTIVE TEACHERS PAY ATTENTION TO STUDENT THINKING: THE EFFECTS OF AN ONLINE DISCUSSION BOARD

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Some educators have recently suggested taking a new orientation toward teacher education, in which prospective teachers (PTs) are taught the skills and dispositions needed to improve their teaching over time through a systematic study of their own practice. One method for this improvement of teaching, in which teachers use a framework of setting learning goals, teaching a lesson and collecting evidence of student achievement of those goals, creating hypotheses about how instruction helped that achievement, and revising the lesson accordingly, was outlined by Hiebert, Morris, Berk and Jansen (2007). This method ("lesson experiments") is similar to the practices of lesson study, but does not require a collaborative group of educators and may thus be easier to implement for individual teachers.

Emerging research (e.g. Spitzer et al., 2010) suggests that classroom interventions can help PTs improve their lesson experiment skills. However, the time needed for interventions may be difficult to integrate into already-full teacher education programs. Thus, online formats, in which PTs can interact with each other and collaboratively improve their skills outside the classroom, have an obvious advantage. This study aims to investigate the effectiveness of an online discussion board in helping PTs improve their analysis of a lesson.

To investigate this question, we used an online discussion board to help PTs (N = 15) learn to evaluate evidence of students' learning and form hypotheses about the success of a lesson. PTs were given a learning goal and asked to watch a videotaped lesson online. Then, PTs posted initial reactions analyzing the effectiveness of the lesson. Next, PTs read classmate's posts and responded with a reflective comment. Finally, PTs read all posts and then wrote a short paper about their final judgment of student learning and the success of the lesson. Data were analyzed in terms of how well PTs we able to look past surface features of the lesson and instead attend to students' mathematical thinking related to the learning goal.

Initial results suggest that participation in the discussion board activity helped PTs improve their lesson experiment skills; specifically, PTs moved toward evaluating the lesson in terms of student achievement of the learning goal. Specifically, by the end of the intervention (i.e. after the discussion board participation), a higher percentage of PTs' claims were mathematical and relevant to the learning goal (47% compared to 38% in their initial responses). PTs' focus on irrelevant mathematics declined (22% of claims compared to 30%), indicating that they made the learning goal more of a focus in their analysis. However, PTs continued to reference non-mathematical ideas in their evaluation of the lesson (23% of claims in initial responses and 24% at conclusion), suggesting that the PTs' beliefs about teaching, rather than a focus on student mathematical thinking, continued to play a major role in their analysis of lessons. These results suggest that an online discussion board can be used to help improve PTs' lesson experiment skills, though room for improvement remains.

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FACE-TO-FACE AND SYNCHRONOUS ONLINE DISCOURSE AND TEACHERS' KNOWLEDGE ABOUT TEACHING VARIABILITY WITH TECHNOLOGY

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There is a growing emphasis on the development of prospective mathematics teachers' technological, pedagogical, content knowledge, but little is known currently about how discourse in mathematics education courses may affect the construction of this specialized knowledge. Knowing more about how prospective teachers may engage and interact with one another and the knowledge they construct in face-to-face and online settings may reveal important similarities and differences for teacher educators to consider. With variability being the content focus of this study, the TPSK framework (Lee and Hollebrands, 2008) is used to identify knowledge disclosed through discourse in face-to-face and synchronous, online settings.

Research Question and Methodology

- What is the relationship between discourse and the development of knowledge for teaching variability with technology in face-to-face and online mathematics education courses?

Because discourse and constructed knowledge cannot be completely identified and analyzed quantitatively, a qualitative research methodology was selected (Patton, 2002). The author taught a five-week unit on teaching data analysis and probability with technology to middle and secondary prospective teachers as part of a *Teaching Mathematics with Technology* methods course. The face-to-face group (n=25) and an online group (n=17) which met live, using the web-conferencing tool *Elluminate Live* used the same curriculum (Lee, Hollebrands, & Wilson, 2010). Each class session was video-taped or recorded. In addition, a small focus group was selected to watch closely during small group discussions during class and each member of the focus groups was interviewed twice during the study. A pre/posttest was given to obtain information about prospective teachers' statistical and probabilistic knowledge, particularly as it related to variability.

Discussion

Data for this study is under analysis. However, data from a pilot study with face-to-face (n=14) and online (n=8) groups revealed discourse patterns between prospective teachers and misconceptions regarding box plots, residuals, and the law of large numbers. This poster will present: (1) three tasks, given to prospective teachers in the current study, aligned to address those misconceptions, and (2) a comparative analysis of discourse from both environments and prospective teachers' knowledge about how to teach variability with technology.

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IMAGES, ANXIETIES AND ATTITUDES TOWARD MATHEMATICS

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Images of mathematics as fallible or infallible can have potential negative and positive impacts on pedagogy of mathematics with the subsequent change of attitude towards mathematics as positive or negative (Belbase, 2010; Goolsby, 1988; Lakoff and Nunez, 2000; & Ma and Kishor, 1997). An image of mathematics due to a lack of comfort that someone might encounter when required to perform mathematically may result into a mathematics anxiety (Wood, 1988), or the feeling of tension, helplessness, and cognitive disorganization one may face when required to manipulate numbers, symbols and shapes (Richardson and Suinn, 1972; Tobias, 1978). In a long run, mathematics anxiety can harvest different feelings such as like/dislike, happiness/worrisome, and motivation/fear (Hart, 1989; Wigfield and Meece, 1988).

A theoretical-analytical model with combinations of images (fallible, infallible), anxieties (high, low self-esteem), and attitudes (positive, negative) toward mathematics resulted eight possible outcomes representing philosophical-psychological-perceptions about mathematics. These combinations are- (1) infallible, high self-esteem, and positive attitude; (2) infallible, high self-esteem, and negative attitude; (3) infallible, low self-esteem, and positive attitude; (4) infallible, low self-esteem, and negative attitude; (5) fallible, high self-esteem, and positive attitude; (6) fallible, high self-esteem, and negative attitude; (7) fallible, low self-esteem, and positive attitude; and (8) fallible, low self-esteem, and negative attitude (Belbase, 2010). Some of these combinations (1, 4, 5, & 8) seem to be theoretically and practically viable, but others (2, 3, 6, & 7) can be non-practical because high self-esteem and negative attitude and low self-esteem and positive attitude toward mathematics seem contradicting. Such analyses may be helpful to understand possible consequences of these relationships with viable research and pedagogical implications.

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TEACHERS' BELIEFS ABOUT THE MEANINGS OF THE DERIVATIVE

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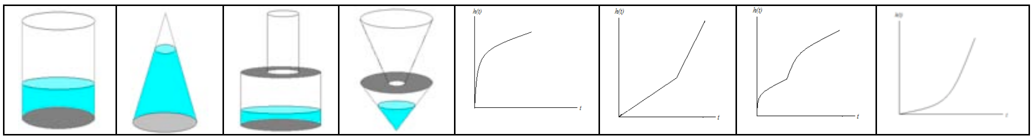
In this report we researched about the beliefs calculus teachers in Mexico have about meanings associated to the derivative function. We designed a questionnaire containing problems: one of them qualitative related to the filling of containers and the other one conceptual related to the analysis functions. Their written and spoken arguments allowed us to categorise their beliefs, determine their strength, coherence and the relations that they establish individually which internally form individual systems that are called Internally Coherent Systems of Beliefs.

Introduction

One of the most serious problems that Mathematics teaching faces in Mexico is a heavy operative load (De la Peña, 2002). This contributes very little to construction, to the comprehension of the meaning of concepts and it deteriorates the conceptual part (Amit y Vinner, 1990; Schoenfeld, 1985; Hiebert y Lefevre, 1986). Particularly the derivative function, we have a concept with many meanings associated to it, his comprehension is more complex because it is not the same learning derivative techniques, understanding its meanings. This doesn't mean that calculus teaching at High School level must be aimed at a strict presentation that takes the place of meanings, not by any means. But the direction depends on the teacher because he is the one who in the end chooses teaching topics and decides on their organisation moved of course by his or her beliefs, ideas and attitudes (Philipp, 2007; Sivunen & Pehkonen, 2009). But, ¿Do teachers understand and distinguish between qualitative and conceptual parts and the management of ideas? ¿What are their beliefs about those ideas? There is very little research about such questions in the field of calculus (Törner, 1999; Badillo, 2003; Speer *et al.* 2005; García, *et.al.* 2006 Hähkiöniemi, 2006;). Exploring the beliefs of teachers is important since recognising such beliefs has a great potential to inform about educational research and practice. According to MaaB & Schlöglmann (2009), identifying them is an essential prerequisite, mostly in relation to central concepts contained in the curricula of school mathematics in order to change and improve teaching in schools. This is our research's rationale whose objective is to describe and interpret the beliefs calculus teachers at a High School level have and the relationships such teachers establish among them, about the meanings of the derivative function. The questions that guide it are: ¿Do calculus teachers notice that the derivative function allows the study of instant change rate? ¿Is the adequate nature of the derivative function clear for calculus teachers?

Methodology, Questionnaire and Participants

The research we did is a qualitative one and is a study of multiple cases. We chose the survey. The instrument we used was a questionnaire whose design responds to the warning made by Tirosh (2009) in the sense that the design of reliable research instruments or tools is one of the challenges that research in mathematical education generally faces. Besides this, we brought about conversations between pairs considering that it demands proof and refutation about their arguments. Those conversations allowed us to prove the effectiveness of the intervention in the creation, modification and elimination of one of the teachers' beliefs. The questionnaire contains two problems that are no routine: the first of them involves an activity of matching seventeen containers that can be either cylindrical, conical or both in which water is poured at a constant rate. It contains graphs showing water height variation in function of time



The second problem involves obtaining the derivative of a function:

$$f(x)=\begin{cases}\sqrt[3]{x} & 0\le x\le 1\\ -\left(x-\dfrac{7}{6}\right)^2+3 & 1<x\le \dfrac{13}{6}\\ 2x-\dfrac{7}{3} & \dfrac{13}{6}<x\le 4\end{cases}$$

This type of functions simulate the filling of combined water containers with different radius and have been utilized in other research studies (Hirst, 1972; Hohensee, 2006). Participants are made up of twelve teachers that belong to three different teaching communities: four engineers who studied at a Polytechnic Institute , four engineers who studied at public Universities and four engineers who studied at normal schools. In Mexico there is no institution devoted to the formation of mathematics teachers for high school levels. Regarding experience in calculus class, teachers Roldán, Buendía, González, López and Camacho are fairly experienced, whereas Zedillo, Rosas, Matías and Zavala show an intermediate level of experience. On the other hand, Gutiérrez, Montoya and Elizarrás show very little or no experience.

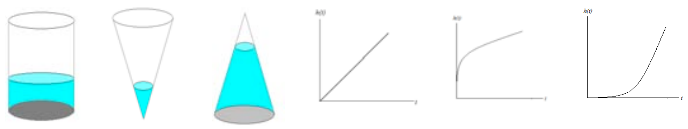
Theoretical Framework

In this theoretical framework it is assumed, to begin with, that beliefs exist in the subject related in a system which is logical to its bearer (internal coherence) and probably 'illogical to the researcher' (inconsistency); the reason for this lies in the explanation that beliefs being highly psychologically loaded take the place of logic. In this sense, the dimensions of Green's (1971) metaphor about beliefs are used: *strength* (central or not peripheral central), *quasi logical relations* (basic or not basic) *and cluster grouping,* along with Thagard's (2000) coherence theory as theoretical references.

Results

When the fact that teachers have different mathematical qualifications comes to mind, one could pose a hypothesis stating that differences between beliefs would be significant; however, reality shows the opposite.

one variable (shape)

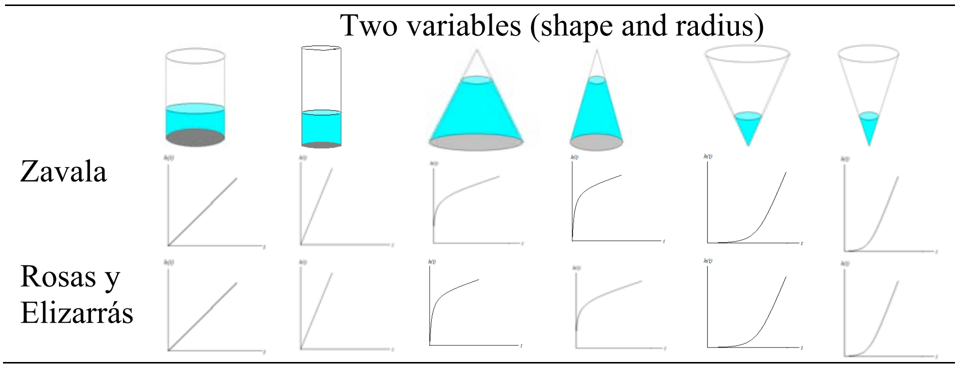


Everybody makes associations correctly like this: cylindrical container-straight line and conical container-curve and in the latter not everyone does it. Most of them use visual and geometric arguments for instance, Rosas

$V = \frac{\pi}{3} r^2 h$ 1) La altura se mantiene constante 2) La variación cambiara con respecto al radio 3) Se mantiene constante al inicio 4) Cambiara su variación y despues crecera de manera casi constante
Trad: 1).The height remains constant 2). Variation changes in relation to the radius 3) It remains constant at the beginning 4) It changes its variation and after that it will grow in an almost constant way

Or physical arguments like López who talks about celerity and speed:

López: First of all the graphs were considered to have two axes *t and h(t)*. That is *t=time; h(t)=speed or filling celerity*. In the first exercise it is possible to appreciate that since the shape is a cylinder, the speed is constant and so is time, for which reason a function of a straight line appears and that is answer C. In exercise two it is a cone where the base is at the top, wherefore, the celerity is pretty high at the beginning; as time passes the filling speed becomes slow and the graph that sticks to this is option A. In exercise three it is a cone where the part of the base (wide) starts filling, so the filling is slow and as time passes the filling is faster because the container is narrower at the top, the answers is B.



When the radius of the containers is changed, teachers' answers show an advance in the cognitive route because they notice the direct relationship that exists between straight line inclination and radius variation. They establish the functional relationship: straight line slope-base radius, a function like $f(t)=f'(t)t$. When it comes to conical tanks the situation changes a little Zavala, Rosas and Elizarrás identify the curve as the graph that simulates but fail to associate the corresponding concavity. Particularly when analysing cones standing on their vertex, Zavala is also unable to notice the celerity with which the height changes in the containers when $t \to 0$. For the third activity involving mixed containers, in the case of cylindrical assembled containers all of the teachers establish the correct and coherent association: container-broken line, even when such containers are made up of a conical section, container-smooth curve. The following dialogue between Roldán and Gaby shows the change of beliefs regarding a smooth curve resulting from the solution of this task

Gaby: But now in these ones, I mean for instance here we obtain a smooth curve because this base is the same as the other, and also here and also here (pointing at some graphs that correspond to containers that join with the same radius).

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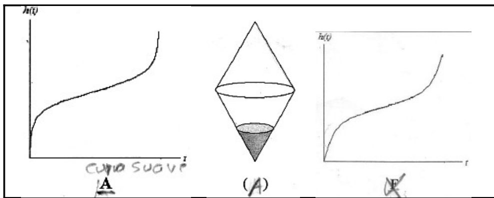
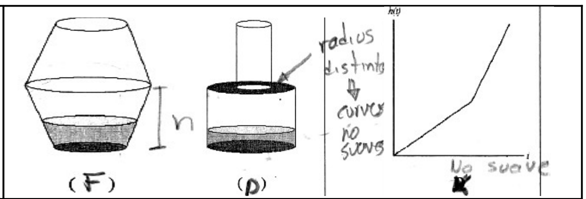
<tr><td></td><td></td></tr>
<tr><td>Trad:
Smooth curve</td><td>Trad:
Different radius, not smooth curves</td></tr>
</table>

Gaby: Then where the shape changes where they join. Lets say this thin cylinder and this big part, the curve is not smooth any more.

Roldán: And this one no, it is the curve that is not smooth…

Gaby: That means, C, D and H, mmmm

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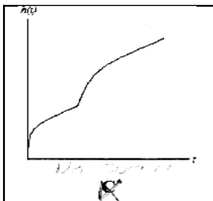
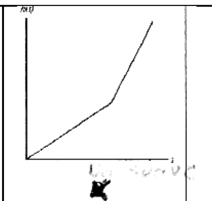
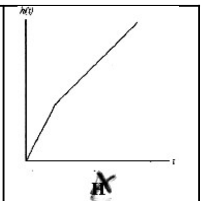
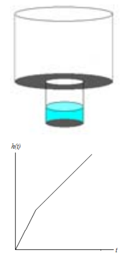
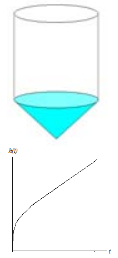
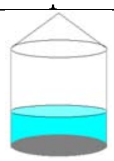
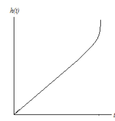
<tr><td></td><td></td><td></td></tr>
<tr><td>Trad:
No smooth</td><td>Trad:
No smooth</td><td></td></tr>
</table>

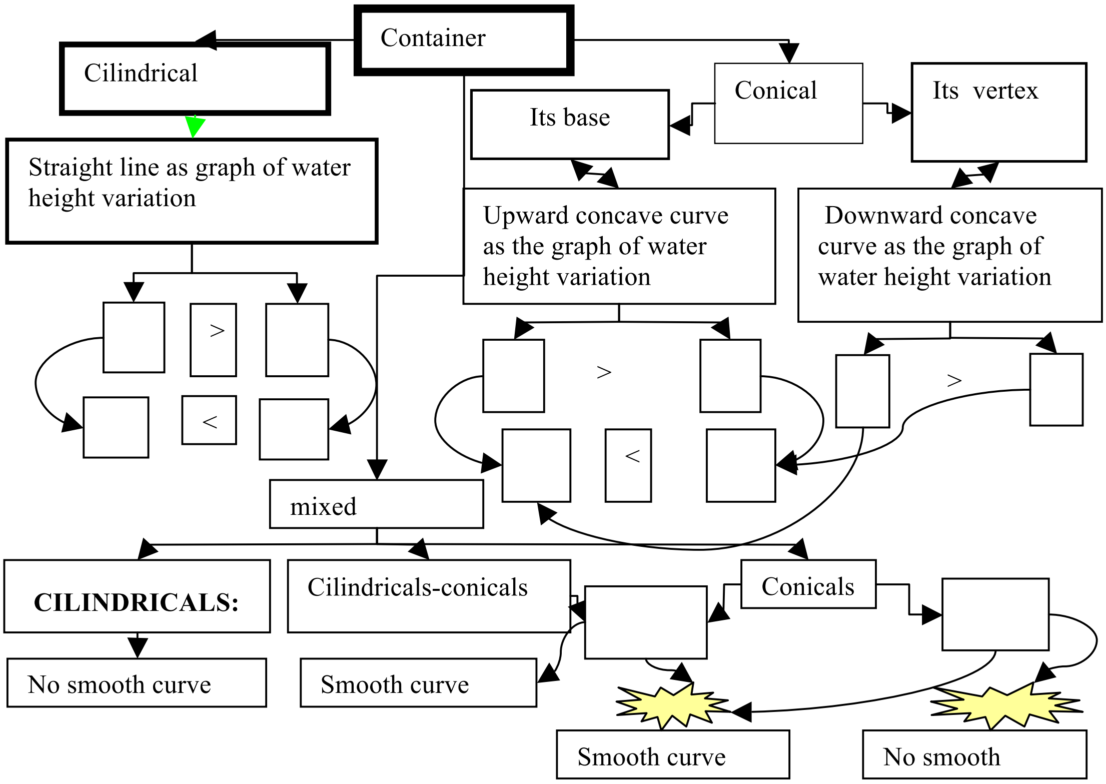
Roldán: Well, they are not physically smooth where they join they are from different areas.

For some teachers like Lopez, it only meant the combination of the two previous activities

el ejercicio 1 son 2 cilindros i gráfica del cilindro delgado y la del grande juntas (H)		el 3 juntas (B) **Trad:**		

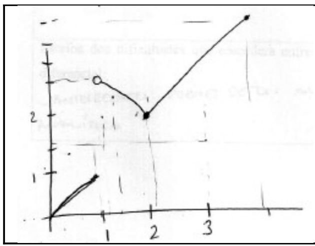
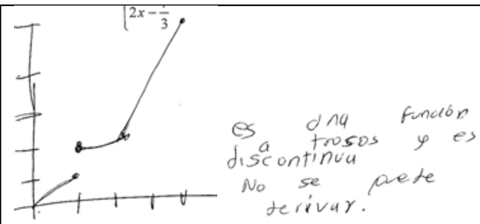
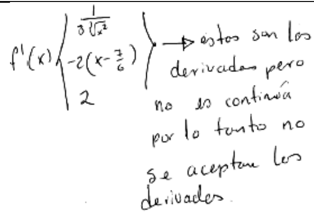
Trad: Exercise 1 consists of two cylinders, thin cylinder graph and big cylinder graph together		The three together		 la 4 juntas (E) **Trad:** Four…Together

The following chart shows a net of concepts established by them

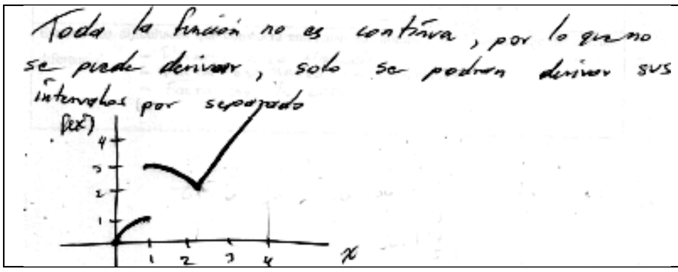


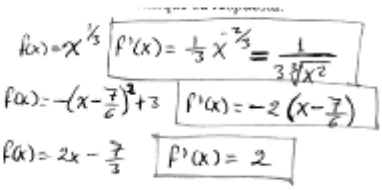
Net of beliefs in the problem of container-graph association

In the task of calculating the derivative of the function defined by pieces, most teachers notice singular characteristics of the function regarding continuity of the function and its relation with its capability of differentiation. They obtain formulae parting from the formulae with which it is defined by using derivation techniques disdaining what the graph visually offers to them. Buendía and López provide two sample cases where definition intervals simply disappear.

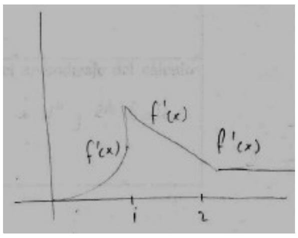
	
	
Trad: These are the derivatives but it is not continuous, therefore the derivatives are not accepted.	**Trad:** it is a function by pieces and is not continuous. It can't be derived

Cedillo comes a little close to a good reasoning but ends up doing the same:

	**Trad**: all the function is not continuous, that's why it can't be derived. Only its intervals can be derived separately



others like Montoya apply derivation rules and even sketch out the “derivative of the function”



$$f'(x)=\begin{cases}\frac{1}{3\sqrt[3]{x^2}}\\ -2\left(x-\frac{7}{6}\right)\\ 2\end{cases}$$

So the task of obtaining the derivative of these functions defined by pieces is reduced to obtaining formulae from the formulae that define the function. This is a very strong belief. Roldán's dialogue shows this strong belief as well but trough the dialogue, from which an extract is presented, he is able to relate continuity-smoothness of the curve-differentiability.

Roldán: But the criteria to say that a function can be derived is that…it is continuous ¿isn't it? It is the first criterion ¿isn't it? Er…¿Because of the limit or? …(silence) since in my classes I never get to these… from mi classes at university I don't remember it divides in intervals… er, by pieces. I don't remember functions by pieces because the teachers I think they were cheating, I don't remember them teaching me a function by pieces; then um… it must be continuous…

Guide: I don't know.

Roldán: They must be united smoothly and to do this we have to get the same slope in the value that I am considering, which in this case is one and in this, for these two, it is thirteen sixths. Then I'm going to replace the derivative... of all of them in the number… well of the two first ones in number one,…in number one the result of the first exercise is one.

Conclusions

Teachers have a qualitative "understanding" of the rate of change which is independent from any meaning of the derivative function. What can be perceived is that the concept of differentiability of the function is an encapsulated concept without any relations with real situations as long as they don't make clear the need for using it. Coherent association in all the cases is due to good perception more than to the posing of an analytic strategy since the arguments offered respond more to what is believed and felt. This is confirmed in the second problem where teachers are coherent with their ideas about derivation. They simply apply techniques without paying attention to the intervals that define the function. Any function defined by formulae believes that it must have a derivative defined by other formula. This research allows us to obtain a general conclusion: beliefs are an independent variable in teacher formation and their experience. It has two separate beliefs clusters, the one of the concepts and another one, the one of perceptions both of which are incompatible. However, this doesn't stop teachers from being sure about the coherence between elements that they cause to intervene. In this sense the teacher and the general subject have an Internally Coherent Beliefs System (ICBS). A secondary conclusion emerging from this research has to do with the design of didactic situations which show the beliefs of the subjects and provide precise information which allows to pronounce, just as a good doctor would do, a precise diagnose in order to design the appropriate treatment. The intervention project to eliminate, modify or create certain beliefs, in this sense the dialogue, is the technique that allows this. Roldán illustrates this situation because he shared many of the beliefs at first, he agreed on their strength, on the quasi logical relations and even on the clusters. Intervention by means of "dialogue" shows that it is possible to modify those beliefs.

Discussion

This research leads to consider deeply teachers beliefs. Perhaps attention should not be paid in looking for didactic situations but first address the need of changing teachers' beliefs about concepts into real knowledge.

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MATHEMATICS TEACHERS' BELIEFS AND THEIR TEACHING PRACTICES

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This study focuses on exploring in detail the relationship between teachers' professed beliefs and their classroom practices. The goals of the paper are to examine the kinds of beliefs a mathematics teacher states and to compare those professed beliefs with two teacher's enacted beliefs as seen through their teaching practices. Professed beliefs were elicited through interviews with the teachers. To compare professed beliefs with enacted beliefs, I then analyzed episodes of their classroom teaching. Analyses suggest that a teacher's professed beliefs may differ from his or her enacted beliefs and the relationship between professed and enacted beliefs is complex, and mediating factors need to be further explored.

Teachers' beliefs about teaching, learning, and the nature of mathematics play a major role in the ways that teachers shape their classroom practices. Yet, as David Cohen's study of "Mrs. Oublier" demonstrates, there is not necessarily a clear relationship between teachers' professed beliefs and their beliefs as they appear to be enacted in classroom practices (Cohen, 2009). Accordingly, this study set out to explore in depth the relationship between professed beliefs and classroom practices.

Although there is no universally agreed upon definition of beliefs in education research, there are many common threads between different definitions. For the purposes of this study, I adopted the following definition, which is largely consistent with general conceptions of beliefs in the literature:

> [Mathematical beliefs are] interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior (Schoenfeld, 1992, p. 358).

Most studies of teachers' beliefs use questionnaires and surveys. However, data from survey methods can be limited in how far it allows a researcher to examine beliefs in detail. Also, data from questionnaires are often superficial. Therefore, in order to create an accurate representation of teachers' professed beliefs, I conducted in-depth recorded interviews with two teachers. In order to compare these professed beliefs to teachers' enacted beliefs, I analyzed recorded episodes of the teachers' classroom teaching as well. The video data is drawn from a lager study, the mathematics component of the Strategic Education Research Partnership. Analyses began with a transcription of all interviews and teaching videos. I then developed a coding scheme for each teacher separately. From this coding, I generated a summary of all teacher beliefs that appeared from what they stated in their interviews. Upon generating categories of teachers' professed beliefs identified in the interview, I then applied these two teacher-specific coding schemes to analyze their enacted beliefs in the classroom.

Analyses suggest that a teacher's professed beliefs may differ from his or her enacted beliefs. Some of their professed beliefs were strongly consistent with their enacted beliefs. Others were weakly consistent with their enacted beliefs. Furthermore, some professed beliefs were not enacted at all in their teaching practices. Findings may have some implications for mathematics teachers' professional development, pre-service teacher education, and research about teachers' beliefs. The examples of these results and findings will be discussed in the poster.

There are some limitations in this study. Interviews may not sufficiently capture teachers' beliefs and I may need to examine their beliefs more deeply with other methods. Furthermore, the teaching videos that I analyzed are limited cases for examining their beliefs. Therefore, expanded cases and mediating factors of the relationship between professed beliefs and enacted beliefs need to be further explored.

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CONSTRUCTING EQUITABLE TEACHING PRACTICES: AN ANALYSIS OF MATHEMATICS TEACHERS' CONVERSATIONS

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This poster traces teachers' evolving narratives of student performance that arise in their conversations. Analysis of the conversations showed teachers' frames became more nuanced in their representation of mathematical competence, disentangling issues of ability from school-savvy. Overall, teachers' frames shifted from less agentic to more agentic ways of understanding problems of practice as co-occurring with development of more equitable teaching practices.

Poster Summary

How does teacher community matter to teachers' development of equitable teaching practices? Teachers' collegial conversations about student ability not only provide important frameworks for collective understandings but shape consequential decisions about teaching (Horn, 2007). Teachers' colleagues thus play an influential role in constructing notions of and potential solutions to problems of practice. The same situation can be narrated differently—a student "flunked out" or "was not yet ready to learn"—and each interpretation positions teachers differently in relationship to less agentic and more agentic ways of understanding such problems.

This poster represents an analysis of conversations among 5 mathematics teachers at urban Clark High School who sought to raise the achievement of students in first year college preparatory mathematics classes. The project aimed to help teachers reconceptualize student competence while they made major changes to their teaching practices to combat the struggling student problem, shifting away from direct instruction toward a particular equity-centered groupwork method. The focal department had 50% failure rate in these classes before the project began. After one year of deliberate work, the failure rate fell to 29%. State exam scores demonstrated increased student learning, especially among underperforming student populations.

To understand teachers' learning throughout this process, we traced the evolving narratives of student performance that arise in their conversations over the year using a case study design and ethnographic methods. Out of a dataset of 32 audiotaped meetings, we focused on 4 meetings that took place in October, January, March, and May of the school year. Using discourse analytic concepts, particularly Goffman's (1974) idea of frames, we analyzed the shifting understandings evidenced in teachers' talk.

Analysis of the meetings showed that many initial framings did not provide teachers with an obvious actionable response. For example, the teachers diagnose students' academic troubles in the October meeting as stemming from students "choosing to fail" or "being flaky." In contrast, during the May meeting, frames emerge that allow for clearer action, such as students "having low academic status" and "not knowing how to do school." While the former frames place the onus for achievement primarily on the students, the latter provides teachers a means for action. As such, the teachers' frames became more nuanced in their representation of mathematical competence, disentangling issues of ability from school-savvy. Overall, teachers' frames shifted from less agentic to more agentic ways of understanding problems of practice as co-occurring with development of more equitable teaching practices.

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PROMOTING MATHEMATICS TEACHERS' DISCOURSE-BASED ASSESSMENT PRACTICE IN LOW SOCIO-ECONOMIC STATUS SCHOOLS IN TAIWAN

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A large achievement gap of mathematics learning between high and low socioeconomic status (SES) students has been observed in Taiwan, possibly coming from the difference of SES-different teachers' instruction (Chen, Crockett, Namikawa, Zilimu, & Lee, accepted). Research (Crockett, Chen, Namikawa, & Zilimu, 2009) has suggested that improving teachers' discourse-based assessment practice (DAP) helps enhance the weak link between teacher professional learning and their students' learning. This article aims to bridge this gap by reporting a research project that tries to improve mathematics teachers' DAP in low-SES schools in Taiwan. The teacher professional development program for DAP was conducted from the end of August 2010 to mid-April 2011. In the study, the first author facilitated participating teachers to view videotapes and transcripts of their own lessons, to analyze the quality of teacher-student discourse with a focus on questioning and feedback, and then to discuss their own findings with one another. Mehan's (1979) ethnomethodological approach, a parsimony framework for analyzing DAP (Chen et al., accepted), was applied to identify IRE and extended sequences.

Research findings indicate that after professional development, the participating teachers: (a) allowed more wait time for their students' responses; (b) were more aware of the quality of questioning; (c) used more high-level questions to cultivate their students' mathematics understanding; and (d) invited greater student participation in teacher-student discourse. In addition, students' learning benefits from their teachers' professional learning. They actively participated in classroom discussions more than before and began to pose questions and debate on mathematics ideas with one another.

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TRANSFORMATION OF MATHEMATICS COACHES: EXAMINING CHANGES IN MATHEMATICAL KNOWLEDGE FOR TEACHING

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This presentation examines changes in mathematical knowledge for teaching of mathematics coaches participating in an ongoing, embedded professional development program. This in-progress study aims to determine whether the coaching program is making statistically significant change as it relates to coaches' mathematical knowledge for teaching.

The Mathematics Coaching Program (MCP) is an ongoing professional development program aimed at improving student understanding of mathematics using mathematics coaches embedded in low-performing schools throughout the state of Ohio (Brosnan & Erchick, 2010). heir intensive model provides sustained professional development of teachers within their own classrooms. Additionally, MCP provides two days of professional development every month for their mathematics coaches. Regionally located facilitators provide two additional days of coach professional development each month.

Over the course of roughly six weeks, MCP coaches provide daily support to participating teachers; helping them to examine and understand their students' mathematical thinking and how to use this knowledge to guide classroom instruction. Dialogues between coach and teacher improve practice through collegial interactions and personalized support (Becker, 2001). Throughout the course of both the coaches' and teachers' involvement in MCP, growth in mathematical knowledge for teaching (Ball, Hill, & Bass, 2005; Hill, Schilling, & Ball, 2004) as well as transformation of pedagogical methods may occur.

Data will be drawn from the past six years of MCP participants, using the University of Michigan Learning Mathematics for Teaching (LMT) Project's instrument. All participating coaches are analyzed for development of mathematical knowledge for teaching over time. Coaches are tested at the beginning of their first year of participation, and again at the end of their first, second, and third years of participation. Average pre-post change for each group of coaches (first year, second year, third year), as well as p-values and effect sizes will be displayed. Interpretation of said analysis will occur and be presented on the poster. Preliminary results from the first three years of data indicate growth in all three LMT content areas, most notably in algebra.

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METAPHOR TO METONYMY OF SLOPE

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In this paper, the nature of an intermediate algebra student's use of a metaphor and a metonymy is discussed. A metaphor of slope is used in the description of the concept and the same metaphor became a metonymy in responding to mathematical questions.

Introduction

The concept of slope is essential to college mathematics learning, especially for those covered in courses calculus and up. Because of the importance of slope in higher mathematical understanding, it is necessary to make sure that students gain an accurate understanding of the basics of the concept before taking advanced courses. Many students in entry-level courses do not have a background of mathematical knowledge from which to attempt to make sense of newly introduced mathematical information (Sfard, 1997).Thus they turn to their existing knowledge, which mainly originates from everyday experiences. This paper reports on a metaphor originating from everyday experiences and a metonymy formed as a result of classroom instructions, which are used to construct an understanding of the slope ideas.

Metaphor

A metaphor can be defined as an implicit analogy (Presmeg, 1997). Presmeg adds that a metaphor has both ground and tension. According to her, similarities between concepts constitute the ground and differences constitute the tension. For instance, for the mathematical statement, "A is an open set," the tension of a metaphor may be the physical idea of openness (an open space view) without a boundary and the mathematical idea of an open set with a boundary. Consider a set of all points that satisfy the inequality, x2+y2<1. Here, the open set is bounded by the unit circle (XXXX, 2007). Considering that similarities between concepts (source and target) are mainly determined by students based on past experiences, rather than being given to them (Sfard, 1997), students, in this case, may apply the no boundary characteristic of the source concept, and come to a conclusion that the particular set is not open since it has a boundary (XXXX, 2007). Moreover, the students' misconception may further be strengthened if initially introduced examples of mathematical open sets are those without boundaries.

Metaphors play an important role in reasoning in mathematics (Lakoff and Nunez, 2000; Presmeg, 1997). Presmeg illustrates how one may reason with metaphors in her example of a high school student who used "Dome" metaphor to reason in solving the question of finding the sum of the first 30 terms of a sequence (5, 8,11,…). This high school student's personal metaphor seemed to play a significant role in solving the problem. The student whose interview excerpts are reported here also appeared to reason with his metaphor in describing slope.

Metonymy

Work on metonymies mainly focuses on them as literary devices, rather than cognitive constructs that are used to encode information. Presmeg (1998; 1997) and Lakoff &Johnson (2000) on the other hand view metaphor and metonymy as cognitive structures. The act of using one object to stand for another is considered as functioning with metaphors or/and metonymies. Presmeg (1998; 1997) considers two types of metonymies. One of which, namely metonymy proper, is defined as "*a figure by which one word is put for another on the account of some actual relation between the things signified*" (Webster). An example of this kind is "*We studied Gauss*." Here, the word "*Gauss*" is used to indicate Gauss' work (*Gauss Gauss' work*). Moreover, mathematical symbols can be put for various mathematical entities such as number families. The symbol "x" for example can represent real numbers (*x real number*) even though the symbol x and the numbers are two unrelated objects.

Second type of metonymy is considered as figure of speech. In this type, a part is used to represent the whole or vice versa (Presmeg, 1998). An example of this kind may come from the sentence, "*I've got a roof over my head.*" Here, the part "*roof*" stands for the whole "*house*" (*roof house*). An illustration of a circle taken to represent the class of all circles can also be considered as the metonymy of this kind. Presmeg (1997) however argues that this example may go beyond the figure of speech type to metonymy proper for the signifier may not be an element of the class represented. In other words, because the elements of classes are mental constructs, and an act of interpretation by an individual is involved in setting up the metonymy, individual may use the illustration to consider a class of circles that are not closely related to the figure. Hence, the illustration may become an example of a metonymy proper.

Methodology

Data comes from an intermediate algebra student's interview conducted as part of a study that gathered concept maps (Novak and Gowin, 1984) and interpretive essays along with interviews to investigate students' conceptualizations as they were introduced to the various aspects of the mathematical slope and function concepts throughout the fall 2003 semester from two intermediate algebra sections offered at a four year midsize Southwest University in the United States. The intermediate algebra course at the University provides basic mathematical concepts and skills that are prerequisite for college mathematics courses such as pre-calculus and calculus. Both sections of the course were traditional in the sense that its material was covered through a lecture by instructor mode. We refer to the student, whose interview responses reported in this paper, with a pseudo name "SAL" (Hispanic-American Male).

Each interview was videotaped and lasted about an hour and a half. Interviews began with more personal questions, and proceeded to questions on the slope or function concepts and concept maps. Students' inaccurate responses were not corrected throughout the interviews in order to further understand the sources of their mistakes, and to eliminate the possible influence of the interviewer's opinion on students' thought processes. Two mathematics graduate students transcribed the interviews, which were then analyzed by the same graduate students and the investigator in order to identify emerging patterns applying a qualitative approach, namely the constant comparison method (Glaser, 1992).

Table 1. SAL's metaphor of slope.

Source	Attributes	Target
Arcs of houses Edges of buildings Sides of roofs	Meaning of "slope" in English Inclination of physical structures Inclination of common examples of graphs of linear functions (lines)	Mathematical slope concept

RESULTS

Metaphor of Slope and Student Description

During the interview, student SAL provided descriptions in both everyday and mathematical settings. Descriptions came about when SAL was asked to introduce and teach the ideas of slope to the interviewer, and to talk about what he thinks of when he hears the word "slope" as well as to address the question, "what a slope is."

Table 1 outlines SAL's metaphor of slope. As seen in the table SAL compares similarities between houses/buildings and graphs of non-horizontal linear functions to form a mathematical understanding of slope. The setting for the metaphoric comparison appeared to have emerged from his instructors both from high school and early college courses giving examples from daily life.

When student SAL was asked to tell the first time he was introduced to slope, he talked about his high school and early college teachers introducing "slope" as the tilt of a line or physical entity that is used in construction (see excerpts below). These examples appeared to have encouraged SAL to consider houses as an example of having "slopes" in the sense of the edges of roof tops, a physical attribute. Even though SAL also considered slope as something used to determine how tilt lines are, the physical feature of slope was more dominant in his knowledge and reasoning.

Physical objects like houses/construction buildings seemed to encourage a formation of a physical entity of slope in the case of SAL. That is, examples provided by his instructors appeared to have acted as analogies for SAL in forming an understanding of the mathematical concept slope. The similarity between the inclined edges of a building described by the English word "slope" may have further enhanced SAL's consideration of the physical attribute of "inclined lines" as being "slopes."

In the following responses, SAL is clearly comparing the aspects of buildings/houses with lines. His phrases "depend how **tilt** it is" and "it's like **a way** to know how **tilt** the line is" provide the clues for his comparison. He is focusing on the tilt/inclination of edges of buildings, beams and houses in describing the mathematical aspects of the slope. Specifically, it is clear in his third response below that he is in fact focusing especially on one attribute of the examples given in the source domain, which is the inclination of structures.

> *SAL: OK, slope? During my second year in high school? Yeah, It was basically said some about, you know, lines, depend how **tilt** it is. That's **the slope**. So like, you didn't get into it really that much. That's about the first time I heard.*

> *SAL: Yeah, he* [his college instructor] *said slope was something that it could be used on almost anything, even like constructions, even buildings, things like that. So it's like **a way** to know how **tilt** the line is used in the field of engineers.*

*SAL: Oh, yeah. I guess when the teacher said it could be used as an engineering tool and as a way of beam stuff, yeah, it makes sense, you know, everything has, I mean, **an inclination**, also you can see the inclination, so, I started looking at the stuff. so…house has **inclination**…*

Metonymy *(Slope ⟶ Tilted Line)* and Student Reasoning

Later in the interview when asked why he included "house" in his concept map, SAL described more of a metonymic aspect of the concept than a metaphoric one. It is evident with two phrases "You know "**house looks like two slopes"** and **"**have a **house slope"** that, for SAL**,** the term **"slope"** solely stands for **"an inclined line."**

*SAL: because I don't, when I … I don't perceive slope, but house is just an example like you know. You know **house looks like two slopes**.*
*SAL: I guess architecture, houses, even anything a house they can have a **house slope**…*

In answering mathematical questions, SAL's preference appeared to be the particular metonymy (slope ⟶ tilted line) over his metaphor. SAL used this metonymy consistently throughout the interview. For instance, when he was asked to give an example for the slope of a linear function, he drew an inclined line and said:

SAL: OK. This could be a slope [draws an x-y coordinate with a line on].

He further stated that a horizontal line is not a slope. This response leaves no doubt that during the interview SAL used the term "slope" as a metonymy to stand for inclined lines.SAL provided further explanation to why he did not consider a horizontal line as a slope. In fact, his explanation below reveals that he is associating slope with a characteristic of lines, that is their "*tiltedness*," and through this association tilted lines are becoming slopes for him**.**

*SAL: Because the line is **tilt**…*
*SAL: So I know it's sloppy. I know. I think I know that any **tilt line is a slope**. That line* [pointing to a horizontal line drawn on a paper] *doesn't, it isn't tilt at all… and **doesn't just slope**.*

Later in the interview he was asked about the role of a numerical value that the interviewer called "a slope" of a linear function. His response (given below) to this question is further verifying his metonymic use of slope as tilted line. He in fact declares that the value can be graphed, clearly meaning a line.

> *I: I guess I was wondering and just curious why it's that we need to get that number* [referring the slope value written on a paper].
> *SAL: Which?*
> *I: This number.*
> *SAL: Oh, you know because so you can* ***graph it.***
> *I: Oh, I see. Graph what?*
> *SAL: You can graph* ***this slope****.*

During the interview, SAL was aware of the slope formula and was able to calculate the slopes of lines provided that two points on the graphs of lines are given. He however did not appear to hold the meaning of slope as "a measure of rate of change."

> *I: What is that formula for?*
> *SAL: ... This formula is, when they give you two points, so you* ***can get a slope on those two points****. And this one, the one with "m," like y equals to m and and b. This is to get a, just to get a slope.*

Lack of the knowledge of slope as a measure of rate of change can further be seen clearly in his response below. When SAL was asked whether he could obtain a function value for a given x-value provided that another point and the slope value of the function are given, he stated that, with the given information, he could get the intersection points of the graph of the function. He also added that, with slope and a point, he only taught of a line and a point, and nothing more. It is evident in his phrase "this point will intersect with the slope" and the excerpts below that he is clearly not considering slope as a measure of rate of change, but using the term "slope" to mean a line.

> *SAL: Ok, the slope is 2 and the point is… we can get the intersection of the, the line….*
> *SAL: This point…yes, it is just this point. I don't know****, what all that I see is a line and a point****, that's all I see…*
> *I: Which line do you see? Which line?*
> *SAL: Oh, the* ***slope****, anyway, it's supposed to go…*
> *SAL: I'm thinking maybe this point will intersect with* ***the slope****, but, it's just some, maybe, I don't know.*

Discussion and Conclusion

Student SAL used one kind of metaphor in his description of the mathematical slope concept. SAL furthermore, later in the interview, turned this metaphor into a metonymy of “slope” for “tilted lines.” Mathematical concept formation appeared to be embedded in one’s social, cultural and pedagogical activities. SAL’s understanding of slope in fact does not solely come from classroom activities but from the cognitive processing of his social and cultural experiences integrated with classroom experiences.

SAL formed an understanding of slope as tilted lines or “*tiltedness*” of lines by adopting the similarities between two domains; source domain of inclined edges (that are called slopes in English) of buildings/houses and the target domain of the inclination of the graph of linear functions (tilted lines). The fact that “slope” term is used to describe both the inclined edges of physical objects and to describe the mathematical concept seemed to have encouraged SAL to form only a visual (physical) understanding of the mathematical slope ideas.

It was apparent in the interview responses that SAL’s metonymy of slope standing for tilted lines began with his metaphor of slope. After the initial metaphoric understanding of slope, SAL turned this metaphoric understanding to metonymic reasoning. In fact, when it came to answering mathematical questions, he consistently preferred the metonymic aspect of slope. Recalling that student SAL had instructors using “construction/houses” as analogies to introduce the mathematical slope concept one implication of the findings for the teaching and learning of mathematics may be that the teachers of mathematics need to be cautious with the use of analogies such as “beams/bridges” and “ski slopes” in introducing the concept. As Max Black indicates, “*Similarity is created in the mind of conceivers of the metaphor rather than being given to them*” (reported in Sfard (1997, page 342)). What is tension in the eye of a teacher may become the ground for students. It is an unavoidable fact that students bring their everyday experiences into mathematics classroom, and consider them during the process of conceptualizing a newly acquired mathematical concept. Teachers may need to explicitly cover the relevant similarities between source and target objects especially when there is a potential of applying irrelevant or incomplete aspects by students. In the case of slope, the rate of change aspect needs to be emphasized. Furthermore, the similarities and differences between the examples from the source domain and the mathematical slope concept from the target domain need to be covered explicitly to make sure students do not consider irrelevant aspects and/or only one relevant feature such as the physical attributes to form an understanding of the subject. This paper reported findings of a single student’s metaphoric understanding and metonymic reasoning. They by no means can be taken as generalization to all mathematics learners. The findings reported in this article though may become the springboard for future investigations with larger sample groups.

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DEVELOPING TEACHER UNDERSTANDINGS OF EARLY ALGEBRAIC CONCEPTS USING LESSON STUDY

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This poster illustrates the use of lesson study as a professional development tool. In particular the poster focuses on the way in which the teachers increased their understanding of how tasks, classroom activity and teacher actions scaffolded student learning of early algebraic reasoning of equivalence and the commutative principle. Teacher voice is used to illustrate how lesson study cycles caused the teachers to reflect and review their own understandings of early algebraic concepts and how their students considered them.

This poster reports on the use of lesson study as a professional development tool to facilitate a group of teachers enhanced development of their algebraic 'eyes and ears' (Blanton & Kaput, 2003). Lesson study is a form of professional development which aims to increase teachers' knowledge about mathematics, knowledge about ways of teaching mathematics, and knowledge about the ways to engage learners in, and make sense of, mathematics (Fernandez & Yoshida, 2004). A specific focus of the poster is on how teachers increased their understanding of how tasks, classroom activity and teacher actions scaffolded student learning of early algebraic reasoning of equivalence and the commutative principle through the lesson study process.

Research studies investigating children's development of early algebraic reasoning covers a wide field including those which focus on classroom practices which scaffold student justification and generalizations. However, in this poster, focus is placed on how lesson study supported teachers to understand how their students constructed early algebraic reasoning in two areas of early algebra; equivalence (equality) and the commutative principle. Participants included two groups of elementary teachers who worked as a professional learning community within their own schools. 'Study lessons' were planned and taught in one classroom and observed by the research group. In-depth analysis and discussion followed observations of the study lesson and subsequent iterations as it was re-planned, re-taught and re-observed in different classrooms as part of the lesson study cycle.

In the poster, teacher voice is used to illustrate how lesson study cycles caused the teachers to reflect on and review their own understandings of early algebraic concepts. The opportunities to closely observe student responses provided a foundation for them to build understandings of how their students approached tasks which challenged their understandings of the commutative principle, and the equal sign. The teachers recognised the pivotal role they held in making links between the early arithmetical and algebraic reasoning and pressing the students towards situations of generality.

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EXAMINING SHIFTS IN TEACHERS' CLASSROOM PRACTICE

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An analysis of how a teacher's instructional practice changed over time as a result of participating in professional development is presented. The specific questions addressed in this analysis include an analysis of the shifts that took place over a three year period. A framework for analyzing teacher's practices as a useful tool for professional development is presented.

Supporting teachers to make shifts in their instructional practice towards reformed approaches that support conceptual understanding is not an easy task. Even though, teachers attend professional development sessions, understanding how it impacts their practice is critical for designing effective professional development. Teachers have to make pedagogical decisions that are adaptive to the learning situation. Recent research on teacher professional noticing (Gamaron, Sherin, Jacobs & Phillips, 2010) point out that teachers' ability to make professional decisions is influenced by their ability to notice and interpret what is going on in the classroom.

Method

Three classroom episodes of a teachers instruction over three years is analyzed using Strauss and Corbin (2001) Constant Comparative method. This teacher participated in a larger Math Science Professional development project in a western state.

Results

The results revealed that shifts in teacher questioning changed along with her beliefs of teaching math. The initial class discussion data revealed that the teacher asked direct questions aimed at getting the "correct" answer such as "So ' ¾ is equivalent to how many 28th?" Students directly answered teacher comments. The teacher questioning shifted along with student participating. She asked more questions aimed at understanding student reasoning and understanding. In addition, students started to respond to each other's comments.

Discussion

We argue that professional developer's ability to notice shifts in teachers' instructional practices and break practice into core components is an important part of designing professional development tasks for teachers. We agree with Gamaron, Sherin, Jacobs & Phillips (2001), that teachers' ability to professionally notice their practice is an important part of improving teaching.

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TEACHER QUESTIONING: TRYING NOT TO REINVENT THE WHEEL

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In this poster presentation we discuss the theoretical and practical issues that we have encountered in trying to conceptualize and document teacher change by examining teacher questioning. Our work is situated within the context of a mathematics-science partnership project, conducted as a 3-year Master's degree program, that engages teachers in intensive coursework and practical experiences designed to enhance their content and pedagogical content knowledge in mathematics and science. The 22 teacher participants are from a public school district in a mid-sized city in the Midwest.

This poster traces our efforts to utilize questioning frameworks that are prominent in the mathematics and science education literature to characterize changes in teachers' questioning practices. Our focus on questioning, per se, was motivated by the IICC teachers' interest in this aspect of their instructional practice. Accordingly, we sought a framework that could be useful from both theoretical and practical perspectives. We examined the literature for models that could inform our analysis of teachers' video-recorded lessons, hoping that an existing framework might be applied to our data, directly or with only slight modification. At the same time, we sought a framework that would be accessible and meaningful to teachers in improving their questioning practices. Drawing on the extant literature to meet the dual requirements of theoretical and practical utility posed certain challenges.

A recurring concern was that of grain size–whether frameworks were detailed enough to detect subtle changes in questioning, yet expansive enough to offer teachers insights into the nature of their questioning. For example, Oliveira (2010) used a broad echoic/epistemic – teacher-centered/student-centered framework. In our analysis we found that the categories were not easily distinguishable and did not provide enough detail to identify subtle changes in the teachers' questioning. Chin's (2007) four overarching categories of questions that promote productive thinking addressed many of our concerns, yet we encountered issues in distinguishing between nonproductive and productive questions. Boaler and Brodie's (2005) framework enabled us to detect differences in the types and frequency of questions teachers posed and could assist teachers in making micro-level changes in specific *questions*, however it might not necessarily support them in making macro-level changes in *questioning*. We conjecture that teachers require a different kind of framework to further promote their development, one that acknowledges the flow of discourse rather than focusing on specific questions only. That is, teachers need to consider what is asked, how it is asked, when it is asked, and how it is followed up; essentially the context within which a specific question occurs.

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A FRAMEWORK FOR ANALYZING TEACHERS' USE OF DATA DURING COLLABORATIVE INQUIRY

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Objectives

We present a theoretical framework for understanding teachers' uses of student learning data in collaborative inquiry processes. The framework is useful to researchers interested in describing and analyzing the ways teachers conceptualize, collect, and analyze student learning data, as well as how teachers discuss results of their analyses

The framework is grounded in five years of analyses of multiple teacher groups at multiple sites. We address the term inquiry stance in the context of a teacher *group* to more fully capture the dialogic aspects of collaborative inquiry. We have found that a group of individuals, even with various stances, do construct a describable group stance toward collaborative work, and our analysis considers the nature of the *interactions* amongst group members rather than any individual teacher's orientation. We further narrowed our own perspective to that which seemed most pressing to the work of the teacher group – their conceptualization and use of student mathematics learning data. We detail the two core dimensions of our framework below.

Framework

For the purposes of this paper, we describe an **epistemological stance** as a way of thinking and being in relation to a particular phenomenon. In our specific case, we are analyzing mathematics teacher interactions in the context of collaborative inquiry processes that incorporate student learning data. Our framework for considering and discussing teachers' epistemological stance toward data lies on a continuum from proving to improving. Teachers with an improving stance toward student data seek to surface limitations in classroom practice through an examination of the data. Teachers with a proving stance toward student data attempt to verify strengths in their practice by uncovering changes in student achievement. The second dimension of our theoretical framework involves the ways in which teachers **position and interact with each other** in the context of collaborative uses of student data. When differing perspectives are surfaced and made explicit, these may be questioned, tested against evidence, compared and contrasted to others' ideas and experiences, and potentially transformed. We characterize these types of interactions as *negotiation*, during which cognitive conflicts are surfaced and explored by group members. At the other extreme, *not negotiation* is evidenced by conversational turns that may be superficially connected through sharing related stories about teaching or students, or teachers taking a task-oriented approach to data. Conversational turns may be related to each other in short sequences, but remain at the descriptive level.

Findings

This study is part of a larger NSF-funded research project involving 8 secondary mathematics and science teacher groups. The above framework was continually refined and described in a matrix used to code transcripts of collaborative inquiry meetings across three years. These data reveal descriptive statistics on the nature of talk and uses of student learning data inside a wide range of teacher groups. Our analysis also focused on the kinds of data teachers accessed during interactions related to student learning. Findings were also related to broader contexts, such as extrenal mandates and school or teacher leadership.

Challenges in Teaching in Applied Mathematics Classes in Urban Schools

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Secondary school mathematics teachers, mathematics department heads, curriculum leaders, and administrators from 11 schools, with support from a local university, participated in the project Collaborative Teacher Inquiry. In this presentation, we will focus on challenges that teachers face when they teach grade 9 applied mathematics.

INTRODUCTION

The transition from Grade 8 to Grade 9 is particularly challenging in many school systems (Galton, 2009). In Ontario, students are promoted to the next grade regardless of their level of performance until Grade 8. In addition, Grade 9 students in Ontario are subject to streaming, being categorized into academic, applied, or essential levels (Ontario Ministry of Education, 2007). As a result, Grade 9 is a critical year for students and teachers. This is especially challenging for urban populations such as the Greater Toronto Area (GTA), where there is a large multicultural population with the highest percentage of foreign-born students in Canada (Statistics Canada, 2005). In fact, 44% of the student population of the GTA has a first language other than English. In this paper, we report the findings to the research question: What challenges do Grade 9 Applied Mathematics teachers face? By identifying the difficulties encountered by mathematics teachers, this research promises to move forward teaching mathematics in applied classrooms, by energizing teams of teachers within schools to activate and guide the teacher improvement process.

FINDINGS AND DISCUSSIONS

We will discuss the various challenges faced by teachers in teaching grade 9 applied mathematics. Due to time limitation, we selected only few categories. There are a number of challenges in urban mathematics education in greater toronto schools. There were a number of situations when challenges identified were multifold. Sometimes teachers did not believe in their capability to achieve success under these circumstances. Teachers said that these negative attitudes along with the students' lack of mathematical background were major barriers to foster student success. Teachers mentioned that they can not be successful alone if their students and their parents do not believe themselves in succeeding.

A mix of behavioral issues and lack of attitudes toward learning was another area of concern. Family situations of students differ considerably and these contexts also brought many issues for several schools. Teachers emphasized the complexity of students being in the same time with behavioral issues and having individual education plans. Lack of time for teachers to prepare their lessons, students need to learn the basics first, concerns about student behavior; and the concerns of preparing students for provincial tests were common themes in both studies. At times, teachers' professional development programs had their own limitations. Implementing reforms that the new curriculum required was challenging by many teachers. In addition, some new teachers had problems in class management. By contemplating on teachers' views in our study, some teachers faced considerable pressure when teaching for applied.

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EXAMINING MATH COACHES' IMPLEMENTATION OF A GRAPHIC ORGANIZER

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This poster shows one piece of a larger Research and Development effort to design a professional development focused on math discourse in elementary classrooms. This professional development evolved from the idea that merely exposing teachers toimportant instructional moves was not enough to support high quality implementation in mathematics classrooms. Rather, we claim that it is necessary to instruct teachers on howto implement instructional moves that support math discourse.

Through collaboration with experts in literacy, the professional developmentincorporates literacy instructional models that support classroom discourse into mathematics teaching and learning. The project has a focus on fully engaging all studentsin mathematics discourse, including English Language Learners (ELL). We contend thatthese models are beneficial to all students (Saunders & Goldenberg, 1999).

Here we focus on the results of presenting one graphic organizer, the Frayer model, to a group of elementary math coaches. One important feature of a classroom rich in mathematical discourse is appropriate use of mathematical vocabulary. Thus, weproposed to the coaches the use of the Frayer model as one tool to launch a discourse-rich mathematics lesson. Using the model can support the assessment of prior knowledge ofvocabulary, allow the class to discuss multiple representations of concepts, and develop acommon understanding of the meaning of a word. We engaged the coaches in completing a Frayer model, with an emphasis on how the model could be used to launch a lesson rich in Math Talk. We then asked them to plan how to take the tool back to their respectiveschools. Coaches were given the option to implement the tool with teachers or students.

In examining the work implemented by the coaches, we addressed the following research questions: What did coaches consider appropriate adaptation of the Frayer Model? What issues emerged for the coaches as pros and cons of using the Frayer Modelin a math lesson? For this study coaches' written reflections and audio-recorded group debriefings were analyzed. Among the fifteen coaches' implementations, a majorityimplemented the model directly with students. Those who implemented with teacherschose to do so as they had experienced the model. The implementations of the modelvaried, with common themes including: student engagement, teacher adoption and adaptation, and the tool as an assessment. The analysis of this data has implications for how graphic organizers are presented in the professional development.

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MATHEMATICS TEACHERS' INVESTIGATIONS OF REASONING THROUGH COLLABORATIVE ACTION RESEARCH

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The National Council of Teachers of Mathematics published *Focus in High School Mathematics: Reasoning and Sense Making* (NCTM, 2009) to address the need for high school mathematics programs to align with the K-12 curriculum goals that were set forth by *Principles and Standards for School Mathematics* (NCTM, 2000). This focus document places emphasis on developing students' reasoning and sense making abilities in order to develop mathematically literate citizens who are prepared for an array of postsecondary work or study options. This study investigates the experiences of seven mathematics teachers responding to these curricular recommendations through individual action research.

Teacher action research is a democratic form of professional development in which teachers can critically examine curricular recommendations generated by outsiders (Stenhouse, 1975). In addition to professional development, action research values the 'insider' stance that teachers hold in their own classroom, and positions teachers as knowledge generators rather than simply recipients and consumers of the knowledge of others (Cochran-Smith & Lytle, 2009). Through collaborative action research, teachers can investigate their individual inquiries while collaborating around a common theme.

A group of mathematics teachers was formed by recruiting teachers interested in investigating their practice through action research. We initially agreed upon the theme of *Reasoning and Sense Making* (NCTM, 2009) as the focus of our work together. We met regularly throughout the school year, a total of nine times, as teachers read *Reasoning and Sense Making* and related practitioner literature that aligned with their chosen foci for their action research. As teachers learned about action research, they individually selected specific actions to take in their practice to incorporate reasoning and sense making. Meetings served as a time for them to share their goals, dilemmas, and successes.

This study used a narrative inquiry approach to investigate the ways teachers incorporated the recommendations for *Reasoning and Sense Making* into their practice, how they interpreted the impact of their actions, and the challenges and opportunities that they encountered throughout the process. Data collected include teachers' conversations, journal reflections, interviews, and classroom observations. This poster presentation will present the preliminary analysis of this study, sharing the variety of ways that teachers focused their action research inquiries, and how their choices were a reflection of those aspects of the recommendations that held meaning for them within the context of their teaching and their visions for their practices. Dilemmas that teachers identified and the lessons they learned will also be shared.

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SECONDARY SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF PROOF BY CONTRADICTION

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The goal of this study is to inquire into secondary school mathematics teachers' understanding about logical structure and validity of proof by contradiction. Three proof tasks were given to 23 teachers and they were asked to evaluate the tasks and to provide reasons. Sixteen teachers were unable to find the premises and conclusions of the statement which were really proved in proof by contradiction, and only a few teachers could explain the validity of the method of proof.

Researches on proof by contradiction which has a specific logic structure of proof, that focus on students' difficulties seem to be greater than those related to direct proof. A proof by contradiction is different from a direct proof which infers a straightforward conclusion from the premise of a given statement. Thus a proof by contradiction shifts from the given statement to another one in which the hypothesis is the negation of the given statement and the conclusion is a contradiction. This research examined teachers' understanding of proof by contradiction, a topic that has not received adequate attention (Antonini & Mariotti, 2008).

This research focused on the following questions: How do teachers understand the logical structure of proof by contradiction and how do teachers understand its validity? Three proof tasks that were purportedly written by students were given to 23 teachers, and they were asked to evaluate tasks and to provide reasons.

After investigating participants' understanding about the hypotheses and conclusions of the new statements which were proved in the format of direct proof, their hypotheses were identified as p q, p ~q and ~q. Their conclusions were identified as a contradiction of the premise, a contradiction of the conclusion for the given statement, and a false proposition. Seventy percent of the teachers were unable to find the correct hypotheses and conclusions of all tasks. On the other hand, six of the participants recognized the valid logical structure of proof by contradiction. To validate the proof method, four of them used truth-table, contraposition and partial contraposition which make a contrapositive statement with one premises and conclusion in case a given statement contains two premises and one conclusion. Only a few teachers could explain the validity, but their responses were inconsistent with the contradiction which occurred in the premise or false proposition. Furthermore, their understanding of the validity of the method of proof was an unconscious one for the majority of the teachers.

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PRESERVICE ELEMENTARY TEACHERS' SUSPENSION OF SENSE-MAKING WHEN SOLVING PROBLEMATIC WORD PROBLEMS

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In this paper we investigate the phenomenon of "suspension of sense-making" (Schoenfeld, 1991) with preservice elementary teachers (PETs) when solving word problems whose solution is not obtained by the straightforward application of the arithmetic operation suggested by the numbers given in the problem. A paper-and-pencil test was administered to 68 PETs enrolled in two sections of a content mathematical course. The test consisted of 8 experimental items and 4 buffer items. The experimental items were adapted from Verschaffel and De Corte's (1997) study. Prospective teachers performed poorly on the experimental items. The number of realistic responses varied from 5 to 58. One of the two lowest numbers of realistic responses was for the problem "Sven's best time to swim the 50 m breaststroke is 54 seconds. How long will it take him to swim the 200 m breaststroke? One of the two highest numbers of realistic responses was for the problem "228 tourists want to enjoy a panoramic view from the top of a high building that can be accessed by elevator only. The building has only one elevator with a maximum capacity of 16 persons. How many times must the elevator ascend to get all the tourists on the top of the building?" In summary, 178 (33%) out of 544 responses were correct or involved a realistic comment. Prospective teachers' difficulties with modeling problematic word problems is also examined in Contreras and Martínez-Cruz (2007). Possible explanations about mature adults' suspension of sense-making when solving problematic word problems will be provided in the poster presentation.

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PROSPECTIVE ELEMENTARY AND MIDDLE SCHOOL TEACHERS' INITIAL ABILITIES TO POSE MATHEMATICAL PROBLEMS WITHIN GEOMETRIC CONTEXTS

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The continuous search to find solutions to mathematical problems has extended the frontiers of mathematical knowledge since antiquity. Creating mathematical problems is conceived by both mathematicians (e.g., Halmos, 1980) and mathematical educators (e.g., Silver, 1994) as a fundamental mathematical activity. Professional organizations (e.g., NCTM, 2000) and the National Research Council (Kilpatrick, Swafford, and Findell, 2001) include problem posing as a core element of mathematical proficiency. NCTM (2000) for example, calls for teachers to regularly ask students to pose and solve interesting problems based on a wide variety of situations. However, we know very little about both prospective and practicing teachers' abilities to pose mathematical problems and their dispositions to engage their students in problem-posing activities.

One of the goals of our research project is to document prospective teachers' initial abilities to pose mathematical problems. To this end, and as a starting point, we asked a group of 25 prospective elementary and middle school teachers to complete four problem-posing tasks within geometric contexts. The students were enrolled in a geometry course for elementary and middle school teachers in spring 2011. The students were told verbally that they will have about 45 minutes to complete each problem-posing task, but they spent about 20 minutes on each problem-posing activity. The written responses are the data sources for examining their initial problem-posing abilities.

In the first problem-posing task, the diagonals task, students were asked to observe a quadrilateral, a pentagon, and a hexagon with all of their diagonals. The diagrams were accompanied with the statement "We notice that a quadrilateral has 2 diagonals, a pentagon has 5 diagonals, and a hexagon has 9 diagonals". The directions asked students to examine the examples and think about other analogous diagrams and to write down as many different mathematical problems or questions as they could.

For the diagonal task, the students generated 63 responses out of which 56 were problems. The students generated 13 extended problems, 14 general problems, 1 proof problem, 2 converse problems, 13 further extended problems involving formulas or patterns, and 13 additional further extended problems of other kind. A more complete description of the data coding, analysis, and discussion of the results for each of the four problem-posing tasks will be reported in a longer version of this paper.

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INSERVICE TEACHERS: CONCEPTUAL UNDERSTANDING OF DIVIDING FRACTIONS

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This study is part of a larger study on designing and implementing professional development. Research indicates that teachers' understanding of fractions is limited (Squire & Bryant, 2002) with few teachers being able to explain reasons and meanings for division of fractions (Ball, 1990). "Greer (1992) argued that though people were usually able to solve division sums involving fractions written mathematically, they were not able to use the concept to represent real life situations" (Rizvi & Lawson, 2007). Similarly, Toluk-Ucar (2009) found pre-service teachers to be unable to provide appropriate representations and explanations in given fraction situations.

The central focus of our study is to investigate elementary in-service teachers' abilities to make connections between algorithmic representations and word problem representations with division of fractions. This study examines the teachers' ability to pose problems based on a given algorithm and generate and solve an algorithm from a given word problem. The problems are listed below:

1. Write a story problem for the following division problem: 5 1/4□ 1/2.
2. Nancy has 6 2/3 meters of material. It takes 5/6 of a meter to make her fabulous fancy hair ribbons. How many fabulous fancy hair ribbons can she make?

Of the 35 participants in the sample, 8 successfully wrote a word problem from a given algorithm. 31 participants translated a word problem into an algorithm and successfully solved it. All 8 participants successful in generating a word problem were also successful at solving a given word problem. A sign test was calculated on this data and yielded significant results (□). The analysis of a sign test revealed that participants are more likely to successfully solve a given word problem than generate a word problem from a given algorithm.

These results emphasize the importance of problem posing in teaching division of fractions for professional development. Teachers must have a conceptual understanding of fractions in order to accurately pose problems within a context. Therefore, problem posing cannot only serve as valuable and necessary teaching tool, but also as an assessment tool.

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INVESTIGATING MIDDLE LEVEL TEACHERS' MATHEMATICAL CONTENT KNOWLEDGE FOR TEACHING

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Transformative mathematics teaching and learning is predicated upon high-quality instruction which requires sophisticated, professional knowledge that goes beyond pedagogical skill. Hill, Rowan, and Ball (2005) demonstrated that first- and third- grade teachers' specialized content knowledge significantly predicted the size of student gain scores. Clotfelter et al. (2008) found evidence that teacher credentials, particularly licensure and certification, positively impacted high-school student achievement on statewide end-of-course exams in comparison to their lateral entry peers. However, less is understood about the mathematical content knowledge necessary or background characteristics of teachers that might influence instructional quality in the middle-school grades. The current study uses data from a five year NSF-funded project whose goal was to improve teachers' specialized mathematical content knowledge for teaching.

The sample included urban and rural middle-level mathematics teachers ($N = 66$; 89% female) in a Midwestern state with elementary ($n = 11$), middle ($n = 19$), and secondary ($n = 27$) certification. Two longitudinal linear mixed models were estimated in SAS PROC MIXED to examine the relationship between teachers' certification level as well as knowledge of Number & Operations (N&O) and Patterns, Functions, & Algebra (PF&A) across three testing occasions. Mathematical Content Knowledge for Teaching (MCK-T; Hill, Ball, & Schilling, 2004) is a pre-post measure that contains both of these subscales. We hypothesize algebra requires sophisticated knowledge of numbers and operations and will predict content knowledge in that area. We tested this by comparing two models. In Model 1, N&O was treated as the outcome and PF&A a time-varying predictor. Model 2 reversed this treatment. Both models included teacher certification level as a time-invariant predictor. We constructed piecewise slopes to observe the effect of time.

In Model 1, the intra-class correlation indicates 34% of the variance is cross-sectional. The unconditional baseline model suggests there are individual differences in the linear rate of change in initial N&O subscale scores. We found two main effects: for every 1-unit higher than the mean PF&A subscale score, N&O increased significantly by .55, $p < .0001$; and for every 1-unit change from baseline score, N&O increased significantly by .32 at that testing occasion, $p < .001$.

In Model 2, the intra-class correlation indicates 58% of the variance is cross-sectional. Through an examination of linear mixed models we determined the model that best represented the data was a compound symmetry model that examined the between person effects of N&O on the intercept and slopes, with main effects of initial N&O, change from initial N&O, and certification level. We found greater initial N&O scores lead to greater initial PF&A subscale scores ($p<.001$). We found for N&O scores between the first two exams, PF&A scores were statistically unchanged, $p > .05$. For N&O scores between the second and third exams, PF&A scores increased by .19, $p = .05$.

There is still a significant amount of variance to account for within both of the final models. Future directions include analysis of the Geometry subscale of the MCK-T, for a more holistic portrait of how knowledge in one area of mathematics might be related to different branches. Other teacher background characteristics will be included as predictors.

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INFLUENCE OF CONTENT KNOWLEDGE ON TEACHERS' INTERACTIONS WITH STUDENTS

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This study examined how two teachers handled unanticipated student questions during typical classroom interactions. I grounded this study in the literature of the specialized knowledge for teaching mathematics and the activities of teaching mathematics. To answer the research questions, I observed and interviewed two experienced teachers of accelerated secondary school mathematics. Each teacher was observed for one unit of instruction and interviewed four times. From the observations and interviews, I found that these teachers responded to unanticipated student questions in four ways. When responding to the student questions, these teachers followed one of three approaches.

How does a teacher use his or her own mathematical knowledge? In particular, when an unexpected moment occurs during the normal course of teaching, how does he or she apply what he or she knows? This study investigated how teachers of students in an accelerated secondary mathematics course applied their mathematical knowledge in unexpected moments during classroom discussions. The research questions that guided this study are as follows:

- What mathematical knowledge does a teacher call upon when presented with an unanticipated/unplanned student question?
- What are the approaches a teacher uses when responding to a student who has posed an unanticipated question?

There are two parts to this study, observations and interviews. I observed teachers in their typical classroom environment in order "to illuminate an understanding of the culture, not to predict future behavior" (Pirie, 1997, p. 82). I used two recording devices for the classroom observations: one device focused on the teacher, and the captured additional student discussions in another part of the classroom. I observed the participants for one unit of instruction. In addition to the observations, I interviewed both teachers four times as part of this study: once before the observation, twice during the observation, and once after the observation. I interviewed each teacher for a total of three hours.

I identified four ways these teachers handled unanticipated student questions or comments. I used three of Fernandez's (1997) categorizations: posing simpler or related questions, providing counterexamples, and following through with a student's thought. The fourth category, acknowledging challenging questions, reflected gifted students' ability to ask teachers complex questions quickly (Park & Oliver, 2009). The teachers' content knowledge built from prior experience explained the infrequent occurrences of these episodes. Future studies would examine teachers' responses to unanticipated student questions in longer observations in a variety of mathematics courses.

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A RESEARCH SYNTHESIS OF PRESERVICE TEACHERS' KNOWLEDGE OF MULTIPLYING AND DIVIDING FRACTIONS

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This poster describes a synthesis conducted to determine what research says regarding preservice teachers' understanding of fractions and to identify the gaps in their existing fraction knowledge. Specifically, this poster will address a smaller portion of the synthesis and report the findings from fraction multiplication and division topics. Results indicated that preservice teachers' understanding of fraction multiplication and division is limited and largely based on rote procedures. Implications for teacher education programs and future research studies are provided.

Teachers need a "solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power" (Conference Board of the Mathematical Sciences (CBMS), 2001, p. xi). In particular, the National Mathematics Panel affirmed the "proficiency with fractions" as a major goal for K-8 mathematics education because "such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped" (p. xvii). Therefore, developing such proficiency in preservice elementary teachers is a critical task for mathematics educators. In this poster, we discuss the main findings from a research synthesis of existing studies on preservice elementary teachers' fraction knowledge to identify critical directions for future research specifically in the area of fraction multiplication and division.

Research regarding preservice teachers' mathematical content knowledge illustrates that preservice teachers have a rule-based conception of fraction multiplication and division. Within the topics of fraction multiplication and division, misconceptions result from overgeneralized rules from other number systems, such as multiplication always makes bigger, or result from not understanding algorithms for multiplying and dividing fractions. Other difficulties preservice teachers have with fraction multiplication and division stem from not having a conceptual understanding of the mathematics. Thus, when asked to provide contextualized situations for multiplication and division problems, preservice teachers tend to create situations not related to the original problem or are unable to generate a situation at all.

By understanding preservice teachers' knowledge of fraction multiplication and division, future studies and improvements in teacher education programs can start to investigate the ways in which preservice teachers overcome their misconceptions to develop the mathematical understandings needed to be an effective teacher. Virtually every study suggests that strong teacher education programs and improvements to teacher education courses are needed, however little has been done to document the types of experiences preservice teachers need. There is still not enough for teacher educators to have an adequate understanding of how preservice teachers think.

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EXPLORING THE EFFECTS OF USING GRAPHIC CALCULATORS IN TRIGONOMETRY

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The aim of this study is to investigate the effects of students learning of trigonometric functions through the use of TI-Nspire technology. The effects of students learning are assessed in terms of (a) students ability to explore trigonometric functions, to develop mathematical modeling skills and to solve problems involving trigonometric functions concepts with the use of TI-Nspire CAS technology; (b)students ability to use the TI-Inspire CAS technology to engage in mathematical discussions and reflections and to communicate their understanding of the key concepts of trigonometric functions; and (c) students attitudes towards the use of TI-Inspire CAS technology in exploring and learning about trigonometric functions in the grade 12 Advanced Functions course.

The theoretical framework of this study is The Ten dimensions of Mathematics Education Framework which breaks down the essential components of a successful mathematics education program (Ross, McDougall, Hogaboam_Gray, & LeSage, 2003).

The mixed methods triangulation design procedure, specifically the convergence model is used to collect and analyze different but complementary data on the effects of using TI Nspire technology in the process of learning trigonometric functions.

The findings of this paper might disclose significant factors that contribute to improve the teaching and learning or Trigonometric Functions with the use of TI Nspire technology. For example, it might reinforce some educational research conclusions that simulations of different real life situations with the use of technology help students extend their mathematical understanding and enhance their critical thinking either through helping them visualize mathematics or by applying the mathematics to everyday settings.

This study guides and informs educators about the potential benefits and limitations of using TI Nspire CAS technology in mathematical explorations, investigations and problem solving. It also helps reveal strategies and techniques to be incorporated or avoided in the teaching of Trigonometry in grade 12 Advanced Functions course.

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ONE CLASSROOM IN TWO LOCATIONS: USING DISTANCE LEARNING TECHNOLOGY TO PROVIDE EQUITY

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Access to advanced mathematics courses is often denied to students in low SES schools, which negatively impacts their ability to attend more selective universities and obtain technical jobs in STEM fields. This poster details an innovative program that used technology to link a high and low SES school to provide access to AP Calculus BC. The students at both schools attained statistically significant AP examination scores and cited rigor, motivation, and collaboration as main aspects of the program that led to success in a challenging mathematics class. This poster candidly describes this program and discusses both the benefits and concerns in the hope of increased research into and application of technology to address equity issues.

When discussing equity in education, there is often a focus on providing students with equal access so that the average student in one community has the same resources and affordances as the average student in another community. Rarely does the discussion extend to the needs of the high achieving student. The National Council for Teachers of Mathematics writes that equity, "demands that reasonable and appropriate accommodations be made and appropriately challenging content be included to promote access and attainment for *all* students" (NCTM, 2000). The emphasis on all students, italicized in the original document, highlights the belief that equity is not simply an issue for students who are performing below standards, but rather an issue for all students at all levels of performance. As such, an important part of achieving equity concerns students who have the potential to succeed in higher level mathematics classes (whom we will term "high achieving students") but lack physical access to these classes or to a highly qualified teacher.

This poster presentation documents a two year project designed to address the equity disparity faced by a group of high achieving, underrepresented high school students in the southern United States through the use of distance learning technology. By adapting teaching practices to include such elements as teacher focused cameras and microphones, two high school teachers at two different schools were able to collaborate and bring Advanced Placement (AP) Calculus BC to high achieving students who previously had no access to such a course. This case study presents to the mathematics education community an example of how practitioners in the field are utilizing advances in technology in order to achieve equity in advanced mathematics courses.

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TECHNOLOGY INNOVATION IN SECONDARY MATHEMATICS: TEACHERS' STAGES OF CONCERN

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Technology has the potential to transform the teaching and learning of mathematics (NCTM, 2000). For many teachers however, technology brings fears and feelings of uneasiness. Thus teachers and administrators seek out professional development opportunities to learn about technological innovations before trying them in a classroom. But with little consistency in professional development or time to implement new technology, "the average individual is doomed to a cycle of continual technology implementation" (Straub, 2009, p. 643) without thought of its impact on student learning. Thus, teachers "must be able to locate and take advantage of appropriate professional development" (Heid, 2005, pp. 364-365) in order to create a paradigm shift for technology readiness.

The research presented in this poster is part of a three year professional development initiative involving 34 teachers from rural districts in the Northwest. This geographic area is agrarian, has a high population of ELL students, and is economically disadvantaged. Teachers were immersed in 96 hours of professional development involving a week long summer institute and three two-day follow up seminars throughout the school year and ongoing regular classroom observations. The professional development introduced teachers to dynamic software and a variety of Web 2.0 applications and asked them to consider how they might be able to implement this technology in their mathematics classrooms to improve student learning.

The Concerns Based Adoption Model (CBAM) (Hall, 2010; Hall& Hord, 2001) is an innovation adoption model that examines teachers' concerns and describes their innovation levels of use, innovation configurations, and facilitator leadership styles. Teachers in this study were measured on the first construct, the Stages of Concern, which examines their feelings and perceptions as they relate to a particular innovation they are facing. Teachers were given a 35 item questionnaire which indicated a score of relative intensity for each of the seven stages of concern. Teachers will have some level of concern for each of the seven stages at any given time and stages are not intended to be hierarchical (Straub, 2009). This poster will present an analysis of the teachers' stages of concern regarding the implementation of new technology in their mathematics pedagogy. Results indicate that as teachers adopted and used the technology, their concerns shifted from self-based at the beginning toward more task- or impact-based as they progressed. These results help fill the gap regarding the process teachers go through as they are presented with new tools and face pedagogical change as a result of new technological tools. By understanding teachers' feelings and perceptions, teacher educators can work to provide successful professional development that impacts student learning and provide support that seeks to meet the teachers' needs and concerns.

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EXPLORING MIDDLE SCHOOL PRESERVICE TEACHERS' MATHEMATICAL PROBLEM POSING

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The National Council of Teachers of Mathematics [NCTM] *Principles and Standards for School Mathematics* (2000) suggested that teachers integrate problem solving into the context of mathematical situations by choosing specific problems because they are likely to prompt particular strategies and allow for the development of certain mathematical ideas. Skinner (1991) defined a problem as a question that engages someone in searching for a solution and a problem cannot exist separately from the person who poses and solves it. Problem posing allows students to generate or to reformulate new problems based on a given situation or problem (Gonzales, 1994). In addition, NCTM encouraged teachers to incorporate problem posing as a means of instruction to engage students with problems involving higher order mathematical thinking. Therefore, the objective of this study was to identify the types of problems the preservice teachers (PT) reformulated based on a framework by Stickles (2006). In addition, the study examined the PT' reflections on posing a new problem.

In this research an open-ended mathematical problem was presented to 40 middle school PT at a southwestern public university. Using the Polya's approach (1973), PT were required to solve the problem. Later, based on the given problem, PT were asked to pose a new problem with the same or higher level of difficulty. After some sessions of problem posing interventions, PT were asked to reformulate the original problem based on Brown and Walter's (2005) approach called "What-If-Not". Moreover, using a reflexive journal, the PT' perceptions of posing new problems were investigated. The data were analyzed using both quantitative and qualitative approaches. The classification scheme (Stickles, 2006) was used to categorize the problem posing statements in terms of problem types. Qualitatively, the journal entries were organized by categories of statements and ideas.

Results revealed some significantly different types of problems before and after the intervention. Also, the findings from the reflective journals enabled the researchers to study the differences in PT' perceptions in creating their own problems. In the poster, we will present the details of the problem posing study. Examples from the data analysis of preservice teachers' strategies when posing the problem before and after the "What-if-not" approach will be presented.

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DEVELOPING A HAPTIC ENVIRONMENT FOR MATHEMATICAL ACTIVITY

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We are investigating the impact of integrating haptic devices into mathematics classrooms. During the first year of our three-year project we developed a set of activities for 32 elementary and undergraduate students that involved the use of a haptic device. Haptic devices are physical machines that provide force-feedback to users in order for them to interact with computer-simulated objects.

In our investigations we employed the use of the SenseAble PHANTOM Omni® haptic device. We define a *haptic environment* as consisting of the haptic device, *hapticons*, and users and their discourse during interactions within the environment. Hapticons (i.e. "haptic icons") are dynamic visual representations (e.g. a sphere or flat surface) that become tangible through their ability to provide force-feedback to a user via a haptic device. We have defined two classes of hapticons, *objective* and *relative*. Objective hapticons link force-feedback to interactions with a specific visual representation within the haptic environment, e.g. putting a sandpaper-like surface on a box. Relative hapticons, however, link force-feedback to a phenomenon in the haptic environment, such as a force that is a function of the distance between two spheres. The visual representation for a hapticon, in either class, may be hidden.

Informal interviews were conducted with elementary and undergraduate students while they interacted within our haptic activities. Hapticons within the activities included perpetually moving spheres that "bumped" into students, and impenetrable walls impeding students' movements. Through analysis of the students' discourse during these interviews, we identified four hapticons that elicited rich use of metaphors stemming from student reactions to the hapticons: *break point*, *lock-on*, *viscosity* and *pulse*. Break point is an objective hapticon, in that it provides resistance to a student's movement while they are in contact with an object, until the student exerts enough force and the resistance disappears. Lock-on is also an objective hapticon, consisting of a magnetic force attracting the user to an object in the haptic environment (e.g. line, sphere). It is also considered a *composite hapticon*, because a break point acts as the object's magnetic field, keeping users in contact with an object until they provide sufficient force to escape the field. The final two hapticons, *viscosity* and *pulse*, are relative, and affect a user's movement through the haptic environment. While a user moves through the environment, the viscosity hapticon provides a motion-opposing force, such that the user feels as though they are moving through a viscous material, while the pulse provides a vibrotactile force.

These hapticons provide the basic haptic elements that will be incorporated into our mathematical haptic activities. The first mathematical activity created using these hapticons focused on the concept of reflective symmetry. The activity takes place within a 2D haptic environment, in which students are shown a point and must find its reflection across a given line of symmetry. Aiding in this goal is a viscosity hapticon, where the opposing force is a function of the student's perpendicular distance from the line of reflection. While the student's goal in the activity is to correctly identify the placement of a reflected point, the pedagogical goal is for students to abstract a relationship about the opposing forces and the distance from the line of reflection. This activity will be implemented during our main intervention with elementary students in the spring of 2011, as part of a suite of mathematical haptic activities on symmetry.

ONLINE MATHEMATICS FOR BASIC SCHOOL AND HIGH SCHOOL STUDENTS IN MEXICO

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Veracruz is a Mexican state placed along the Gulf of Mexico, its natural geography includes many changes in topography: from narrow coastal plains to mountains of over 5,000m height. Veracruz has small arid zones, valleys, jungles, small lagoons and many rivers (more than 40 and tributaries) so that its climate varies from hot and humid to snow. Temperatures range from 0ºC to 40ºC. All these differences make it very difficult to cover the whole state with schools in order to give education to every child in the state. To solve this problem Veracruz's government called for projects to develop an on line education network and cover every city and town of the state (Consejo Nacional de Ciencia y Tecnología, 2009).

In 2009 we started a project co-financed by National Council of Science and Technology (named CONACYT for Consejo Nacional de Ciencia y Tecnología) and the Government of the State of Veracruz in Mexico (Authors, 2009). The project named "Design, development and generation of on line didactic materials for teaching mathematics in the school system of Veracruz" has two main goals. The first is to encourage the design and production of on line didactic materials to teach mathematics and sciences in basic schools and in high schools in the State of Veracruz in Mexico. The second is to create videos and demonstration material for teacher's training in the use of on line didactic activities.

The project is divided in four stages. After the first stage we have designed 20 activities applying the Theory of Didactic Situations (Brousseau, 1997) to teach trigonometry to basic school students (Authors, 2010). Those activities were designed to be used online only because they are programmed in Java®. In this poster we describe and show some of those activities and a few remarks we have received from teachers and students.

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MATH + SCIENCE PERFORMANCE FESTIVAL

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Traditionally, "mathematical performance" is conceptualized as pertaining to the domain of assessment and evaluation. However, we assume a view of *mathematical performance* as the process of communicating mathematics using the performance arts (Gadanidis & Borba, 2008). Exploring mathematical ideas through performance can offer a way to challenge stereotypes around mathematics learning. Learning mathematics can be viewed as an aesthetic and human experience (Higginson, 2006) rather than an impersonal and unpleasant activity. By creating music or skits, math performers can provide surprises to their audiences and communicate deep mathematical ideas in a creative way. Digital technologies play a significant role in mathematical performance. *Digital mathematical performances* are digital media (e.g., video and audio files, flash animations, and virtual objects) used to communicate mathematics through music, cinema, theater, poetry, storytelling, and so forth. Gadanidis, Hughes, and Borba (2008) suggest thinking about mathematics as performance may lead to new ways for (a) students' learning with technologies and (b) sharing mathematics beyond classrooms. Since 2008, students and teachers have been submitting digital mathematical performances to the *Math + Science Performance Festival* (see www.mathfest.ca). This Festival is based on a virtual environment (a website) where digital performances are published and shared. Every year, Canadian celebrities (e.g., musicians, song writers, poets, TV presenters) and mathematicians select their favorite performances in terms of: (a) depth of the mathematical ideas; (b) creativity and imagination; and (c) quality of the performances. Gadanidis and Geiger (2010) have referred to this Festival as "one example that helps bring the mathematical ideas of students into public forums where it can be shared and critiqued and which then provides opportunity for the continued development of knowledge and understanding within a supportive community of learners" (p. 102). In this poster presentation, we highlight a case study to illustrate how Brazilian students created digital mathematical performances for the Festival. Based on the video recordings of students' skit performances, activities in learning sessions, and interviews, we suggest mathematical performance can offer ways to: (a) disrupt students' negative images and stereotypes of learning mathematics; (b) produce mathematical meanings and knowledge with creativity and imagination; and (c) share mathematics beyond classrooms.

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AN INVESTIGATION OF A TEACHER'S USE OF TI-NAVIGATOR TO FACILITATE STUDENTS' UNDERSTANDING OF QUADRATIC EQUATIONS

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Students can move between representations with the help of technology such as graphing calculators (NCTM, 2000). Using instructional strategies with handheld calculators provides dynamically linked representations and observation of immediate changes, which leads to deeper conceptual understanding. The relationship between the coefficients of a quadratic equation and its graph is a good illustration. Representational fluency is the ability to translate between representations as well as make meaningful interaction and generalizations between different representations (Zbiek, Heid, & Blume, 2007). In the present study, we used representational fluency as the analytical framework to examine one teachers' instructional practice. Through this analysis, we demonstrate how a teacher used the TI-Navigator™ to help students' developing representational fluency skills that facilitate the development of in-depth understanding of quadratic equations.

We used four principles of effective mathematics instruction incorporating Classroom Connectivity Technology (CCT) (Pape et al., 2011) to analyze three videotaped teaching episodes. Additionally, we examined the teachers' post-observation interviews and student focus group interviews using an inductive approach to explore the teacher's instructional strategies. Findings revealed that the teacher used different instructional strategies and processes to provide a context in which students were supported to develop representational fluency including students working in groups to model a parabola in a real-world context by translating and transforming among verbal, symbolic, and graphical representations. In one lesson, students submitted group equations to match the curve of a necklace projected on a screen. A whole-class discussion led to an examination of their responses and suggestions for changes to more closely match the curve. Representational fluency was developed as students made conjectures based on their prior knowledge, checked their conjectures by making changes to the equations, and were able to see the results of these changes immediately. We will focus on the classroom discourse that provided the students the opportunity to analyze the graphs and make generalizations about the impact of changing each of the coefficients in the quadratic equations on the parabola. We argue that this discourse resulted in active student engagement.

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THE INFLUENCE OF TECHNOLOGICAL TOOLS ON STUDENTS' MATHEMATICAL THINKING IN SECONDARY CLASSROOMS

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Research has found that the mathematical thinking that students do while engaging with instructional tasks has important consequences for student learning (Stein and Lane, 1996), but few studies have examined how technology influences students' mathematical thinking in general. This study investigates the role of digital cognitive technologies in supporting students' mathematical thinking while engaging with classroom instructional tasks. Data was collected in three secondary and one middle school mathematics classrooms via classroom observations, collection of student work, and post-lesson teacher interviews. The *Mathematical Tasks Framework* (Stein, Smith, Henningsen, & Silver, 2009) is employed to evaluate the opportunities for mathematical thinking present in the mathematical tasks chosen by these teachers, and how those opportunities were realized or not during implementation. The use of technology was coded with respect to whether it served to *amplify* students' thinking by making students' work more efficient or accurate without changing the nature of the task, or whether it was used to *reorganize* students' thinking by shifting their cognitive focus (Pea, 1987). Results suggest a general association of the level of cognitive demand of a task with the way in which technology is used. When technology is used as an amplifier, it generally has no influence on or relationship with the cognitive demand of the task. On the other hand, teachers generally used technology as a reorganizer to set up high level tasks, but three of the four teachers failed to maintain the high level demand during implementation on a consistent basis. The role of technology in the decline or maintenance of high level thinking during implementation is discussed. In particular, the Vygotskian notion of *tool use*, and the process of *instrumental genesis* (Drijvers & Trouche, 2008) by which students construct meaning for and with tools, is used to explain the results. I hypothesize that students' ability to use technology as a reorganizer while engaging in high level mathematical tasks is related to a trajectory of instrumental genesis.

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USING CHILDREN'S MATHEMATICAL THINKING TO PROMOTE TEACHER LEARNING: A COMPARISON OF FIVE PROJECTS

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Representatives from five quasi-experimental projects using professional development approaches for K-6 teachers modeled after those of the Cognitively Guided Instruction project and the Early Algebra project will compare the features of their professional development designs. Teacher assessment and student achievement data will promote a discussion of the challenges of documenting the impact of professional development.

When Wayne, Yoon, Zhu, Cronen & Garet (2008) advocated experiments and quasi-experiments as fruitful designs to explore which specific features of professional development matter and what happens to a PD program when implemented across a range of settings by a variety of trainers. Our discussion group will include representatives from five quasi-experimental PD projects focusing on helping elementary school teachers reform their mathematics teaching practices. Our discussion of our PD and our data should provide some of the specificity that Wayne et al. proposed was necessary to guide PD design. All of the projects include a focus on children's mathematical thinking approaches similar to that of the Cognitively Guided Instruction (CGI) project (Carpenter, Fennema, Peterson, Chiang & Loef, 1989) and its sequel, the Early Algebra Project (Jacobs, Franke, Carpenter, Levi & Battey, 2007). All five projects, funded by the Improving Teacher Quality program under the auspices of the California Post Secondary Education Commission, span three years of intensive work with teachers and reflect the best practices identified by Darling-Hammond and Richardson (2009). Foundational to each project is the idea that children have the capacity to develop their own strategies. While this idea comes from extensive research, teachers tend to question whether the students in their classrooms are capable of this. In our discussion group, we will explore similarities and differences among the five projects to determine how each facilitates teachers' learning about students' mathematical thinking and how each project helps teachers' come to believe that these ideas pertain to their students.

While descriptive accounts in the literature attest to the success of CGI oriented professional development (PD) in a variety of settings (see for example, Franke & Kazemi, 2001), these have been limited to what Wayne and his colleagues (2008) called *efficacy trials,* in which the developers of the PD are directly involved with its implementation with teachers. In existing mathematics education literature we were not able to find *effectiveness trials* that document the success of CGI approaches "when delivered in typical settings by those not involved in the development of the program (Wayne et al, p. 470)." Three of our projects claim to implement

CGI with K-2 teachers, and vary in the degree to which PD providers are affiliated with the CGI project designers. In one case, the project director worked as a graduate student researcher on the CGI project, in the second case the project director was an advisee of a former graduate student researcher on the CGI project and in the third case project leaders do not have direct connections to CGI developers. While each of these three settings was in some ways "typical" of California, they were distinct from one another and quite distinct from those of the original CGI project. An examination of teacher and student data from each of these three projects will allow us to explore the *effectiveness* of a CGI approach in CA at this point in time.

We also plan to discuss the professional development design of each project to explore the ways in which each project addressed the recommendations prevalent in the research on PD. Ball and Cohen (1999) in making their case for practice based PD advocated the use of "records of practice" including student work, videotapes of student activity and curriculum activity. They alluded to the challenge of choosing which records to focus and pointed out that "the pull of the personal and the immediate in the case of one's own classroom can mitigate against reflection, analysis, and investigation of alternative perspectives and courses of action. Moreover, the current norms of teacher interaction and discourse do not readily support the kinds of joint consideration of one another's practice that would be useful. (p. 24)." They suggested that analyzing artifacts generated in a stranger's classroom allows teachers to critique and generate questions that would be too awkward to raise about a peer's classroom. On the other hand, analyzing artifacts generated from teachers' own and their colleagues' classrooms has an authenticity that contributes to the meaning and utility of the analysis. Kazemi and Hubbard (2008) noted that records of practice can be considered boundary objects that serve as a conduit between teachers' classrooms and the PD. Each of our projects differed in the types of records of practice utilized and the origins of the record enabling us to explore Kazemi and Hubbard's question "Whose practice is represented in the PD context and with what kinds of records? (p. 431)"

Wayne, et al. (2008) noted research is needed to examine the assumption that PD is best when school-based. School or district-based PD might contribute to the creation of strong working relationships between teachers which breaks the isolation teachers often experience which tends to inhibit their growth (Darling-Hammond, Wei, Andree, Richardson & Orphanos, 2009). Our PD projects had three distinct contexts; two were school-based, one was based in a small district and the other two included many districts. This diversity of settings will allow us to examine the benefits and constraints of each of these contexts.

All of the PD projects served teachers working in "under-performing" schools who faced pressures to use the state adopted textbook and to prepare their students for the state standardized tests. Some teachers had to adhere to administrative mandates including district benchmark assessments and pacing guides that they perceived limited their freedom to experiment with new instructional practices in their classrooms. Our differing contexts led each project to address these pressures differently and we will share our approaches and reflections on the success of our efforts.

Each discussion session will include some brief presentations of data and artifacts from each project. We hope to spend the bulk of our time in discussion of the above issues. We hope that others who have conducted similar PD will join us to discuss ways in which we can establish the efficacy of our work through research and improve our practices.

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HOW CAN MODELS OF MATHEMATICAL DEVELOPMENT BE STRUCTURED, REPRESENTED, COMMUNICATED AND USED IN FORMATIVE ASSESSMENT?

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The purpose of this working group is to identify ways in which teachers and students can use models of mathematical development productively as part of a formative assessment process. In the sessions, and starting from some existing "seed" examples, participants will identify different ways of representing, communicating, and using these models in formative assessment in the classroom, generate new examples, and draft a research agenda investigating the efficacy of these approaches. Follow-on activities include expanding, piloting, and evaluating the different approaches through collaborations among researchers and practitioners, and convening again to share and learn from our collective experiences.

Background

Formative Assessment

Formative assessment is "a process used by teachers and students during instruction that provides feedback to adjust ongoing teaching and learning to improve students' achievement of intended instructional outcomes." (CCSSO, 2008). There is a large body of research linking the consistent and systematic use of formative assessment to improvements in student learning (Black & Wiliam, 1998, Brookhart, 2005, Nyquist, 2003). Wiliam (2004) stated that "in order for assessment to function formatively, it needs to identify where learners are in their learning, where they are going, and how to get there" (p.5). Most current work on formative assessment draws attention to both the teacher role and the student roles, emphasizing the importance of bringing students into the formative assessment process (Heritage, 2010). There is a small but developing body of research that suggests that within the process of formative assessment, teachers struggle particularly when it comes to determining "how the get there" i.e. the next instructional steps they should take with their students based on the assessment evidence (Heritage, Kim, Vendlinski & Herman, 2008).

Learning Progressions

Heritage (2010) citing Black and Wiliam (1998) reminds us that one requirement of formative assessment "is a sound model of students' progression in the learning of the subject matter, so that the criteria that guide the formative strategy can be matched to students' trajectories of learning" (Black & Wiliam, 1998, p. 37). The notion is that in order for teachers and students to identify what the next learning goal should be, they need to be able to assess where a student currently is on a path of learning. The next step on the path is then a natural learning goal for that student. Progressions define a series of points or states of understanding as an ordered sequence. By knowing a students' current state and the ordered sequence, teachers and students can target the next significant state of understanding that the student should move towards. While progressions can be used to determine the next state of understanding, they do not necessarily define what a student, with teacher support, needs to do to move into this next state of understanding, i.e. they define where to go, but not how to get there.

Within mathematics, Daro, Mosher, and Corcoran (2011) state that the term *learning trajectory* is more commonly used in mathematics, and allows for the possibility that the a trajectory includes the learning and teaching strategies that are needed for shifting students' understanding to the next level.

In our work on learning in mathematics, we have used the term *model of mathematical development* to make clear that the characterization of different levels of understanding is essentially a descriptive theory of the different states of understanding, and is predictive to the extent that the sequence of states captures common sequences of changes in understanding. We define a model of mathematical development as describing the major shifts in understanding that generally occur over medium to large time periods (typically months to years) as students grapple with mathematical knowledge. These shifts describe how a variety of types of knowledge (e.g. conceptual, procedural, contextual and metacognitive knowledge) change over time with respect to central ideas of the domain as students' competency improves due to learning within the domain and related domains, as well as more general cognitive development. Developmental stages can include shifts in view and belief, as in the shift from an understanding of the equal sign as a "calculate something" sign to an understanding of the equal sign as a truth-value statement of equality or balance between two mathematical quantities or objects. Developmental stages also can include increases in generality, integration, and robustness of existing knowledge, as in increases in procedural or representational fluency (Harris & Bauer, 2009).

Two progressions are included in table 1 and table 2. In an in-progress literature review, the authors identified 18 mathematics concepts that have associated candidate models of mathematical development (Bauer, Graf, Harris, Haberstroh, Attali, Wylie, and Leusner, in preparation). The first example, a model characterizing different understandings of central tendency is drawn from Watson and Moritz (1999), and Callingham and Watson (2003). Their proposed a theory of development for the concept of average and measures of central tendency (median and mode) is based upon the SOLO taxonomy Biggs and Collis, 1991) which is a neo-Piagetian framework for characterizing learning and development. The Watson and Moritz (1999) model focuses on how students develop an understanding of average as a measure of representativeness and draw conclusions about a variety of types of data. Table 2 provides a description of each of the levels (excerpted from Watson and Moritz, 1999) and examples drawn from Callingham and Watson, 2003, and Watson and Moritz, 1999).

Table 1
Central tendency developmental levels

Level	Description	Example
Preaverage (P)	Students have no term for average, even in a colloquial sense. When asked to solve problems involving concepts of central tendency, they typically provide tautological responses or idiosyncratic stories about the context.	To a question about the relation between average wage and median home price in Australia, an example response from a 9th grader at this level is “average means it is small and good for little families” (Watson & Moritz, 1999, p. 27)
Single Colloquial Usage for Average (U)	Students use colloquial terms for average, such as *normal* or *okay* and sometimes refer to "add up" colloquially but not in a calculation sense. They do not have an understanding of how to calculate an average.	Student use terms such as “same as others,” “okay,” “normal.” (Callingham & Watson, 2003, p. 31 Avg 2.2) and refer to median of set of measures as “the most accurate value.” (Watson & Callingham, 2003, p. 40 ME 13.1).
Multiple Structures for Average (M)	Students use at least one, often two or three ideas for the concept of central tendency including “most”, “middle”, and the add and divide algorithm for the mean to describe average in straightforward situations. However, they fail to consolidate multiple concepts meaningfully. They rarely use more than one of these ideas in complex questions. Students sometimes acknowledge conflict between incorrect calculations of mean and idea of mode.	“mean is a lot of numbers added up and dived by how many numbers there were to start.” (Watson & Moritz, 1999, p 28).
Representation With Average (R)	Students refer to add-and-divide algorithm for the mean to describe average in straightforward situations. They can distinguish between the concepts of mean and median. They are aware of the association of the decimal form with the algorithm for mean. Students express some ideas related to the representative nature of average (e.g., prediction, estimation, or representing whole data set) and refer to “most” to describe data distributions compatible with mean or offer mode as alternative average concept. They know the algorithm for the mean but do not successfully apply it in complex contexts without prompting;	“not an average which would include all the extremes…out of 100 it would include 49.5 above it and 49.5 below it.” (Watson & Moritz, 1999, p. 28).

A second example is provided in Table 2 which is a model of students’ understanding proposed by the authors of equations and expressions with respect to two underlying concepts: equality and variable. The table begins with a description, that we call level 1, that represents the most basic level of understanding of both of these concepts addressed by the table, and then proceeds through four more levels of understanding. At each level there is a description of how a student thinks about equality and also the notion of a variable, and by combining two understandings, a description of what that student would be able to do and not do at that level of understanding Before looking at the table, which combines both idea, it is worth describing major conceptual changes in student understanding of equality and variable separately.

For equality, the primary distinction that is made is between thinking *operationally* about equality and thinking *relationally* about equality (Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006). The *operational* thinker treats the equal sign as a signal to do something – add or subtract or whatever action was called for on the left hand side of the equation. This way of thinking likely arises from students seeing many more arithmetic problems in math written as 5+3=__ compared to __=3+5 and often teachers will say “five plus three gives you what?” When a student begins to think *relationally,* he or she understands that the expression on the left hand side of the equation has the same value as the expression on the right hand side i.e. they are in balance. And furthermore, when a student understands *relational equivalence*, he or she will understand that equality in an equation is maintained provided you perform the same operation on both sides of an equation and will be able to judge equivalence in many cases without having to calculate values for each side of an equation by using techniques such an compensation and cancellation (Rittle-Johnson. Taylor, Matthews., & McEldoon, in press).

For variable, an important distinction is between students thinking of a variable as representing a *specific unknown*, i.e. as a placeholder for one and only one number, and as a *generalized number* i.e. that a variable can take on many values and be part of more complex mathematical relations as in a linear function (Booth, 1984; Weinberg, Stephens, McNeil, Krill, Knuth, and Alibali, 2004). Prior to these two understandings, students may have one or more contextually dependent beliefs about variables such as an algebraic letter must be the same as the first letter of what it stands for (e.g. ‘b’ for brownies) (Küchemann, 1978; Booth, 1984).. The table below combines these understandings of equality and variable into five levels of understanding overall.

Table 2
Equality and Variable: Equations and Expressions

At **level 1**, students have a superficial understanding of the concept of variable that does not include the idea of variable representing an "unknown." Instead, students may hold one or more beliefs about variable including, for example, that variables represent objects (e.g. that the variable "b" stands for "brownies" in a word problem because it starts with a b). Alternatively students may treat variables like individual digits of a number (if $2x=24$, x must equal 4), they may ignore variables in an equation only operating on the numbers, or separate variables from numbers in an equation or expression. At this level students also have an *operational* sense of equal sign. Students with this type of understanding believe that the equal sign is a signal that indicates that there is s problem to be solved or computation expressed on the left of the equal sign and that the answer should be placed to the right of the equal sign. Given these understandings of equality and variable, students should have trouble solving word problems, even with informal methods, algebra solving follows misconceptions as described in model of equality.
At **level 2** students still have an operational understanding of equality as in level 1 above. Their understanding of variable is different. Students have an understanding of variable as a "specific unknown" in which letters stand for one and only one number. For example, at this level if asked "in the expression 2n+3, can the variable n stand for the number 4?" the, student will say yes and but, then if asked if the variable n can stand for 37 they will say "no. Students can apply informal methods to solve simple algebraic word problems, but will continue to display misconceptions when attempting to solve symbolic equations. Students can attempt to solve equations using guess and check strategy and other approaches that do not involve substantial symbolic manipulation
At **level 3**, students have "specific unknown" level concept of variable as in level 2. What's different is their understanding of equality. They have a more sophisticated understanding of the equal sign as expressing that the quantity to the left of the equal sign has the same value or is in balance with the quantity expressed on the right of the equal sign (described sometimes as a relational understanding of equality). They also know some simple procedures for algebraic manipulation. Students can solve problems with one variable represented in text symbolically using algebraic manipulation. Because students have few ways of understanding algebraic expressions and equations, they may apply less efficient methods to solve equations. Expressions are only understood correctly if they can take on a single numerical value (i.e., consistent with students' understanding of variable as specific unknown).
At **level 4**, students' understanding of equality is richer than in level 3. They understand relational equivalence in most contexts (meaning that they not only understand that both sides of an equation are equivalent, but they also understand that performing the same operation to both sides of the equation maintains the equivalence) and can readily use transformations to balance equations to solve for an unknown. Students still have the same understanding of variable as in level 3 i.e. the "specific unknown" level concept of variable. Students do not yet have a "generalized number" concept but because of relational equivalence they can solve many equations with one unknown algebraically
At **level 5,** students have "generalized number" level concept of variable, in which a variable can function as a pattern generalizer for arithmetic, as in the statement a + b = b + a, or can express relations among sets of numbers as parameters and arguments as in the linear equation x = mx + b. Students have the same understanding of equality as in level 4, i.e. they understand relational equivalence and can readily use transformations to balance equations to solve for an unknown. Students can solve symbolic equations, parsing expressions to flexibly apply operations on to solve equations efficiently.

There is substantial research and development still needed to understand the characteristics of effective and practical formative assessment. Additional research is needed also in identifying, describing, and evaluating models of mathematical development. In this working group, in order to help advance both fields, we will explore the mutual constraints that these two areas provide. We believe answers to a variety of useful questions can be developed by making explicit constraints that models of mathematical development provide to the field of formative assessment, and the constraints that formative assessment can provide to the creation of models of mathematical development. To make this idea clearer, the aims and plans of the working group are described next.

Aims of the Working Group

Models of mathematical development are central to formative assessment. They provide testable theories of how student understanding changes in light of instruction and time. They are needed to define gaps between students' current understanding and learning goals in a way that makes closing the gap feasible. If the gap between two states of understanding has been empirically validated, is present within the population of learners, and many students have moved from one state to the other in the past, teachers and students should be confident that this unit of learning is feasible. There has been research on the feasibility of developing interim and summative assessments to collect evidence and make statistically valid inferences about student levels within models (Watson and Callingham, 2005; Weaver and Junker, 2004; Wilson, 2008). There has been less work on how best to represent models for use by teachers and students, (but see Lesh, Lamon, Gong, & Post, 1993; Wilson, 2005; Watson and Callingham, 2005). Still less is known about how to use models of mathematical development in the service of deciding on next instructional steps to be taken by teachers and students (Heritage, Kim, Vendlinski & Herman, 2008).

Wiggins (1998) provides an example of formative assessment within a welding classroom. In the example, students are learning to weld a ninety degree corner joint. When students think they have completed the weld to industry standards, they compare their welded corner joint to a set of joints to determine to which joint theirs matches best. The joints are ordered in terms of quality and each has specific characteristics relevant to beginning welders In doing this comparison, students are a) assessing at what level they are performing b) determining their next learning goals, and most importantly c) understanding what specific next things they need to do to improve. In other words, a student can consider his or her next instructional steps by, for example, realizing that on the next joint he or she works on, the two pieces to be joined must be more precisely positioned, or that gaps need to be filled, or bubbles removed. In this example, the connections from student performances to level of competency and next instructional step are transparent. Teachers and students can compare student performance to benchmark performances and determine what they need to do next.

Unfortunately, this sequence of inferences is not so transparent in mathematics learning. How does the Wiggins welding example translate to a mathematics context, when it is not skill in a directly observable work product that is evolving and improving, but rather knowledge and understanding around abstract concepts such as the meaning of "average" or "variable"? While there has been some work in this area (e.g. Forster and Masters' work on the Developmental Assessment Resource for Teachers (DART) (2004), much still needs to be learned. How can models of mathematical development be structured to best support formative assessment? Is a linear sequence of levels an adequate model to characterize student understanding to inform formative assessment? Are there circumstances in which model in the form of a network of states of understanding could provide greater value? Is a model in the form of a set of understandings better in some circumstances? What other structures are possible and advisable? How can models of mathematical development be represented and presented to make learning goals and instructional approaches as transparent as possible to teachers and students? What additional information will be helpful to provide? The aim of this working group is to begin to develop answers to these questions.

Plan for the Sessions

The sessions will focus on three work products 1) a list of different ways of representing models of mathematical development with examples that might be especially useful to teachers and students 2) descriptions of the use of these representations in making next instructional choices or adjustments transparent along with descriptions any additional materials and 3) a research agenda for expanding, piloting, and evaluating the use of different representations of models of mathematical development in formative assessment along with specific work individuals or groups will be doing in the coming year with respect to this agenda.

In session 1, the organizers will present the aims of the group and the session, provide some examples of models of mathematical development, a common definition and explanation of formative assessment, example tasks, and a video of the use of one model to support formative assessment as part of a classroom discussion. The bulk of the session will be a facilitated brainstorming session on different structures and representations of models for use in formative assessment. One area of interest to the organizers is how to include students (following from the welding example) in the use of the models of development. Much of the writing around formative assessment has emphasized not just the teacher role in the process, but the centrality of students to the process. Given that models of development may span several years of student development, there are questions about when it is appropriate to be explicit about the developmental model, and how a model can be of most value to students.

In session 2, the working group will select a set of the representations and break into small groups to flesh out examples of their use with respect to specific models of mathematical development. In the later portion of this session, the working group will reconvene to discuss and contrast the examples and potential value of the different representations.

In session 3, the working group will identify research questions and associated approaches and activities for addressing them based upon the work in sessions 1 and 2. Participants will discuss their interest and willingness in engaging in continued research collaborations to address these questions and sketch out next steps for the working group.

Follow-On Activities

Throughout the year, participants may explore the use of the representations of models and ideas for use in formative assessment in their research or in their practice. It is the hope of the organizers that early pre-proposal explorations are carried out. And one or more research proposals may be generated. The work will also inform an aspect of an existing grant on which the organizers are working. The organizers are currently working with a group of approximately 15 middle school teachers, and will have opportunities to pilot some of the emerging ideas from this working group session and will provide feedback to the working group participants on their ideas. The intent is to reconvene at the next PME-NA to share experiences on specific collaborations that developed, to discuss progress on aspects of the work, and to plan and focus continued efforts in this area.

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WHY DO UNIVERSITY TEACHERS NOT WANT TO LEARN?

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Training has never been a priority for teachers as it is for other activities for which society demands that someone must be trained to be able to practice. We can't accept buying bread from a baker who never learned how to make bread, neither to be operated on by a surgeon who never knew how to operate, nor to accept a haircut from hairdresser who never learned to use scissors. Teaching has always been considered as natural, easy and possible without necessarily knowing what it is about. Some years ago, this idea changed, mainly because of the multiple results of education and psychological sciences that revalued teaching as an activity that can build the individual and showed further the deep consequences on the individual practice, thinking, working, and all facets of his life. As a result, training teachers of schools (all primary schools and some high schools) became a necessity in many countries. However, training university teachers remains an exception so far in many parts of the world if not all over the world. Aren't we committing, again, the same mistake by considering that university teaching is easy and automatic?

1. The teacher-student relationship

Beyond the socioprofessional obstacles which cause students to fail in mathematics, there exist the psychological impacts related to the failure in mathematics which can create a negative behaviour against this science. You can find many individuals who have graduated in other fields, who could succeed easily in mathematics, but who have kept the fear and the uncomfortable feeling about mathematics from their early school years. Many of these negative beliefs are closely related to the teacher behaviour.

If we do define exactly the teacher's role in the teaching process, it's clear that this latter is very important and even decisive. In addition to being the knowledge transmitter, the teacher is a very important element in the whole system besides curricula, students, knowledge, books, teaching methods ... The teacher's impact on the learner is very strong. Everyone remembers at least one of his teachers and carries certainly with him many memories related to the science and the learning process and many other good or bad details. In fact we learn from a teacher a lot of other things than science.

Many studies show that students identify the science to the teacher, which is very dangerous. Most of teachers aren't aware enough of this question; they don't realize how every move, every side of their personality, and every thought shapes the science for the student. Many students chose to study mathematics because they had teachers who could make them like it, however many decided to never study it after having teachers who were not able to show them what mathematics is besides many complicated and weird formulas. As a result, we receive many students at the universities believing strongly that mathematics is a senseless science full of abstract notions that have no existence in reality. They need time and especially teachers who have to focus on those wrong ideas to change and correct them; otherwise those students will never learn how to write proofs and how to understand mathematics. But it can happen, if these students are unlucky, that they never get such teachers and continue to learn mathematics blindly as a set of nonsense notions.

As the university is the last school, if the wrong beliefs are not corrected there, they can be kept a long time later or forever.

The Relation Evolution

The teacher-student relationship began with a traditional model presenting the teacher as the

1 The elements of teaching: James M. Banner, Jr. and Harrold C. Cannon.

knowledge source who transmits it to the students and assesses them at the end. This relation was qualified as unidimensional, vertical or descending. Later, this traditional model changed because many elements came out, such as:

- The importance of environment made us aware that all is related. Individuals understood that knowledge is not only some written information in some books but it's more focused in the real world and in our daily real life with our health, relationships, thoughts, eating habits, childrising, economic issues, and many other fields.
- The group phenomena focused on leaning and working in groups, in order to serve many students better than learning only with the teacher. Group learning became a part of the learning process.
- The different interactions with others sciences, with the society and with the media make learning became a continuous process and information can be found elsewhere than with the teacher.
- The knowledge conception changed from being a special activity among special people to a normal and a daily activity to anybody. Everyone can learn anything at any time of his life.
- C competency took more and more space beside the knowledge; we value more someone competent than someone highly educated with no competency. Knowledge became more realistic and applicable than before.
- Evaluation changed as competency was taken more into consideration. So, we evaluate more the individual's competency than their ability to get or memorize information.
- The learner 's role changed from being the center of the teaching-learning process to be the principal actor in it.

So, a new model has been created. It's a modern one, by which the student-teacher relationship is different. A data bank exists somewhere outside of the teacher. Both teachers and students have access to it. They collaborate together so that the student transforms the information to knowledge. The student becomes more independent and handles his learning and training himself.

As a result, the relationship is multidimensional:

- Ascending and interactive: from the student to the teacher.
- Transversal and interactive: between the students.
- Multiform and interactive: the student with the whole environment.

This brought the teaching-learning process to be a dual, and a face-to-face meeting for both teachers and students who share continuously the role of teaching and learning at the same time. The teaching process focuses more in integration into the reality, professional actions of comprehension and the understanding of oneself. The result of the pedagogic action is no more in the objects (knowledge) but in building a person made by a complex set of competencies (cognitive, psychological, social, ...).

Difficulties of the Relationship

This teacher-student relationship, even with strong action, has many problems to manage in order to reach the goals.

- Sharing the power or not.
- Managing the great individual diversity.
- Dealing with delicate individuals.
- Managing the age difference between the teacher and the students.
- The teacher personality.
- Managing time.
- Labeling teachers.
- The big humain investment.

Beginning with new students especially in the first year at the university is not easy; the teacher should create above all a confident climate in which he will work on their complex against mathematics, correct and better interpret their wrong beliefs. He should also show that he's not the student's enemy; this complex, long lasting with the students, must disappear to revalue the teacher as a guider and a saver.

A big part of the learning-teaching process rests on the mistake. How to present it, how to use it and how to deal with it has been the subject of much research. It has also been an obstacle for many teachers and many students. The fear and the uncomfortable feeling about the mistake can lead both students and teachers to bad results. It's more present with teachers than with students, so many questions come up. Is it possible to learn without making mistakes? Do we have to avoid mistakes or correct them? Many students have long thought that writing a problem solution comes directly without review by correcting or adjusting some points and some steps; this comes from observation of teachers who have been writing the solution directly. Many teachers also thought that making mistakes at the blackboard when making some calculations would ruin their image and their career in front of their students and so become hostile to a student who notices spontaneously a small mistake of a sign or whatever hidden somewhere.

Another point, not of less importance, is the dialogue between the students and the teacher. A teacher talking to his students and listening to them would understand better their difficulties and the origin of their weaknesses. Most of the time, we think that we found the method that works, but the same teaching method can give different results with different kind of students. The teacher should ask and talk to his students to see how it works with them. Many students hate a teacher that always points out their failures. They need encouragement very often even for the least they do. It's important to a teacher to know that learning is one of the most difficult processes that an individual can undertake. Particularly learning mathematics is often far from simple and delightful. Encouragement will prove to the students again and again how wrong are their prejudices about learning mathematics such as: mathematics is just possible for geniuses, or solutions must be accepted and not understood, or students will never be able to find solutions unless they become teachers.

Believing in students' abilities has always been a strong stimulation to their desire to learn and we need to keep in mind that a student who believes that he or she is bad, will be bad even if it's not true. Even adults need someone to encourage them and appreciate their efforts.

A Study about the Student-Teacher Relationship

The following quiz has been given to almost 250 students in different Algerian universities. It contains three questions about the teacher-student relationship and how students define good and bad teachers according to them. These questions were given to check if the good-bad teacher definition of students is the same, nearly the same or completely different from the definition held by the teachers. We were motivated by noticing that many experienced teachers with high degrees getting less appreciation from the students than some others with fewer degrees and much less experience. We also want to set a training teacher program in our university. With this study, we could know more about which fields should be focused on. We also wanted, throughout this questionnaire, to know what the new generation of students wants in their teachers. With time, everything has changed as the whole world changed, but how has the teacher role or status towards the students evolved? We wanted it mainly as an update for teachers.

1. Who is a good teacher for you?
2. Who is a bad teacher for you?
3. What are your expectations of the student-teacher relationship?

I need to remark before giving the results; first, they were beyond our expectations. Second, we got nearly the same results with the same percentages in different universities. One of them welcomes students from all over the country, so we got the opinion of students from nearly all parts of Algeria. Third, the students gave some requests about point they felt necessary to be given even with no questions.

The answers to the questions are given by decreasing percentage. The answers have been gathered by me under the same idea which is given as a title. The students' exact words are given without any change.

1. The answers to the first question about a good teacher.

- Explains very well: goes slowly, repeats as many times as we ask, teaches without being boring, sure about his information, writes less and talks more, methodical, has simple ideas, spontaneous.
- Answers to the students' questions: even the silliest ones.
- Close to the student: shows that he cares for him, listens to him, talks to him, guides him, corrects his mistakes, considers him as his son, puts himself in his place, pushes him to work even if he has a weak level and tries to understand why, encourages him especially when he has bad grades, trains him, prepares him for the future.
- Modest and nice : smiling, cool, kind, calm, doesn't scare, doesn't give complexes, doesn't show that he is superior because he just has got more information than us, doesn't yell, doesn't became nervous.
- Honest and sincere: doesn't cheat and keeps his promises.
- ∑ Deals with the students in the same way: treats the students equally even if they are at different levels, girls equally with boys, punishes those who do not work and rewards those who work hard.
- Does his best so that the students succeed.
- Masters the class and imposes silence.
- Deals positively with students' mistakes: without yelling, no hard feelings, no intimidation.
- Has clear handwriting.
- Likes his course and his work.
- With a respectable physique.
- Fights social problems: drugs, smoking, disrespectful clothing, and any undisciplined form.
- Is an example for the students.

These two answers came many times:

- When we like the teacher, we inevitably like the course.
- The teacher gives 50% of courage and wills to the student to succeed.

2. The answers to the second question about a bad teacher.

- Haughty: who gets the truth, considers himself as perfect, genius, crack.
- Intimidates and disparages students: treats them as ignorants, of no value, silly, stupid, laughs at them, breaks them down, disheartens them, diminishes their dignity.
- Becomes nervous: yells, sulks, angry, wicked, monster, bad, mocking, never laughs, blames for anything and nothing.
- Doesn't answer the students' questions.
- Imposes himself: doesn't listen, stops students to talk, and imposes his idea even when it's not the best one, not sure about himself, memorizes everything.
- Fast: uses difficult words, skips the work, makes a lot of mistakes, just recites memorized information.
- Doesn't keep his promises.
- Works only with some students: those who are good or those who are sit in the front rows.
- Has no work method: works in a non ordered way, with confused ideas, muddle-headed, complicates even the simple things, wastes time, doesn't prepare his courses, goes

frequently off topic.

- Shows no interest to the students: considers them as objects, talks alone at the blackboard, comes just to write down without even looking at the students.
- Doesn't accept the criticism and doesn't admit his faults and mistakes.
- Refuses to adapt his course and method to the students' level.
- Considers the student as a genius, as a teacher like him.
- Doesn't stop writing: especially when he copies from a sheet that he never leaves.
- Doesn't stop talking about finishing the program and the course: doesn't care about the students' comprehension.
- He's not realistic: doesn't understand the real students' situation, pays attention to nothing.
- Works for money: doesn't like his work, fills the emptiness, makes the students work at his place.
- Gives points to some students and not to others.
- Doesn't give zero to the student that has cheated on the exam.
- Gives very difficult exams.
- Smokes in the classroom.

3. The answers to the third question about the student-teacher relationship.

- Kinship, teachers as parents.
- Fraternal relationship, as brothers and sisters.
- Friendly relationship, as friends but without private life details.
- Respect, exchange, communication, sincerity, reciprocity, familiar ambiance, love.

Conclusion

It's very amazing to notice that nearly all the answers are not related to the scientific side. This is completely opposite to what we teachers do and think. Most of our efforts and thoughts are about preparing the course scientifically. We never pay attention to other sides; if it happens that we do, we always think that it's not so important for the students. We might have some behaviour for years that students hate all that time. We also learned that the teacher behaviour comes first for the students and then the information and then knowledge side; however, teachers think that the opposite is true.

Students consider that the pedagogic relationship is very important and can have a big influence on their learning, on their ability to make the necessary efforts and on their results. They expect their teachers to be competent not only in the scientific side, but also and particularly in the pedagogical side. They consider that the teacher is a guide, a parent, an idol. If we have to train teachers, we need to take this into consideration.

The qualified university teacher has a price to pay, the consideration and the respect of the student-teacher relationship beyond any caprice in full discipline and dignity. In such perspective, it's the one who gets power who should show maturity and make the first steps not to seduce more but to better understand and to reach evolution and improvement.

2. Multiple reforms

Algeria, like many other countries, has made many systemic reforms in many levels especially these last ten years. Many changes have been made in the high schools programs, in primary schools programs, and in the past five years, in the university programs. Difficulties of how to adapt the teaching styles to the new programs came up even for the most experienced teachers. All teachers became beginners after each reform.

For instance, for this last university reform, changes were focused on reducing the scheduled time for each chapter and course; that is, if you are used to teaching some course in four sessions, then with the new program, you have to do the same but in only two sessions. How and in which way?

The teacher has to decide which are the more important and the less important topics, about

the priorities, as when you deal for example with sequences, group theory or integrals. What are the sacrifices to make and at which level; what about the students learning?
Administrators and politicians who supported this new reform have always directed teachers to teach the most important and in a light way and it's up to the teachers to change their teaching method so that they can teach the same materials in less time than before. I asked almost fifty teachers two years after this reform about how they manage their time. Here are the questions and the teachers' answers.
The teachers were divided, according to their answers, into three sets when they were asked about how they manage their time with their courses:

- Those who think that it's easy and fast and even done with less effort than before. However, they guarantee nothing about the students' learning as they don't feel concerned about that now that the administrators do not stop controlling the program progress during the academic year.
- Those who think that it's impossible to realize and even ridiculous as it's in opposition to the didactic contract about learning and teaching.
- Those who are in between and seek so far the best way to deal with it.

I need to define some notions at this point as our reform concerns mainly time. Didactic time, according to Chevallard and Mercier, is the cutting of a knowledge through a duration of time. For instance, if I teach differentiation, I need three courses, and I need to decide what to do in the first, the second and the third. However, if I have been given only two courses, I'll decide again about the cutting and decide again what to put in the first and the second course. I may put the first and the second courses of the first time in the first course of the second time and teach the third course for the entirety at the second course. I may also decide to teach the two first courses in one and let only the last one for the second course. So, many options are offered, but it's not easy to decide. What exactly do we base the decision on? That's why this didactic time management is one of the difficulties in a teacher's work both for beginner teachers and the most experienced ones every time that they are faced with new data. The problem is that this time is closely related to watch time and we have to manage both. It can happen that the watch time advances and didactic time doesn't advance, for instance when some material takes more time than expected. So, the cutting through that period of time didn't work or was not suitable.
Many other questions came up after this reform, such as:

- With a short time, what is can be the most important part in a mathematics course?
- Is it definitions, or theorems and results?
- What should we cut down? What about the proofs?
- Is it possible to make a mathematics course without any proofs?
- If not, which proofs can we cut down?

I asked the same teachers as before how they make their courses with shorter time. I got two sets of answers.

- Those who work in a horizontal way; they were the majority: they do:
  - almost all definitions,
  - some theorems and some results,
  - no proofs.
- Those who work in a vertical way, they do:
  - some definitions,
  - the theorems and the results that correspond to the past definitions,
  - selected proofs.

Other questions arise about the exercises that correspond to both methods and assessment for each one. But still we need necessarily time to be familiar with some notions and notations, let some ideas mature, assimilate the material, learn how to write in mathematics, develop ideas (try to make connections between them), ask and answer questions, take some distance backwards to see things clearly, and just to talk.

Conclusion

Time management and how to adapt any change to teaching style taking into consideration results of both student learning and mathematics teaching is not so easy, neither natural or automatic. Taking all the time where it's necessary and going faster where it is necessary; that's the dilemma in order to be efficient and to reach quality.

3. Evaluation systems

When talking about evaluation, we can't consider any assessment method unless it's faithful. That is, it gives as closely as possible the truth about the students' level, the students' efforts and the teachers's teaching methods. Thirty years ago, the evaluation results in Algeria, in all levels, were not doubted by anyone. Society trusted completely that the results were translating true and real results. Nowadays, the least I can say about evaluation is that the situation is upset and all of society questions it. The worst is that we trust nobody and nothing. If your son gets a very high grade, you know that you can't be so happy, just because you would not to be surprised that this same son will get a worse mark in the next exam or in other course. In the university, the most frequent situation is that almost all new university students (in the first year) come to the university with good and even very good results, but two to three months later, the first exam results show that almost all of them are bad. This situation also exists with every first exam in each year with newcomer students (in a higher level). Another frequent situation that is to be questioned is when many good students who worked hard for a long period get bad results in their exams.

One of the evaluation definitions is (Stufflebeam, 1975) 'assessing a student is measuring the distance which separates his acquisitions from the objectives set and followed by the teacher or the institution'. How is this definition realized with our students and our teaching method?

It's clear that evaluation is managed by many factors. The main question is how to choose and set priorities to realize a faithful evaluation. It's important to know how to make an exam subject, how to set teaching objectives, how to adapt the programs to the given data, what is to be evaluated, how and which way, and who to manage time during an exam. In all exams in any field, (sport competitions, cooking tests, medicinal exams, driving exams,...) the candidates know the evaluation rules, the mistakes to be avoided and know mainly what is required to succeed. So, what about the students in mathematics exams, is it the same? According to what we read in our students' exams, the answer is clearly no!

According to a study made with the first year university students in Algeria, many principal factors were revealed to be directly related to the students exams results, I just give them without any comment to not make the paper longer than it should be.

- The students' comprehension is most of the time wrong before the exams; they correct most of their notions after the exams, but late after paying that by bad grades. The preparation for the exams reveals this important point, or how do we prepare the students for the exams?
- Writing style in mathematics can be a cause of failure even with a student who masters his results or at least thinks he is doing so. We all know that in a mathematical formal sentence, switching the existential quantificator with the universal one gives two different meanings. So, how do we teach our students to write and to read formal sentences, in which level, and do we pay attention to the writing details?
- The grading scale or method. Students generally do not know about the teacher's grading scale. It becomes worse when teachers have all different methods. For instance, a teacher may consider an exercise half correct if the method is correct with a calculation mistake,

but another one may consider it entirely false. For an integral with two parts, a teacher may give half of the credit if one of the parts is correct whereas another one may give nothing considering it all false.

- The mistake is the main element during the evaluation. But how is this element dealt with in the teaching and learning process; do we teach the students mistakes or do we avoid them making mistakes and then give them bad grades when they make them in their exams? It is a full dilemma for the students.
- Time management takes a large part during the examination. A slow student can correctly solve two exercises out of four and get the same score as another one who gets only the simple questions in all the exercises. A frequent situation is to have students who didn't get enough exercises and so not enough practice before the exam. On the exam day, they are still slow and can not solve all the questions.
- The psychological side of the students is also not less important in this matter. We get different results with the same exam given to encouraged students and discouraged ones. And with students treated with hostility and stress versus other students treated with confidence.

Conclusion

Assessment is itself a science. Many factors need to be managed and to be taken into consideration. But the most difficult question remains how to adopt an evaluation which can give true results. Without that, we can never progress, at least not in any good way. How can we strengthen the link so that we teach to evaluate and evaluate to teach?

4. Mathematics education research

We can not teach mathematics without being in touch with what's happening in mathematics education. Research is dealing with almost all sides and all fields of teaching and learning mathematics. The multiple studies and the multiple experiences can serve any teacher. When dealing with, for instance, a course about limits, or a course on rings, you can find many possibilities with good and bad sides for each: you can work on the right examples and avoid the bad ones and choose the right exercises and avoid the wrong ones to reach some objectives and so on. This can make the teaching and the learning process progress. When you find the same mistakes with the same level in many studies, you gain confidence and learn to question and so you progress. Mathematics education research can reveal many secrets, solve many problem situations and explain many difficult enigmas. Nowadays, simply, a training in mathematics education for any teacher is a necessity.

5. What about the principles of the teaching mission?

If you ask many teachers about the definition of teaching, you will get as many definitions as people you ask. Isn't that weird? I wondered one day, when I noticed that we teachers all teach in different ways. You'll say that's a good and not a bad thing; it may be a richness. I would agree if our all different ways were not opposite. The students learn different and opposite principles with their teachers at different levels.

A student can be asked for a long time to accept all the results and one day, he will be asked to prove all the results. He can be told all the time to not ask questions and with some teacher, he'll be asked to ask all the questions, if he doesn't, he'll find some of those questions in the exam. He can never know, for a long time, what mathematical comprehension is and one day he will be asked all the time 'did you understand' without knowing how to answer this question; if it is yes, almost or no as he didn't learn how and what to understand. How is it that we are all different teachers? Shouldn't we have common points that students know for sure that they must exist with every teacher? In a bestseller that I read [1], the writers who consider that teaching is not only a science but rather an art, redefined teaching by the most important elements that should exist with any teacher dealing with any science, giving ~~fruit to a long experienc~~e and a deep study. I'm glad to cite them here to revalue the teacher and to destroy the bad image that has been formed these last years especially about the university teacher as ‘someone who is able to write and to read what’s written at the blackboard’.

Nine principles have been associated to a teacher to accomplish his noble mission:

1. learning before teaching,
2. authority,
3. ethics,
4. order,
5. imagination,
6. compassion,
7. patience,
8. character,
9. pleasure.

Conclusion

I’ll conclude with some quotes from the cited book.

‘It’s much easier to talk or write about teaching than it is actually to teach. The aspects of teaching that we examined in this book are ideals –ends always to be sought if only rarely gained, their pursuit one of life’s most difficult tasks. Those who have never taught probably cannot fully imagine the demands on energy, patience, and will imposed by classroom work It’s exacting labor, often lacking clear and tangible results and requiring teachers to begin all over again what they have already tried to do; and it has been made more demanding today by trying social conditions that all too frequently challenge teachers and their students.’

‘Teaching is also the gift of one person to another. It is a compassionate extension of self in acknowledgment of the needs and aspirations of someone else, usually but not always younger than we are and always, for a time at least, depending on us for some kind of knowledge. In that gift of self consists teaching’s greatest satisfaction –the giving not so much of knowledge, which each person must acquire, as of habits of mind and heart and powers of thought.’

Universities need not only trained teachers, but teachers who challenge the continuous change of the world. We need to prepare and to offer to any society professional citizens, balanced, mastering perfectly their work and knowledge, and ready to give more and better, especially citizens who can handle again and again the permanent challenge of life.

Endnote

1 The elements of teaching: James M. Banner, Jr. and Harrold C. Cannon.

HOW CAN MODELS OF MATHEMATICAL DEVELOPMENT BE STRUCTURED, REPRESENTED, COMMUNICATED AND USED IN FORMATIVE ASSESSMENT?

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The purpose of this working group is to identify ways in which teachers and students can use models of mathematical development productively as part of a formative assessment process. In the sessions, and starting from some existing "seed" examples, participants will identify different ways of representing, communicating, and using these models in formative assessment in the classroom, generate new examples, and draft a research agenda investigating the efficacy of these approaches. Follow-on activities include expanding, piloting, and evaluating the different approaches through collaborations among researchers and practitioners, and convening again to share and learn from our collective experiences.

Background

Formative Assessment

Formative assessment is "a process used by teachers and students during instruction that provides feedback to adjust ongoing teaching and learning to improve students' achievement of intended instructional outcomes." (CCSSO, 2008). There is a large body of research linking the consistent and systematic use of formative assessment to improvements in student learning (Black & Wiliam, 1998, Brookhart, 2005, Nyquist, 2003). Wiliam (2004) stated that "in order for assessment to function formatively, it needs to identify where learners are in their learning, where they are going, and how to get there" (p.5). Most current work on formative assessment draws attention to both the teacher role and the student roles, emphasizing the importance of bringing students into the formative assessment process (Heritage, 2010). There is a small but developing body of research that suggests that within the process of formative assessment, teachers struggle particularly when it comes to determining "how the get there" i.e. the next instructional steps they should take with their students based on the assessment evidence (Heritage, Kim, Vendlinski & Herman, 2008).

Learning Progressions

Heritage (2010) citing Black and Wiliam (1998) reminds us that one requirement of formative assessment "is a sound model of students' progression in the learning of the subject matter, so that the criteria that guide the formative strategy can be matched to students' trajectories of learning" (Black & Wiliam, 1998, p. 37). The notion is that in order for teachers and students to identify what the next learning goal should be, they need to be able to assess where a student currently is on a path of learning. The next step on the path is then a natural learning goal for that student. Progressions define a series of points or states of understanding as an ordered sequence. By knowing a students' current state and the ordered sequence, teachers and students can target the next significant state of understanding that the student should move towards. While progressions can be used to determine the next state of understanding, they do not necessarily define what a student, with teacher support, needs to do to move into this next state of understanding, i.e. they define where to go, but not how to get there.

Within mathematics, Daro, Mosher, and Corcoran (2011) state that the term *learning trajectory* is more commonly used in mathematics, and allows for the possibility that the a trajectory includes the learning and teaching strategies that are needed for shifting students' understanding to the next level.

In our work on learning in mathematics, we have used the term *model of mathematical development* to make clear that the characterization of different levels of understanding is essentially a descriptive theory of the different states of understanding, and is predictive to the extent that the sequence of states captures common sequences of changes in understanding. We define a model of mathematical development as describing the major shifts in understanding that generally occur over medium to large time periods (typically months to years) as students grapple with mathematical knowledge. These shifts describe how a variety of types of knowledge (e.g. conceptual, procedural, contextual and metacognitive knowledge) change over time with respect to central ideas of the domain as students' competency improves due to learning within the domain and related domains, as well as more general cognitive development. Developmental stages can include shifts in view and belief, as in the shift from an understanding of the equal sign as a "calculate something" sign to an understanding of the equal sign as a truth-value statement of equality or balance between two mathematical quantities or objects. Developmental stages also can include increases in generality, integration, and robustness of existing knowledge, as in increases in procedural or representational fluency (Harris & Bauer, 2009).

Two progressions are included in table 1 and table 2. In an in-progress literature review, the authors identified 18 mathematics concepts that have associated candidate models of mathematical development (Bauer, Graf, Harris, Haberstroh, Attali, Wylie, and Leusner, in preparation). The first example, a model characterizing different understandings of central tendency is drawn from Watson and Moritz (1999), and Callingham and Watson (2003). Their proposed a theory of development for the concept of average and measures of central tendency (median and mode) is based upon the SOLO taxonomy Biggs and Collis, 1991) which is a neo-Piagetian framework for characterizing learning and development. The Watson and Moritz (1999) model focuses on how students develop an understanding of average as a measure of representativeness and draw conclusions about a variety of types of data. Table 2 provides a description of each of the levels (excerpted from Watson and Moritz, 1999) and examples drawn from Callingham and Watson, 2003, and Watson and Moritz, 1999).

Table 1
Central tendency developmental levels

Level	Description	Example
Preaverage (P)	Students have no term for average, even in a colloquial sense. When asked to solve problems involving concepts of central tendency, they typically provide tautological responses or idiosyncratic stories about the context.	To a question about the relation between average wage and median home price in Australia, an example response from a 9th grader at this level is "average means it is small and good for little families" (Watson & Moritz, 1999, p. 27)
Single Colloquial Usage for Average (U)	Students use colloquial terms for average, such as *normal* or *okay* and sometimes refer to "add up" colloquially but not in a calculation sense. They do not have an understanding of how to calculate an average.	Student use terms such as "same as others," "okay," "normal." (Callingham & Watson, 2003, p. 31 Avg 2.2) and refer to median of set of measures as "the most accurate value." (Watson & Callingham, 2003, p. 40 ME 13.1).
Multiple Structures for Average (M)	Students use at least one, often two or three ideas for the concept of central tendency including "most", "middle", and the add and divide algorithm for the mean to describe average in straightforward situations. However, they fail to consolidate multiple concepts meaningfully. They rarely use more than one of these ideas in complex questions. Students sometimes acknowledge conflict between incorrect calculations of mean and idea of mode.	"mean is a lot of numbers added up and dived by how many numbers there were to start." (Watson & Moritz, 1999, p 28).
Representation With Average (R)	Students refer to add-and-divide algorithm for the mean to describe average in straightforward situations. They can distinguish between the concepts of mean and median. They are aware of the association of the decimal form with the algorithm for mean. Students express some ideas related to the representative nature of average (e.g., prediction, estimation, or representing whole data set) and refer to "most" to describe data distributions compatible with mean or offer mode as alternative average concept. They know the algorithm for the mean but do not successfully apply it in complex contexts without prompting;	"not an average which would include all the extremes…out of 100 it would include 49.5 above it and 49.5 below it." (Watson & Moritz, 1999, p. 28).

A second example is provided in Table 2 which is a model of students' understanding proposed by the authors of equations and expressions with respect to two underlying concepts: equality and variable. The table begins with a description, that we call level 1, that represents the most basic level of understanding of both of these concepts addressed by the table, and then proceeds through four more levels of understanding. At each level there is a description of how a student thinks about equality and also the notion of a variable, and by combining two understandings, a description of what that student would be able to do and not do at that level of understanding Before looking at the table, which combines both idea, it is worth describing major conceptual changes in student understanding of equality and variable separately.

For equality, the primary distinction that is made is between thinking *operationally* about equality and thinking *relationally* about equality (Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006). The *operational* thinker treats the equal sign as a signal to do something – add or subtract or whatever action was called for on the left hand side of the equation. This way of thinking likely arises from students seeing many more arithmetic problems in math written as 5+3=__ compared to __=3+5 and often teachers will say "five plus three gives you what?" When a student begins to think *relationally,* he or she understands that the expression on the left hand side of the equation has the same value as the expression on the right hand side i.e. they are in balance. And furthermore, when a student understands *relational equivalence*, he or she will understand that equality in an equation is maintained provided you perform the same operation on both sides of an equation and will be able to judge equivalence in many cases without having to calculate values for each side of an equation by using techniques such an compensation and cancellation (Rittle-Johnson. Taylor, Matthews., & McEldoon, in press).

For variable, an important distinction is between students thinking of a variable as representing a *specific unknown*, i.e. as a placeholder for one and only one number, and as a *generalized number* i.e. that a variable can take on many values and be part of more complex mathematical relations as in a linear function (Booth, 1984; Weinberg, Stephens, McNeil, Krill, Knuth, and Alibali, 2004). Prior to these two understandings, students may have one or more contextually dependent beliefs about variables such as an algebraic letter must be the same as the first letter of what it stands for (e.g. 'b' for brownies) (Küchemann, 1978; Booth, 1984).. The table below combines these understandings of equality and variable into five levels of understanding overall.

Table 2
Equality and Variable: Equations and Expressions

At **level 1**, students have a superficial understanding of the concept of variable that does not include the idea of variable representing an "unknown." Instead, students may hold one or more beliefs about variable including, for example, that variables represent objects (e.g. that the variable "b" stands for "brownies" in a word problem because it starts with a b). Alternatively students may treat variables like individual digits of a number (if $2x=24$, x must equal 4), they may ignore variables in an equation only operating on the numbers, or separate variables from numbers in an equation or expression.

At this level students also have an *operational* sense of equal sign. Students with this type of understanding believe that the equal sign is a signal that indicates that there is s problem to be solved or computation expressed on the left of the equal sign and that the answer should be placed to the right of the equal sign.

Given these understandings of equality and variable, students should have trouble solving word problems, even with informal methods, algebra solving follows misconceptions as described in model of equality.

At **level 2** students still have an operational understanding of equality as in level 1 above. Their understanding of variable is different. Students have an understanding of variable as a "specific unknown" in which letters stand for one and only one number. For example, at this level if asked "in the expression 2n+3, can the variable n stand for the number 4?" the, student will say yes and but, then if asked if the variable n can stand for 37 they will say "no.

Students can apply informal methods to solve simple algebraic word problems, but will continue to display misconceptions when attempting to solve symbolic equations. Students can attempt to solve equations using guess and check strategy and other approaches that do not involve substantial symbolic manipulation

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At **level 3**, students have "specific unknown" level concept of variable as in level 2. What's different is their understanding of equality. They have a more sophisticated understanding of the equal sign as expressing that the quantity to the left of the equal sign has the same value or is in balance with the quantity expressed on the right of the equal sign (described sometimes as a relational understanding of equality). They also know some simple procedures for algebraic manipulation.

Students can solve problems with one variable represented in text symbolically using algebraic manipulation. Because students have few ways of understanding algebraic expressions and equations, they may apply less efficient methods to solve equations. Expressions are only understood correctly if they can take on a single numerical value (i.e., consistent with students' understanding of variable as specific unknown).

At **level 4**, students' understanding of equality is richer than in level 3. They understand relational equivalence in most contexts (meaning that they not only understand that both sides of an equation are equivalent, but they also understand that performing the same operation to both sides of the equation maintains the equivalence) and can readily use transformations to balance equations to solve for an unknown. Students still have the same understanding of variable as in level 3 i.e. the "specific unknown" level concept of variable.

Students do not yet have a "generalized number" concept but because of relational equivalence they can solve many equations with one unknown algebraically

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At **level 5,** students have "generalized number" level concept of variable, in which a variable can function as a pattern generalizer for arithmetic, as in the statement a + b = b + a, or can express relations among sets of numbers as parameters and arguments as in the linear equation x = mx + b. Students have the same understanding of equality as in level 4, i.e. they understand relational equivalence and can readily use transformations to balance equations to solve for an unknown.

Students can solve symbolic equations, parsing expressions to flexibly apply operations on to solve equations efficiently.

There is substantial research and development still needed to understand the characteristics of effective and practical formative assessment. Additional research is needed also in identifying, describing, and evaluating models of mathematical development. In this working group, in order to help advance both fields, we will explore the mutual constraints that these two areas provide. We believe answers to a variety of useful questions can be developed by making explicit constraints that models of mathematical development provide to the field of formative assessment, and the constraints that formative assessment can provide to the creation of models of mathematical development. To make this idea clearer, the aims and plans of the working group are described next.

Aims of the Working Group

Models of mathematical development are central to formative assessment. They provide testable theories of how student understanding changes in light of instruction and time. They are needed to define gaps between students' current understanding and learning goals in a way that makes closing the gap feasible. If the gap between two states of understanding has been empirically validated, is present within the population of learners, and many students have moved from one state to the other in the past, teachers and students should be confident that this unit of learning is feasible. There has been research on the feasibility of developing interim and summative assessments to collect evidence and make statistically valid inferences about student levels within models (Watson and Callingham, 2005; Weaver and Junker, 2004; Wilson, 2008). There has been less work on how best to represent models for use by teachers and students, (but see Lesh, Lamon, Gong, & Post, 1993; Wilson, 2005; Watson and Callingham, 2005). Still less is known about how to use models of mathematical development in the service of deciding on next instructional steps to be taken by teachers and students (Heritage, Kim, Vendlinski & Herman, 2008).

Wiggins (1998) provides an example of formative assessment within a welding classroom. In the example, students are learning to weld a ninety degree corner joint. When students think they have completed the weld to industry standards, they compare their welded corner joint to a set of joints to determine to which joint theirs matches best. The joints are ordered in terms of quality and each has specific characteristics relevant to beginning welders In doing this comparison, students are a) assessing at what level they are performing b) determining their next learning goals, and most importantly c) understanding what specific next things they need to do to improve. In other words, a student can consider his or her next instructional steps by, for example, realizing that on the next joint he or she works on, the two pieces to be joined must be more precisely positioned, or that gaps need to be filled, or bubbles removed. In this example, the connections from student performances to level of competency and next instructional step are transparent. Teachers and students can compare student performance to benchmark performances and determine what they need to do next.

Unfortunately, this sequence of inferences is not so transparent in mathematics learning. How does the Wiggins welding example translate to a mathematics context, when it is not skill in a directly observable work product that is evolving and improving, but rather knowledge and understanding around abstract concepts such as the meaning of "average" or "variable"? While there has been some work in this area (e.g. Forster and Masters' work on the Developmental Assessment Resource for Teachers (DART) (2004), much still needs to be learned. How can models of mathematical development be structured to best support formative assessment? Is a linear sequence of levels an adequate model to characterize student understanding to inform formative assessment? Are there circumstances in which model in the form of a network of states of understanding could provide greater value? Is a model in the form of a set of understandings better in some circumstances? What other structures are possible and advisable? How can models of mathematical development be represented and presented to make learning goals and instructional approaches as transparent as possible to teachers and students? What additional information will be helpful to provide? The aim of this working group is to begin to develop answers to these questions.

Plan for the Sessions

The sessions will focus on three work products 1) a list of different ways of representing models of mathematical development with examples that might be especially useful to teachers and students 2) descriptions of the use of these representations in making next instructional choices or adjustments transparent along with descriptions any additional materials and 3) a research agenda for expanding, piloting, and evaluating the use of different representations of models of mathematical development in formative assessment along with specific work individuals or groups will be doing in the coming year with respect to this agenda.

In session 1, the organizers will present the aims of the group and the session, provide some examples of models of mathematical development, a common definition and explanation of formative assessment, example tasks, and a video of the use of one model to support formative assessment as part of a classroom discussion. The bulk of the session will be a facilitated brainstorming session on different structures and representations of models for use in formative assessment. One area of interest to the organizers is how to include students (following from the welding example) in the use of the models of development. Much of the writing around formative assessment has emphasized not just the teacher role in the process, but the centrality of students to the process. Given that models of development may span several years of student development, there are questions about when it is appropriate to be explicit about the developmental model, and how a model can be of most value to students.

In session 2, the working group will select a set of the representations and break into small groups to flesh out examples of their use with respect to specific models of mathematical development. In the later portion of this session, the working group will reconvene to discuss and contrast the examples and potential value of the different representations.

In session 3, the working group will identify research questions and associated approaches and activities for addressing them based upon the work in sessions 1 and 2. Participants will discuss their interest and willingness in engaging in continued research collaborations to address these questions and sketch out next steps for the working group.

Follow-On Activities

Throughout the year, participants may explore the use of the representations of models and ideas for use in formative assessment in their research or in their practice. It is the hope of the organizers that early pre-proposal explorations are carried out. And one or more research proposals may be generated. The work will also inform an aspect of an existing grant on which the organizers are working. The organizers are currently working with a group of approximately 15 middle school teachers, and will have opportunities to pilot some of the emerging ideas from this working group session and will provide feedback to the working group participants on their ideas. The intent is to reconvene at the next PME-NA to share experiences on specific collaborations that developed, to discuss progress on aspects of the work, and to plan and focus continued efforts in this area.

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THE GENDER AND MATHEMATICS WORKING GROUP: REDEFINING RESEARCH AGENDAS AROUND GENDER, INTERNATIONAL CONTRIBUTIONS, AND TEACHING

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The Gender and Mathematics Working Group (GMWG) reconvenes this year at PME-NA to: discuss current work on gender at the national and international levels; examine feminist and gender study theories that inform our work; and connect gender and mathematics research findings to practice in schools and communities. Members share seven short summaries within topical subgroups to discuss theoretical frameworks, research questions, and results; and to make recommendations for further work. After sharing out results from the topical subgroups, the GMWG discusses future directions and makes plans for collaborations during the coming year. Collaborations include support in developing research agendas, collaborative writing, and pursuit of funding opportunities. The GMWG is open to any PME-NA members interested in issues of gender and mathematics; one does not have to have been a past participant.

Introduction

The Gender and Mathematics Working Group (GMWG) of PME-NA suspended its activity so that members and leadership could take the time to pursue agendas defined by the group. The GMWG members reconvened at PME-NA XXXIII to discuss current work done by GMWG participants and in the field at large and to organize additional work. In the paragraphs that follow we share a brief history of the working group, explain the emphases of this year's meetings and how they relate to PME-NA goals, and outline our group's agenda.

Brief History of the Working Group

The GMWG met annually at PME-NA from 1998 until 2007, except for the year of the joint meeting with the International Group for the Psychology of Mathematics Education in 2003. The work of the group began with reviews of gender and mathematics scholarship and an effort to identify absences in the research strands reviewed. Committing to an integration of our collective scholarship on gender and mathematics, we defined future directions for research and for the working group. An early result was a visual representation, a graphic, of our conception of the field of gender and mathematics, and the complexity of the elements with(in) which we work (Damarin & Erchick, 1999; Erchick, Condron & Appelbaum, 2000).

After the first meeting of the GMWG, we continued to gather annually at each PME-NA meeting, sharing our scholarship on gender and mathematics, collectively and individually redefining our directions and purposes. We supported each other and sought feedback from the GMWG participants and the PME-NA membership at large. Forming peer groups of individuals with common interests and related research efforts, we reviewed, critiqued, and discussed our body of scholarship. It was the support of the individuals' work that defined our group, as we mentored, critiqued, and collaborated with each other across the varied directions members pursued. This approach to our work exemplifies a way in which the group practiced a particular element of the scholarship connected to gender – agency. As agency became an emphasis of our group, it determined the most appropriate approach for us in our work together.

At the 2004 PME-NA sessions in Toronto, the GMWG members began moving our work into new spaces. In these sessions we explored ways in which we can more deeply examine the relationship between gender and mathematics, with reflection upon international perspectives and critical theory, connected work in gender and technology, and critical perspectives on pervasive, recurring questions about the place for gender work in mathematics education (Erchick, Applebaum, Becker, & Damarin, 2004). At the 2005 PME-NA sessions in Roanoke, we discussed topics across the range of the scholarship of our members, and found a unifying framework emerging in our work. That framework was the role and development of social agency in women and girls' experience as students, teachers, and researchers in mathematics education and led our way until we temporarily suspended our work together after 2007.

Connection to PME-NA Goals

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians and mathematics teachers.
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

The first of these goals, involving international contacts and exchanges, was one of the areas for which our group had just begun to make gains. We intend through the 2011 meetings to reignite our international initiative with presentations in the sessions and efforts to extend and develop the international aspect of the work of this PME-NA group. The second and third goals are in part addressed by our GMWG goals to utilize the theory of feminist and gender studies in our research, to focus more of that research on teaching practice, and then to apply more of those findings to the classroom setting, at all levels. Both the second and third goals are also addressed in part by our third area of interest, the advancement of theory around concepts related to the study of gender and mathematics (e.g. voice, agency, epistemology), both for its broad theoretical advancement and for its application to teaching and learning. Finally, we encourage emergent agendas that arise within this work, especially those that support the goals of PME-NA.

Plan for Active Engagement of Participants

As has always been the case with the GMWG, the sessions we conduct this year are intended to be active with discussion, decision-making, and work activities. We remain committed to an initiative that depends upon participant voices for direction and support. In this year's sessions, we began with introductions and a very brief presentation of a multi-faceted framework to help inform directions in this year's GMWG meetings. As participants discuss theory, research, and application to practice in the topical subgroups, we asked they consider the framework elements to help guide their work. The framework elements are:

- Recent calls to incorporate additional perspectives such as queer theory, and to clarify the definition of *gender* (see more at *Sexual Identity and Mathematics Education Research and Practice* and *The Importance of Defining "gender"* below);

- Recent findings on international studies on gender and mathematics (see more at *An International Survey of the General Public's Views on Gender-related Issues and Mathematics Learning* and *International Perspectives on Gender and Mathematics Education* below); and
- Attention to the focus of research funding possibilities: Discussion of Gender in Science and Engineering program (GSE), Joanne Rossi Becker.

The GSE programs seeks to develop the nation's knowledge, social, and human capital aimed at broadening the participation of girls and young women in STEM education. The GSE program covers grades K-undergraduate education with particular attention to transition points in women's educational experiences. The program also is broad enough to include research and diffusion activities that focus on men and boys underrepresented in STEM fields. The GSE program funds three types of projects:

1) Research projects to investigate:
   i) Gender-based factors that affect the learning in STEM education and choices students make to choose STEM education and careers
   ii) Societal, formal and informal systems and how they interact with individuals to encourage or discourage interest and persistence in study or careers in STEM fields.
2) Diffusion of research-based innovation projects to:
   i) Provide a mechanism to engage a wider audience of practitioners with research findings and strategies for changing educational practice relative to gender issues
   ii) Pilot, scale up or disseminate
3) Extension services to create a cadre of agents who can inform and assist practitioners in adopting proven gender-inclusive policies and practices in pedagogy (train the trainer model).

We encouraged interested members of the GMWG to consider collaboration and possible grant applications. Discussion supported generating questions for research, questions for diffusion, and models of extension services.

These three themes (gender theory, international perspectives, and funding possibilities) cut across the discussion in the topical subgroups listed below. After the brief framework discussion, the group spent the rest of its time in day one and half of its time in day two working in the topical subgroups, sharing and discussing each others' current work (in various stages of development), feedback on research questions, and recommendations to further the work. Sharing out across subgroups, and making decisions on directions for the year complete day two. Day three was dedicated to participants actively planning research for the upcoming year.

Topical subgroups will be organized around the PME-NA and GMWG goals as follows:

- International Contributions to the Study of Gender and Mathematics
- Utilizing Multiple Theoretical Perspectives in the Study of Gender and Mathematics (Feminism, Postmodernism, and Gender Studies)
- Taking Gender and Mathematics Research Findings into Teaching (School, Teacher Education, and Out-of-School Experiences)

The sharing of work in the subgroups is brief, interactive, and introduces questions to the whole group for further discussion. In the following paragraphs we describe work contributors will share work as a part of the subgroups' discussions.

Introduction to Shared Work

The following sections summarize the work contributors shared in groups and discussion questions to help facilitate the topical groups. Additional participants joining the group at the

PME-NA meeting, also shared their work in discussions receiving feedback and support as well.

Helen Forgasz, Monash University
An International Survey of the General Public's Views on Gender-related Issues and Mathematics Learning

Among many factors found to contribute to gender differences favoring males in mathematics learning outcomes is the perception that mathematics is considered a male domain. Gender-stereotyped beliefs associated with mathematics learning include, for example, beliefs that boys are "naturally" better than girls at mathematics and that males are more suited to careers in mathematics-related fields (e.g., science, computing). While the views of students, teachers, parents, and other stake-holders have been gathered fairly regularly and in a range of countries, less often are the opinions of the general public determined. Using an innovative approach, participants in a survey aimed at gauging the general public's views on gender stereotyping and mathematics learning were recruited using the social networking site, Facebook. The target audience for the advertisement about the survey was limited to those over 18 years of age (an ethical consideration), and the penetration was global. Responses were received from over 300 from 54 different countries. The general findings were that the majority of respondents did not hold gender-stereotyped views. However, among those that did, the traditional male-stereotype persisted.

Discussion questions

- Considering the progress that has been made in many aspects of women's lives in many parts of the world, why is the male stereotype so pervasive with respect to mathematics learning?
- What can or should be done to address the persisting view that mathematics is a male domain? To whom should action be targeted?
- What additional research is needed in this field?

Olof Steinthorsdottir, North Carolina State
International Perspectives on Gender and Mathematics Education

This short report focuses on the recently published book *International Perspectives on Gender and Mathematics Education* edited by Helen J. Forgasz, Joanne Rossi Becker, Kyeong-Hwa Lee, and Olof Bjorg Steinthorsdottir (Forgasz, Becker, Lee & Steinthorsdottir, 2010). This book emerged from five years of working groups at the international conferences of PME. Separated into four sections, the book provides papers on: history, policy and non-school factors affecting gender and mathematics; foci on the status of gender research in various countries; discussions of high achievers and mathematics education; and reports on tertiary students.

The first section begins with a reprint of a 1979 paper by Teri Perl on the Ladies Diary and the mathematics therein published in the US from 1704-1941. Remaining chapters focus on how parents do math with their children, review out-of-school programs, discuss a longitudinal study of youth in New Zealand, and examine how attention paid to gender equality in Australia has changed over time. The second section has a focus on the state of gender research in a variety of international venues, with two chapters from Mexico, a synthesis of several large-scale international studies, and work from Iceland and Germany. The third section of the volume focuses on research on high achievers in mathematics, with work from Korea, Australia, and the US. The final section of the book includes three chapters that focus on gender and mathematics beyond high school, with particular work from Canada, the US, and a variety of other countries.

Discussion questions:

- ∑ Are gender differences in mathematics actually diminishing? Is the situation the same in all countries or just in developed countries?
- ∑ What are some critical areas of research to benefit from cross-cultural investigation?
- ∑ What theoretical frameworks make sense in various cultural contexts?

Lynda R. Wiest, University of Nevada, Reno

Sexual Identity and Mathematics Education Research and Practice

According to Alexander (2008), “Queer theory is a collective of intellectual speculations and challenges to the social and political constructions of sexualized and gender identity” (p. 108). The sciences are charged with being heterosexist in nature with narrow, essentialist perspectives grounded in biological determinism and systems for classifying living things into “neat” categories, and other assumptions (Toynton, 2007). Interestingly, research shows a relationship between sexual orientation and spatial abilities (specifically, mentally rotation of objects), with heterosexual men performing better than bisexual or homosexual men and the reverse being true for women (University of Warwick, 2007). It is thus curious that the field of gender and mathematics education has paid so little attention to sexual identity in research and practice.

Rands (2009) notes that subject areas other than mathematics have incorporated attention to sexual identity and asserts, “The time has come to queer elementary mathematics education” (p. 182). Suggestions for “queering” mathematics education include “adding gays and stirring,” where one has gay people and topics in otherwise heterosexist activities, such as using a story problem about children with two moms, constructing a timeline of gay history, and doing mathematical explorations with gay symbols such as the pink triangle. “Queering” also includes analyzing social statistics on topics such as census figures, same-sex marriage, and the school and everyday experiences of sexual-minority individuals (e.g., school attendance, hate crimes); and challenging fundamental assumptions and perspectives, such as binary male/female and masculine/feminine distinctions in which mathematics is mainly associated with male/masculine and presumptions about opposite-sex attractions and distractions in single-sex schooling (Copping, 2011; Mendick, 2006; Rands, 2009).

Discussion Questions

1. What relationships may exist between sexual identity and mathematics?
2. What research objectives and methods do you suggest for investigating the nature of the relationship between sexual identity and mathematics?
3. What obstacles are researchers and practitioners likely to face when incorporating sexual-identity topics/content into mathematics instruction? How might they address these challenges?
4. Should tests that disaggregate data by other social identities, such as gender, race/ethnicity, socioeconomic status, and language proficiency, also disaggregate data according to students’ sexual identity? Why or why not?

Diana B. Erchick, Ohio State University at Newark

The Importance of Defining “gender”

What do we mean by “gender”? Damarin and Erchick (2010) raise the issue of defining what we mean by gender in their JRME article “Toward clarifying the meanings of *gender* in mathematics education.” They note Glasser and Smith’s (2008) call to the education research community at large, and reviewed all JRME issues between 2000?? and 2007 in their analysis of the meanings of gender therein. What they found is one of the areas we need to keep in our view as we proceed with our work around gender and mathematics. From Fennema and Hart's 1994 JRME article in which those scholars identified the need for bringing cognitive and feminist perspectives to bear on the study of gender and mathematics, to biological perspectives, to postmodern and socio cultural approaches, it is clear that we are not clear about what we mean by gender. Damarin and Erchick argue "it is time to take a new turn by acknowledging that the relationships of gender to mathematics education are much more complex than once thought and bringing theories of gender to bear on research. In this rich context, clarity must supplant vagueness in the meaning of gender in mathematics education research" (p. 321).

Recommendations from Damarin and Erchick (2010) include deconstructing the binary; attending to power, agency and voice; considering what it is about mathematics that might contribute to our struggles to define gender; and considering what it is about sex/gender that makes it different from or similar to other biological/social constructs such as race. One might consider Burton's 1995 seminal work "Moving towards a feminist epistemology of mathematics" to inform our thinking about the nature of mathematics. And consider the collection of works in *International Perspectives on Gender and Mathematics* (Forgasz, Becker, & Lee, 2010) in exploring sociocultural connections. Overall, the sex/gender system, introduced to the first GMWG by Suzanne Damarin in 1998 (Erchick & Damarin, 1999), has a complexity that both challenges us and incites us to define, each in our own work, what we mean by gender.

Discussion questions:

- How can consideration of feminist epistemology help inform our work?
- How can gender identity help us broaden our understanding of gender?
- Do multiple definitions of constructs in the sex/gender system clarify or compound the definitions of gender?

Judith Olson and Melfried Olson, University of Hawaii at Manoa

Nonverbal Communication When Parents and Children Work on Mathematics Tasks

In the project, *The Role Of Gender In Language Used By Parents And Children Working On Mathematical Tasks,* the authors investigated gender-related differences in verbal language used by parents and children working on mathematical tasks. We analyzed three ten-minute videos for each of 114 dyads, mother-son, mother-daughter, father-son, and father-daughter. Additionally, 66 ten-minute videos for 22 dyads (five boys, with both mother and father, and 6 girls, with both mother and father) were analyzed for gender-related differences in nonverbal communication.

Nonverbal communication is sending and receiving wordless messages, both intentional and unintentional. In everyday situations we often use verbal and nonverbal channels simultaneously to convey intent. Nonverbal messages constitute an integral part of speech and provide visual referents, punctuate key points, add structure to words, and synchronize body and thought. Nonverbal communication can be used in a variety of ways, such as to repeat and strengthen a verbal message; complement a verbal message; send opposing or conflicting message; regulate conversation; substitute for a verbal message; or alter interpretation of a verbal message.

While much nonverbal communication is based on arbitrary symbols that differ from culture to culture, a large proportion is to some extent iconic and may be universally understood. Our investigation examined eight different nonverbal actions, each with two subareas: 1. body positioning (proximity or action toward another; proximity or action away from another), 2. gesturing (finger or hand pointing; moving of objects in workspace), 3. head movements (nodding or shaking head or shrugging shoulders; tilting while focused on work area), 4. attempted eye contact (seeking approval and expecting a response; monitoring actions with no response expected), 5. making eye contact (to give or seek approval or a response; face to face discussion); 6. facial expressions (demonstrating emotion; disengaged); 7. smiling (happy; discomfort), and 8. laughing (happy; discomfort). All but the facial expression actions were coded by dyad according to whether a mother, father, daughter or son initiated the action. Hence, when we recorded DF, we recorded actions for the daughter-father dyad, but for which the daughter initiated the action. Facial expression was coded only by dyad, not by who initiated. Preliminary results and initial analysis were shared in the GMWG.

Discussion Questions:

- Of the actions and categories we have identified, which might make significant contributions to gender research?
- What variations are there in interpretation of the nonverbal actions?
- What is the relevant theoretical base for gender-related nonverbal interactions among parents and children working together on mathematical task?
- Does each nonverbal action have communication implications?

Manjula P. Joseph, Ohio State University

Supporting Student Learning In A Girls' Mathematics Summer Camp Classroom

This study examined pedagogy in a mathematics summer camp for girls conducted at a large mid-western university in the United States. The day-camp is for female students in the middle grades and is conducted for a period of five days with twenty teaching sessions that cover five units of middle school mathematics. Camp curriculum and video-recordings of the teaching sessions were analyzed to find correlations between curricular aspects and equity pedagogy and teaching practices that support a gender-equity agenda. Curriculum was examined to identify those practices that most support equity and activities that enhance certain equity practices.

The research questions for this study included:

- What curricular and pedagogical decisions in a middle school mathematics camp setting support girls' learning?
- How do teachers' perspectives, and practice support equitable teaching in the classroom?
- How does a teacher's ethnicity/race/gender influence classroom practice for equity?

Study participants are teachers from the summer camp. Transcriptions of video-recordings of sessions and follow-up interviews with teachers comprise qualitative data for the study. Session transcriptions were coded for those particular equitable practices that attend to gender-related issues in mathematics teaching and learning. The camp's curriculum was studied to identify connections between aspects that provide opportunities for teachers to enhance student learning and participation for girls in their classrooms. Data from the interviews with participants helped make connections between teacher perspectives and beliefs and their classroom practice.

Discussion Questions:

- What questions should be asked to identify "equitable practices" for teaching girls?
- What interview questions inform understanding teacher perspectives on girls' learning?
- Is it worthwhile to explore teacher race/ethnicity/gender as influencing classroom environment/culture in an all-girls classroom?

Katrina Piatek-Jimenez, Central Michigan University
Women's Choices of Mathematical Careers

For years, women have been earning nearly half of the mathematics baccalaureate degrees in the United States; but, they continue to make up a smaller percentage of those pursuing advanced degrees in mathematics or entering mathematics-related careers. Although these statistics are well-documented, the literature does not thoroughly address *why* many women who choose to study mathematics at the undergraduate level do not continue with careers in the field.

During the spring semesters of 2006, 2007, and 2008, I conducted individual interviews with a total of 12 undergraduate women mathematics majors at two different universities in the Midwest. The goal of the interviews was to investigate what influenced these women to choose to study mathematics at the undergraduate level and how they had determined their future career plans. I asked each of these students if they would be interested in participating in a follow-up interview three to five years after the initial interviews. During the Spring semester of 2011, I conducted follow-up interviews with the majority of these women to learn what career decisions they made after graduating and to learn what influenced these decisions.

Results suggest a variety of reasons influenced participants' choice to study mathematics at the undergraduate level. They all considered themselves to be good at mathematics and felt they had a strength in the subject. All of these women also had at least one significant person (family member or teacher) who supported and encouraged them in the field of mathematics. Many of them liked the idea of majoring in mathematics because it meant, to them, that they were breaking gender boundaries. Finally, for some of the women intending to become high school teachers, the promise of marketability also influenced their decision to major in mathematics.

The most predominant theme that influenced their career aspirations, while they were still in college, was the desire to have a career where they could help people and make a difference in the world. For those not earning degrees to become high school teachers, their lack of knowledge of mathematical careers also played a large role in their future career goals.

Discussion Questions

- How can we learn about patterns of discouragement/encouragement of women entering mathematics?
- How does perception of mathematics as a male domain interact with discouragement patterns – both sending and receiving?
- What mathematical careers can be framed as more helping?

Michael Dornoo, The Ohio State University at Newark
Addressing Gendered Needs in Mathematics Learning Through Social Networking

Mathematics educators interested in issues of gender equity have designed and implemented awareness-raising experiences for parents, counselors, and teachers as well as special programs to encourage female students in mathematics learning (Karp & Niemi, 2000). Despite these efforts, findings vary about how best to address gendered needs because female students remain less likely than male students to persist in mathematics study (Ansell & Doerr, 2000; Lubienski, McGraw, and Strutchens (2004). Leder (1992) found that gender differences in mathematics achievement between males and females are small compared to differences within gender groups. Ansell and Doerr (2000) and Lubienski et al. (2004) have identified differences between males and females in multiple areas of mathematics, including the strategies that students employ to solve problems and performance on computation tasks. In addition to differences in mathematics achievement, Ansell and Doerr (2000) documented gender differences related to affective factors, including students' attitudes, self-concepts, and self-confidence. For instance, while male students tend to have a more positive attitude and self-concept toward mathematics, females tend to lack confidence and have negative attitudes (Lubienski et al., 2004).

Damarin and Erchick's (2010) cultural/historical construction model, in the context of women and work, has gender and technology as a co-constructed foundation of the model. As technology becomes more prevalent, and advancements make working interactively under a global network more feasible, the advancements add complexity to the study of gender and mathematics and other STEM fields. How do interactive technologies that emphasize online collaboration, social collaboration and resource sharing among users free of charge support or hinder the gendered development of girls in mathematics? Web 2.0 could be an appropriate technology to build many global virtual communities among students, teachers, educators and social workers from every corner of the world, but does it help girls in terms of their relationship with mathematics? Or does it instead reinforce barriers between girls and STEM fields.

Discussion Questions

- How can a new generation of mathematics educators address gendered needs in mathematics learning through social networking?
- Will modifications in the delivery of mathematics instruction by integrating technology provide experiences that bridge gaps in gender groups or create another barrier?

Anticipated Follow-Up Activities

The following work will continue following the PME-NA meetings:

- Members will continue supporting the development and implementation of individual and collaborative projects clarified or identified in the GMWG sessions.
- Members so inclined will pursue funding opportunities in gender and mathematics research, individually or collaboratively, with grant development support as available.
- Members will expand the functionality of the PME-NA GMWG website. The website currently is used primarily as a repository for prior PME-NA GMWG proceedings papers. We expect to expand it to include a reference list of gender and mathematics scholarship, both from our members and from scholars outside of our group; a password-protected component for group discussions in between annual meetings; and a repository for PME-NA proceedings papers from our GMWG members.

Closing

In pursuing research on gender and mathematics, the 2011 PME-NA Gender and Mathematics Working Group participants continue prior work that was a part of the GMWG in its first years, and readily reconvene to again collaborate with and support each other. The most recent work on international perspectives, syntheses and new questions around gender study and gender identity, and renewed views on pedagogy inspire our work. We welcome the challenges of addressing the complexities emerging in the literature and as we commit to an interpretation of the field of gender and mathematics as complex and nonlinear. We choose, too, to do our work in support of each other, consistently respecting and welcoming the voices that emerge.

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rich.

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SUPPORTING MATHEMATICS TEACHERS TO INCREASE RETENTION

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National reports have identified the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education. However, providing high quality mathematics education for all students goes beyond the recruitment of knowledgeable teachers. This working group is designed to offer an opportunity to examine the role that professional development and support play in the work and retention of mathematics teachers. Retention focuses on new teachers, especially those in urban areas and mathematics teachers in hard-to-hire settings. Discussions concentrate on the study of interventions through Professional Development and Support Models. Efforts to deepen our understanding of the complex and multifaceted picture of why teachers leave and why they stay, and how efforts to retain teachers impact their work in the classroom and their decisions to stay or leave, are developed through the sharing of research designs, data collection, and on-going results. This working group is appropriate for anyone who has work to share or who is thinking about supporting a retention project. Throughout, we address this very complex task in terms of both the opportunities and challenges for mathematics education researchers to provide quantitative and qualitative input on a major political issue. It is hoped that this working group will enrich the dialogue about a national crisis in mathematics education.

Brief Overview of the STIR Working Group

The Supporting Teachers to Increase Retention Working Group was launched in 2009 at PME-NA 31 in Atlanta, GA to investigate the relationship between Professional Development/Support and the retention of mathematics teachers. A second meeting in 2010 at PME-NA 32 in Columbus, OH elaborated on this initial ground work and directions identified for participants to focus on. Over the past two years, a total of 16 participants attended the working group sessions aimed at identifying the gaps in the research on mathematics teacher retention in order to move forward in tackling this complex national issue. The emerging dialogue was based on the results of research studies and ideas of the participants. Participants shared their backgrounds and interests in retention issues, and discussed potential research directions with the ultimate goal of addressing key research issues that would constitute a research agenda for the group. Participants were interested in examining the on-going preparation, support and retention of grade 7-12 mathematics teachers from a variety of angles, including: (1) Impact of Professional Development on Teacher Retention ; (2) Relationship between Content Knowledge and Retention ; (3) Obtaining Research that examines the Dimensions of Professional Development and Support that impact Retention. The overarching question of Mathematics Teacher Retention is often overlooked under the assumption that effective professional development would in essence lead to increased retention. However a closer look at what type of support helps teachers stay in their school, let alone their profession is necessary.

At PME-NA 32 in Columbus, OH, the group elaborated on the research agenda that was outlined at PME-NA 31 in Atlanta, GA. To stimulate discussions, members of the group provided summaries of on-going projects to be discussed during the group meeting sessions. These projects fall under the following categories: (1) Role of technology in support/retention; (2) Professional support community that reflect the building of networks and contacts to support work, decisions, challenges, and opportunities that arise in the teaching of mathematics, including lesson study and electronically-based communities of practice; (3) Role of leadership and/or career enhancement in retaining mathematics teachers, including the PD directed towards new teachers entering the field through alternative certifications, those coming from other careers and shifts in PD needed as new teachers move through the challenges of their first 5 years of teaching; (4) Content-based Professional development with emphases on conceptual linking and problem solving; and (5) Research issues that arise in examining teacher retention.

Issues of Psychology of Mathematics Education to be Focused on

The study of the relationship between Professional Development and Support Models on the work and retention of mathematics teachers in grades 7 – 12 merits careful examination. Several national reports have pointed to the need to increase the pool of highly qualified mathematics teachers as a way to improve mathematics education and maintain the United States' economic competitiveness (National Academy of Sciences, 2007; Glenn Commission, 2000). However, providing high quality mathematics education for all students goes beyond the recruitment of mathematically knowledgeable teachers to encompass issues of teacher support, professional development, and retention. Over the past two decades, analyses of teacher employment patterns reveal that many new recruits leave their school and teaching a short time after they enter. Ingersoll, using data from the School and Staffing Survey concluded that in 1999-2000, 27% of first year teachers left their schools. Of those, 11 percent left teaching and 16 percent transferred to new schools (Smith and Ingersoll, 2003). Earlier research revealed that teachers who leave first are likely to be those with the highest qualifications (Murnane and other 1991; Schlecty and Vance 1981). This "revolving door" problem is even worse in large urban districts; for example, 25% of the teachers new to Philadelphia in 1999-2000 left after their first year and more than half left within four years (Neild and other 2003). In Chicago, an analysis of turnover rates in 64 high-poverty, high-minority schools revealed that 23.3 percent of new teachers left in 2001-2002.

Reasons for the lack of retention of new teachers and teachers in high-poverty schools are often related to "working conditions" and lack of support (Ingersoll, 2001; Smith & Ingersoll, 2004; Johnson et al., 2004), though pay also plays a role (Hanushek, Kain, & Rivkin, 2001). This support includes professional and collegial support such as working collaboratively with colleagues, coherent, job-embedded assistance, professional development, having input on key issues and progressively expanding influence and increasing opportunities (Johnson 2006). Preparation, support, and working conditions are important, because they are essential to teachers' effectiveness on the job and their ability to realize the intrinsic rewards that attract many to teaching and keep them in the profession despite the relatively low pay (Johnson & Birkeland, 2003; Liu, Johnson, & Peske, 2004).

A status report on teacher development focusing on professional development and support of teachers (Darling-Hammond et al, 2009) summarized findings and put forth recommendations for effective professional development. The basis for the paper included national surveys with self reported data, a meta-analysis of 1,300 research studies, and specific studies. The conclusion is that “well designed” professional development can influence teacher practice and student performance. The paper focuses on what is or could be regarded as well designed. One strand of the paper is that of effective support for new teachers. Although half of the states require support for new teachers (Education Week, 2008) it was found that rates of participation in teacher induction programs varied by school types with highest rates in schools with the least poverty and lowest in schools with high levels of poverty. Beyond the rates of participation and availability of support, there is the question of what is effective support. The paper cites an on-going large-scale research project currently underway, which aims to measure the impact in terms of classroom practices, student achievement and teacher mobility. Initial results seem to reflect the difficulty in identifying the impact of support.

Another study presently in its fourth year, Supporting Teachers to Increase Retention (STIR) is studying the relationship between retention and support of mathematics teachers across the state of California. This five-year study is looking through the lens of 10 sites with different support models to relate retention to content knowledge, classroom practices, professional communities of support, leadership and needed support. Initial results are complex but are showing relationships between sustained professional development and support and teacher retention. Data collected to establish a base line for retention across a five-year period 2002-2006 preceding STIR shows that yearly attrition averaged 20% across all 10 sites. For the five-year period the attrition average was 54% with sites reporting an attrition of mathematics teachers as high as 73%. In the first year the attrition dropped from 20% to 14%. Three years of data and results of intervention are now available for sharing. But, what is the relationship between the support and the retention? One of the 10 sites from this study observed that success of a retention initiative takes root in a variety of needs: the need to know your District and its teachers – a need that is often addressed by established, long-term relationships between the university and district leaders; the need to offer sustained support as opposed to punctual interventions in order to break the isolation of beginning teachers and to create a sustainable community; the need to establish relevance of the professional development activities proposed by engaging participants in deep introspection of their own knowledge gaps; the need to involve all players in the community to prevent miscommunication from annihilating attempts made towards change; the need to nurture the community created by moving its members forward into roles and responsibilities they are ready to take on; and last but not least, the need to refine even successful models to keep the momentum (Felter & Faughn, 2009). These findings align with prior research emphasizing that support must be specific in addressing the needs of teachers in their particular context (Fulton et al., 2005).

As indicated in the comments above, support comes from multiple sources. Another recent study from Peabody College, Vanderbilt University, finds that principals play a critical role in the support of new mathematics teachers (McGraner, 2009). This confirms Ingersoll’s recent analysis of Mathematics and Science teacher turnover from the past two decades indicating a steady increase in the phenomena. Since the existing pool of Mathematics and Science teachers is not as well supplied as in other non-STEM disciplines, this increase is not easily absorbed by individual schools, even though enough newly qualified teachers are produced each year to cover increases in students’ enrollment and the effects of retirement. Indeed, upon leaving their teaching position, Mathematics and Science teachers are more likely to opt for non-educational professions than teachers in other fields. According to data gathered by the National Center for Education Statistics with the School and Staffing Survey and the Teacher Follow-up Survey, the provision of useful Professional Development is one of the organizational factors influencing choices of Mathematics teachers to leave or remain in their positions. Another factor is the degree of individual classroom autonomy (Ingersoll & May, 2010). But how do we successfully involve principals in supporting professional development that increases retention?

Finally, an additional aspect of the issue at stake is the retention of mathematics teachers entering the profession through alternative certification as brought up by one working group participant. In a presentation to the group at PME-NA 2009, Brian Evans emphasized retention issues within the New York City Teaching Fellows program and provided us with the following literature review: “Teachers leave teaching in New York City for three reasons (Stein, 2002): retirement, leaving the profession, and transferring to a school outside New York City. [...] A concern with alternative certification is lack of retention, especially in large urban areas such as New York City (Darling-Hammond, Holtzman, Gatlin, & Heilig, 2005). Sipe and D’Angelo (2006) found when surveying Fellows that about 70% of them intended to stay in education. NYCTF reports that 89 percent of Fellows begin a second year of teaching after the completion of their first year (NYCTF, 2008). Boyd, Grossman, Lankford, Michelli, Loeb, and Wyckoff (2006) reported that about 46% of Teaching Fellows stay in teaching after four years as compared to 55% to 63% of traditionally prepared teachers. Kane, Rockoff, and Staiger (2006) found that Teaching Fellows and traditionally prepared teachers have similar retention rates. Further, Tai, Liu, and Fan (2006) claim that alternative certification teachers, in general, have comparable commitment to the teaching profession as do traditionally trained teachers. In a survey of 31 Teaching Fellows, 90 percent said they were considering leaving their high needs schools for better schools in or outside of New York City, or leaving the teaching profession altogether (Stein, 2002). [...] Similar to results found in other studies (Costigan, 2004; Cruickshank, Jenkins, & Metcalf, 2006; Evans, 2009), teachers were very concerned with student behavioral problems and unsupportive administration.” (Evans, 2009) During the 2010 meeting, Evans elaborated on these findings by incorporating information about the Teach of America Program: “Darling-Hammond, Holtzman, Gatlin, and Heilig (2005) cautioned that, upon becoming certified, many TFA teachers leave teaching. This is in contrast to Teach for America’s own report of retention of TFA teachers on their website. TFA claimed that about two-thirds of TFA teachers stay in the field of education upon completing their time in the program, and half of those remain in teaching. This means about one-third of all TFA alumni stayed in the classroom upon fulfilling their commitment, and another one-third maintained non-teaching roles in education, such as in administration or advocacy (TFA, 2008). As of 2010, there were 17,000 TFA alumni (TFA, 2010). According to TFA, over 5600 TFA members remained teaching in the classroom after their commitment ended (TFA, 2009).” (Evans, 2010)

This working group is designed to offer a comprehensive, multifaceted examination of the ongoing preparation, support and retention of 7-12 grade mathematics teachers based on the results of research studies and ideas of the participants. It is hoped that this working group will enrich the dialogue relating the “support gap” and the work and retention of teachers of mathematics. It is also expected that this working group will propose areas ripe for further research. In light of national efforts to close “poor performing schools” this work to identify ways to improve retention of mathematics teachers becomes especially critical.

Year 1 & 2 of the Working Group: Summary

A Summary Report from the first two years of the working group is included below. This year's proposal builds upon the proposal from last year by integrating participants' research interests and focusing the discussions around four major themes relating to teacher retention. A reference list produced by combining the references from the proposal and work of the first two years is found at the end of this proposal.

Summary Report: Activities of Working Group PME-NA meetings 2009 & 2010

<u>**Participants:**</u> *Douglas Owens (Ohio State University, OH); Drew Polly (UNC Charlotte, NC); Candice Ridlon (UMES, MD); Brian Evans (Pace University, NYC); Christine D. Thomas (Georgia State University, GA); Ellen Clay (Drexel University, PA); Allyson Hallman (UGA, GA) ; Michael Meagher (CUNY, NYC); Barbara Pence (SJSU, CA); Axelle Faughn (WCU, NC); Nancy Schoolcraft (IN); Terran Felter (CSUB, CA); Molly Fisher (University of KY); Jacqueline Leonard (UC Denver, CO); Sharilyn Owens (ASU, NC).*

Participants were asked to consider the question "Can support impact teacher retention?", and more specifically "What are the different aspects of support? How is impact measured? What are opportunities and challenges encountered when researching teacher retention?" The major task for our first session consisted of coming up with directions to work and identifying missing foci in literature. Participants organized the questions and interests into four main categories that emerged from the initial brainstorming: Impact of Professional Development on Teacher Retention; Content Knowledge and Retention; Research issues and Retention; Equity and Retention. A detailed list of questions arranged by interests under the four major themes can be found in the 2009 Working Group Summary (Faughn, Pence, Thomas, 2010).

Further discussions underlined that PD must be sustained, long-term, and involve a community of learners: Mentoring necessitates careful pairing, could be done through videotapes and reflections; Decreasing the number of preps for beginning teachers could provide more time for planning and reflecting; We need to connect retention to student learning: Is seeing students succeed part of teacher perceived success that could help with retention? i.e., would evidence of increased students' performance help build confidence & a sense of competence? Finally, advantages of "whole school" reforms and developing leadership skills of individuals to bring PD back to their site were emphasized to increase onsite presence through lesson study, lesson planning, online community, coaching, and/or videotapes.

In addition to identifying interest and gaps from the literature, participants shared the following models of support:

∑ Brian Evans – NYCTF literature review
∑ Drew Polly – UNC Charlotte - Researcher-beginning teacher onsite mentorship
∑ Candice Ridlon – PD through reform curricula in Utah
∑ Axelle Faughn – Addressing content knowledge through higher education courses
∑ Barbara Pence – CMP STIR 10 sites, 10 models
∑ Christine Thomas – Developing an online community of support

The core question of the relationship between Professional Development/Support and retention was significant and central to all discussions but due to lack of directly related research, formed the springboard for lists of questions. The 2010 working group built a more focused and active research based on active engagement of participants in productive reflection on the issues across projects. After a review of previous work for new participants, two main breakout groups aimed at lay the foundations of publishable collaborative work. Returning participants were asked to inform the group of any progress they made in their respective projects. Ideas were then shared on further work and dissemination in order to better define the direction of our efforts. In response to the question "Where do we want to go from here?", several options were put forward, including developing research and papers, participation in future conferences (New Teacher Center; AERA; 2012 Symposium on Mathematics Teacher Retention), establishment of a website/webbase on teacher retention, draft of a monograph on teacher retention, write up of a joint paper for an editorial journal (NCTM/AMTE), and ultimately publication of a book on Professional Development (Math content, Technology, Communities of Practice, Leadership) aimed at teacher educators who focus on bridging research and practice. This brainstorming was followed by a detailed overview of breakout groups by group leaders.

Description and Summary of 2010 Breakout Sessions Listed by Category

(1) Category A: Role of technology in support/retention (Axelle Faughn).

We discussed issues pertaining to the role of technology in empowering teachers both in the classroom and the larger mathematics education community. Questions to address include: (1) What model of PD has been provided that engaged participants in technology use? (2) What trends are you noticing in technology use by teachers? (3) Are you able to relate technology to Retention or Leadership? (4) What constitutes technology in the work of mathematics teachers? (5) What are issues related to equity when working with instructional technologies?

Background: Teachers constantly ask for engagement strategies, including games, technology, etc. Students live in a technology-embedded world; how does this affect their learning schemes? How can teaching take this into consideration? Teachers are often less technology-savvy than their students. How do we support teachers in increasing confidence and competence on learning the instrumentation, shifting from learning to use technology to using it for teaching math, staying up-to-date with new technology (at T^3 last year many teachers were looking for TI-84 talks, but most presentations focused on the TI-NSpire)

Questions/Issue:

- How can we best support teachers to acquire TPACK?
- Can technology help with student engagement/collaborative learning, student learning for understanding, teacher leadership, teacher retention, and access to higher cognitive demand tasks?
- What different types of technology are involved in the work of mathematics teachers and how do they support teaching: Technology for presentation purpose and/or planning; Technology for conceptual learning; Technology for Community Building (Teachers, students)
- How do we collect evidence of support through technology? (Use TPACK qualitative framework and TPACK stages of development)
- What type of technology do teachers tend to embrace first?

Work was shared about teachers using the TI-Navigator, looking at discourse through case studies (CCMS).

- Other issues raised in the discussion: How do we bring communities of practice into an online environment? What is the promotion step beyond being a Master Teacher except for leaving the classroom? (Coach, Higher Ed.)

Work in Progress: AERA paper to be presented in April 2011 in New Orleans, LA, on four CMP-STIR sites.

(2) Category B: Professional support community that reflect the building of networks and contacts to support work, decisions, challenges, and opportunities that arise in the teaching of mathematics, including lesson study and electronically-based communities of practice (Christine D. Thomas).

We share our evaluation and synthesis of the literature on Second Life, professional development within a PLC, and distance learning used in our participatory action research study of this Second Life project. The Second Life project that spans August 2009-May 2011 aims at: (1) sustaining mathematics teachers who are attempting to improve their teaching and students' learning and (2) conducting research for the dissemination of knowledge on retention of secondary mathematics teachers in these schools. We use this as an opportunity to share our understandings, interpretations, and analysis of the literature on the topic as it relates to the goals, and needs of the urban community and the mathematics teachers we serve. In addition, we share our beginning analysis, and initial findings of the study based on the baseline data collected.

Background: Description of work with an Online Professional Community of Teachers in Second Life in Atlanta, GA, allowing for more convenience and flexibility in the offering of PD, and enhancing opportunities for teacher leadership.

Themes: Online learning, social theories of learning (situated learning), Peer leadership, Adult learners.

Questions/Issues:

- Finding a framework for looking at the dimensions of teacher retention through online professional learning communities, especially for teachers in high need schools – Suggestion was made to investigate Collective Efficacy within Communities of Practice
- How do we protect users through modeling and training on the ethical use of technology?
- How do teacher leaders reinforce retention? How can teachers become more integral parts of the school leadership?
- Other issues raised: What type of mathematics is discussed among teachers? Is generating a professional community of mathematics teachers online different from other such communities in other disciplines, in other words, how do you provide math PD online?

Work in Progress: One paper in press and one paper to be presented at AERA 2011.

(3) Category C: Content based Professional development with emphases on conceptual linking and problem solving and establishing collaboration with mathematicians (Imre Tuba)

Background: Professional Development and Mathematical Understandings

- Mathematical Knowledge (Kathy Heid) –Mathematical Thinking – Problem Solving
- Mathematical Knowledge for Teaching – D. Ball
- Integrated Pedagogy and Math Content

Questions:

- Does content *driven* PD result in improved mathematical understanding?
- Can the PD content be linked to classroom practices?
- How does this effect retention?

Data:

- Pretest/Posttest on Content
- Beliefs about content understanding relative to content to be taught (Confidence, Competence)
- Classroom practice (Self reports, Classroom Observations

Work in Progress: Specific to Imperial Valley site

- Content tests linked with self reported changes in classroom practices
  - Overview of PD
  - Pre/Post results (Full test, Scales)
- Classroom observations and self reported reflections relative to the PD
  - Confidence and Competence

Plans for the Third Year of the Working Group

In advance of the conference, last year's participants will be surveyed to identify any questions or shifts in foci that occurred this year. Additional people will also be contacted. The Supporting Teachers to Increase Retention 2011 working group comes in conjunction with the working group organizers' endeavors to promote discussion and raise national awareness of the issue of Mathematics Teacher Retention through a Mathematics Teacher Retention Symposium to be held March 22-24, 2012 in Los Angeles. In preparation for this symposium, several documents have been drafted that will support the work of this working group, namely a clear statement of the current problem in Mathematics Teacher Retention, a list of guiding principles in attempting to address this national issue, and a list of guiding questions aimed at advancing research-based knowledge. All these will be shared with working group participants ahead of time, and will inform the directions and foci for the 2011 meeting.

Additionally, recent work by Ingersoll and May (2010) on the current national trends in mathematics teacher retention will guide some of the discussions and questions for the group to ponder in light of their own on-going projects. In particular Ingersoll and May investigated mathematics and science teachers' turnover using data from the Teacher Follow Up Survey through the lenses of **Magnitude, Destinations,** and **Determinants.** In preparation for this Working group, participants will be asked to read and report on these findings.

Regarding Professional Development and Support, we will continue to explore the themes of Technology, Online Professional Communities, and Content Knowledge. Additionally, models of Communities of Practice through Lesson Study, and establishment of leadership roles for teachers to support retention will specifically be addressed. Participants interested in examining the Lesson Study theme will address implementation strategies and techniques for implementing lesson study. Questions to be discussed include: (1) What is the timeline for planning, teaching and re-teaching? (2) How do teachers define/identify misconceptions by students? (3) How and what do teachers decide in changing their lesson? (4) Who is involved in the Lesson Study planning and implementation? We also plan on exploring the role of early-career leadership and/or career enhancement in retaining mathematics teachers, including the shifts in PD needed as new teachers move through the challenges of their first five years of teaching.

The general outline of the three days of the working group will include:

Day 1 • Introductions and Review of past years' work and update on work by those who are attending, including review of new literature
- Overview of MTRS and reflection/focus on work by Ingersoll and May in small groups

Day 2 • Introductions of new attendees
- Breakout groups around new and old themes chosen by participants
- Brief discussion of issues addressed in each group
- Small group work focused on challenges and knowledge base

Day 3 • Breakout group reports and future directions

The productivity of the working group will be a function of the advanced organization. But at this point, we anticipate that the breakout groups will either continue to address the questions generated from the previous years' discussions, or attempt to tackle new directions in light of recent research findings.

Anticipated Results of the Working Group

To continue addressing the particular issues of retention and technology, retention and community, retention and leadership, and/or retention and content-based professional development, each participant will share a description of the work to date in their project, the stage of development the project is in, research design and instrumentation, and a summary of current findings. Joint collaboration was already successful through the efforts of this Working Group to disseminate findings at the 2011 meeting of AERA. As noted earlier, the organizers of the working group plan to solicit more papers emerging from this collaborative work, and possibly the development of a monograph synthesizing our work on mathematics teacher retention and support.

The network will also be included in the development of a Mathematics Teacher Retention Symposium (MTRS), sponsored by the CMP-STIR group, to take place in spring 2012.

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ADDRESSING EQUITY AND DIVERSITY ISSUES IN MATHEMATICS EDUCATION

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As the title suggests, this Working Group has a dual focus on issues of mathematics teaching and learning and issues of equity and diversity. This includes topics discussed at the Working Group in 2009 and 2010, namely: pre-service mathematics teacher education for social justice, culturally relevant and responsive mathematics education, supporting mathematical discourse in the linguistically diverse classroom, creating observation protocols around instructional practice, and examining student experiences. This work attempts to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships, including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Brief History

This Working Group builds on and extends the work of the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME is a group of emerging scholars (new faculty and graduate students) who graduated from, or in some cases are still studying at, three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA). The Center was dedicated to creating a community of scholars poised to address some critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).

The DiME Group (as well as subsets of that group) has already engaged in important scholarly activities. After two years of a cross-campus collaboration dedicated to studying issues framed by the question of why particular groups of students (i.e. poor students, students of color, English learners) fail in school mathematics in comparison to their white (and sometimes Asian) peers, we presented a symposium at AERA 2005 (DiME Group, 2005). This was followed by the writing of a chapter in the recently published *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007). Further, in an effort to bring together and expand the community of scholars interested in this work, DiME, at AERA in 2008, sponsored a one-day Professional Development session examining equity and diversity issues in Mathematics Education. In addition, DiME members have joined with other scholars in joint presentations and conferences. A book on research of professional development that attends to both equity and mathematics issues has recently been published (Foote, 2010). Many DiME members as well as other scholars contributed to this volume.

Moreover, the Center has historically held DiME conferences each summer. These conferences provide a place for fellows and faculty to discuss their current work as well as to hear from leaders in the emerging field of equity and diversity issues in mathematics education. Since the summer of 2008, the DiME Conference opened to non-DiME graduate students with similar research interests from other CLTs such as the Center for the Mathematics Education of Latinos/as (CEMELA), as well as graduate students not affiliated with an NSF CLT. This was initially an attempt to bring together a larger group of emerging scholars whose research focuses on issues of equity and diversity in mathematics education. In addition, DiME graduates, as they have moved to other universities, have begun to work with scholars and graduate students including those with connections to other NSF CLTs such as MetroMath and the Urban Case Studies Project in MAC-MTL whose projects also incorporate issues of equity and diversity in mathematics education.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Theoretically we have been building on the work of such scholars as Marta Civil (Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Megan Franke (Franke, Kazemi, & Battey, 2007), Eric Gutstein (Gutstein, 2006), Danny Martin (Martin, 2000), Judit Moschkovitch (Moschkovich, 2002), and Na'ilah Nasir (Nasir, 2002). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), and again Megan Franke (Kazemi & Franke, 2004), as well as many others outside of the field of mathematics education.

A significant strand of the work of the DiME Center for Learning and Teaching included implementing professional development programs grounded in teachers' practice and focusing on equity at each site. The research and professional development efforts of DiME scholars are deeply intertwined, and much of the research thus far produced by members of the DiME Group addresses issues of equity within Professional Development. Additionally, since the majority of the DiME graduates, as new professors, along with a number of current Fellows, are engaged in teaching Mathematics Methods courses, the integration of issues of equity with issues of mathematics teaching and learning in Math Methods has become a site of interest for research. We have learned through experience that collaboration is a critical component to our work.

We were pleased for the opportunity offered by the first two years of being a Working Group at PME 2009 and PME 2010 to continue working together as well as to expand the group to include other interested scholars with similar research interests. In 2009 an ambitious number of sub-groups were formed. Some of these sub-groups worked together throughout 2009-20 10; others had less success in accomplishing goals set out at PME 2009. During PME 2010, several of the sub-groups were collapsed so that a smaller number of sub-groups evolved out of the 2010 Working Group Meeting. These groups have set agendas for collaboration during 2010-2011 and work is in progress. We were encouraged that our efforts were well received; more than 40 scholars from a wide variety of universities and other educational organizations took part in the Working Group in 2009 and an equal number did so in 2010.

Under the umbrella of attending to equity and diversity issues in mathematics education, researchers are currently focusing on such issues as teaching and classroom interactions, professional development, pre-service teacher education (primarily in mathematics methods classes), student learning (including the learning of particular sub-groups of students such as African American students or English learners), and parent perspectives. Much of the work attempts to contextualize the teaching and learning of mathematics within the local contexts in which it happens, as well as to examine the structures within which this teaching and learning occurs (e.g. large urban, suburban, or rural districts; under-resourced or well-resourced schools;

and high-stakes testing environments). How the greater contexts and policies at the national, state, and district level impact the teaching and learning of mathematics at specific local sites is an important issue, as is how issues of culture, race, and power intersect with issues of student achievement and learning in mathematics.

Existing research tends either to focus on professional development in mathematics (e.g., Barnett, 1998; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Kazemi & Franke, 2004; Lewis, 2000; Saxe, Gearhart, & Nasir, 2001; Schifter, 1998; Schifter & Fosnot, 1993; Sherin & vanEs, 2003), or professional development for equity (e.g., Sleeter, 1992, 1997; Lawrence & Tatum, 1997a). Little research exists, however, which examines professional development or mathematics methods courses that integrate both. The effects of these separate bodies of work, one based on mathematics and one based on equity, limits the impact that teachers can have in actual classrooms. The former can help us uncover the complexities of children's mathematical thinking as well as the ways in which curriculum can support mathematical understanding in a number of domains. The latter has produced a body of literature that has helped to reveal educational inequities as well as demonstrated ways in which inequities in the educational enterprise could be overcome.

To bridge these separate bodies of work, the Working Group has begun and will continue to focus on analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for students of color to have access to mathematical trajectories in school and beyond. Further, asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This research begins to take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy.

Some of the research questions that the Working Group will consider are:

Teachers and Teaching

- What are the characteristics, dispositions, etc. of successful mathematics teachers for *all* students across a variety of local contexts and schools? How do they convey a sense of purpose for learning mathematical content through their instruction?
- How do beginning mathematics teachers perceive and negotiate the multiple challenges of the school context? How do they talk about the challenges and supports for their work in achieving and equitable mathematics pedagogy?
- What impediments do teachers face in teaching mathematics for understanding?
- How can mathematics teachers learn to teach mathematics with a culturally relevant approach?
- What does teaching mathematics for social justice look like in a variety of local contexts?
- What are the complexities inherent in teacher learning about equity pedagogy when students come from a variety of cultural and/or linguistic backgrounds all of which may differ from the teacher's background?
- What are dominant discourses of mathematics teachers?
- What ways do we have (or can we develop) of measuring equitable mathematics instruction?

Students and learning

- What is the role of student academic and mathematics identity in achievement?

- How do students' out-of-school experiences influence their learning of school mathematics?
- What is the role of perceived/historical opportunity on student participation in mathematics?
- *Policy*
- How does an environment of high-stakes standardized testing affect whether and how teachers teach mathematics for understanding? How does this play out across a variety of local contexts? How can we support teachers to teach mathematics for understanding in that environment?
- How do we address issues of tracking/ability grouping and in particular the grouping of students by test designation?

Plan for Working Group

The overarching goal of the group continues to be to facilitate collaboration within the growing community of scholars and practitioners concerned with understanding and addressing the challenges of attending to issues of equity and diversity in mathematics education. The PME Working Group provides an important forum for these scholars to come together with other interested researchers who identify their work as attending to equity and diversity issues within mathematics education in order to develop plans for future research. Our main goal for this year, then, is to continue a sustained collaboration around key issues (theoretical and methodological) related to research design and analysis in studies attending to issues of equity and diversity in mathematics education.

In order to support this collaborative research, smaller research groups were formed from participants in the large Working Group. The plans and goals for these sub-groups are detailed in the final section: Previous Work of the Group and Anticipated Follow-up Activities.

Much of our work is qualitative in nature and we recognize that one way to increase the number of participants is to conduct research across several sites. In order to do this, we need to use the same protocols for data gathering. One of our sub-groups has taken as its charge to develop an observation protocol. Another sub-group is collecting vignettes of practice. We intend that, during what we hope to be the long life of this Working Group, research protocols and other materials that can support researchers, teacher educators, and teachers may be developed and used across a variety of research and professional development (including preservice teacher education) projects.

More specifically, for PME 2011 we will proceed as follows.

SESSION 1:

- Review and discussion of goals of Working Group.
- Introduction of participants.
- Reports of the work of sub-groups during 2010-2011
- Initial brief meeting of sub-groups

SESSION 2:

- Sub-group meetings to discuss and delineate further plans to address and expand research questions initially developed at PME 2009, and refined at PME 2010.

SESSION 3:

- Sharing and discussion of work from Session 2.
- Planning for further collaboration.
- Developing a tentative agenda for future Working Group meetings.

Previous Work of the Group and Anticipated Follow-up Activities

The Working Group met for three productive sessions at both PME 2009 and PME 2010. In 2009, we identified areas of interest to the participants within the broad area of equity and diversity issues in mathematics education. Much fruitful discussion was had as areas were identified and examined. In 2010, sub-groups were reorganized to streamline the work, as it became apparent during 2009-20 10 that participants could not participate in more than one subgroup during the year. During 2010, participants met in an identified sub-group of their choice. Within these sub-groups, rich conversations ensued regarding theoretical and practical considerations of the topics. Plans and goals for the 2010-2011 academic year were developed. What follows are the plans and goals of the sub-groups following the 2010 PME meeting. The activities stated in these plans are currently in progress.

Pre-service Teacher Education that Frames Mathematics Education as a Social and Political Activity

This sub-group is interested in teacher education that frames mathematics as a social and political activity, including multicultural education, teaching math for social justice, funds of knowledge, ethnomathematics, issues of equity and diversity in mathematics, and so forth. Our goal is to share resources to improve our own work as teacher educators and to support each other in our research. Our long term goals include developing an annotated list of articles, developing an annotated collection of resources (lessons, activities, syllabi, etc.), writing a paper about our differing meanings and approaches to teacher education that frames mathematics as a social and political activity, and conducting research about doing this work across contexts.

This sub-group acknowledges tensions in our work focusing on equity and social justice in relationship to reform mathematics. Frameworks are needed to understand these issues. These can build on work in culturally relevant pedagogy (Ladson-Billings, 1995), teaching for social justice (Gutstein, 2003), funds-of-knowledge (González et al., 2001) as well as more general issues of equity, diversity, social analysis, and critical pedagogy. We need to begin by defining what we mean by these terms (e.g. reform mathematics, social activity, political activity); and how we recognize them in the classroom.

Goal for 2010-2011. Each month (or so) one person will be responsible for engaging this sub-group in some fashion that that person will determine, and then recruiting someone to take over for the next month. This could be as simple as uploading a few materials and letting people know about them or it could involve asking others to read and respond to an article, task, and so forth.

Culturally Relevant and Responsive Mathematics Education (CRRME)

The title reflects our view that mathematics education needs to be both culturally responsive and culturally relevant and a primary goal of this group is develop a comprehensive collection of the scholarship we draw on to define these terms. We are interested in language, discourse, ethnicity, ways of interacting, family, community, experiences, generational issues, expectations (not high and low, but individual or community's expectations). We want to examine what we mean by social justice. Issues can be examined, such as teaching (a) about social justice (the context), (b) with social justice (status and participation), and (c) for social justice (power and question). Various aspects of CRRME include (a) local contexts, (b) local associations, (c) using cultural referents, (d) ethnomathematics, (e) critical pedagogy, and (f) teaching "classical" math. We are concerned as well with how literature on culturally relevant pedagogy is grounded in existing theory and research on culture and social constructivism.

Goal for 2010-2011. Two key goals of this group are to (a) develop a comprehensive collection of the scholarship we draw on, and (b) develop a collective definition of what Culturally Relevant and Responsive Mathematics Education means. The second goal requires an examination of what social scientists say about culture and how we view the relationship of culture to research in mathematics education.

Creating Observation Protocols around Instructional Practices

This group is developing a protocol that can measure instructional practice AND be a tool to help teachers improve their instructional practice. Our focus is on the importance of improving instruction for students of color; this is our goal. We recognize that protocols have limits. For example, protocols do not necessarily look at microgenesis, teacher change, structural issues, dispositions. At the past PME-NA we reviewed various protocols to examine discuss existing protocols posted on google groups to stimulate rich discussions around questions such as: What do we like? What are they missing? How might we revise, combine, and extend them? Since group members were using various measures and it was not viable to have everyone use the same one, we began to develop a dimension that could be added on to the different protocols. To focus the work we decided to develop a dimension on expanding notions of competence in the classroom.

Goal for 2010-2011. The observation protocol group has two goals for the year. First, we plan to develop the dimension on expanding notions of competence in the classroom. Second, we want to begin field testing this dimension and compare results across various school sites including variation in terms of geography, class, race etc. Our longer-term goal is to develop multiple dimensions that are theoretical grounded in order to produce an observation tool that does not ignore issues of race, class, and so forth. We also hope to develop a paper or report focused on the validity, reliability, and utility of this tool.

Language and Discourse Group: Issues around Supporting Mathematical Discourse in Linguistically Diverse Classrooms

This sub-group is interested in examining language diversity in the mathematics classroom. They think about this broadly and want to be inclusive of the perspectives of teachers, students, and parents.

Goal for 2010-2011. The language and discourse group has identified two tasks for 2010-11. One is to identify vignettes embedded in academic articles or book chapters on (a) teachers' experiences with language during mathematics instruction, (b) involvement of parents who are native speakers of a language other than English, and (c) experiences of students who are native speakers of a language other than English or are bilingual speakers of English and another language. We will explore the possibility of categorizing these vignettes in such categories as (a) social benefits of bilingualism, (b) cognitive benefits of bilingualism, (c) language ideologies, and (d) policy initiatives. The second task is to network in order to identify others (teachers, researchers, students) who would be willing to write-up their experiences/journeys as they relate to the use of language in mathematics activities in and outside of the classroom.

A Critical Examination of Student Experiences

This sub-group is interested in examining the intersectionality of students' experiences as learners of mathematics and in mathematics classrooms. This involves considering students' mathematics identity in relationship to one or more of their racial, social, cultural, and gender identities. This also includes understanding how structural inequalities shape students' mathematics experience, particularly students from non-dominant groups.

*Goal for 2010-2011.*The primary goal of the student experiences group for this year is to develop our capacity for collaboration. First, we will reach out to researchers in mathematics education interested in issues at the intersection of student identity/identities and mathematics learning experiences and establish a network to facilitate the sharing of our empirical work, data collection and analysis resources (e.g., interview and observation protocols), and theoretical resources. Based on the connections that develop, we will then consider potential joint products for next year—e.g., literature review, conference symposia, edited volume.

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DEVELOPING INVESTIGATIONS OF LIVED AND LIVING MATHEMATICAL EXPERIENCES: THE DIME WORKING GROUP

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The "Developing Investigations of Mathematical Experience" (DIME) working group is focused on building a research-based understanding of the interiorized experiential world of idiosyncratic mathematics. Goals of this research are to characterize individual experiences in ways that acknowledge a person's active and reflective thinking efforts within mathematical contexts linked to emotive dimensions of their lived and living mathematical experiences, in order to inform mathematics teaching practices. To probe these largely unexplored complex domains, we are using both and phenomenological and constructivist theories and methods. We seek to identify and relate expressed indicators of intended and actual mathematical experiences, as they appear to occur in the complex relationships of participants involved in mathematics teaching and learning. Because today's goals and strategies for a sound mathematical education embody a major emphasis upon stimulating, nurturing, and demonstrating high quality thinking arising from particular kinds of intended experiences, we must now seek to understand more clearly the nature of what students and teachers are actually experiencing in their "math lives." These working group sessions will build upon the DIME Research Team meetings conducted during the initial planning conference of the Wyoming Institute for the Study and Development of Mathematics Education (WISDOM^e) held September 2010 at UW. The primary goal of the PME-NA Working Group will be to involve additional interested scholars in the ongoing work of the team, through further conceptual analyses related to the phenomena of lived and living mathematical experiences, to extend our epistemological and psychological perspectives for characterizing critical elements, and to identify and address methodological issues inherent in investigating interiorized mathematical experiential phenomena.

Brief History of the Group

A core membership for this new PME-NA Working Group was established through the initial invitational "Planning Conference for WISDOM^e" conducted September 8-10, 2010 at the University of Wyoming. The background context for that conference is the establishment of three collaborative, interdisciplinary Research Teams connected to four new Ph.D. program research identities: quantitative reasoning, mathematical modeling, technology tools and applications, and lived/living mathematical experience. Each of the teams consists of UW mathematics education, mathematics, and educational cognate faculty, mathematics education doctoral students, and four-to-five young, active Mathematics Education researchers from a variety of national and international universities. Following four invited plenary papers prepared and presented at the conference by key active senior researchers, each team met to frame an initial program of collaborative work. The DIME (Developing Investigations of Mathematical Experience) Team currently includes six UW faculty members, six UW Mathematics Education Ph.D. students, and five faculty members from as many other universities.

Through the opportunity to conduct Working Group sessions at PME-NA, we seek an important venue to continue and expand discussions of the background perspectives and issues related to investigations of lived and living mathematical experience, to share information, issues, and problems related to ongoing research, to promote interest and potential participation in furthering these and other disciplined inquiries into these phenomena, and to provide continuing support to the team members to collaborate within and across ongoing and future research.

Perspectives on *Lived and Living Mathematical Experience*

Today, there is surely an implicit and implied emphasis globally upon high quality mathematical experiences within the goals and expectations for a sound mathematical education. Yet, within our literatures and advocacies for curriculum, for teaching, for learning and development, and for research knowledge there seems to be very little explicit attention to the phenomenal data of an individual's lived/living mathematical experiences. This may be partially explained by the nature of what most mathematicians and mathematics educators seem to focus upon within mathematically productive activity: cognition, thinking, sense making, reasoning, solving, explaining, etc. Research to build models of such focal emphases should surely be a fundamental cornerstone of mathematical education research. But, to further our knowledge of actual lived/living mathematical experience set against such mathematically productive activity frames an important goal DIME research efforts.

Some fundamental questions that we are seeking to investigate include the following:

- What is the nature of lived and living mathematical experiences?
- What are the particular qualities (essences) of human experience that make it a "mathematical experience?"
- Are these experiential qualities that are unique to mathematics, or are they also found more broadly in other human experiences?
- How do identifiable qualities of lived/living mathematical experiences relate to learning mathematics and to psychological models to describe mathematical thinking and feeling?
- In what ways do lived/living mathematical experiences vary or differ among humans, and why?
- What factors related to the individual or the experiential context might frame or affect the actual phenomenon of a high quality "living mathematical experience?"
- In what ways do the experiences of individuals vary within mathematical contexts, such as classroom situations that are intended to engender the "same" kinds of experiences for all participants?
- How do lived mathematical experiences accumulate within differing individuals, and with what more general intellective and emotive consequences?
- In what ways do thinking and feeling aspects of mathematical experience interact within a living experiential context, and with what kinds of impacts or outcomes?
- In what ways might a mathematics educator use what can be learned about lived/living mathematical experiences of students, parents, teacher educators, mathematicians, or others in society to improve educational practices?

These are some of the questions that serve as "starting points" for DIME research. In addition, we are confronting many research methodological questions that are challenging much of our initial struggles to conceptualize viable approaches to such phenomena.

If we want educational outcomes that mirror the empowerment that a deep knowledge and proficiency of mathematics can afford, we must now seek to move beyond our current perspectives and approaches that still seem to produce (for too many of our students) much less than we seek. If we are to understand deeply why so many of our students still achieve very poor understandings of the mathematical ideas and processes after experience with our reformed curricula and our improved teaching methods, materials, and tools, we must begin to penetrate beyond more superficial indicators, such as test scores or written work, or even observed classroom behaviors into the interior world of the student's actual lived/living mathematical experiences where we may be able to identify deeper explanatory aspects of the individual's progressive growth (or deficits) toward greater mathematical knowledge and proficiency.

There is one other rationale that seems increasingly important. Too many of our citizens bear negative feelings toward their own past mathematical education (Hersch & John-Steiner, 2011), but for us educators to help our current students avoid such debilitating beliefs, feelings and attitudes we must begin to understand the nature of their experienced emotions that occur and develop within the mathematical contexts we offer. That is, not only must we understand the intellective dimensions of a person's constructions and reconstructions of their mathematical knowledge and proficiency, but we must also understand the origins and dynamics of emotional dimensions of their experiences, and how the interplay of thinking and reasoning actually functions within their live experiential feelings, emotions and affective schemas in relation to engaged, productive mathematical thinking. It is time to seek to understand the "whole mathematical life" of our students.

The goal of such new understandings is the assumed potential that from such knowledge we mathematics educators will all be helped to engage in improved forms of mathematical education in which enhanced qualities of lived/living mathematical experiences occur for all persons. We seek a pedagogy that understands and honors the experiences of the other (surely a primary intention of humanist philosophy and constructivist epistemology).

Further perspectives on mathematical experience can be found in the limited literature on this subject. One of the notable senior scholars of mathematics education, Hans Freudenthal (1983) claimed an early phenomenological perspective. In his review of Freudenthal's book, Usiskin (1985) identified a general view of mathematics as an abstraction of reality, where mathematical ideas come from commonalities inherent in a wide variety of situations. Through his "didactical phenomenology," Freudenthal sought to return the mathematically minded to the world from which it was abstracted. In greatly detailed ways that form an analytical compendium of most school mathematics concepts, Freudenthal discussed analytical connections between an idealized conceptual world and the complex world, which relates to it. It is these connections, Usiskin explained, that form Freudenthal's personal "phenomenology" of the mathematical structure, and it is the implications for the instruction of students that forms his "didactical phenomenology."

Brown (1996) described the mathematics classroom from the perspective of social phenomenology where it is seen as an environment of signs, comprising things and people, which impinge on the reality of the child. His framework sees mathematical ideas as contained and shaped by the child's personal phenomenology, which evolves through time. He introduces the notion of "personal space" as a model for describing how children proceed through a classroom environment of phenomenal experiences toward their idiosyncratic "sense making." The report provides interesting data from children solving symmetry tasks, and demonstrates how Brown uses his framework to interpret the child's functioning in experiential terms.

Roth (2001) critically questioned some of these views and approaches in his review of Brown's (1997) book, an extensive attempt to relate many important aspects of mathematics learning and teacher education to phenomenology, hermeneutics, post-structuralism, and semiotics. In particular, Roth's central concern is whether all mathematical activity and experience can be reduced to text, based upon Brown's unequivocal assertion that "There is no experience outside the text" (p. 226).

One interesting study, reported by Francisco (2005), involved the elicited reflections of five high school students in their 12th year of a longitudinal study in which their school mathematics from grade one had been developed in ways claimed to be consistent with a constructivist approach (called "Rutgers Math"). Maher (1987) had described these learning conditions to include: collaborating in work, exploring patterns, conjecturing, testing their own hypotheses, reflecting on extensions and applications of the concepts, and explaining and justifying their reasoning. Steffero's (2010) study of one female student's beliefs and mathematical activities that occurred over a seventeen-year period in "Rutgers Math" and beyond constitutes an enormous record of her mathematical experiences from grade four through graduate school. Handa (2006, 2011) examined ways of knowing and doing mathematics using phenomenological reflections to construct views of experiential relationships with the subject matter of mathematics.

Perhaps there are good reasons why so little explicit focal attention is given to "mathematical experience." Thompson (1991) observed that while the authors of papers in an ICME conference are concerned with the epistemological foundations of mathematical experience, they offer no information about "what is a mathematical experience." Indeed, in DIME we fully acknowledge a serious lack of clarity of constructs in this domain, but through systematic efforts to study lived/living mathematical experience, we will seek to build new, grounded clarities of meanings.

It is the goal of the DIME research program to begin to study intentionally the phenomena of lived/living mathematical experiences. Why bother? Are not the traditional views emphasized in intended curriculum, instruction, learning, assessment, evaluation, and research on learning and teaching adequate? Indeed, it is exactly because the goals and strategies of a sound mathematical education today embody a major emphasis upon stimulating, nurturing, and demonstrating high quality thinking arising from particular kinds of intended experiences that we must now seek to understand more clearly the nature of what students and teachers are actually experiencing in the "flow of their math lives" (Csikszentmihalyi, 1990).

Issues in a Psychology of Mathematical Experience

We each know the centrality of our own lived experiences. Even modest personal reflection can lead an individual to a sense of realization that it is specific experiences that shape "who they are, what we know, how we think" in powerful and fundamental ways. This includes what is experienced "outside," in the so-called "real world" that involves our interactions within our proximal environment (our sensory-based physical experiences) and within our interactions and relationships with other minds (our socially-based logical and emotional experiences). But, this also includes interior experiences that occur in our mind within our "interiorized" constructed world, where our thinking and our feeling "being" (person) is shaped and functions in thought (mind).

In DIME, we acknowledge the significance of the typical "mathematical" environment that includes students, teachers, parents, discourse, textbooks, technology tools, classrooms in schools, tasks and tests, the structured contexts of lessons, local, state and national curriculum frameworks, professional preparation and development, mathematicians, societies and cultures, governance, politics---all of the usual cultural and social elements in and around a person's mathematical education today. In doing so, we also accept that the meaning structures for any or all of these elements is a totally idiosyncratic construction for any of the multitudes of persons attending to matters of mathematical education. Now, we seek to "look into" the interior phenomenal world of lived experiences where each individual dynamically encounters and processes their idiosyncratic "mathematical experiences that lead to "who they are and become," mathematically. As educators, perhaps a most important aspect of this involves the dynamics of lived/living experiences that might characterize how experience might possibly bring about change (transformations) in an individual's thought and feeling with respect to mathematics.

Phenomenology begins in the lived world, and seeks to bring to reflective awareness the nature of the events of lived experience (Hegel, 1977; Husserl, 1970, 1982). The principle of intentionality acknowledges an inseparable connection to the world (in our focus, the world of mathematical education) wherein "…we question the world's very secrets and intimacies which are constitutive of the world, and which bring the world as world into being for us and in us" (van Manen, 1990, p. 5). In DIME, we are choosing to study thematic meanings, adopting themes and conducting thematic analyses within our particular orientation to the phenomenon of mathematical experience as "people of mathematics:" all being teachers, teacher educators, students of learning and teaching, mathematically educated, and interested is pedagogic theories. We are trying to be explicit about our individual and shared intentions and orientations as a preparatory anchoring step in our research process.

As such, phenomenology accepts the curriculum of being and becoming (paideia), pursuing understanding of the personal, the individual, set against the background of an understanding of the other, the whole, the communal, or the social. It seeks to explicate phenomena as they present themselves to consciousness---the only access humans have to the world. But, consciousness cannot be described directly (the fallacy of idealism); the world cannot be described directly either (the fallacy of realism); real things in the world are only meaningfully constituted by conscious human beings, and these constructed meanings can only be revealed by the constructing human as inferences.

In our formative research approaches we presume the nature of lived/living experience to be fundamentally an internal construction/re-construction that emphasizes a consciousness of "sense-making," attempted within the unique idiosyncratic mental operations, schemas and constructive mechanisms as they exist and function in the mind of the individual within that lived/living experience. To emphasize Piaget's (Piaget & Inhelder, 1969) theoretical conclusions that both intellective and affective aspects are involved within experience leading to development, we seek to study both as a seamless whole, even as we reject many other dichotomies as false (such as "thinking versus feeling") typical of a strictly modernist structuralism.

Among the constructivist focal constructs we want to consider in our study of lived/living mathematical experience are representation and re-presentation, reflection and reflective analysis and abstraction, intuition and intuitive reasoning, and perturbation and equilibration, and particularly to search for where and how they may be found to function in, impact upon, and in turn be affected by particular experiential contexts. For example, in today's curricular frameworks one sees a major attention given to "representations" and representational activities; students are expected to learn to understand, use, and make various canonical mathematical representations. Yet, what from a study of lived/living mathematical experiences can we find about a student's actual conceptions and views of representations, and especially how they experientially use such images in problem contexts to re-present the conceptual ideas they are presumed to represent? Moreover, von Glasersfeld's (1991) view of cognitive functioning sees representation, re-presentation, and reflective abstraction as inseparable aspects; can we find this exhibited in lived/living mathematical experience?

Against these theoretical lenses, we have identified a variety of issues to confront as we attempt to study lived/living mathematical experiences. Some of these are identified below, and these will be addressed in the activities of the Working Group described next.

1. In what ways is it possible to study the lived/living mathematical experiences of anyone: one self? The "other?" We have found helpful literature that addresses important theoretical perspectives and methodological strategies for this question [e.g., Dewey, 1938; Hegel, 1977; Hurlburt & Akhter, 2006; McLeod, 1964; Moustakas (1994); Petitmengin, (2006); van Manen (1990)]. We have identified these guiding principles:

A. A research methodology for studying lived/living experience includes the theoretical precepts behind the methods---the values and assumptions of phenomenology and constructivism as briefly discussed above. Within a focus on actual experience, we are willing to yield upon procedures or techniques that certain methods, even in qualitative research, attempt to objectify or make more standardized.

B. Our overall framework is adapted from van Manen's (1990) structure "…seen as a dynamic interplay among six research activities:

(1) Turning to a phenomenon which seriously interests us and commits us to the world;
(2) Investigating experience as we live it rather than as we conceptualize it;
(3) Reflecting on the essential themes that characterize the phenomenon;
(4) Describing the phenomenon through the art of writing and rewriting;
(5) Maintaining a strong and oriented pedagogical relation to the phenomenon;
(6) Balancing the research context by considering parts and whole." (p. 30-31)

C. Some views of phenomenology aim at being "presuppositionless," warding off a tendency to construct or enact a predetermined set of fixed procedures or techniques that would rule-govern the research. We will engage our observations and analysis with general acceptance of this view, while also making efforts to stipulate and articulate, a priori, as many of our individual values, assumptions, beliefs, and attitudes about the phenomenon of "mathematical experience as we can and seems relevant. One view (van Manen) ---*our problem is not that we know too little about it, but that we know too much!* (p.46). As such, we are predisposed to interpret the nature of the phenomenon before we have even come to grips with the phenomenological questions.

D. In those research contexts where our aim will be to stimulate new experiences within real-time, unfolding events, we adopt the dynamics of constructivist orientations but set aside "teaching or learning" aims, per se (e.g., where questions from the researcher-teacher would seek to provoke particular kinds of mathematical thinking or productions). Rather, within an initial problematic situation posed to provoke or engender lived mathematical experiences, we honor the paths as determined and taken by the person.

E. We seek to address the phenomenon of mathematical experience in a variety of ways. In doing so, we will seek attentively to orient to the phenomenon as we strive to deepen our formulation of the phenomenological questions.

F. We are building a team approach with a purposeful aim of including a variety of perspectives and voices, and this brings opportunities beyond research conducted by one, or even two collaborating scholars. As such, we have adopted views and tactics to mirror what we perceive as formative, developmental research, and to be alert to elements of our inquiry that includes aspects of team building, per se.

2. Whose experiences should be studied, and why? Who are to be the subjects of the research? How are they chosen? Because mathematics as a human endeavor in society and globally is so pervasive and seemingly universal, we foresee a full range of research participants who experience mathematics in a wide array of situations and for a diverse set of reasons. Of course, one orientation we bring to this is the "enterprise" of mathematical education, and this will greatly influence our choices for whose experiences we will try to investigate, and also determine how we frame the contexts and the templates of analysis and interpretation. We want to include at least these in our sampling of lived/living mathematical experiences---ourselves, mathematicians, mathematics teacher educators, mathematics education researchers, pre-service and in-service mathematics teachers across levels of school mathematics, mathematics students across levels of school mathematics, and parents of the students. Different types of participants will allow us to address a variety of aims; specific individuals will be chosen in terms of particular aims and purposes.

3. What are sources or forms of "data of lived/living experience?" How can these be generated in ways that yield penetration into phenomena? Be seen to be accurate (true to the phenomena)? Valid? Reliable? Viable (Steffe, 2010)? We accept the following views about the nature of "data of lived/living experience." The world of lived/living mathematical experience is for us both the source and the object of our research. We each bring strong (yet varying) orientations to it. But, we share a fundamental assumption: *experiential accounts are never identical to lived/living experience itself.*

We already see that the sources and forms of our "data of lived experience can be rich and varied. Yet, we will pursue in each the step of generating written descriptions, and these may be of two kinds: (a) an immediate description of the "*life-world as lived*," or (b) an intermediate (or a mediated) description of the "*life-world as expressed in symbolic form*." While we accept that in this step there occurs interpretation, among team members we will share in an analysis of the description as produced, and engage in intentional interpretation (hermeneutic) to produce a "second-generation description" that purposefully seeks to identify and describe "essences as deeper meanings of the lived mathematical experience."

Subsequently, through team interactions, we will question the accuracy of these descriptions and interpretations to assess if they seem "true to the phenomena" (as such, are they valid?). We will try to address some sense of reliability, at least in terms of a quality of internal consistency among individuals in the team. To deepen or extend the possible derived meanings arising from group work, we will try to view the expressed perspectives of individuals in the team compared to their initial written description of "a priori conceptions about mathematical experience noted earlier [a way of possibly using "bracketed" aspects to clarify meanings]. Husserl (1970) acknowledged that "common sense" pre-understandings, suppositions, assumptions, and existing bodies of knowledge can lead one to premature influences in one's effort to interpret, to "make sense." How can we best suspend such influences? Husserl (who was at first a mathematician) used the notion of "bracketing" to describe how one must try to place outside of the present phenomenon one's prior or extant or presumed knowledge about the phenomenon.

4. In what ways can research subjects be directly engaged in experiential mathematical situations while informing the researcher about what they are experiencing? It is one of the basic assumptions of phenomenology that experience will be changed within an attempt to introspect---to "rise above" and give attention to the experience while it is occurring, and that the distinctions between what is introspection and retrospection are blurred (thus our use of "lived/living"). While we accept this assumption in theory, we also want to explore this phenomenon. Petitmengin (2006) used an interview method aimed at helping a person to become aware of her subjective experience and to describe it with great precision. Hurlburt and Akhter (2006) used a “descriptive experience sampling” method to explore inner experience. Their subjects were prompted by an electronic beeper which they carried as they moved in their natural environments. When the beep sounded, they were trained to “capture their inner experience and jot notes about it; they discussed it during a later expositional interview.

As researcher-observers interactively engaged in mathematical situations we’ve posed for the purposes of engendering active involvement by, say, a student, we intend *to become a part of the student’s experiences*, per se---to get into “the flow” of what the student is experiencing (Csikszentmihalyi, 1990). “Being there” can mean (to the student) that, as a part of their unfolding experience, we ask questions. While these questions will primarily focus upon their activity and their “thinking aloud” verbalizations related to it, at times we will ask a question pointing more directly to their conscious reflection upon their experiences, per se. In some sense, we anticipate that such a question can result in a kind of interruption of the flow of experiences related to the posed mathematical situation; we intend to ask the student later about the effects of such questions upon their perceptions of the flow of their experience. We anticipate there will be variable impacts reported by different persons, but we also expect that across time and successive observational interviews, individuals will develop a greater capacity for minimizing (or perception of) the disruptive effects of such questions.

5. When or how might a research intervention or analysis or interpretation influence or alter or “contaminate” an experiential context under study? How can this be avoided? This concern admits that there may be ways in which our research methods could negatively impact upon what is being experienced, both in immediate or longer-term ways. Indeed, one characterization of instruction are the intentional impacts that alter otherwise natural experiences of students. As such, these would not be avoided.

Here is one nexus of conflict within the aims and processes of phenomenological research and mathematics education. We accept this tension, and seek to probe and penetrate into the lived mathematical experiences while realizing the potential invasive violations we may induce into what “the other” is experiencing. We believe our resolution of this conflict resides in the distinctions to be found in “natural, every-day” experiences and “intended” experiences of mathematics education, though we realize that as effective teachers we strive for there to be a quality of “natural” in what and how a student might experience and build-up their mathematical knowledge. Here again, as explorers we will remain open, yet exercise surveillance to this concern.

6. How do we, as researchers, conduct analysis and interpretation of data to build accurate portrayals of lived/living mathematical experience? Key to phenomenological or constructivist research is analysis and interpretation as an observer, or teacher-researcher. Critical to either are the struggles to maintain, as much as possible, open thinking in which one consciously acknowledges potential biases and avoids “projecting” one’s own experiences onto the situation. Intentional “bracketing” is attempted; members of our team are trying to describe, *a priori* to individual or shared acts of analysis and interpretation, our individual perspectives on what we each may “see” in the phenomena of lived mathematical experience---we will refer to these as our “initial construct views” (ICV). We will share these written ICV descriptions, and try to use these when we are subsequently engaged in analyses and interpretations of our observations and descriptions. We are unsure about exactly how we will use these, but one could be when we disagree about what we “see” in a particular protocol or description; we may be able to find reasons for a researcher's interpretations in the anchoring viewpoints they expressed in their ICV.

We believe that through a team approach in which multiple descriptions can be generated independently and then discussed and debated, we will likely achieve more sensitively accurate interpretations of the phenomena---"negotiated meanings." Across experiential episodes we will look for consistencies as well as variation, thus being attuned to elements of cross-validation of the qualities to be found in a person's lived mathematical experiences. Also, in these we will look for how the nature of experiences for each subject may change; again, as educators we seek and expect change--growth and development as a consequence of something we call "mathematical experience."

In our studies, we will focus not only on indicators of change in relation to mathematical knowledge, but also on the emotional elements within experience and how these occur within the dynamics of various qualities of lived experience. Further, we will be alert to how the person's capacities to reflectively abstract in relation to their lived experiences might change, and what impacts that may have upon their future awareness of themselves within, and after, lived mathematical experiences. In some studies, we will explore how the subject might be given our description of their experiences, and asked to provide critical review, reaction and feedback about it; this may lead to new insights into the "essences" of it, and result in another form of "negotiated meanings."

7. In what ways can we "make sense of" our study of the observed lived/living mathematical experience in relation to its implications and potential applications? This question speaks to the important intent that our research results lead beyond information about the interior "mathematical life," although such research-based information is generally lacking today. Our goals include the possible implications for such new insights and understandings about lived mathematical experience to impact upon future mathematical experiences. We foresee possible benefits to the ways that mathematics teachers seek to stimulate and engage their students to engender certain qualities of experiences they might have. Information about what is occurring in the lived mathematical experiences of students may prompt changes in curricular topics, placements, or treatments. Deeper understandings from studies of experience may raise implications for testing and assessment strategies related to mathematics learning.

Specific Plans for DIME Sessions at PME-NA 2011

Overall, the sessions will be conducted in "workshop" format, structured by an overview presentation (20 minutes) followed by four brief (10-12 minutes each) plenary "highlighting" presentations to pose research issues and problems, guiding written resources distributed to all participants, "breakout" sessions for four working sub-groups, guided sub-group task-oriented discussions, and closing plenary summarizing presentations and discussions. Feedback and input from all discussions will be collected; post-conference written summaries for each sub-group will be prepared at UW and distributed to all WG participants, and posted to the public Institute website.

The four sub-groups will focus upon one of the following domains of discussion (while respecting that these are not a partitioning, so overlaps and interactions among the issues and questions of the focal domains will likely occur):

A. Framing 'problems and questions' for research on lived/living mathematical experience---what to study?

B. Addressing 'issues and concerns' for research methodologies to be used---how it is to be studied?
C. Discussing techniques for using and transforming data to describe, analyze, and interpret in ways that illuminate and inform---how is data to be evidentially used?
D. Discussing approaches to reporting what is found and to applying to inform improved educational practices---how results are to be reported and used?

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REPRESENTATIONS OF MATHEMATICS TEACHING AND THEIR USE IN TRANSFORMING TEACHER EDUCATION: CONTRIBUTIONS TO A PEDAGOGICAL FRAMEWORK

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Discussion document for the Working Group "Representations of Mathematics Teaching"

(a) Brief History of the Working Group

This is the second meeting at PMENA of this RMT working group. The idea of this working group emerged during a series of three-day conferences on representations of mathematics teaching held in Ann Arbor, Michigan in 2009 and 2010,(and earlier workshops in 2007 and 2008) organized by ThEMaT (Thought Experiments in Mathematics Teaching), an NSF-funded research and development project directed by Herbst and Chazan. ThEMaT originally created animated representations of teaching using cartoon characters to be used for research, specifically to prompt experienced teachers to relay the rationality they draw upon to justify or indict actions in teaching. The original workshops were conceived to begin disseminating those animations to be used in teacher development . The (Representations of Mathematics Teaching) RMT conferences in 2009 and 2010 gathered developers and users of all kinds of representations of teaching to present their work and discuss issues that might be common to them. These conferences included users of video, written cases, dialogues, photographs, comic strips, and animations. An outcome of the 2009 RMT conference was a special issue (Volume 43, issue 1, 2011) of the journal *ZDM—The International Journal of Mathematics Education,* guest edited by Herbst and Chazan Outcomes of the 2010 Conference included two sessions at the 2011 NCTM Research Presession. The 2011 Conference (June 13-15) will also work toward the goal of creating events related to the use of representations of mathematics teaching in other conferences. In proposing a continuation of the working group for PMENA 2011 we are interested in continuing the discussion and work we had in Columbus during PMENA 2010 around the elaboration and investigation of a pedagogical framework for teacher development that makes use of representations of teaching and work toward an edited book on the subject.

(b) Issues in the Psychology of Mathematics Education that Will Be the Focus of the Work

Review of Existing Work Relating the Theme of the Working Group to the Field

The use of representations of mathematics teaching, particularly those that are maintained in a digital form, calls for specialized pedagogical practices from teacher developers. They also open new areas for investigation of how future professionals learn to practice and the role that various technologies play in scaffolding that learning. In the 2010 PMENA discussion paper, Herbst, Bieda, Chazan, and Gonz‡lez (2010) briefly reviewed the literature on the use of video records and written cases in teacher education. We also noted that classroom scenarios sketched as cartoon animations have begun to be utilized for those purposes and argued that they have affordances that are distinct from those of video and written cases (see also Herbst, Chazan, Chen, Chieu, & Weiss, 2011) . We also noted existing literature on the use of written and video cases in teacher education and cited examples that concern mostly face-to-face facilitation. We argued that the increased capabilities of information technologies for creating, manipulating, and collaborating over multimedia point to a promising future for teacher development assisted by representations of practice. In the special issue of ZDM referenced above , several new articles have added to this literature. In particular Ghousseini and Sleep (2011) and Nachlieli (2011) describe the facilitation of face -to-face discussions around representations of practice and provide two views on what makes these effective for studying practice. Yet the features of novel media and their use with digital technologies may require other pedagogical strategies for teacher education that have not been sufficiently identified and explored.

In this document we complement the previous year's review by briefly accounting for three areas of emergi ng scholarship: (1) information technologies that support teachers' learning from representations of practice; (2) the particular challenge of helping prospective teachers understand students' thinking; and (3) research and theory about what is important or possible to achieve in having prospective teachers look at or work with representations of teaching.

Information Technology and Teacher Education

The field of teacher education is becoming more adept at using technology in teacher education classes. Communication technology is connecting teacher education students, university faculty, and mentor teachers through shared access to classroom videos (Price & Chen, 2003; Whipp, 2003), as well as providing another (sometimes more effective) mode of communication between teacher education students and university faculty (Derry, Seigel & Stampen, 2002; Reasons, Valadares & Slavkin, 2005). Teacher education is also leveraging communication technology to support the development of teacher communities (Farooq et al, 2007; Gomez et al, 2008). These communities increase teacher candidates' access to resources and interaction with peers.

Digital video technology allows for prospective teachers to interact with practice in more meaningful ways than in the past. In addition to watching videos, teacher education students can now annotate and edit video quickly and easily (Chieu, Herbst, & Weiss, 2011; Pea, Mills, Rosen & Dauber, 2004; Rich & Hannafin, 2008, 2009). These technologies allow for faster communication as well as discussions that focus on specific moments of an instructional episode and therefore on the particulars of teaching practice. Video and animations have also been used effectively to illustrate to prospective teachers how the theoretical principles that they learn in university courses can be put into practice in the classroom (Moreno & Ortegano-Layne, 2008).

The availability of new technologies to represent teaching for its use in teacher education offers an interesting challenge for research on the pedagogy of teacher preparation and teacher candidates' learning to teach: These issues are of importance to the PMENA community. Scholars have noted, for example, that in addition to the functionalities afforded by various technologies, it is paramount to consider the tasks that teacher candidates engage with, and the support that teacher education students receive from the course instructor (Lockhorst, Admiraal & Pilot, 2002). Other scholars have looked more in depth at technology-enabled instructional interventions and expanded upon features that could be situated into Lockhorst et al's list of important features. Chieu, et al. (2011) looked at the use of online forums containing embedded animations of classroom episodes and argued that this access allows for richer conversations among teacher candidates. Llinares & Valls (2009, 2010) focus on the tasks posed to teacher candidates and the effect of these tasks on learning These studies all highlight the role of pedagogy in the use of new technologies in teacher education. As technology develops, teacher educators need to continue to develop their pedagogies to make the best use of the technology.

The Challenge to Help Teacher Candidates Understand Students' Thinking

There continues to be a need for improving the connection between research on students' thinking and learning and the work of teacher development. What kind of cognitive research on students is useful for teaching and teacher education ? How might teachers come to learn it and use it? How can new media assist in this work? Researchers have addressed those questions by 1) using multimedia to show examples of actual student responses to research-based tasks (e.g. Franke et al., 1998); 2) drawing upon research to develop representations of student thinking (e.g. Balacheff, 1988); and 3) using such research to analyze pedagogical moves in classroom dialogues created by teacher candidates (Crespo, Oslund, & Park s, 2011).

Professional development connected to the research on Cognitively Guided Instruction (CGI) pioneered the use of records of student work to engage teachers with research on students' thinking (Carpenter, Fennema & Franke, 1996). By showing teachers video of students solving mathematical problems, as well as providing students' written responses for further analysis, the CGI project aimed to build upon teachers' existing knowledge of student thinking by systematizing and enriching this knowledge. The responses from participants of that program highlighted the value of having teachers analyze student work as a way to build knowledge that they could draw upon to make instructional decisions.

Beyond records of students' work, research on students' thinking has also been produced other representations of students' work, though less is known about the effects of their use in teacher education. Balacheff (1988) provides an early example, using a comic strip to represent responses provided by two students as they worked on the prompt: "provide a means of calculating the number of diagonals of a polygon when you know the number of vertices it has" (p. 220). This representation, included in a book for teachers, suggested a new way to acquaint teachers with student thinking in story form. Students' work has also been represented in narrative cases embedded within cases of teaching episodes (e.g. Stein et al., 2000). It is important to continue to look for ways of making students thinking accessible to teacher candidates and to find out what teacher candidates learn from these different representations of students thinking.

What is Possible to Achieve with Representations in Teacher Education

The combination of the use of representations of teaching and new technologies has facilitated the goals of teacher education in several areas. In particular, teacher educators have more diverse access to several types of representations of teaching (Grossman et al, 2009) and are able to use these in more flexible ways to support the learning of teaching "in, from, and for practice" (Lampert, 2010). Two important goals of teacher education are to increase teachers ' ability to notice aspects and events in the instructional environ ment (Rosaen et al, 2008; Sherin & Han, 2004; Sherin & van Es, 2009) and to afford teachers the skills and opportunities to reflect on their practice (Santagata & Angelici, 2010; Stockero, 2008). Two related goals are to enable teachers to learn from their own practice (Borko et al, 2011; Santagata & Guarino, 2011) and to alter or improve their teaching practices (Lampert et al, 2010; Polly & Hannafin, 2010, 2011; Van Zoest & Stockero, 2008).

A goal still ahead is to improve the connection between theories and conceptualizations of mathematics teaching and specifications of the curriculum of mathematics teacher education: Can we expect theories of teaching to determine the curriculum of teacher education? For example, in the context of the Teacher Education Initiative at the University of Michigan an innovative program for elementary teacher preparation has been emerging with a practice-centered curriculum. This curriculum prescribes teacher candidates' learning of specific practices of teaching that are deemed high leverage—these include "leading whole-class discussions of content" and "recognizing and identifying common patterns of student thinking in a content domain" (Curriculum Group, 2009). This curriculum gathers those practices in domains such as "assessing students" and "enacting instruction" (Curriculum Group, 2008). While it is possible to tie each of these practices and domains to the existing literature, it is not evident how one could argue that they constitute all the domains and practices that teacher candidates need to learn. A theory of practice would be useful to systematically generate all the domains and practices, to provide an underlying coherence to these domains and practices, and eventually to limit or at least modularize expansions of the curriculum.

An example of a theory of teaching that could provide such systematization is our theory of instructional exchanges in which we conceptualize the work of teaching as centered on the problem of exchanging students ' mathematical work for claims on their knowledge of the mathematics at stake within classroom environments that respond to four obligations (to the discipline, the individual students, the groups of students, and the school institution; see Herbst, 2010). But while this theory provides a conceptual basis for subject-specific descriptive models of the regulations of the work of teaching (with constructs such as situations, norms, and dispositions) it has no t yet operationalized the work of teaching in ways that support professional education or development. There is a pervasive need in the field to articulate prescriptive theories of teacher education with descriptive theories of teaching—the latter can give completeness and validity to the former while the former can operationalize the latter.

Toward a Pedagogy for Teacher Development Assisted by Representations of Practice

Building on the proposal from last year , the working group's purpose is somewhat ambitious: to design a pedagogy for teacher development that meets the goal of helping teacher candidates learn teaching in, from, and for practice by taking advantage of representations of practice and new technologies. This development includes conceptual developments, for example in articulating connections between descriptive theories of teaching and the prescription of a curriculum for teacher education. Our work will include asking questions such as, is it possible to further develop the descriptive norms (e.g. when a teacher expects students to do a proof she will provide them with 'givens' and the 'prove') into an operational specification of what a teacher candidate needs to learn to do, say, not only to be able to proficiently comply with such norms but also to negotiate tasks that depart from the norm? The working group will discuss the articulation between theories of teaching practice and practices of teacher education.

The convenors of this working group are also particularly interested in exploring how cartoon-based representations of practice facilitate teacher learning. Our thinking of how to make such representations usable for teacher education has been influenced by Lampert 's (2010) notion of learning teaching in, for, and from practice and by Grossman et al. (2009) account of professional education 's use of representations, decompositions, and approximations of practice. We surmise that cartoon-representations of teaching are sufficiently malleable to create not only representations of practice (that prospective professionals can view and annotate) but also to create decompositions of practice that prospective professionals can study from as well as approximations of practice in which prospective professionals can practice their skills. In this sense, cartoons can fashion virtual settings for teacher learning in, for, and from practice before the teacher candidates are ready to learn in real settings.

The LessonSketch environment (www.lessonsketch.org) is one example of a set of tools and resources that can exemplify what such a virtual setting for teacher development could be like. We are interested in having the RMT working group use LessonSketch to engage concretely in applying the elements of a pedagogical framework for teacher education assisted by representations. We expect that such work will help improve the framework and further develop specifications for technologies that respond to the needs of the field. With that purpose in mind, this document sketches current and envisioned features of LessonSketch as it lays out an updated version of a pedagogical framework for teacher education assisted by representations of teaching (building on last year's working group document; see Herbst et al.,2010).

This revised pedagogical framework considers the need for a larger library of representations of practice. While in the past we had only included representations of lessons in which teaching and learning mathematics were integrated in scenarios of classroom instruction, the notion of decomposition of practice proposed by Grossman et al. (2009) suggests the need for two more sets of representations. One of those sets of representations consists of representations of students' work: Depictions of how students solve problems, indexed in a database that permits searches by problems, operators, representation systems, and controls (Balacheff & Gaudin, 2010). The other new set of representations consists of representations of practices, strategies, tactics, and techniques in teaching, depicted through commented scenarios of instruction and also indexed in a database. We surmise that the cartoon medium can be useful for researchers on students' cognition in the PME-NA community to reach teachers with representations of how students think about specific conceptions; likewise the cartoon medium can be useful to support the learning of teaching skills by prospective teachers.

The remaining elements of the framework expand on what we offered last year, taking advantage of the materials we brought to the meetings of the working group and the feedback received from participants. Last year we proposed that a pedagogy of teacher preparation assisted by representations of practice needed at least four categories of elements: open ended expressions, activity types, problem types, and technology tools or screens.

The first element of the framework we call *open-ended expressions*. These are terms and expressions that can be used in transactions between teacher developers and their clients without needing to be completely defined; tokens that teacher developers and their clients may be able to take as shared so as to negotiate activities, problems, and representations. One such expression is "mathematical action" which we have observed being used in a geometry class for future teachers and in the context of having the students watch an animation of geometry instruction. Other open-ended expressions that can serve comparable purposes are "student thinking," "teaching move," "instructional goal," "resources," etc. We surmise that these boundary objects may be useful to mobilize the work with representations-- enabling explorations of practice that may lead to shaping more precise meanings.

A second element of the framework proposed in last year's document is a taxonomy of activity structures or activity types for mathematics teacher education—behavioral configurations that describe the formal division of labor between instructor and students. Some of these activity types could be the same as those found in K-12 classrooms, such as mini-lecture, homework review, etc. (see Lemke, 1990). But there are activity types that are particular to the work of mathematics teacher education assisted by representations of teaching. A quite common activity type could be described as “working on the math:” Quite often, mathematics teacher developers who intend to show a video that displays students working on a mathematical task will first have their clients work on the mathematics problem that will be featured in the video. Another activity type we have used in teacher education is a form of review of homework in which clients enact scripts of action that they conceive outside of class in response to practical problems of teaching such as how to explain a step in a procedure. Usually those enactments give some clients practice enacting teaching moves they had planned; other clients give feedback, and the event supports raising more general questions about the task of teaching being learned. In general different activity structures involve clients in interactions with (manipulate, annotate, etc.) the representations being used and for that reason they become quite important in the design of technological tools and interfaces. A partial list of activity types, which we brought to the meeting of the working group last year, is shown in Figure 1.

Annotating a representation with free written comments	Creating a mix or mash of existing representations
Discussing a representation with peers in class, chat, or forum	Searching for a representation that meets some conditions
Responding individually or in groups to a question about a representation	Scripting an event or lesson
Using a rubric to comment on a representation	Enacting scenarios or scripts of action publicly
Providing a verbal rejoinder to a representation	Rehearsing a practice, strategy, tactic, or technique given a script and a rubric
Viewing (or reading) a representation	Enacting scenarios or scripts in a chat room
Comparing two or more representations	Enacting a response to students at a specific moment
Introducing or framing a representation	Creating examples of actions that illustrate a practice, strategy, tactic, or technique given a rubric
Creating a new representation given an existing representation one (transcript, video)	Tagging a representation with elements of a rubric or coding scheme
	Adding to a representation events that might come before, after, or in between

Figure 1. Activity types

A third element of the framework consists of *problem types*. By this we mean specific intellectual work that participants do within an activity type involving representations—problem types specify the perspective or the goal with which clients confront a representation of practice. Different problem types may engage different kinds of thinking and doing on the part of clients and as a result different problem types might need to be chosen and articulated to promote different kinds of learning. A *noticing* problem might lead to developing the capacity to spot opportunities to probe student thinking, and a *norm ativity* problem might be used to engage the client in designing how to do so. Specific problem types are described in Figure 2.

What else could [the teacher/a student] do at this moment? (Alternativity)
What would you call that [teaching move/mathematical action/etc.]? (Generality)
What should be done (what should the teacher/student do) in these circumstances? (Normativity)
Which alternative is more appropriate? (Normativity)
How would you have felt (what would you have done) if those were your students? (Projectiveness)
What do you think about this episode? (Reflectiveness)
How would you interpret this episode/action?(Reflectiveness)
At what moment would you say [such thing] happened? (Temporality)
What do you predict came before/after/in between? (Probability)
Which alternative would be more typical/probable? (Usuality/Probability)
Which alternative is more desirable according to [some standards]? (Desirability)
What did you see students/teacher doing? (Noticing)

Figure 2. Problem types

The final element of this emerging framework addresses the technological affordances needed to realize this pedagogy of mathematics teacher education. Clearly one could do many of these activities having only a video projector and playing media off a single computer. But there are important pedagogical considerations associated with more technology-intensive environments. Chieu, et al. (2011) show evidence that clients' comments in forum or chat benefitted from having an embedded screen for the animation being discussed, which they could access at the same time as they interacted with peers in a forum or chat. This media-enabled-forum is one of several functionalities available in LessonSketch. We expect the working group will be able to explore these functionalities, and the potential combinations that could be made with them; we also expect the working group to have suggestions of new functionalities to add. Figure 3 includes the tools and resources that we considered in last year's meeting.

Show media	Enable the viewing and commenting of others' attached media
Show commented media	Enable the viewing and editing of others' attached media
Show media and enable comments	Show media and enable pinning moments
Pose an open ended question	Enable the attachment of media to answers to questions
Pose a multiple choice question	Show media and enable marking intervals
Enable exporting comments, pins or intervals to forums or documents	Request a decision between representation based alternatives which direct users to different paths
Enable the revision of answers	Request the sketching of a scenario with Depict
Request an entry in a forum	Enable the creation of a multimedia document
Request a diagram or picture	Request an entry in a discussion thread with embedded media
Invite to a chat room	Enable the attachment of media to forum posting
Offer a text to read	Respond to an individual user's comment or answer

Figure 3. Technology tools and functionalities

We envision that instructional actions in a pedagogy of teacher education will integrate the elements of those categories in experiences in which clients use technology tools to work on problems (that resemble one problem type) involving boundary objects and in the context of an activity type supported by a particular representation of practice. These experiences are geared to help them learn about or learn to do a particular practice of teaching.

(c) Plan for active engagement of participants in producti ve reflection on the issues

The plan includes starting with a brief exposition by the authors of the structure and contents of the present framework. We'll engage the audience in creating sketches on paper of sessions they'd like to engage their clients in. The idea is to use the collective planning of these sessions to enrich the framework by adding more items to the lists considered, and possibly also adding new categories of elements. The first meeting of the working group will involve a discussion of this document and the demonstration of the current framework. Participants will then form groups and spend the second half of the first session and the first half of the second session creating exemplars. Then the second half of the second session and the closing session will be dedicated to sharing these exemplars and improving the framework.

(d) Anticipated follow -up activities

By the time this working group meets we will have had the third conference on Representations of Mathematics Teaching in Ann Arbor (June 13-15, 2011). We will be proposing a session slot at the AMTE Annual Meeting in 2012 to continue this work. We plan to use that slot to mirror the work done at the PMENA meeting and to engage in further work on (1) improving the exemplars and (2) using the exemplars to improve the taxonomies. We hope we will be able to use those products to continue this working group at next year's PMENA.

Endnote

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DYNAMIC AND INTERACTIVE MATHEMATICS LEARNING ENVIRONMENTS (DIMLE)

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The proposed working group will review some basic definitions on the features and affordances of Dynamic and Interactive Mathematics Learning Environments (DIMLE) and explorative learning. First, we suggest setting a stage to explore, identify, and conceptualize the features and affordances of DIMLE. Then, we aim to conceptualize explorative learning, dynamic learning, visual learning, and multiple-linked representations in the context of DIMLE. It is our goal to organize this discussion so that it will deepen our understanding of the explorative activities in learning mathematics and develop possible strategies for increased implementations of and our expectations from existing and prospective DIMLE. It is also our intention to establish a working group to continue exploring the topic after the conference and to contribute to the academic studies with a new publication.

Introduction

The innovations in technology have accelerated the evolution of mathematics education and mathematics itself, and this evolution is well-documented in the literature (e.g., Moreno-Armella, Hegedus, & Kaput, 2008). Not surprisingly, mathematicians, mathematics educators, and cognitive scientists have been interested in exploring, discussing, and improving their understanding and possible implementations of technology use in mathematics education at various national and international conferences such as ICME, ICMI, PME, and PME-NA (Arcavi, 1999; Duval, 1999; Galbraith, 2006; Hitt, 1999; Hoyles, 2008; Kaput, 1999). Moreover, some specific working groups, i.e., ICMI 17 Study Group and the Working Groups at PME-NA, have been established to take these discussions even further (Hoyles &Lagrange, 2010; Karadag & McDougall, 2009; Radakovic, Karadag, McDougall, & Stoilescu, 2010).

In addition to the growing interest in this topic in scholarly community, many well-argued articles and reports on the descriptions and needs of learning habits of N-Gen'ers (Attewell, 2001; Caron & Caronia, 2007; Freiman & Lirette-Pitre, 2009; Martinovic, Freiman, & Karadag, 2011; Ministry of Children and Youth Services, 2007; Newman, 2008) have motivated us to propose a working group for PME-NA201 1. The focus of this proposed working group will be on Dynamic and Interactive Mathematics Learning Environments (DIMLE), a term coined by Martinovic and Karadag (2010), and explorative learning, which are at the heart of technology use in mathematics education.

The following sections will briefly address the theoretical framework, which is also closely related to the literature that we plan to share with the audience at the conference, the rationale and the goals of the working group, the format of the proposed meeting, and a future proposal to publish an international book in the area of discussion.

Theoretical Framework

The contemporary technological tools provide DIMLE for learners to explore, learn, and model mathematics. These environments provide many intuitive affordances such as multiple-linked representations to engage learners in explorative learning. By explorative learning, we also mean experimentation, inductive reasoning, and visual learning. Of course, explorative learning in DIMLE may also include deductive learning with a proper guidance. The following themes are intended to spark the discussants' interest and initiate a collaborative effort to explore and discuss DIMLE and explorative learning more deeply.

Explorative Learning

A literature review searching for the definition and possible implementations of explorative learning usually results in either discovery learning or computer science applications (e.g., simulations) while some people use discovery learning and explorative learning interchangeably. For example, Njoo and de Jong (1993) argue that "exploratory or discovery learning" (p. 821) is based on free exploration in an open learning environment, where learners can interact with learning objects to uncover the relationships among them and to track the processes to create and/or employ them. Then, they claim that explorative learning triggers learners' inductive reasoning through exploration. Similarly, Bass (2011) puts exploration and experimentation together by claiming that: "we explore and experiment with the context" (p. 4). He, then, argues that "mathematics is not merely a descriptive science" (p. 5) because learners need to explore the concept or the process to be able to develop an insight for the mathematics behind them. For other scholars, this development of insight as a process and as an outcome is the abstraction itself (Gray & Tall, 2002; Hershkowitz, Schwarz, & Dreyfus, 2001).

Presently, we have enough evidence that the emergence of contemporary digital tools affects learning habits of 21st century youth and that this youth, also called N-Gen'ers, prefer explorative approaches to learn about the world around them. Therefore, the shift in the learning habits of youth motivates curriculum developers and policy makers to seek new approaches (Martinovic, Freiman, & Karadag, 2011; Ministry of Children and Youth Services, 2007; Newman, 2008). Kondratieva (2011) speculates on the results on these attempts by saying that: "Modern curriculum is moving from a formal approach towards more exploration based and inquiry-based study of mathematics." (p. 376)

Multiple-linked representations. Multiple-linked representations usually refer to different representations that are accessible through a computer and that are linked by the software (Kaput, 1992). For example, learners can easily access algebraic, graphical, and numeric representations in GeoGebra (see Figure 1) and similar DIMLE, and navigate from one representation to another (Martinovic, Freiman, & Karadag, 2010).

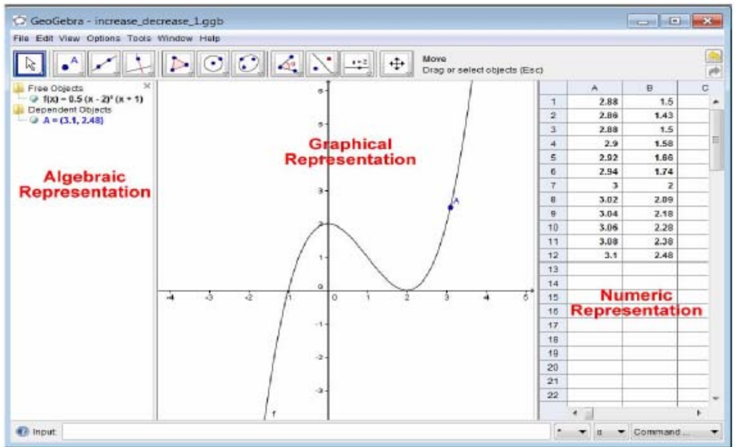

Figure 1. A snapshot of algebraic, graphical, and numeric representations in GeoGebra

In such a context, learners have the opportunity not only to explore one specific representation, but also to compare, contrast, and analyze different representations (e.g., algebraic, graphical, and numeric representations as illustrated in Figure 1), and conjecture on how each is related to another. Furthermore, once a conjecture is created, they can test it through experimentation and see if it holds in all cases.

Experimentation. The process of experimentation in mathematics education refers to the student interaction with mathematical objects such as exploring, comparing, and contrasting to create and test conjectures (e.g. Bass, 2011). The ultimate goal of experimentation is to help students identify patterns, to generate rules based on their observations, and to build upon their existing knowledge or further their own understanding.

Computer technology is specifically conducive to experimentation, because users can work on open-ended problems and easily re-do them by changing parameters and navigating among different representations. Borba and Zullatto (2006) mention that different technologies provide for different levels of experimentation intensity. What additions, if any, would make DIMLE the highly intensive experimentation spaces?

Inductive reasoning. Inductive reasoning refers to "examining particular cases, identifying relationships among those cases, and generalizing those relationships" (Greenes & Findell, 1999, p. 128). Students studying mathematics in DIMLE have the opportunity to benefit from multiple representations and to generate conjectures or test conjectures.

Visual learning. Visualization, visual learning, and visual thinking have different meanings depending on the discipline: "Visualization is multifaceted: rooted in mathematics, the field has important historical, philosophical, psychological, pedagogical and technological aspects" (Zimmermann & Cunningham, 1991, p. 7). Its potentiality 'urges the student to decide on the representations to be displayed in his/her interface so as to best support his/her investigation' (Buteau & Muller, 2007, p. 1117).

As many others, Zimmermann and Cunningham (1991) consider that visual representation is equivalent to graphical representation and is valuable if linked to algebraic and numeric representations. In a more recent publication, Sinclair and Yurita (2005) argue that "dynamic features of *Sketchpad* promoted a shift in the nature of questions and actions along the way, placing the visual in a central role in the process of exploration" (p. 5). In contrast to considering visual representation and visual thinking as a complete dichotomy, Giaquinto (2007) suggests taking them as a spectrum. Is it so? We may speculate on the existence of multiple visual representations within the same representation means (see Figure 2) to explore the boundaries between various representations.

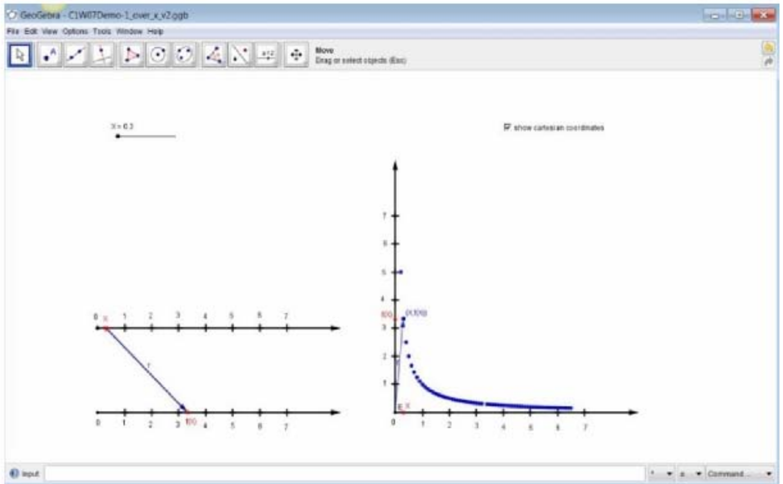

Figure 2. Two different graphical representations of the function f(x)=1/x created in GeoGebra

As seen in the figure 2, it is also possible to integrate other representational systems (algebraic representation in this example although numeric representation is also possible). The perception in this context could be more complex than one may assume. For example, what does f(x) on the second axis of dynagraphs represent? A number or a distance from zero?

Dynamic and Interactive Mathematics Learning Environments (DIMLE)

Having these questions in mind, we aim to conceptualize the features and affordances provided by DIMLE. Dynamism and interactivity seem to be most striking affordances of DIMLE and will be briefly discussed below.

Dynamism. New technologies may help Net Gen'ers in handling mathematical problems in new contexts and in new ways. Without these technological means, at the school level, many sophisticated mathematical results and procedures could not be explained, applied, and efficiently obtained. Dynamism used in software packages is not a pure dynamism in the sense we understand in science, rather it is a quick transition among discrete states of the mathematics system; states that depend on the user action (i.e., input), to present transition from one image to another as a continuous and dynamic process. As emphasized by Laborde (2007), "the meaning constructed by the individual is not only affected by the features of the representations available but also by the possible ways to use them. Mathematical activity requires manipulations of and operations on these representations" (p. 71).

Moreover, this perception of continual change in mathematical objects may affect the users' understanding of mathematics concepts and lead them to develop a new type of learning, what we call *dynamic learning.* Does dynamic learning and dynamic perception provide opportunities to learn mathematics in a natural way? According to those who write about "hyperproof" (Barwise & Etchemendy, 1991) and "visual proof" (Gravina, 2008), it is possible. Therefore, we suggest exploring dynamic learning as a theme in the working group meeting.

Interactivity. In the context of computer software, interactivity provides a kinaesthetic experience to the user and is often seen as some kind of a feedback to the user's action. The interaction mechanism is actually an action-reaction-new action iterative process, because when users perform action (i.e., virtual manipulation), they get reaction from the software (in terms of changing representation), and consequently perform another action (self-regulated action).

According to Buteau and Muller (2007), "the potentiality of interactivity encourages the student to make explicit the parameters that could play a role in the investigation of his/her conjecture or real-world situation in such a way that they are accessible from the interface" (p. 1117). In this way, through use of learning objects and simulations, students apply inductive reasoning and conjecturing; interactivity of the software may enhance students' inductive reasoning by providing opportunities to build their own mathematical understandings through experimentation.

The Rationale and Goals of Working Group

The goals of this proposed working group are to explore and discuss the features and affordances of DIMLEs and explorative learning and their dimensions. Furthermore, we shall suggest discussing the consequences of a possible shift in designing mathematics learning environments and formal mathematics learning in the context of college and university mathematics. We shall invite invite participants to explore the relevancy of explorative learning for N-Gen'ers.

Objectives

- To perform a comprehensive review of concepts regarding DIMLE and explorative learning
- To explore and conceptualize the features and affordances of DIMLE
- To explore and conceptualize the dimensions of explorative learning
- To identify possible effects of explorative learning in formal mathematics education.

Questions to frame our discussion

- What are characteristics of DIMLE?
- Which representations are more compatible with the nature of N-Gen'ers' learning?
- How can DIMLE alleviate algebra being a barrier to progressing through school mathematics?
- How can we observe or track the effects of visual versus symbolic representations in learning mathematics and their effects on concept development?
- How could the DIMLE be effective in explorative learning of concepts and processes?

Format for the Discussion Group Meeting

Day 1: Identifying the features and affordances of DIMLE

We propose to begin the discussion with a brief introduction of the topic, and then, ask the participants to reflect on these definitions in small groups and share their reflections with the whole group. The second hour of the discussion will be devoted to generating a workingdefinition for DIMLE and engaging the participants to brainstorm on the features of DIMLE by demonstrating examples from GeoGebra, Geometer's SketchPad, and Cabri.

- Introducing group members
- Introducing discussion group topics and goals
- Reviewing related literature on definitions and engaging participants in reflecting on these definitions
- Brainstorming on the features and affordances of DIMLEs
- Brainstorming on the various forms of representations used in DIMLEs
- Re-assessing the features of DIMLE suitable for addressing the needs of N-Gen'ers
- Conceptualizing the features and affordances of DIMLE.

Day 2: Discussing explorative learning and its components

We will start the meeting by briefing previous day's discussion and by outlining themes emerging from the discussion. Then, we will set the stage for the use of DIMLE in mathematics education, that is, explorative learning and its dimensions. We will demonstrate a couple of dynamic worksheets created in GeoGebra and ask participants to develop scenarios in their small groups on how to use these dynamic worksheets in learning mathematics and to share their scenarios with the whole group. During the last half hour, we will encourage the participants to discuss the connection of explorative learning to formal mathematics curriculum.

- Summarizing previous discussions
- Revisiting the concept of DIMLE
- Exploring the dimensions of explorative learning
- Brainstorming on the various forms of representation ideas in the same mode of representation such as dynagraphs and Cartesian coordinate systems in graphical representation
- Exploring the relation between explorative learning and the learning habits of N-Gen'ers.

Day 3: Summarizing the discussions and setting goals for the future collaboration opportunities

- Summarizing previous discussions
- Forming an international working group on the topic
- Setting an agenda for the future.

Possible Future Agenda Items

- Set up an international collaboration on DIMLE and explorative learning
- Create possible research questions to explore in the field
- Discuss opportunities to disseminate the outcome of the discussions, likely a book publication.

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MODELS AND MODELING WORKING GROUP

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In this Working Group, we will continue to reflect on a Models and Modeling Perspective (MMP) to understand how students, teachers, and researchers learn and reason about real-life situations. During the Conference, we will continue ongoing discussions of the role of Model-Eliciting Activities (MEAs) and model-centric Design Experiments in our work in Student Development, Teacher Development, Assessment, Curriculum Development, Problem Solving, and Research Design. We will also dedicate special attention to two emerging themes: the application of MEAs in Elementary Grades, and the exploration of a Mathematics of Collections/Collectives, bridging the perspectives offered by Statistical and Complex Systems approaches to modeling phenomena.

Introduction

The *Models & Modeling* Working Group of PME-NA has been a significant presence at the Conference since its inauguration in 1978. Over that time it has been successful in supporting the pursuit of high-quality research projects and programs, and it has acted as a means of convening a diverse set researchers to work collaboratively on larger projects and research problems. The purpose of the Group is to discuss and extend the ways in which models are used both to learn mathematics and applied science and to study those very learning processes in action.

In this context, models are considered as conceptual systems that are expressed using external representational media or notation systems, and that are used to construct, describe, or explain the behaviors of other systems. Models are thus conceptual and representational tools that have the additional advantage of providing illumination into how students, teachers, researchers, and other educators learn, develop, and apply relevant mathematical concepts (Lesh & Doerr, 2003; Lesh, Doerr, Carmona, & Hjalmarson, 2003).

The study of model development is a robust area of research: it appears in a variety of settings and is generally a complex, dynamic, and iterative process. In the classroom, the tradition of Model-Eliciting Activities (MEAs) has grown over the years, reaching an ever-broadening range of learning settings. Such MEAs generate rich information and *thought-revealing artifacts* that can both extend our understanding student thinking and learning and provide the basis for various nuanced forms of assessment. At other units of analysis for educational research, model development has also yielded fruit as a framework for inquiry. In particular, Design Experiments of various kinds have enabled researchers in the Models and Modeling Perspectives (MMP) tradition to study the types of models that students, teachers, and researchers develop (explicitly) to construct, describe, or explain mathematically significant systems that they encounter in their everyday experiences (Lesh, Hoover, Hole, Kelly, & Post, 2000).

We would like to invite participants to this Working Group to foster the continued development of a community of researchers and practitioners to enrich the field of research on the ways in which models are used in Problem Solving, Curriculum Development, Student

Development, Teacher Development, Assessment, and Research Design. In addition, this year we intend to give special emphasis to two emerging themes, described briefly below.

Emerging Theme One: Model-Eliciting Activities in the Elementary Grades.

The first theme concerns the application of Models, Modeling, and Model-Eliciting Activities (MEAs) in elementary grades. For example, Data Modeling with learners of this age group has been an area of particular focus for Lehrer and Schauble at Vanderbilt (see, e.g., Lehrer & Kim, 2009; Lehrer & Schauble, 2010). Also, during the past year, other leaders among the *Models & Modeling* Working Group have been using videoconferencing seminars to work with more than forty faculty members and graduate students in countries ranging from Turkey, Cyprus, and Israel, to Australia and Mexico – as well as the United States. The use of modeling activities with primary-grade children has been one of several key foci for this group. For primary-grade children and pre-schoolers, working group members' research shows that the use of children's *stories* often sets the context for productive inquiry–similar to the way "real life" contexts have functioned for older students. At PME-NA, we hope to share detailed results from this work and encourage expanded participation and collaboration across these discussions and research efforts.

Emerging Theme Two: A Unified Mathematics of Collections/Collectives.

Our second theme would bring together researchers in both statistics and complex systems/dynamic systems education, in order to explore common issues and themes concerning the "mathematics of collections/collectives" – that is, mathematics that concern the measures, models, data, and trends that describe collective or systemic phenomena. The consideration of complex systems is certainly not new to the Models and Modeling Working Group (see, for example, the proceedings of the Group at PME-NA XXIV and XXVIII for themes that engaged complexity as a subject area and as a fruitful perspective on educational systems, respectively); and a focus on statistical thinking has been strong in the Group from its beginnings. This year, however, we propose to explore ways in which *bridging* these two areas might be generative for both pedagogical and research purposes.

Examples may include work exploring student or teacher understanding of the emergence of exponential growth from probabilistic and multiplicative roots (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009; Confrey & Smith, 1995; Doerr, 2000); of the encapsulation and quantification of behaviors enacted at multiple levels (Wilkerson-Jerde & Wilensky, 2010); of inference from measures of sampled populations (Abrahamson & Cendak, 2006; Garfield, 2002; "Developing Reasoning about Informal Statistical Inference" [Special Issue], 2011); and of nonlinear behavior stemming from collective feedback, limiting behaviors, and other characteristics of complex systems ("Dominance of Linearity in Mathematical Thinking" [Special Issue], 2010).

The rationale for bringing together an interdisciplinary group to discuss the mathematics of collectives has both pragmatic and theoretical dimensions. At a pragmatic level, literate and effective citizens require increasingly sophisticated skills in reasoning about the behavior of collectives. While these skills are not currently well represented in the curriculum, our experience is that many of them can be introduced to students at quite a young age. Thus, it would be opportune for the community to propose and study options for integrating these subjects, for consideration and research. At a theoretical level, while there is a great deal of research on student learning within each of the individual components of the mathematics of collectives, many exciting avenues for inquiry might emerge in addressing the challenge to design and implement curricula and learning environments that address these components.

Continued focus on the historically-successful themes of the Group.

In addition to the two Emerging Themes described above, the Working Group will foster the ongoing discussion of the cross-cutting strands of inquiry— Problem Solving, Curriculum Development, Student Development, Teacher Development, Assessment, and Research Design—that have provided a framework for numerous collaborations among Group members over the years.

Advisory Group

Beyond the principal organizers, we propose to draw upon an Advisory Group to engage interested researchers, advocate participation, and support the Working Group's activity both during and after the conference sessions. This Advisory Group will include the following members (listed alphabetically):

- Lyn English, Queensland University of Technology
- Stephen Hegedus, University of Massachusetts, Dartmouth.
- Joan Garfield, University of Minnesota
- Beste Gucler, University of Massachusetts, Dartmouth
- Richard Lehrer, Vanderbilt University
- Bharath Sriraman, University of Montana
- Uri Wilensky, Northwestern University

During the course of the summer leading up to the conference, we will confer with members of this group, as well as with other leading researchers who have participated frequently in the Working Group's meetings, to ensure that the subgroups around each of our themes are well-supported and positioned for effective work both during the Conference and afterwards.

Goals for the Working Group at PME-NA Reno and Beyond

The Working Group will convene for three sessions during the Conference. In the first session, an overview of each of the theme-based subgroups will be provided, and members will select the subgroup in which they wish to participate. Subgroup leaders will facilitate initial planning discussions and identify objectives for the three-day discussion sessions. At the end of the first day, the subgroup leaders will gather to share objectives statements and identify points of common interest that may merit connections between the subgroups during the following two days. On the second day and third days' sessions of the Conference, the subgroups will work independently toward the objectives they have set for themselves, with subgroup leaders maintaining communications.

The Working Group will reconvene as a whole for the final two hours of the third day's session, to share findings and opportunities for collaboration following the Conference. In addition, mailing lists and subgroup Wikis will be put in place by the organizers during the Conference. These tools will be employed during the work sessions and will also be explicitly demonstrated in the final session, to ensure that all members are familiar enough with them to employ them as supports for the work that will continue after the Conference.

Accomplishments of the *Models & Modeling* Working Group

The Models & Modeling Working Group has been one of PME-NA's most active working groups—beginning with the very first PME-NA at Northwestern University in 1978. The Group has a long history of encouraging research collaborations among group members — and of including junior colleagues in a variety of research and development activities. Consequently, since the founding of both PME and PME-NA, models and modeling perspectives (MMP) have provided useful frameworks for a large number of books, journal publications, funded projects, and conference presentations which featured collaborations among multiple working group members.

During the past year, leaders in the Models and Modeling Working Group have been using videoconferencing seminars to work with more than forty faculty members and graduate students in countries ranging from Turkey, Cyprus, and Israel, to Australia and Mexico – as well as the United States. The focus of this work has been on four on areas: (a) modeling activities for primary grade children, (b) modeling activities focusing on integrated approaches to mathematics for high school students, (c) modeling-based activities for teacher development, and (d) modeling-focused design research methodologies. We intend for such fruitful and cross-cutting collaborations to continue into the future, and to continue to feed into the Working Group community's annual discussions.

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WISDOM[e]: QUANTITATIVE REASONING AND MATHEMATICAL MODELING WORKING GROUP

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The Quantitative Reasoning and Mathematical Modeling (QRaMM) working group is one of three research strands initiated by the Wyoming Institute for the Study and Development of Mathematical Education (WISDOM[e]). QRaMM brings together researchers from multiple universities across the country to share ideas and conduct research on quantitative reasoning and modeling within an interdisciplinary context. As part of the work of QRaMM and in collaboration with the National Science Foundation MSP Pathways in Environmental Literacy Project, a national virtual seminar QRaMM in Science was initiated in Spring 2011. The seminar engaged 15 speakers from across the country in the online video conference, including experts in quantitative reasoning and modeling from the fields of science, computational science, mathematics, science education, and mathematics education. The context for the discussion of quantitative reasoning and modeling was environmental literacy. The issues in psychology of mathematics education for the seminar included students' development of quantitative reasoning and mathematical modeling from 6th to 12th grade, creation of parallel QRaMM learning progressions for those being created by the MSP Pathways project for environmental literacy, and study of the impact and interplay of QR and modeling on students' development of environmental literacy. The national experts informed a QRaMM Science research team consisting of faculty and graduate students who developed Environmental Science QR interview protocols based on MSP Pathways learning progressions. Data from these protocols will lead to the creation of Environmental Science QR assessments and research-based professional development that supports learning environments that lead to environmentally literate citizens capable of making informed decisions about the grand challenges facing the next generation. Participants in the QRaMM working group will join members of the QRaMM research team in critiquing environmental science QRaMM learning progressions, interpret data on QR in the sciences, vet QRaMM assessment items, and debate the QRaMM theoretical framework.

History of WISDOM[e]

A core membership for this new PME-NA Working Group was established through the initial invitational **Planning Conference for WISDOM[e]** conducted September 8-10, 2010 at the University of Wyoming. The background context for that conference is the establishment of three collaborative, interdisciplinary Research Teams connected to four new Ph.D. program research identities: quantitative reasoning, mathematical modeling, technology tools and applications, and lived/living mathematical experience. Each of the teams consists of UW mathematics education, mathematics, and educational cognate faculty, mathematics education doctoral students, and four-to-five young, active Mathematics Education researchers from a variety of national and international universities. Following four invited plenary papers prepared and presented at the conference by key active senior researchers, each team met to frame an initial program of collaborative work. The QRaMM (Quantitative Reasoning and Mathematical

Modeling) Team currently includes faculty from multiple universities and a cadre of mathematics education graduate students at the University of Wyoming.

The QRaMM research team is working in collaboration with the National Science Foundation Pathways in Environmental Literacy Project. The Pathways Project is focused on creating learning progressions for three content strands that are fundamental to the development of environmentally literate citizens from sixth to twelfth grades: carbon cycle, water cycle, and biodiversity. The Pathways QR Theme team is exploring the impact of quantitative reasoning on the development of environmental literacy. These two teams have come together through an online virtual seminar, where researchers from mathematics, mathematics education, science, and science education discuss QR and mathematical modeling from their respective points of view. The research papers and presentations are informing the development of QR and mathematical modeling learning progressions, as well as assessment items for clinical interviews and the creation of written assessments of QR abilities in the context of environmental science.

Through the opportunity to conduct Working Group sessions at PME-NA, we seek an important venue to continue and expand discussions of the background perspectives and issues related to investigations of quantitative reasoning and modeling, to share information, issues, and problems related to ongoing research, to promote interest and potential participation in furthering these and other disciplined inquiries into these phenomena, and to provide continuing support to the team members to collaborate within and across ongoing and future research.

Issues in Psychology of Mathematics Education

The primary research question for the Pathways project is: What are the characteristics of a 6th to 12th grade learning progression aimed at developing environmentally literate citizens? The purpose of the QR Theme is to determine the quantitative reasoning aspects of the learning progression leading to the primary QR research question: What are the essential quantitative reasoning abilities that are required for the development of environmental literacy? Specific research questions include the following.

- How does quantitative reasoning integrate and interact with environmental literacy?
- What is the relationship between learning progressions for quantitative reasoning and environmental literacy – should they be integrated or is a separate learning progression for quantitative reasoning required?
- What is the current state of quantitative reasoning supporting environmental literacy at the 6th grade level (defined as the lower anchor for the learning progression)? What is the required level of quantitative reasoning for an environmentally literate citizen at the 12th grade level (defined as the upper anchor)?
- How does the learning progression inform professional development supporting the development of environmental literacy?

The Pathways project is based on the theoretical framework of learning progressions. The Consortium for Policy Research in Education defines learning progressions as follows:

> Learning progressions are hypothesized descriptions of the successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time (Corcoran, Mosher, & Rogat, 2009, pg 37).

The panel identified essential elements of learning progressions to be:

- Upper Anchor: target performance or leaning goals which are the end points of learning progression and are defined by societal expectations, analysis of the discipline, and requirements for entry into the next level of education.
- Progress Variables: dimensions of understanding, application, and practice that are being developed and tracked over time.
- Levels of Achievement: intermediate steps in the developmental pathway(s) traced by a learning progression.
- Learning performances: tasks students at a particular level of achievement would be capable of performing.
- Assessments: specific measures used to track student development along the hypothesized progression.

The Pathways learning progressions for environmental literacy are based on research in science education and cognitive psychology, foundational and generative disciplinary knowledge and practices, and strive for internal conceptual coherence. The QR in environmental literacy frameworks build on these characteristics, incorporating mathematical and statistical frameworks.

The Pathways learning progressions have a lower anchor which is the typical accounts of environmental issues given by students at the upper elementary and middle school level (Anderson, 2009). These accounts are empirically tested through a cyclic research process of clinical interviews informing the learning progression and leading to the development of written assessments given on a large scale. The Pathways learning progressions upper anchor is based on experts views of what a scientifically literate citizen should know and be able to do by the 12th grade. The upper anchor is much like a NSTA or NCTM standard, but learning progressions differ from standards in that the lower anchor and intermediate achievement levels are research-based, reflecting the actual trajectory of student learning. A limited number of achievement levels (4 or 5) are identified as plateaus in students' development of more sophisticated ways of thinking about enduring understandings, concepts, and processes. The progress variables for the Pathways project at the meta-level are the carbon cycle, water cycle, and biodiversity, which are considered areas in which students must develop conceptual understanding if they are to become environmentally literate citizens. Within each of these areas progress variables are identified. For example, in the carbon strand the progress variables are generation-photosynthesis, transformation-food chain/web/biosynthesis, oxidation-cellular respiration, and oxidation-combustion. Learning performances are exemplars drawn from the clinical interviews and written assessments which demonstrate student responses at different achievement levels. Learning progression matrices are created by cross tabulating achievement levels (rows in matrix) with progress variables (columns in matrix). A number of learning progressions in science are currently under development including: tracing carbon in ecological systems (Mohan, Chen, & Anderson, 2009), particle model of matter (Merrit, Krajcik, & Swartz, 2008), modeling in science (Schwarz, Reiser, et. al., 2009), genetics (Duncan, Rogat, & Yarden, 2009), chemical reactions (Roseman, et. al., 2006), data modeling and evolution (Lehrer & Schauble, 2002), explanations and ecology (Songer, Kelcey, & Gotwals, 2009), buoyancy (Kennedy & Wilson, 2006), atomic molecular theory (Smith, Wisner, Anderson, & Krajcik, 2006), and evolution (Cately, Lehrer, & Reiser, 2005). Examples of three of these learning progressions are provided in (Corcoran, Mosher, & Rogat, 2009).

The QR Theme of the Pathways project is using the learning progressions theoretical framework to research the impact of QR on students' development of environmental literacy.

The overarching goal is to study the capacity of students to understand and participate in evidence-based discussions of socio-ecological systems. Three overarching progress variables are hypothesized: quantitative literacy which consists of arithmetic understandings supporting science, quantitative interpretation which is the process of interpreting scientific models to determine trends and make predictions, and quantitative modeling which is the creation of models by the student. Within each of these areas four progress variables are identified that are hypothesized to be critical QR for science (Table 1).

Progress Variables	Quantitative Literacy	Quantitative Interpretation	Quantitative Modeling

Components	Numeracy
	• Number Sense
	• Small/large Numbers
	• Scientific Notation
	Measurement
	• Accuracy-precision
	• Estimation
	• Dimensional Analysis
	• Units
	Proportional Reasoning
	• Fraction
	• Ratio
	• Percents
	• Rates/Change
	• Proportions
	Basic Prob/Stats
	• Empirical Prob.
	• Counting
	• Central Tendency
	• Variation
		Multiple Representations (rule of 4)
		• Written/oral
		• Tables
		• Graphs/diagrams
		• Equations
		  • Linear
		  • Quadratic
		  • Power
		  • Exponential
		Science models
		Statistics
		• Normal Distribution Model
		• Linear Regression Model
		• Statistical plots
		• Correlation
		• Causality
		• Z-scores
		• Confidence Intervals
		Logarithmic Scales
			Logic
			Problem Solving
			Modeling
			• Normal Distribution Model
			• Regression Model
			  • linear
			  • polynomial
			  • power
			  • exponential
			  • logarithmic
			• Logistic Growth
			• conceptual science models
			• table and graph models
			Inference
			• Inference
			• Hypothesis testing

Table 1. QR Progress Variables

The development of learning progressions is an iterative process typical of design-based research. The Pathways environmental literacy learning progressions began with hypothetical frameworks based on theories about reasoning about processes in socio-economic systems, including discourse, practices, and knowledge, as well as linking processes between lower and upper anchors. Challenges for the QR Theme researchers are to determine which of these theories carry over to QR aspects of environmental literacy and to discover potential theories that are more QR-centric. For example, students achieving the upper anchor in the environmental literacy learning progression should function as informed decision makers within the socio-economic system at three levels: discourse, practices, and knowledge. "Knowledge is embedded in practices, which in turn are embedded in discourses" (Anderson, 2009). We all participate in multiple discourses that associate us with communities of practice (Gee & Green, 1998) and as we gain understanding of a phenomena as our discourse around it matures. The lower anchor discourse for the environmental literacy progression is force dynamic, relying on students' theory of the world with a focus on actors, enablers, actor's purposes, conflicts between actors, and settings for the actions (Pinker, 2007; Talmy, 2003). At the upper anchor scientific discourse is essential, with students moving away from actors in settings to laws that govern the work of systems. Certainly there is a parallel development of mathematical and statistical discourse that is essential for quantitative reasoning. There are four practices that are essential for environmentally responsible citizenship: inquiry (What is the problem and who do I trust?), accounts-explaining (What is happening in the system?), accounts-predicting (What are the consequences of my course of action?), and deciding (What will I do?). At the lower anchor explaining and predicting practices are expressed in force dynamic discourse, while at the upper anchor they are expressed using the language and theory of scientific discourse. These practices reflect the public roles (voter, advocate, volunteer) and private (consumer, owner, worker, learner) roles in which decisions are made. Informed explaining and predicting practices often require QR, which allows for connecting observations to patterns and models, as well as analyzing data from a scientific perspective. Finally knowledge is embedded within discourses and practice. At the lower anchor students focus on knowing facts, while at the upper anchor scientific knowledge is applied to create coherent systems of observations, patterns, and models which serve as the basis for scientific inquiry and accounts. QR parallels this trajectory from knowing mathematical or statistical algorithms to an understanding of enduring concepts that allow one to analyze problems quantitatively.

A framework for the Pathways learning progression consists of progress variables (matrix columns), levels of achievement (matrix rows), learning performances (content of matrix cells), and linking processes which are common processes that are recognized by students at all levels of achievement (Anderson, 2009). The carbon framework in Figure 2 provides an example of linking processes within the carbon strand. What are the linking processes for QR? How do they interact with the environmental literacy linking processes? Another aspect of the Pathways learning progressions which is highly quantitative in nature is scaling. Students at the lower anchor often function at a macroscopic or "individual" scale in which they view the world from their sensory purview. Hence, their accounts of environmental issues are based within their daily perceptions at this scale. Moving students to the upper anchor requires that they scale up to a global view, as well as scale down to a microscopic view and even an atomic view. Regardless of the scale, moving up or down the scale becomes intensively quantitative.

Figure 2: Carbon Linking Processes

Carbon-transforming processes	Generating organic carbon	Transforming organic carbon			Oxidizing organic carbon		
Scientific accounts	Photosynthesis	Biosynthesis	Digestion	Biosynthesis	Cellular respiration		Combus-tion
Linking processes	Plant growth		Animal growth		Breathing, exercise	Decay	Burning
Informal accounts	Plants and animals as actors, accomplishing their purposes in life, using their abilities, if their needs (food, water, sunlight, and/or air) are met					Natural process in dead things	Flame as actor consuming fuel

QRaMM Reflection Tasks

The three QRaMM working group sessions will actively engage the participants in discussion of the research questions stated above. Each session will begin with a 15 minute plenary presentation by a member of the QRaMM research team to pose research issues and problems for discussion. Seminal readings related to the presentation will be provided to all participants in the first session, serving as support materials for discussions in later sessions. Breakout sessions for four working sub-groups will be organized around task-oriented discussions. The last session will provide plenary summarizing presentations and report out discussions by the working groups.

The four sub-groups will focus on the following domains of discussion:

1. Framing problems and questions for research on quantitative reasoning and modeling related to learning progressions and assessment.
2. Addressing issues and concerns for research methodologies to be used.
3. Discussing techniques for using and transforming data to describe, analyze, and interpret in ways that will illuminate and inform.
4. Discussion approaches to reporting what is found and to applying to inform improved educational practice related to interdisciplinary aspects of quantitative reasoning and mathematical modeling.

QRaMM Follow-up Activities

We are proposing a follow-up national seminar on QRaMM with virtual presentations by the participants of this working group. With the assistance of the WISDOM^e research group we are proposing a monograph on QRaMM learning progressions and interdisciplinary teaching issues. The monograph would solicit articles from the participants of the working group as well as the WISDOM^e and Pathways research teams.

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NUMBER TALKS: PUTTING MATHEMATICAL REASONING IN THE CENTER OF MATHEMATICAL PRACTICE

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Number Talks are an innovative pedagogical approach that has great potential in shifting mathematics teaching and learning practices to support students' conceptual reasoning. The proposed discussion group will examine the features of Number Talks, how students and teachers learn through this approach, and the variations of its use in classrooms, as well as in pre-service and in-service teacher education settings. The goal of the discussion group is to provide mathematics educators and mathematics teacher educators a context for sharing their work with Number Talks and deepen their understanding of this pedagogical approach.

In 2000, the Mathematics RAND committee proposed that mathematical practices, such as reasoning, generalizing, and making connections, should be at the center of instruction (RAND, 2002), and these practices are now enshrined in the Common Core State Standards (2010). However, engaging students in mathematical practices places a new demand on U.S. teachers. Despite research evidence outlining the need to engage students in activities that also develop conceptual understanding (Fosnot & Dolk, 2001; Fuson, 1992; NCTM 2003), computation in elementary and middle school classrooms has traditionally been taught through memorization of number facts and repeated practice of procedures. One of the reasons that teachers continue to focus on memorization of number facts is that they have not received sustained opportunities for learning how to engage students differently (Ball & Cohen, 1999). Thus, teachers may feel at a loss when trying to teach mental computation and numerical reasoning in a more conceptual manner.

One of the innovative pedagogical approaches for conceptual numerical reasoning is called 'Number Talks', a method that has been used in a small number of cases but with extremely impressive results (Parish, 2010, Boaler, 1998). This method begins with teachers presenting a simple, context free number problem to their class of students. The number problems can target many different levels. For example second grade students may feel challenged by a problem such as 38 + 55; whereas, 8th grade students might initially struggle with problems such as 48 x 27 or 12% of 80. After teachers present a problem to students, they ask students to work out the answer in their heads or on paper but with no discussion. They then elicit different student solution strategies. The problems generally produce multiple solution methods in any class of students. Teachers continue by facilitating student discussion; for example, they may ask students to discuss the ways the methods are similar and different or explain how they could represent the methods visually. Additionally, the teachers may show student methods using numerals or drawings (e.g., number lines or geometric representations) to highlight common mathematical ideas across the methods and ask students to share connections they notice among the different mathematical ways of thinking. In the process, the core mathematical concepts are highlighted and discussed, while students become agents of numerical reasoning. As students work to find answers mentally and then discuss the connections between different methods, they develop both computational fluency and numerical reasoning. For teachers, the discussions offer important opportunities to hear their students' thinking (Schifter, 1998) and to extend their students' understanding to the next conceptual level, working at and with the students' 'zone of proximal development' (Vygotsky, 1978). As teacher educators and mathematics educators, we are interested in understanding the features of this pedagogical approach that advance and enrich students' conceptual pathways (Fosnot & Uittenbogaard, 2007; Huffered-Ackles, Fuson, & Sherin, 2004).

Goals and Activities of the Discussion Group

In this proposed discussion group, we will examine and explore the features of Number Talks, unpack their benefits and challenges, and discuss their potential for mathematics teaching and learning. Different video cases will be presented, and varied Number Talk formats (e.g., Number Strings, Math Talk) will be compared and analyzed. The organizers of this discussion group will present examples of using Number Talks both in pre-service teacher education courses as well as in in-service professional development programs. The goal of the discussion group is for mathematics educators and mathematics teacher educators to have the context to share their work with Number Talks, to deepen their understanding of this pedagogical approach, and to plan for future research on the topic.

The guiding questions for the discussion group are as follows:

1. What are the features of Number Talks that support students' numerical reasoning?
2. How do teachers learn to focus on students' mathematical thinking through Number Talks?
3. What are the different formats of Number Talks, and how do they support students' numerical reasoning differently?
4. What are the challenges of using Number Talks in classrooms?
5. 5. How are Number Talks used to support pre-service and in-service teachers' learning?

Work and Research with Number Talks

Number Talks have been successfully taught to teachers, as well as to teacher education students, with clear evidence of increased student learning and engagement. The method was also used in a research project aimed at re-engaging failing middle school students (Boaler et al., *in press).* Researchers in the middle school study found that students who learned through 'Number Talks' changed their views of mathematics, reporting that they had never before realized that mathematics problems could be solved with more than one method. The students also increased their achievement significantly and improved their subsequent grades in algebra courses (Boaler et al., *in press).* 'Number Talks' represent what Grossman & MacDonald (2008) have called a 'high leverage practice' as they are linked to increased student understanding, but they also have the considerable advantage of being easy to learn (as opposed to many other high leverage teaching practices). They can be added to any curriculum scheme, as they take about 15 minutes of class time.

The current research on Number Talks is limited (Parish, 2010, Boaler, 1998), but there is a wide body of research related to the goals and practices encompassed by Number Talks. We know that students who have learned to reason with numbers in a variety of ways demonstrate a deepened understanding of place value (Fosnot & Dolk, 2001; Kamii, 2000), that students with differing backgrounds are supported by learning communities where talking mathematics is the norm (Spillane, 2000; Hufferd-Ackles, Fuson, & Sherin, 2004), and that equitable outcomes are promoted when students engage in communication and reasoning (Ladson-Billings, 1997; Martin, 2009).

Number Talks have the potential to help teachers shift their teaching to focus more on student thinking and on mathematical connections, while helping students learn foundational mathematics with greater understanding.

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